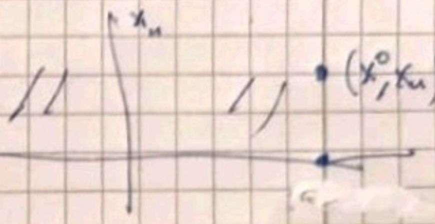
Lecture 11

Theorem (Poisson formula for half space) [Different from Evans]
(The Poisson Semigroup)

$$(1)\quad \begin{cases} \Delta u = 0, & \{x \in \mathbb{R}^n \mid x_n > 0\} =: U \\ u = g, & \text{on } x_n = 0 \end{cases}$$



Hypotheses: $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$

$$(2)\quad u(x) = \frac{2x_n}{n\alpha(n)} \int_{\mathbb{R}^{n-1}} \frac{g(y)}{|x-y|^n}\, dy$$

Then

(i) $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$

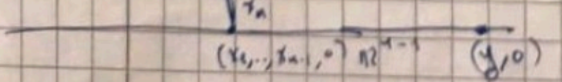
(ii) $\Delta u = 0$; in $\mathbb{R}^n_+$

(iii) $\lim\limits_{x_n \to 0} u(x) = g(x^0)$, $x^0 \in \mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$
uniformly on $\mathbb{R}^{n-1}$ compacts

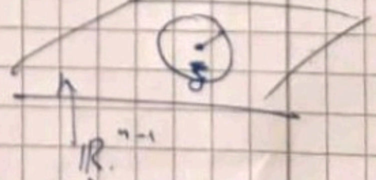
Pf

$$K(x,y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|x-y|^n}, \qquad u(x) = \int_{\mathbb{R}^{n-1}} K(x,y) g(y)\, dy$$

1. $\displaystyle\int_{\mathbb{R}^{n-1}} \frac{1}{|x-y|^n}\, dy < +\infty$ $\quad x$ fixed on $\mathbb{R}^n_+$



$$|x-y|^2 = \sum_{i=1}^{n-1} (x_i - y_i)^2 + x_n^2 =: r^2 + x_n^2$$



$$\lim_{r\to\infty} \int_{|x'-\xi| < r} \frac{1}{(r^2 + x_n^2)^{n/2}}\, dx' = \int_0^\infty \int_{\partial B(\xi,\rho)} \frac{d\sigma}{(\rho^2 + x_n^2)^{n/2}}\, d\rho \qquad \xi = (x_1, \dots, x_{n-1})$$

Note

$$\int_0^\infty \frac{\rho^{n-2}}{(\rho^2+x_n^2)^{n/2}}\,d\rho < \infty.$$

$$\frac{\rho^{n-2}}{(\rho^2+x_n^2)^{n/2}} \sim \frac{\rho^{n-2}}{\rho^n} \sim \frac{1}{\rho^2}, \rho \to +\infty.$$

Hence $u(x)$ is well defined.

Note also

$$\frac{\partial}{\partial x_i}\frac{1}{|x-y|^n} = (-n)|x-y|^{-n-1}\frac{x_i-y_i}{|x-y|}$$

$$\Rightarrow \left|\frac{\partial}{\partial x_i}\frac{1}{|x-y|^n}\right| \leq n\frac{1}{|x-y|^{n+1}}$$

and for $|y-x| \geq 1$

$$\frac{1}{|x-y|^{n+1}} \leq \frac{1}{|x-y|^n}$$

$$\Rightarrow \int \frac{\partial}{\partial x_i}\frac{1}{|x-y|^n}\,dy < +\infty.$$

Thus

$$u(x) \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$$

2. $$K(x,y) = \frac{\partial}{\partial y_n}G(x,y)\Big|_{y_n=0}$$

$$\Rightarrow \Delta_x K(x,y) = \Delta_x \frac{\partial}{\partial y_n}G(x,y)\Big|_{y_n=0} = \frac{\partial}{\partial y_n}\Delta_x G(x,y)\Big|_{y_n=0} = 0.$$

3. Claim

(3) $$\int_{\mathbb{R}^{n-1}} K(x,y)\, dy = 1$$

Notice that this is expected since for $g=1$
$$\begin{cases} \Delta u = 0, & x_n > 0 \\ u = 1 & \text{on } \mathbb{R}^{n-1} \end{cases}$$
has solution $u \equiv 1$

Recall from 1.

$$\frac{2x_n}{n\alpha(n)} \int_{\mathbb{R}^{n-1}} \frac{1}{|x-y|^n}\, dy = \frac{2x_n}{n\alpha(n)} |S^{n-2}| \int_0^\infty \frac{\rho^{n-2}}{(\rho^2+x_n^2)^{n/2}}\, d\rho$$

Notice

$$\frac{x_n\, \rho^{n-2}\, d\rho}{(\rho^2+x_n^2)^{n/2}} = \frac{\left(\frac{\rho}{x_n}\right)^{n-2} d\left(\frac{\rho}{x_n}\right)}{\left[\left(\frac{\rho}{x_n}\right)^2+1\right]^{n/2}}$$

$$\int_0^\infty \frac{\xi^{n-2}}{(1+\xi^2)^{n/2}}\, d\xi = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

Hence (3) checks.

We will use the following theorem on mollifiers:

Th (Approximation of the Identity - because the Fourier Transf of $\delta \leftrightarrow 1$)

Let $\varphi \in L^1(\mathbb{R}^k)$ and $\int_{\mathbb{R}^k} \varphi(\xi)\, d\xi = 1$

$\forall\, \varepsilon > 0$ we define $\varphi_\varepsilon(\xi) = \varepsilon^{-k} \varphi\left(\frac{x}{\varepsilon}\right)$

Suppose

$f \in L^p(\mathbb{R}^k)$, ~~$1 \leq p \leq \infty$~~

(α) $p < \infty$

$$f * \varphi_\varepsilon \xrightarrow[\varepsilon \to \infty]{L^p} f$$

(β) $p = \infty$ and $f \in C(\mathbb{R}^k) \cap L^\infty(\mathbb{R}^k)$

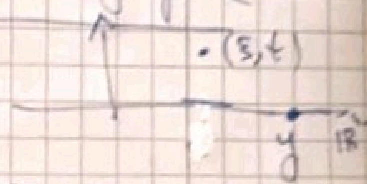
Then

$$f * \varphi_\varepsilon \xrightarrow[\varepsilon \to 0]{} f \quad \text{uniformly on compacts of } \mathbb{R}^k$$

$\square$

See Folland, Real Analysis: Moder techniques, Ch 8, Section 3.

<u>Conclusion of the Pf of Poisson's for half space</u>



Define $\boxed{t = x_n}$, $x = (\xi, t)$

$$P_t(\xi) = \frac{2t}{n\alpha(n)} \frac{1}{(|\xi|^2 + t^2)^{n/2}}$$

$$\Big(K(x,y) = K(\xi, t, y) = P_t(\xi - y) \Big)$$

Then

$$u(\xi, t) = \int_{\mathbb{R}^{n-1}} P_t(\xi - y)\, g(y)\, dy = g * P_t$$

$$P_1(\xi) = \frac{2}{n\alpha(n)} \frac{1}{(|\xi|^2 + 1^2)^{n/2}}$$

$$\boxed{\int_{\mathbb{R}^{n-1}} P_1(\xi)\, d\xi = 1} \iff \int_{\mathbb{R}^{n-1}} K((\xi, 1), y)\, dy \overset{(3)}{=} 1$$

Note

$$P_t(\xi) = \frac{1}{t^{n-1}} P\left(\frac{\xi}{t}\right).$$

Thus by the Approximation of the Identity thm we are set.

$\square$