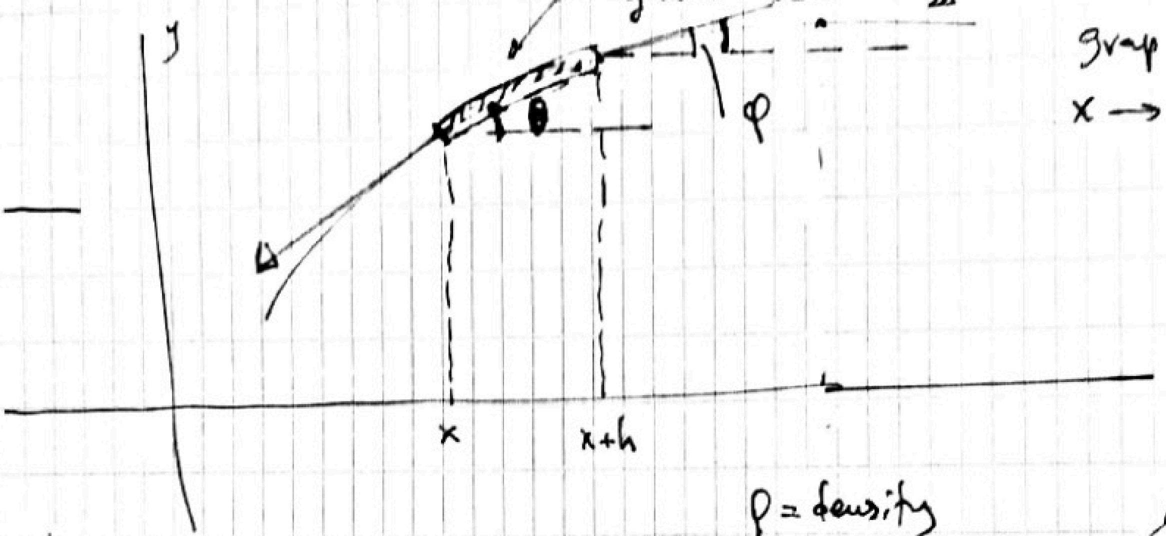
Lecture 17

The Wave Equation

D'Alambertian

(1) $$\Box u := u_{tt} - \Delta u = 0$$

A. Derivation



ρ = density

tension

Newton: $F = \rho h u_{tt}$ F = total force in y-direction

$$= T\sin\varphi - T\sin\theta$$

small deformations of chord $\longrightarrow$ $\simeq T\tan\varphi - T\tan\theta$

$$= T\,[u_x(x+h,t) - u_x(x,t)]$$

$$\simeq T\, u_{xx}(x,t)$$

$(h \to 0)$

$\therefore$

(2)
$$\begin{cases} u_{tt}(x,t) = \dfrac{T}{\rho} u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x) & \text{(initial position of chord)} \\ u_t(x,0) = h(x) & \text{(— " — velocity — " —)} \end{cases}$$

B. Relationship with the $\boxed{\text{simplest}}$ equation describing wave propagation

The Transport equation

$$(3) \quad \begin{cases} v_t + c v_x = 0 \\ v(x,0) = \varphi(x) \end{cases}$$

travelling wave - no change of shape - translation.

$v(x,t) = \varphi(x-ct)$

Take $c=1$

$$(4) \quad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = 0$$

Set

$$(5) \quad v(x,t) := \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x}$$

$\therefore \quad (4) \Leftrightarrow$

$$(6) \quad \begin{cases} v_t + v_x = 0. \\ v(x,0) = u_t(x,0) - u_x(x,0) = h(x) - g'(x) =: a(x) \end{cases}$$

Solving (6) $\Longrightarrow$ $v(x,t) = a(x-t)$ (7)

Thus

(8) $$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = v(x,t) = a(x-t)$$

$\Rightarrow$ (See Exercise below)

(9) $$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} a(y)\,dy + b(x+t)\,, \quad b(x) = u(x,0)$$ $g(x)$

$$= \frac{1}{2}\int_{x-t}^{x+t} (h(y) - g'(y))\,dy + g(x+t)$$

$$= \frac{1}{2}\left[g(x+t) - g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy$$

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Exercise

$$\begin{cases} w_t + c\,w_x = f(x,t)\,, & x\in\mathbb{R},\ t>0\,, \quad f,\varphi\ C^1 \text{ fn's} \\ w(x,0) = \varphi(x) \end{cases}$$

Change of variables : $\xi = t$, $\eta = x - ct$, $\hat{u}(\xi,\eta) = u(x,t)$

$$\therefore \quad \tilde{u}_\xi(\xi,\eta) = f(\eta + c\xi, \xi)$$

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Rk

The general solution of $u_t - u_x = 0$ is $F(x+t)$ travelling wave, left

— " — $u_t + u_x = 0$ is $G(x-t)$ T.W. right

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\left[F(x+t)+G(x-t)\right]$$
$$=\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)G(x-t)$$
$$=\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)G(x-t)$$
$$=0.$$

$\therefore$ General solution of W. E. on $\mathbb{R}$.

C. The Half-Line $\quad x \geq 0$

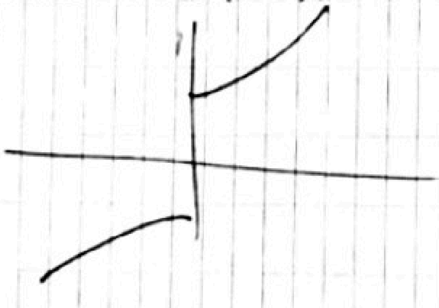
(10) $\quad u_{tt}-u_{xx}=0, \quad (0,\infty)\times(0,\infty)$

$$\begin{cases} u(x,0)=g(x) \\ u_t(x,0)=h(x) \end{cases} \quad C^1 \text{ fn's}$$

reducing it to the whole line case

For solving by reflection need $\boxed{g(0)=h(0)=0}$

Otherwise discontinuity!



$$\tilde{g}(x)=\begin{cases} g(x), & x\geq 0 \\ -g(-x), & x\leq 0 \end{cases}$$

$$\tilde{h}(x)=\begin{cases} h(x), & x\geq 0 \\ -h(-x), & x\leq 0 \end{cases}$$

(11) $\displaystyle \tilde{u}(x,t)=\frac{1}{2}\left[\tilde{g}(x+t)+\tilde{g}(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t}\tilde{h}(y)\,dy$

$$\tilde{u}(0,t) = \frac{1}{2}\left[\tilde{g}(t) + \tilde{g}(-t)\right] + \frac{1}{2}\int_{-t}^{t} \tilde{h}(y)\,dy = 0$$

$$\tilde{u}(x,0) = \frac{1}{2}\left[\tilde{g}(x) + \tilde{g}(x)\right] + \frac{1}{2}\int_{x}^{x} h(\tilde{y})\,dy = \tilde{g}(x) = g(x), \quad x \geq 0$$

$$\tilde{u}_t(x,0) = \frac{1}{2}\left[\tilde{g}'(x) - \tilde{g}'(x)\right] + \frac{1}{2}\left[\tilde{h}(x+t) + \tilde{h}(x-t)\right]\Big|_{t=0} = \tilde{h}(x)$$

$(11) \Rightarrow$

$$(12) \quad \begin{cases} u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy, & x \geq t \geq 0 \\ u(x,t) = \frac{1}{2}\left[g(x+t) - g(t-x)\right] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\,dy, & 0 \leq x \leq t \end{cases}$$

D. The Method of Spherical Means

$$(13) \quad \begin{cases} u_{tt} - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \quad u \in C^m(\mathbb{R}^n \times [0,\infty)) \\ u(x,0) = g(x), & x \in \mathbb{R}^n \qquad m \geq 2 \\ u_t(x,0) = h(x) \end{cases}$$

Define

$$(14) \quad \begin{cases} U(x;r,t) := \fint_{\partial B(x,r)} u(y,t)\,dS_y \\ G(x;r) := \fint_{\partial B(x,r)} g(y)\,dS_y \\ H(x;r) := \fint_{\partial B(x,r)} h(y)\,dS_y \end{cases}$$

Ausatz (Euler-Poisson-Darboux equation)

Let $u(x,t)$ satisfy (13). Then $U \in C^m(\overline{\mathbb{R}}_+ \times [0,\infty))$

$$(15) \quad \begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 &, \mathbb{R}_+ \times (0,\infty) \\ U = G, \ U_t = H &, \mathbb{R}_+ \times \{t=0\}. \end{cases}$$

1. As in the harmonic case

$$(16) \quad U_r(x;r,t) = \frac{r}{n} ⨍_{B(x,r)} \Delta u(y,t)\,dy$$

$\Longrightarrow$

$$(17) \quad \lim_{r\to 0} U_r(x;r,t) = 0$$

$$U_{rr} = \frac{d}{dr}\left[\frac{r}{n} ⨍_{B(x,r)} \Delta u \, dy\right]$$

$$= \frac{d}{dr}\left[\frac{1}{c r^{n-1}} \int_{B(x,r)} \Delta u \, dy\right]$$

$$= \frac{1}{c r^{n-1}} \int_{\partial B(x,r)} \Delta u \, dS_y + \frac{1-n}{c r^n} \int_{B(x,r)} \Delta u \, dy$$

$$= ⨍_{\partial B(x,r)} \Delta u \, dS_y + \left(\frac{1}{n} - 1\right) ⨍_{B(x,r)} \Delta u \, dy$$

$$\lim_{r\to 0} U_{rr}(x;r,t) = \Delta u(x,t) + \left(\frac{1}{n} - 1\right)\Delta u(x,t)$$

$$= \frac{1}{n}\Delta u(x,t)$$

2. $$U_r(x;r,t) = \frac{r}{n} \fint_{B(x,r)} \Delta u \, dy$$

$$\overset{(17)}{=} \frac{r}{n} \fint_{B(x,r)} u_{tt}\, dy$$

$$= \frac{1}{n\alpha(n)} \frac{1}{r^{n-1}} \int_{B(x,r)} u_{tt}\, dy$$

$$r^{n-1} U_r = \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt}\, dy$$

$$\Rightarrow (r^{n-1}U_r)_r = \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt}\, dy = r^{n-1} \fint_{\partial B(x,r)} u_{tt}\, dy$$

$$= r^{n-1} U_{tt}$$

□

<u>The Kirchhoff Solution</u> $n=3$

$$\tilde{U} = rU, \quad \tilde{G} = rG, \quad \tilde{H} = rH$$

<u>Claim</u>:

(18) $$\tilde{U}_{tt} - \tilde{U}_{rr} = 0$$

Verification

$$(19) \quad \tilde{U}_{tt} = r\,U_{tt} = r\left[U_{rr} + \frac{2}{r} U_r\right] \qquad (n=3)$$

$$= r U_{rr} + 2 U_r$$

$$= \left(U + r U_r\right)_r = \tilde{U}_{rr}$$

Also

$$(20) \quad \begin{cases} \tilde{U}(x;r,0) = \tilde{G}(x,r), \quad \tilde{U}_t(x,r,0) = \tilde{H}(x,r,0) \\ \tilde{U}(x;0,t) = 0 \qquad ((17)). \end{cases}$$

Apply (12).

$0 \leq r \leq t$

$$(21) \quad \tilde{U}(x;r,t) = \frac{1}{2}\left[\tilde{G}(r+t) + \tilde{G}(r-t)\right] + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(y)\,dy$$

$$u(x,t) = \lim_{r\to 0} U(x;r,t)$$

$$= \lim_{r\to 0} \frac{\tilde{U}(x;r,t)}{r}$$

$$= \tilde{G}'(t) + \tilde{H}'(t)$$

$\tilde{G} = rG, \quad \tilde{H} = rH$

(9)

$$u(x,t) = \frac{\partial}{\partial t}\left(t \fint_{\partial B(x,t)} g\,dS\right) + t \fint_{\partial B(x,t)} h\,dS$$

Utilizando

$$\fint_{\partial B(x,t)} g(y)\,dS_y = \fint_{\partial B(0,1)} g(x+tz)\,dS_z$$

$\Rightarrow$

$$\frac{\partial}{\partial t} \fint_{\partial B(x,t)} g(y)\,dS_y = \fint_{\partial B(0,1)} \nabla g(x+tz)\cdot z\,dS_z$$

$$= \fint_{\partial B(x,t)} \nabla g(y)\cdot \frac{y-x}{t}\,dS_y$$

$\therefore$

$$\boxed{u(x,t) = \fint_{\partial B(x,t)} \left[t\,h(y) + g(y) + \nabla g(y)\cdot(y-x)\right]dS_y}$$

$x \in \mathbb{R}^3,\ t > 0$