Lecture 13

1. Continuation on Poisson's Formula for the Ball

Recall $y^* = \frac{R^2}{|y|^2} y$. Take $\underline{R=1}$ (unit circle)

Απολλωνιος $\Rightarrow$

$$(1) \quad \frac{|x-y|}{|x-y^*|} = |y| \Leftrightarrow$$

The same holds for the sphere S^{n-1}. (Why?)

Interchanging the roles of x and y (placing charges at x and x^*, view $y \to G(x,y)$)

$$(2) \quad \frac{|y-x|}{|y-x^*|} = |x| . \qquad (y \in \partial B(0,1))$$

Difference between $n=2$, and $n \geq 3$

Place at x^* not a unit charge

but

(3) $\quad |x|^{2-n}$ units charge.

Def (Green's fn)

$$G(x,y) := \Phi(y-x) - \Phi(|x|(y-x^*)),$$

$$x, y \in B(0,1), \quad x \neq y.$$

For $|y|=1$ $\quad \frac{1}{|x-y|^{n-2}} = \frac{1}{\left|\frac{x}{|x|^2} - y\right|^{n-2}} |x|^{2-n}$

hence for $|y|=1$ $y \to G(x,y)$ is zero.

$$\underset{\partial B(0,1)}{\overset{\shortparallel}{|y|=1}}.$$

$$(4) \quad \begin{cases} \Delta u = 0 & , x \in B(0,1) \\ u = g & , x \in \partial B(0,1). \end{cases}$$

From (4), Lecture 9

$$(5) \quad u(x) = -\int_{\partial B(0,1)} g(y) \frac{\partial G(x,y)}{\partial \nu(y)} dS_y$$

<u>Calculation of $\frac{\partial G}{\partial \nu}$</u>

$$\frac{\partial G}{\partial \nu(y)} = \frac{\partial G}{\partial y_i} \cdot \nu_i = \frac{\partial}{\partial y_i} \Phi(y-x) - \frac{\partial}{\partial y_i} \Phi(|x|(y-x^*))$$

$$\frac{\partial}{\partial y_i} |x-y|^{2-n} = (n-2)\, |x-y|^{1-n} \frac{x_i - y_i}{|x-y|} (-1)$$

$$= (2-n) \frac{x_i - y_i}{|x-y|^n}.$$

$$\frac{\partial}{\partial y_i} \left(|x|^{2-n} |y-x^*|^{2-n} \right) = |x|^{2-n} |y-x^*|^{1-n} \frac{y_i - x_i^*}{|y-x^*|} (2-n)$$

$$= (n-2)\, |x|^{2-n} |y-x^*|^{1-n} \frac{x_i^* - y_i}{|y-x^*|}$$

$$= (n-2)\, |x|^{2-n} \frac{x_i^* - y_i}{|x^* - y|^n}$$

Now for $y \in \partial B(0,1)$ by (2)

$$\frac{|y-x|^n}{|y-x^*|^n} = |x|^n .$$

Hence

$$\frac{\partial}{\partial y_i}\left(|x|^2 |y-x^*|^{2-n}\right) = (n-2)|x|^{2-\cancel{n}} \frac{x_i^*-y_i}{|y-x^*|^n} |x|^{2\cancel{n}}$$

$$= (n-2)\frac{|x|^2(x_i^*-y_i)}{|x-y|^n}$$

$$= \frac{(n-2)\left(x_i - y_i|x|^2\right)}{|x-y|^n}$$

On $\partial B(0,1)$, $\nu_i = y_i$ $\quad \left(\nu = \frac{y}{|y|} = y\right)$

$\therefore$

$$\frac{\partial G}{\partial y_i} y_i = \frac{-1}{n\alpha(n)}\left[y_i(y_i-x_i) + y_i(x_i - y_i|x|^2)\right]\frac{1}{|x-y|^n}$$

$$= -\frac{1}{n\alpha(n)}\frac{1-|x|^2}{|x-y|^n}$$

Conclusion

(5) $\Rightarrow$

$$(6)\quad \boxed{u(x) = \frac{1-|x|^2}{n\alpha(n)}\int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n}\, dS_y}$$

Poisson's for the Ball

Conclusion

$$\begin{cases} \Delta u = 0 , & x \in B(0,R) \\ u = g , & x \in \partial B(0,R) \end{cases}$$

$\Rightarrow$

$$(7)\quad u(x) = \frac{R^2 - |x|^2}{n\alpha(n)R} \int_{\partial B(0,R)} \frac{g(y)}{|x-y|^n} \, dS_y \, .$$

A similar theorem to the half space case holds, for u given by (7), $g \in C(\partial B(0,R))$:

$$\begin{cases} u \in C^{\infty}(B(0,R)) \\ \Delta u = 0 \\ \lim\limits_{|x| \to 1} u\left(\frac{x}{|x|}, |x|\right) = g\left(\frac{x}{|x|}\right) \end{cases}$$

$$\left(\begin{array}{l} u = u(r,\theta) \\ \lim\limits_{r \to 1} u(r,\theta) = g(\theta) \end{array} \right)$$

#

Dirichlet's Principle (Calculus of Variations)

(1) $\begin{cases} -\Delta u = f & , x \in U \subset \mathbb{R}^n, \quad U = \text{open, bdd} \\ u = g & , x \in \partial U \end{cases}$

$g \in C(\partial U)$

(2) $I(w) = \int_U \left(\frac{1}{2}|\nabla w|^2 - w f\right) dx$

$\mathcal{A} = \{ w \in C^2(\bar{U}) \mid w = g \text{ on } \partial U \}$

<u>Th</u> (Dirichlet)

If u solves (1) then u minimizes I in $\mathcal{A}$, and conversely, if u minimizes I in $\mathcal{A}$ then it solves (1).

<u>Rk</u> (Physics)

$\int |\nabla u|^2 dx$ is the potential energy of the Electric field $-\nabla u$.

(1) is called the <u>Euler-Lagrange Equation</u> for I.

<u>Pf</u>

1. Suppose u solves (1). For $w \in \mathcal{A}$ we have

$$0 = \int_U (-\Delta u - f)(u - w)\, dx$$

$$\overset{\text{Green}}{=} \int_U \nabla u (\nabla u - \nabla w)\, dx - \int_{\partial U} \frac{\partial u}{\partial \nu} \cancel{(u-w)}\, dS$$

$$- \int_U (u-w) f\, dx$$

$\Leftrightarrow$

$$\int_U (|\nabla u|^2 - uf)\,dx = \int_U \nabla u \cdot \nabla w\,dx - \int_U wf\,dx$$

$$\leq \int_U |\nabla u|\,|\nabla w|\,dx - \int_U wf\,dx$$

$$\leq \int_U \left(\tfrac{1}{2}|\nabla u|^2 + \tfrac{1}{2}|\nabla w|^2\right)dx - \int_U wf\,dx$$

$\Rightarrow$

(3) $$\int_U \left(\tfrac{1}{2}|\nabla u|^2 - uf\right)dx \leq \int_U \left(\tfrac{1}{2}|\nabla w|^2 - wf\right)dx$$

$\Leftrightarrow$ $I(u) \leq I(w)$, $\forall w \in \mathcal{A}$.

2. Suppose u minimizes.
Let $v \in C_c^\infty(U)$, and fix it.

Set

(4) $j(\tau) = I(u + \tau v)$, $\tau \in \mathbb{R}$, $|\tau| < 1$

$u + \tau v \in \mathcal{A}$

$j(0) = \min j$

$\Rightarrow$

(5) $$j'(0) = 0 \Leftrightarrow \frac{d}{d\tau}\Big|_{\tau=0} I(u + \tau v) = 0$$

$$0 = \frac{d}{d\tau}\Big| \int \left[\tfrac{1}{2}|\nabla(u+\tau v)|^2 - (u+\tau v)f\right]dx = \int \frac{d}{d\tau}\left\{\left[\tfrac{1}{2}|\nabla(u+\tau v)|^2\right] \ldots\right.$$

$$= \int_U (\nabla u \nabla v - v f)\, dx$$

$$= \int_U [(-\Delta u) v - v f]\, dx$$

$v \in C_c^\infty(U)$

$$\Rightarrow 0 = -\int_U [\Delta u + f] v \, dx \quad , \forall v \in C_c^\infty(U).$$

$$\therefore \quad \Delta u + f = 0 \quad \text{in } U$$

$$u = g \quad \text{on } \partial U \quad (\text{since } u \in \mathcal{A}).$$

$\square$