## Lecture 15



**Exercise** (The porous medium equation)

A. Consider

(1) $u_t = (u^m)_{xx}$, $u(x,t) \geq 0$, $m > 1$, $x \in \mathbb{R}$, $t > 0$

We seek a similarity solution analogous to the one for the heat equation. Particularizing that procedure for $n=1$ we sought a solution in the form

(2) $$u(x,t) = \frac{1}{t^{1/2}} w\left(\frac{x}{t^{1/2}}\right), \text{ for } u_t = u_{xx}.$$

Show by following an analogous procedure for (1) that the corresponding candidate for solution is

(3) $$u(x,t) = \frac{1}{t^{\frac{1}{m+1}}} f\left(\frac{x}{t^{\frac{1}{m+1}}}\right), \quad f = f(\xi)$$

B. Show that $f$ should satisfy

(4) $$\left((f(\xi))^m\right)'' + \frac{1}{m+1}\left(\xi f(\xi)\right)' = 0$$

C. Show that

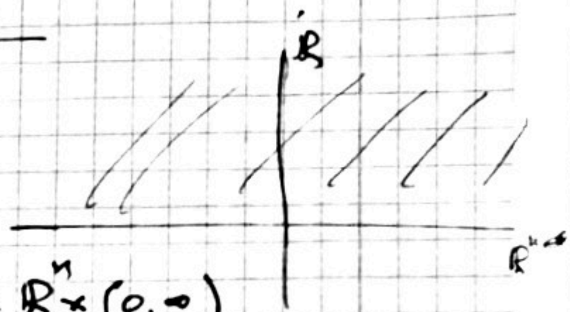
$$f(\xi) = \left(c - \frac{m-1}{2m(m+1)}\xi^2\right)_+^{\frac{1}{m-1}}, \quad \xi \in \mathbb{R}$$

where

$$x_+ := \max(x,0) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

$\square$

## Solution of the Cauchy Problem




Th 1 (The Heat Semigroup)
Consider

$$(1) \quad \begin{cases} u_t - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) \end{cases}$$

Suppose $g \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$.
Then

$$(2) \quad u(x,t) = \Phi_t(x) * f \quad \in C^\infty(\mathbb{R}^n \times (0,\infty))$$

$(\Phi_t(x) := \Phi(x,t)$, (13) p5 Lecture 14 $)$

satisfies (1)(i). If $g$ is bounded and continuous then $u$ is continuous on $\mathbb{R}^n \times [0,\infty)$ and $u(x,0) = g(x)$. If $g \in L^p$ where $p<\infty$, then

$$u(\cdot,t) \xrightarrow[t \to 0]{L^p} g$$

Pf

The proof is verbatim the proof for the Poisson semigroup for half space, Lecture 11, pg 3. The crucial fact is that

$$\left\{ \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} \right\}, \; t>0$$

is an approximation to the identity.

Rk

We will establish later a uniqueness result for the heat equation on the whole space that will imply that the solution provided by Th 1 is unique.

Thus completely analogously to the Poisson semigroup on the upper half space, lecture 12, we have

Exercise (Heat Semigroup) [ $p \in [1, \infty]$ ]

For $t \geq 0$ define

(2) $$S(t)g = u(\cdot, t), \quad u(\cdot, 0) = g$$

Show that $\{S(t)\}_{t \geq 0}$ is a contraction semigroup on $L^p(\mathbb{R}^n)$:

$$S(t)(S(\tau)g) = S(t+\tau)g, \quad t, \tau \geq 0$$

$$\|S(t)g\|_{L^p(\mathbb{R}^n)} \leq \|g\|_{L^p(\mathbb{R}^n)},$$

hence by linearity

$$\|S(t)(f-g)\|_{L^p(\mathbb{R}^n)} \leq \|f-g\|_{L^p(\mathbb{R}^n)}.$$

\#

# The Variation of Constants formula (Inhomogeneous case)

## The ODE Case

Consider the system

(3) $\quad x'(t) = A\,x(t) + f(t)$

$$x = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A = (a_{ij}), \quad f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

constant matrix, $a_{ij} \in \mathbb{R}$

Suppose $f : (0,T) \longrightarrow \mathbb{R}^n$ continuous.

(4) $\quad x(0) = x_0 \qquad$ (Initial Condition).

Claim: The solution to (3) is given by the formula

(5) $$x(t) = e^{At} x(0) + \int_0^t e^{A(t-s)} f(s)\, ds.$$

$e^A$ = matrix exp.

Explanations

For $n=1$ $\qquad x' = ax + f(t) \quad , x(0) = x_0$

$$x(t) = e^{at} x_0 + \int_0^t e^{a(t-s)} f(s)\, ds.$$

For $n \geq 1$ we argue as follows:
To find the general solution of (3) we begin with a particular solution, which we seek in the form

(6) $$x_p(t) = e^{At}\xi(t), \quad \xi : \mathbb{R} \to \mathbb{R}^n$$
to be determined:

We calculate

(7) $$x_p'(t) = Ae^{At}\xi(t) + e^{At}\xi'(t)$$

and demand that $x_p(t)$ solves the equation (not the I.C.).

(8) $$x_p'(t) = Ax_p(t) + f(t) \iff$$
$$Ae^{At}\xi(t) + e^{At}\xi'(t) = A\left(e^{At}\xi(t)\right) + f(t)$$
$$\iff e^{At}\xi'(t) = f(t)$$
$$\iff \boxed{\xi(t) = \int_0^t e^{-As} f(s)\,ds,} \quad (9)$$

The general solution is

(10) $$x_p(t) + \text{General solution of Homogeneous}$$
$$= x_p(t) + e^{At}k, \quad k \text{ an arbitrary const.}$$

Hence

$$\boxed{\begin{aligned} x(t) &= e^{At}k + e^{At}\int_0^t e^{-As} f(s)\,ds \\ &= e^{At}k + \int_0^t e^{A(t-s)} f(s)\,ds. \end{aligned}}$$

Setting $x(0) = x_0$ we obtain $k = x(0)$ (5)