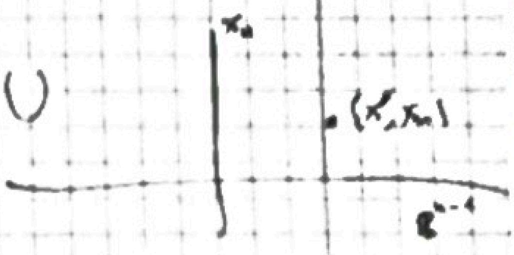
Lecture 12

①

Comments on Lecture 11

A)

$$(1)\quad \begin{cases} \Delta u = 0, & \{x \in \mathbb{R}^n \mid x_n > 0\} =: U \\ u = g & \text{on } x_n = 0 \end{cases}$$



Hyp: $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$

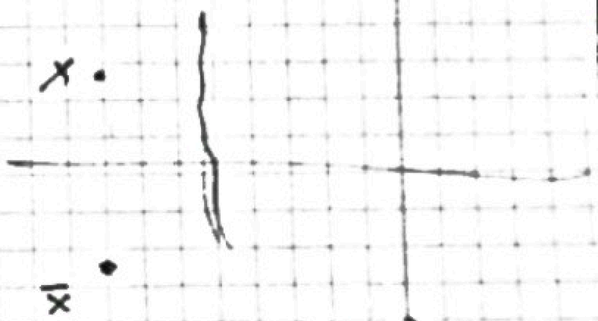
$$(2)\quad u(x) = \frac{2x_n}{n\alpha(n)} \int_{\mathbb{R}^{n-1}} \frac{g(y)}{|x-y|^n}\,dy =: \int_{\mathbb{R}^{n-1}} K(x,y)\,g(y)\,dy.$$

$$(3)\quad \int_{\mathbb{R}^{n-1}} K(x,y)\,dy = 1$$

Have established (3)

— " — $u \in C^\infty(\mathbb{R}^n_+)$

— " — $\Delta u = 0$.



$$G(x,y) = \Phi(y-x) - \Phi(y-\bar{x})$$

$x = (x', x_n)$

$x' = (x_1, \ldots, x_{n-1})$

$\bar{x} = (x', -x_n)$

$$K(x,y) = \frac{\partial}{\partial y_n} G(x,y)\Big|_{y_n = 0}$$

$-\Delta_y G = \delta_x(y)$

$G(x,y) = 0,\ y \in \partial U$

Recall from lecture 9 we have established

$$(4)\quad u(x) = -\int_{\partial U} g(y) \frac{\partial G}{\partial \nu}(x,y)\,dS_y + \int_U f(y)\,G(x,y)\,dy$$

$-\Delta u = f$ in U

$u = g$ on ∂U

Hence (2) follows from (4)

To establish $u \in L^\infty(\mathbb{R}^n_+)$:

$$(5)\quad |u(x)| \le \int_{\mathbb{R}^{n-1}} |K(x,y)|\,|g(y)|\,dy \le \|g\|_{L^\infty} \int_{\mathbb{R}^{n-1}} |K(x,y)|\,dy = \|g\|$$

(since $K \ge 0$).

ON UNIQUENESS

Notice that (2) under the Hyp $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ does not capture all solutions to (1), but by (5) [necessarily] only solutions that are $L^\infty(\mathbb{R}^n_+)$.

For example

(6) $u(x_1, \dots, x_n) = x_n$ satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = 0 & \text{on } \mathbb{R}^{n-1} \end{cases}$$

and is not captured by (2).

It turns out that in the class of $L^\infty(\mathbb{R}^n_+)$ fns (2) gives the unique solution.

c) The harmonic polynomials are not in this class

Ex $(n=2)$

$x^2 - y^2, \ 2xy$ (degree 2)

$x^3 - 3xy^2, \ 3x^2y - y^3$ (— 3)

$x^4 - 6x^2y^2 + y^4, \ 4x^3y - 4xy^3,$ (degree 4)

$\vdots$

D) Poisson's Semigroup (See Folland, p 91, G.)

Recall from p (4) in Lecture 10

$x_n = t, \ x^0 = \xi$ (2) takes the form

(7) $$u(\xi, t) = \int_{\mathbb{R}^{n-1}} P_t(\xi - y) g(y)\, dy = (g * P_t)(\xi).$$

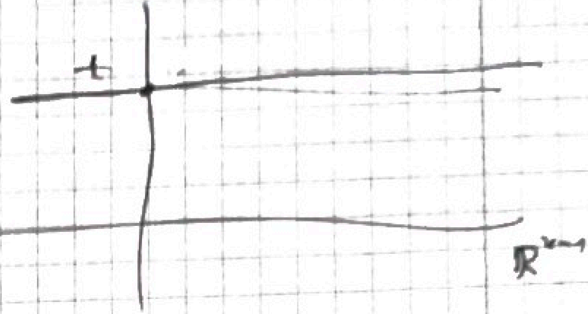
...all also Young's Inequality

(8) $$\|f * h\|_{L^p(\mathbb{R}^k)} \le \|f\|_{L^p(\mathbb{R}^k)} \|h\|_{L^1(\mathbb{R}^k)}.$$

$(L^1 * L^p \to L^p)$ $\qquad (p \geq 1)$

Hence from (7)

(9). $$\|u(\cdot,t)\|_{L^p(\mathbb{R}^{n-1})} \le \|g\|_{L^p(\mathbb{R}^{n-1})} \|P_t\|_{L^1(\mathbb{R}^{n-1})}$$



$$= \|g\|_{L^p(\mathbb{R}^{n-1})}.$$

For $t \geq 0$ define

(10) $$S(t)g = u(\cdot,t), \quad u(\cdot,0) = g.$$

It turns out that in the class of $L^p(\mathbb{R}^n_+)$ fus (2) gives the unique solution

Exercise $(p \in [1,\infty])$

Utilizing the assumed uniqueness show that $\{S(t)\}_{t \geq 0}$ is a contraction semigroup on $L^p(\mathbb{R}^{n-1})$:

$$S(t)(S(\tau)g) = S(t+\tau)g, \quad t,\tau \geq 0$$

$$\|S(t)g\|_{L^p(\mathbb{R}^{n-1})} \le \|g\|_{L^p(\mathbb{R}^{n-1})}$$

Thus by linearity

$$\|S(t)(f-g)\|_{L^p(\mathbb{R}^{n-1})} \le \|f-g\|_{L^p(\mathbb{R}^{n-1})}.$$

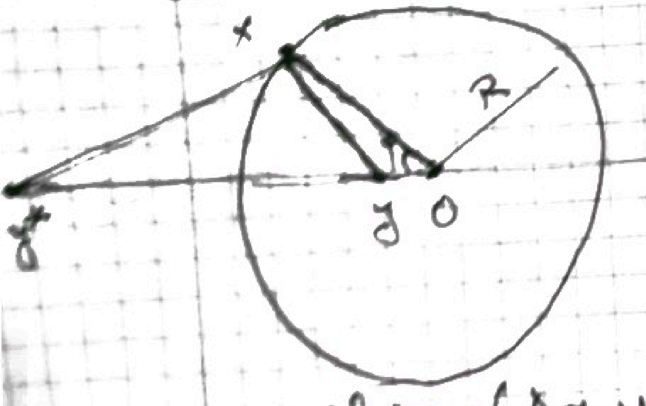
Exercise

(a) Find the fundamental solution for $\frac{d^2}{dx^2}$ on $\mathbb{R}$

(b) Show the Green's fn for $\frac{d^2}{dx^2}$ on $(0,1)$ is $x(y-1)$ for $x<y$, $y(x-1)$, $x>y$.

Poisson's Formula For The Ball

Geometric Fact



Consider R-disc on the plane. Given $y \in D_R$ define $y^* = \frac{R^2}{|y|^2} y$, the reflection of y through the circle

Claim (Απολλώνιος)

Η περιφέρεια είναι ο γεωμετρικός τόπος των σημείων x των ο λόγος των αποστάσεων από y και y^* είναι σταθερός:

$$\frac{|x-y|}{|x-y^*|} = \frac{|y|}{R}$$

Pf

$|x-y|^2 = |x|^2 + |y|^2 - 2x\cdot y$

$|x-y^*|^2 = |x|^2 + |y^*|^2 + 2x\cdot y^* = R^2 + \frac{R^4}{|y|^2} - 2\frac{R^2}{|y|}\cos\varphi$

$= R^2 + \frac{R^4}{|y|^2} - 2\frac{R^2}{|y|^2} x\cdot y$

$= \frac{R^2}{|y|^2} |x-y|^2$

$\left(x\cdot y^* = x\cdot y \frac{R^2}{|y|^2} \right)$

Hence

$$G(x,y) = \left(\frac{1}{2\pi} \ln|x-y| - \frac{1}{2\pi} \ln|x-y^*|\right) - \frac{1}{2\pi} \ln\left(\frac{|y|}{R}\right)$$

vanishes on ∂B_R.