$\boxed{\text{Th (analyticity of harmonic fn)}}$

Assume u is harmonic in U. Then u is analytic in U

i.e. $\forall\, x_0 \in U$

$$u(x) = \sum \frac{D^\alpha u(x_0)}{\alpha!}(x-x_0)^\alpha$$

, series converges absolutely in nbd of x_0 $B(x_0,r) \subset U$

Pf

1. We have established $\quad C_k = \dfrac{(2^{n+1} n k)^k}{\alpha(n)}, \; k = |\alpha|$

$$(1) \quad |D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$$

We notice that

$$\sup_{0<r<r_0} \frac{1}{r^n} \|u\|_{L^1(B(x_0,r))} = \sup_{0<r<r_0} \frac{1}{r^n} \int_{B(x_0,r)} |u|\,dx$$

$$\leq \|u\|_{L^\infty(B(x_0,r_0))} = M$$

, since u cont. in $B(x_0,2r) \subset U$

Hence (1) takes the form

$$(2) \quad |D^\alpha u(x_0)| \leq M \left(\frac{2^{n+1} n}{r}\right)^{|\alpha|} |\alpha|^{|\alpha|}$$

2. Note that

$$e^k = \sum_{\nu \geq 0}^{\infty} \frac{k^\nu}{\nu!} \geq \frac{k^k}{k!}$$

Hence

$$(3) \quad |\alpha|^{|\alpha|} \leq e^{|\alpha|} |\alpha|!$$

Note also that from the expansion of the multinomial

$$n^k = (1+\dots+1)^k = \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!}$$

(Evans, p13, §1.5)

$\Rightarrow$ (4) $|\alpha|! \le \alpha!\, n^k \Rightarrow$

(5) $|D^\alpha u(x_0)| \le M\left(\frac{2^{n+1} n}{r}\right)^{|\alpha|} e^{|\alpha|}\, |\alpha|! \le M\left(\frac{2^{n+1} n^2}{r}\right)^{|\alpha|} \alpha!$

3. Take $x_0 = 0$

$$u(x) = \sum_\alpha \frac{D^\alpha u(0)}{\alpha!} x^\alpha$$

Take as norm $|x| = \max |x_i|$

Claim: If $|x| \le \frac{r}{2^{n+2} n^3 e}$ the series converges absolutely.

$$\sum_\alpha \frac{|D^\alpha u(0)|}{\alpha!} |x^\alpha| \le \sum_\alpha \frac{|D^\alpha u(0)|}{\alpha!} \left(\frac{r}{2^{n+2} n^3 e}\right)^{|\alpha|}$$

$|x^\alpha| = |x_1^{\alpha_1} \dots x_n^{\alpha_n}| \le |x|^{|\alpha|}$

$$\overset{(5)}{\le} \sum_\alpha M\left(\frac{2^{n+1} n^2 e}{r}\right)^{|\alpha|} \left(\frac{r}{2^{n+2} n^3 e}\right)^{|\alpha|}$$

$$= \sum_{k=1}^{\infty} \sum_{|\alpha| = k} \left(\frac{1}{2n}\right)^k$$

separate k objects with $n-1$ dividers – # of partial deriv.

Total # of symbols $n-1+k$

divi

choose $n-1$ symbols

Now the number of α s.t. $|\alpha| = k$ is $\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!\,(n-1)!} \underset{k\to\infty}{\sim} \frac{k^{n-1}}{(n-1)!}$

Have $\binom{n+k-1}{n-1}$

$$\sum_{|\alpha| = k} \left(\frac{1}{2n}\right)^k \sim \frac{k^{n-1}}{(n-1)!} \left(\frac{1}{2n}\right)^k$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{k^{n-1}}{(n-1)!} \left(\frac{1}{2n}\right)^k < +\infty.$$

(Roots test $\left(\frac{k^{n-1}}{(n-1)!} \left(\frac{1}{2n}\right)^k\right)^{1/k} \to \frac{1}{2n} < 1$) $\square$