Lecture 14 The Diffusion Equation

1. Dirichlet (D)

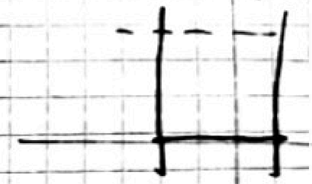
$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u, & (x,t) \in U \times (0,T] \\ u(x,0) = g(x), & \partial U \ni x \\ u = 0 & (x,t) \in \partial U \times (0,T) \end{cases}$$

$U \subset \mathbb{R}^n$

bdd, open

Neumann (N)

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u, & (x,t) \in U \times (0,T] \\ u(x,0) = g(x), & x \in \partial U \\ \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial U \times (0,T) \end{cases}$$



$\nu \perp \partial U$

$$\begin{cases} \text{Cauchy (C)} \\ \frac{\partial u}{\partial t} = \Delta u, & (x,t) \in \mathbb{R}^n \times (0,T) \\ u(x,0) = g(x), & x \in \mathbb{R}^n \\ g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \end{cases}$$

Scaling in $\mathbb{R}^n \times \mathbb{R}^+$ (parabolic)

$$(x,t) \to (\lambda x, \lambda^2 t)$$

$\frac{\partial u}{\partial t} - \Delta u = 0$ invariant, i.e. $u^\lambda(x,t) = u(\lambda x, \lambda^2 t)$, $\lambda \neq 0$ also solution

$$\Big(\frac{\partial}{\partial t} u^\lambda(x,t) = \lambda^2 u_t(\lambda x, \lambda^2 t)$$

$$u^\lambda_{x_i x_j} = u_{x_i x_j}(\lambda x, \lambda^2 t)\lambda^2 \Big)$$

Note: The quantity $\frac{|x|^2}{t}$ is invariant under the parabolic scaling.

Recall that Δ is invariant under orthogonal transformations, i.e. $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \mid \|Ax\| = \|x\|$. We [illegible] that $|x|$ is an important quantity for the Laplacian (Fundamental solution $|x|^{-(n-2)}$

Self-Similar Solution - Fundamental Solution

Seek Self-Similar means same profile at all times modulo rescaling:

(1) $u(x,t) = \lambda^{\alpha} u(\lambda^{\beta} x, \lambda t) \quad \forall \lambda > 0$

setting $\lambda = \frac{1}{t} \Rightarrow$ | Exercise 12, pg 87, Evans

(2) $u(x,t) = \frac{1}{t^{\alpha}} u(\lambda^{\beta} x, 1) = \frac{1}{t^{\alpha}} u\left(\frac{x}{t^{\beta}}, 1\right) =: \frac{1}{t^{\alpha}} v\left(\frac{x}{t^{\beta}}\right)$

Note: We reduced # of indep variables from $n+1$ to n

Substitute in $\frac{\partial u}{\partial t} - \Delta_x u = 0$:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(t^{-\alpha} v\left(\frac{x}{t^{\beta}}\right)\right) = (-\alpha) t^{-\alpha-1} v\left(\frac{x}{t^{\beta}}\right) + \nabla_y v \cdot y \, (-\beta) t^{-\alpha-1}$$
$$= (-\alpha) t^{-\alpha-1} v(y) + (-\beta) t^{-\alpha-1} \nabla_y v \cdot y.$$

$$\frac{\partial u}{\partial x_i} = \frac{1}{t^{\alpha}} \frac{\partial v}{\partial y_i} \frac{\partial y_i}{\partial x_i} = \frac{1}{t^{\alpha}} \frac{\partial v}{\partial y_i} \frac{1}{t^{\beta}} = \frac{1}{t^{\alpha+\beta}} v_{y_i}$$

$$u_{x_i x_i} = \frac{1}{t^{\alpha+2\beta}} v_{y_i y_i}$$

Hence

(3) $\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} \nabla v \cdot y + t^{-(\alpha+2\beta)} \Delta_y v = 0$

Further simplification by taking $2\beta = 1 \Leftrightarrow \beta = \frac{1}{2}$

$\left(\frac{x}{\sqrt{t}}\right.$ natural already noticed $\left.\right)$

(4) $\alpha v + \frac{1}{2} y \cdot \nabla_y v + \Delta_y v = 0$

[...]ther reduction of variables by requiring $\boxed{n \longrightarrow 1}$

(5) $\quad v(y) = w(|y|) \qquad$ (radial)

$$\frac{\partial v}{\partial y_i} = w'(|y|)\frac{\partial}{\partial y_i}|y| = w'(|y|)\frac{y_i}{|y|}$$

$$\sum_i \frac{\partial^2 v}{\partial y_i^2} = \sum_i \frac{\partial}{\partial y_i}\left[w'(|y|)\frac{y_i}{|y|}\right]$$

$$= \sum_i w''(|y|)\frac{\partial |y|}{\partial y_i}\frac{y_i}{|y|}$$

$$+ \sum_i w'(|y|)\frac{\partial}{\partial y_i}\left(\frac{y_i}{|y|}\right)$$

$$\frac{\partial}{\partial y_i}|y| = \frac{y_i}{|y|}, \quad \frac{\partial |y|}{\partial y_i}\frac{y_i}{|y|} = \frac{y_i^2}{|y|^2}$$

$\Rightarrow$ 1st term $= \underline{w''(|y|)}$

2nd term: $\displaystyle \sum_{i=1}^{n}\frac{\partial}{\partial y_i}\left(\frac{y_i}{|y|}\right) = \frac{n|y| - y_i\frac{\partial}{\partial y_i}|y|}{|y|^2}$

$$= \frac{n}{|y|} - \frac{y_i\frac{y_i}{|y|}}{|y|^2}$$

$$= \frac{n-1}{|y|}$$

(4) (5) $\Rightarrow$

(6) $\quad \alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0, \qquad r = |x|$

$\left(y \cdot \nabla v = y_i \frac{\partial}{\partial r} w \frac{\partial r}{\partial y_i} = y_i w' \frac{y_i}{|y|} = r w' \right)$

Note that (6) is an O.D.E. !

$$0 = \underbrace{w'' + \frac{n-1}{r} w'} + \alpha w + \frac{1}{2} r w' = 0$$

radial part of $\Delta = \frac{1}{r^{n-1}} \left(r^{n-1} w' \right)'$

$$= \frac{1}{r^{n-1}} \left(r^{n-1} w' \right)' + \alpha w + \frac{1}{2} r w'$$

$$= \frac{1}{r^{n-1}} \left\{ \left(r^{n-1} w' \right)' + r^{n-1} \alpha w + r^{n-1} \frac{1}{2} r w' \right\}$$

We are seeking a particular solution in closed form, here would be convenient to have

(6) $\alpha r^{n-1} w + \frac{1}{2} r w' r^{n-1} = \frac{1}{2} \left(w r^{n} \right)'$

Thus we take

(7) $\alpha = \frac{n}{2}$

and (5) takes the form.

(8) $\left(r^{n-1} w' \right)' + \frac{1}{2} \left(r^{n} w \right)' = 0.$ (divergence form)

Integrating once

(9) $r^{n-1} w' + \frac{1}{2} r^{n} w = c$

Assume $w \to 0$, $w' \to 0$ fast at ∞ $\Rightarrow c = 0$

(10) $w' = -\frac{r}{2} w$

$\Rightarrow$

(11) $w = b e^{-\frac{r^2}{4}}$

Conclusion

(12) $$\frac{b}{t^{n/2}} w\left(\frac{|x|}{t^{1/2}}\right) = \frac{b}{t^{n/2}} e^{-\frac{|x|^2}{4t}}$$

is a solution of $u_t = \Delta u$ for $t > 0$.

Def (Fundamental Soln)

(13) $$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & t > 0,\ x \in \mathbb{R}^n \\ 0, & t \leq 0. \end{cases}$$

$\boxed{\text{Claim}}$

Φ formally solves:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u, & t > 0 \\ u(x,0) = \delta_0(x) \end{cases}$$

Pf

1. $\int_{\mathbb{R}^n} \Phi(x,1)\, dx = 1$, $\Phi(x,1) = \frac{1}{(4\pi)^{n/2}} e^{-\frac{|x|^2}{4t}}$

2. Hence $\left\{\frac{1}{\varepsilon^n} \Phi(x/\varepsilon, 1)\right\}$ is an approximation of the identity. We take $\varepsilon = t^{1/2}$

$$\frac{1}{t^{n/2}} \Phi\left(\frac{x}{t^{1/2}}, 1\right) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

$\square$