Lecture 16

Mean Value formulas for the Heat Equation

6-10 translate

1. The Heat Ball

$n = 1$

$$\left\{ (x,t) \;\middle|\; \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}} \geq \frac{1}{r} \right\} =: S_r$$

$$y = f_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} = \Phi(x,t)$$

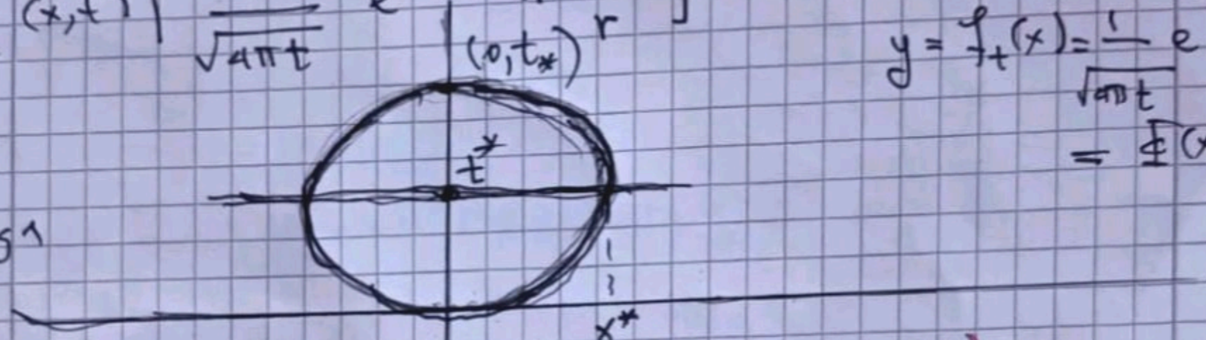


Fig 1

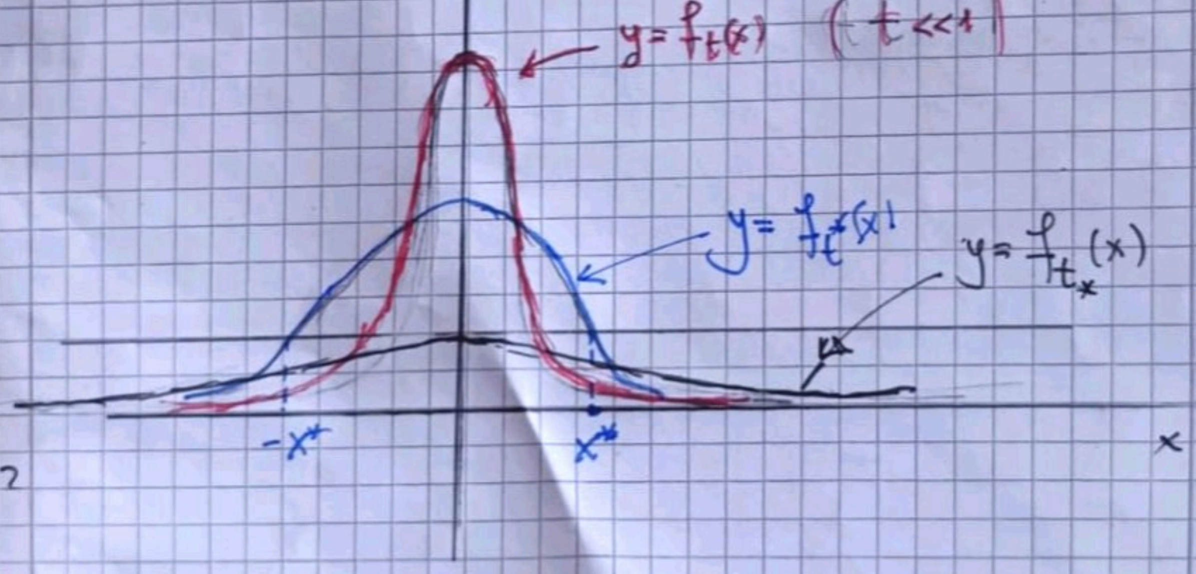


Fig 2

$t_* =$ maximal time, $\{0\} \times (0, t_*) \in S_r$

$$\frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}} \geq \frac{1}{r} \iff \frac{1}{\sqrt{4\pi t}} \geq \frac{1}{r} e^{\frac{|x|^2}{4t}} \geq \frac{1}{r} \Rightarrow \boxed{t^* = \frac{r^2}{4\pi}}$$

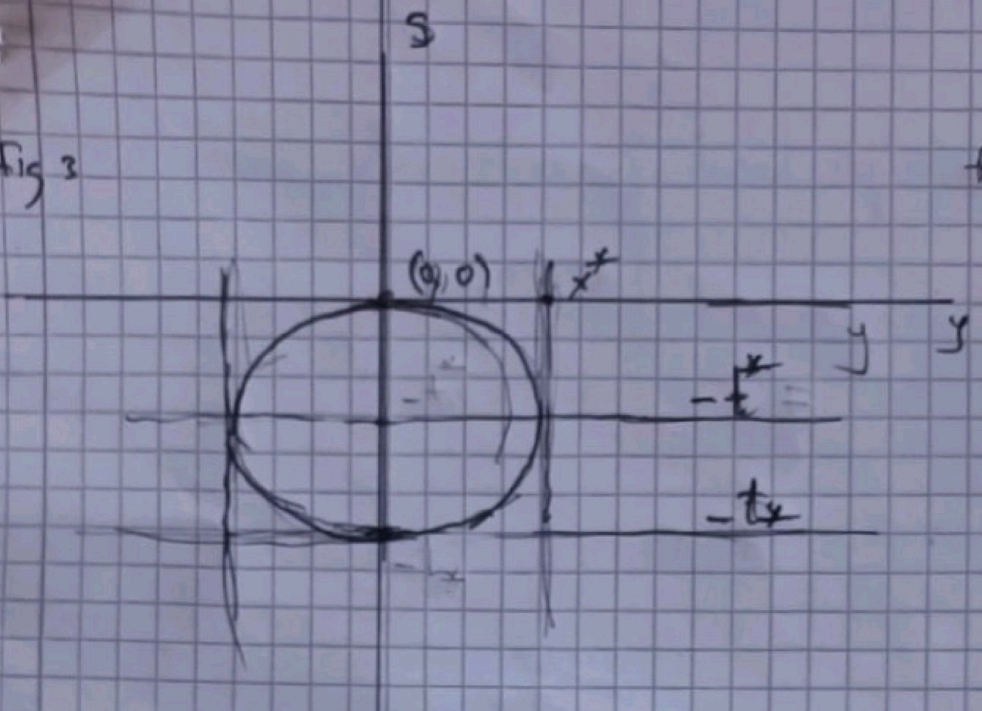
$t^* =$ time that maximizes $|x|$, in $(0, t_*)$

$$\frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}} \geq \frac{1}{r} \iff e^{-\frac{|x|^2}{4t}} \geq \frac{\sqrt{4\pi t}}{r} \iff -|x|^2 \geq 4t \ln\left(\frac{\sqrt{4\pi t}}{r}\right)$$

$$\iff |x|^2 \leq 4t\left(-\ln \frac{\sqrt{4\pi t}}{r}\right)$$

$$|x|^2 \leq \max_{t \in (0,t_*)} [\quad] = 4t^*\left(-\ln \sqrt{4\pi t^*}\right) = |x^*|^2$$

Fig 3



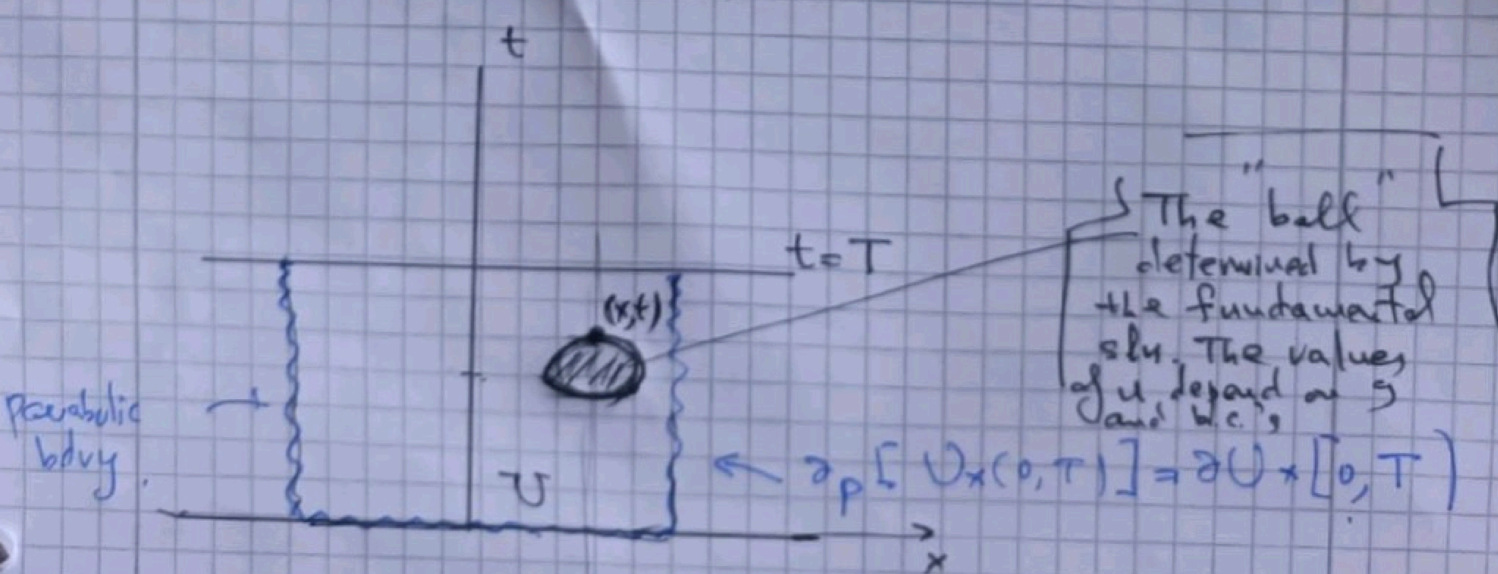
$t \to -t$

$$E(0,0,r) := \left\{ (y,s) \;\middle|\; s \le 0,\ \Phi(y,-s) \geq \frac{1}{r} \right\}$$

2. n-dim

$$E(x,t;r) := \left\{ (y,s) \;\middle|\; s \le t,\ \Phi(x-y,t-s) \geq \frac{1}{r^n} \right\}$$

$$(*) \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u, & (x,t) \in U \times (0,T), \quad U \subset \mathbb{R}^n \text{ bded} \\ u(x,0) = g(x) \end{cases}$$



$$\overline{U\times(0,T)} - \partial_p[U\times(0,T)] = U\times(0,T]$$

<u>Th</u> (Mean Value thm for heat equation)

[N. Watson, 1973]

Suppose $u \in C_1^2(U \times (0,T])$

Then

(2) $$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}\, dy\, ds$$

$$\forall\ E(x,t;r) \subset U \times (0,T]$$

<u>Pf</u>

1. Equation is invariant under translations in x and t, hence enough to establish

(3) $$u(0,0) = \frac{1}{4r^n} \iint_{E(0,0;r)} u(y,s) \frac{|y|^2}{s^2}\, dy\, ds =: \frac{1}{4} \phi(r)$$

2. <u>Change of Variables</u> (parabolic scaling)

$y = \lambda \bar{y}$, $s = \lambda^2 \bar{s}$

(4) $$\phi(r) = \frac{1}{r^n} \iint_{E(0,0;r)} u(y,s) \frac{|y|^2}{s^2}\, dy\, ds$$

$$= \frac{1}{r^n} \iint_{E(0,0,\frac{\lambda}{r})} u(\lambda\bar{y}, \lambda^2 \bar{s}) \frac{|\bar{y}|^2 \lambda^2}{\bar{s}^2 \lambda^4} \lambda^{n+2}\, d\bar{y}\, d\bar{s}$$

$$= \left(\frac{\lambda}{r}\right)^n \iint_{E(0,0;\frac{\lambda}{r})} u(ry, r^2 s)\, \frac{|y|^2}{s^2}\, dy\, ds \qquad \text{(drop the bar)}$$

We take $\lambda = r$

Hence we have

$$(5) \qquad \phi(r) = \int_{E(1)} u(ry, r^2 s)\, \frac{|y|^2}{s^2}\, dy\, ds$$

(simplify notation)

(6) Claim : $\phi'(r) \equiv 0$,

hence

$$(7) \qquad \phi(r) = \lim_{r \to 0} \phi(r) = \left(\iint_{E(1)} \frac{|y|^2}{s^2}\, dy\, ds \right) u(0,0)$$

$$= 4\, u(0,0)$$

Exercise

show that
$$\iint_{E(1)} \frac{|y|^2}{s^2}\, dy\, ds = 4$$

$\Rightarrow$ (3) follows.

Verification of (6) : $\phi'(r) \equiv 0$

$$\partial E(0,0;r) = \left\{ \Phi(y,-s) = \frac{1}{r^n} \right\}$$

Set $\psi := \ln \left[\dfrac{\Phi(y,-s)}{\left(\frac{1}{r^n}\right)} \right]$

$\boxed{\psi = 0 \quad \text{on } \partial E(r)}$ (*) $\qquad E(0,0;r) =: E(r)$

Naturally ψ plays a role in the computation:

$$(8)\quad \psi = -\frac{n}{2} \ln(-4\pi s) + \frac{|y|^2}{4s} + n \ln r$$

$$\phi'(r) = \int_{E(1)} \frac{\partial}{\partial r} \left[u(ry, r^2 s) \right] \frac{|y|^2}{s^2}\, dy\, ds$$

summation convention

reconvert back after the differentiation

$$= \int_{E(1)} \frac{|y|^2}{s^2} \left[u_{y_i} y_i + u_s (2rs) \right] dy\, ds =$$

$|y| = 1 \rightarrow$
$|\bar{y}| = 1$

$\bar{y} = ry, \quad \bar{s} = r^2 s \longrightarrow \boxed{\bar{y} \rightarrow y, \quad \bar{s} \rightarrow s}$

$d\bar{y} = r^n dy, \quad \boxed{y_i = \frac{1}{r} \bar{y}_i}$

$d\bar{s} = r^2 ds$

$s^2 = \frac{1}{r^4} \bar{s}^2$

$$dy\, ds = \frac{1}{r^n} d\bar{y} \frac{1}{r^2} d\bar{s} = \frac{1}{r^{n+2}} d\bar{y}\, d\bar{s}$$

$$\frac{|y|^2}{s^2} = \frac{\left|\frac{1}{r} \bar{y}\right|^2}{\left(\bar{s} \frac{1}{r^2}\right)^2}$$

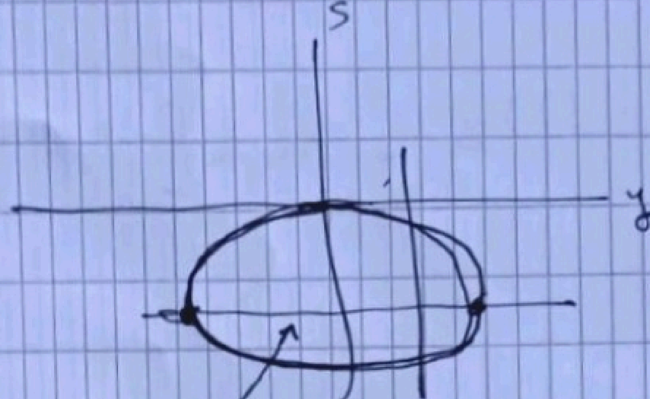
$u_{y_i} = u_{\bar{y}_i} r$

$u_s = u_{\bar{s}} r^2, \quad 2rs = \dfrac{2\bar{s}}{r}, \quad \dfrac{|y|^2}{s^2} = \dfrac{|\bar{y}|^2}{\bar{s}^2} r^2$

$$\boxed{\bar{y} \to y, \quad \bar{s} \to s}$$

$$= \frac{1}{r^{n+1}} \int_{E(r)} \left(u_{y_i}(y,s)\, y_i \frac{|y|^2}{s^2} + 2u_s \frac{|y|^2}{s} \right) dy\, ds$$

$$=: A + B$$



Observation

$$(8) \Rightarrow \quad \frac{2u_s}{s}|y|^2 = 4u_s\, y_i\, \psi_{y_i}$$

(9) $$B = \frac{4}{r^{n+1}} \int_{E(r)} u_s\, y_i\, \psi_{y_i}\, dy\, ds$$

explicit b.c.

(*) (above) + Fig

$$= \frac{4}{r^{n+1}} \int_{E(r)} \Big[\underbrace{\frac{\partial}{\partial y_i}\big(u_s y_i \psi \big)}_{0} - (u_s y_i)_{y_i}\, \psi \Big] dy\, ds$$

$$(y_i)_{y_i} = \sum \frac{\partial}{\partial y_i}(y_i) = n$$

$$= \frac{-4}{r^{n+1}} \int_{E(r)} \left[n u_s \psi + u_{s y_i}\, y_i \psi \right] dy\, ds$$

We will integrate by parts the 2^{nd} term

$$u_{s y_i}\, y_i \psi = \frac{\partial}{\partial s}\left(u_{y_i} y_i \psi \right) - u_{y_i} y_i \psi_s$$

Utilizing once more (*)

$$B = -\frac{4}{r^{n+1}} \int_{E(r)} \left[n u_s \psi + u_{y_i} y_i \psi_s \right] dy\, ds$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} \left[-4n\, u_s \psi + u_{y_i} y_i \left(-\frac{n}{2s} - \frac{|y|^2}{4s^2} \right) \right] dy\, ds$$

$|y| = r.$

$$= \frac{1}{r^{n+1}} \int_{E(r)} \left[-4n\, u_s \psi - \frac{2n}{s} u_{y_i} y_i \right] dy\, ds \;-\; A$$

$\therefore$

$$\phi'(r) = A + B = \frac{1}{r^{n+1}} \int_{E(r)} \left[-4n\, u_s \psi - \frac{2n}{s} u_{y_i} y_i \right] dy\, ds$$

$(u_s = \Delta u)$

$$= \frac{1}{r^{n+1}} \int_{E(r)} \left[-4n\, \Delta u\, \psi - \frac{2n}{s} u_{y_i} y_i \right] dy\, ds$$

Utilizing from (8) that

$$\psi_{y_i} = \frac{y_i}{2s}$$

after integrating by parts Δu, utilizing $\circledast$ once more we conclude that

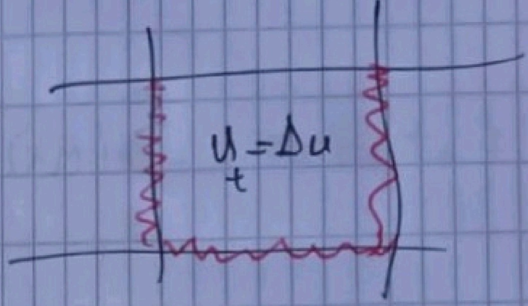
$$\phi'(r) = 0.$$

The proof is complete.

Corollaries

1. Strong Maximum Principle

$$U_T := U \times (0, T], \quad \partial_p \overline{U}_T = \overline{U}_T - U_T$$



Thm

$u \in C_1^2(U_T) \cap C(\overline{U}_T)$ solves

$$u_t = \Delta u \quad \text{in } U_T$$

Then

(i) $$\max_{\overline{U}_T} u = \max_{\partial_p U_T} u$$

(ii) If U connected and $\exists (x_0, t_0) \in U_T$ s.t.

$$u(x_0, t_0) = \max_{\overline{U}_T} u = M$$

$\Rightarrow$

$$u(x,t) \equiv M \quad \text{on } \overline{U}_{t_0} = \overline{U \times (0, t_0]}$$

Pf

1. Suppose $u(x_0,t_0) = M$, $(x_0,t_0) \in U_T$. Let $E(x_0,t_0;r)$ the heat ball in U_T.

(Figure with disconnected U)

By the M.V.T.

(1) $$u \equiv M \quad \text{in } E(x_0,t_0;r)$$

2. Take any (y_0, s_0), $s_0 < t_0$, and consider the line segment connecting (x_0, t_0) to (y_0, s_0) as in the Fig. above; Denote it by L.

Consider

$$(2) \qquad \Sigma = \{ s \geq s_0 \mid u(x,t) = M \quad \forall (x,t) \in L, \; s \leq t \leq t_0 \}$$

Clearly $\Sigma \neq \phi$. Let

$$(3) \qquad r_0 = \operatorname{Inf} \Sigma .$$

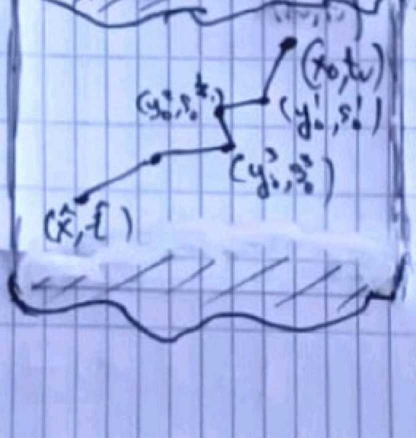
By continuity of u, $u(z_0, r_0) = M$. Since $(z_0, r_0) \in U_T$ we can choose $r > 0$, small enough, so that the $\underbrace{\text{heat ball}}_{E(z_0, r_0; r)}$ centered at (z_0, r_0) is in U_T.

But then by the M.V.T. $u \equiv M$ inside $E(z_0, r_0; r)$ contradicting the definition of r_0.
Thus

$$(4) \qquad u(y_0, s_0) = M .$$

Thus we can reach all the way to $\partial_p \overline{U}_T$. Thus (i) is proved.

3. If U is connected then given $(\hat{x}, \hat{t})$, $\hat{t} < t_0$ we can connect it to (x_0, t_0) with a polygonal line with segments



finite in number, and of the type as in the pf above.

Thus by that argument we conclude that

(5) $$u(\hat{x},\hat{t})=M$$

The proof is complete.

$\square$