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By ALAN H. SCHOENFELD and ABRAHAM ARCAVI

The concept of variable is central to mathematics teaching and learning in junior and senior high school. Understanding the concept provides the basis for the transition from arithmetic to algebra and is necessary for the meaningful use of all advanced mathematics. Despite the importance of the concept, however, most mathematics curricula seem to treat variables as primitive terms that—after some practice, of course—will be understood and used in a straightforward way by most students. Mirroring the textbook presentations, we mathematics teachers are frequently seen at the chalkboard manipulating *a*'s, *b*'s, *x*'s, and *y*'s in an easy and almost automatic way. In fact, it is easy to do so without keeping in mind the multiple connotations, meanings, and uses of the terms we manipulate. As Freudenthal (1983, 469) notes, this ease is only natural:

> I have observed, not only with other people but also with myself . . . , that sources of insight can be clogged by automatisms. One finally masters an activity so perfectly that the question of how and why is not asked any more, cannot be asked any more, and is not even understood any more as a meaningful and relevant question.

This article offers an exercise in unclogging. In it we describe a structured reflective exercise whose intention is to reexamine the notion of variable and, in doing so, rediscover its richness and multiplicity of meanings. In turn, our renewed understanding of the concept may help us to reshape the ways we teach and use the concept in our classes.

Our exercise, structured in three parts, follows. We have used it with a diverse group of people, including mathematicians, mathematics educators, computer scientists, linguists, and logicians. As you work through the exercise, compare your responses with theirs. The depth and variety of responses point to the richness and diversity of the concept. After you have worked through the exercise, you might consider working it with your students.

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Try describing "variable" in one word.

On the Elusive Meaning of Variable

We start with a deceptively simple exercise. One way to try to capture the essence of a concept is to try to define it using very few words. The idea is to free oneself from the burden of creating long verbal descriptions and to concentrate on finding one appropriate descriptive noun that gets at the meaning of the concept. The rules of the game are important here; mere synonyms are not enough. We want terms that capture the heart of the concept. Having said this, we present exercise 1.

> Complete the following sentence:
>
> A variable is a ____________ .
>
> Use *the one* English word that, in your opinion, best captures the meaning of the term *variable*. (If, as we did, you generate a number of possible candidates, please list them in order of preference.)

We recommend that you work at this exercise for a while before proceeding. You will probably generate a small number of terms, most of which seem to be about right but none of which seem truly adequate. For purposes of comparison, here is the list that we produced:

symbol, placeholder, pronoun, parameter, argument, pointer, name, identifier, empty space, void, reference, instance

It is interesting that the term *unknown* was not among the descriptions we generated—a fact we failed to notice until the editor suggested adding it to our list! The reason for the omission is that for each of us, variable means something that does indeed vary, or that has multiple values. The connotation of unknown is something that has fixed value but that you don't yet know. For example, x is the unknown in the equation $3x + 2 = 5x - 4$. In this equation, you may not know what x is; but x isn't variable in that it only has one value. For a discussion of literal symbols (variables, unknowns) see Wagner (1983).

Mathematical meaning is often determined by context rather than by formal rules.

Our group found the process of generating and discussing this list quite instructive. First, we noted that people from different disciplines made strikingly different choices, and the choices served, in a sense, to announce their disciplines. The computer scientist, for example, was fairly pleased with *pointer* and the logician with *pronoun;* yet these terms left the other members of the group cold. Was our disciplinary diversity the reason for the fact that the list doesn't seem satisfactory? That is, are the multiple uses of the term *variable* in different disciplinary contexts responsible for the elusiveness of the term, the refusal of the term *variable* to be captured in a single word? We think that the answer is no. Even if we delete from our list those nouns whose denotations seem to stray from the mathematical meanings of variable, the remaining words hardly converge in meaning. No single word seems quite right.

Perhaps, the group suggested, the single-word restriction is an impossible constraint. Perhaps a central notion does exist, but it is one that can't be captured in just one word. We shall, therefore, abandon the constraint. In part 2 we shall examine some definitions from the literature. As always, however, it's useful to try something on your own before reading how someone else has dealt with it. Before proceeding to part 2, try to produce a definition—of any length and any form you consider appropriate—that, from your point of view, captures the essence of the term *variable.*

On the Variable Meaning of *Variable*

In the following list you will find ten definitions of the term *variable,* all taken from the literature. This list is a representative sample. You could expand it, of course.

1. "Latin—variabilis:'changeable'"
2. *a.* "A quantity that may assume any one of a specified set of values"
   b. "A symbol in a mathematical formula representing a variable: placeholder"
3. "Variable quantities . . . are such as are supposed to be continually increasing or decreasing: and so do by the motion of their said increase or decrease, generate lines, areas or solidities."
4. "A quantity or force which throughout a mathematical calculation or investigation, is assumed to vary or capable of varying in value."
5. "A variable is a symbol that can be replaced by any element of some designated set of numbers (or other quantities) called the domain of the variable. Any member of the set is a value of the variable. If the set has only one

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member, the variable becomes a constant. If a mathematical sentence contains two variables related in such a way that when replacement is made for the first variable the value of the second variable is determined, the first variable is called the independent variable, and the second is called the dependent variable."

6. "A general purpose term in mathematics for an entity which takes various values in any particular context. The domain of the variable may be limited to a particular set of numbers or algebraic quantities."

7. "Variables, which are usually represented by letters, represent an empty space into which an arbitrary element (or its symbol) from a fixed set can be substituted. . . . Variables are useful in two ways: they make it easy to state laws, and the solution of a problem expressed in terms of variables yields the result for arbitrarily many individual cases without new calculations, by mere substitution."

8. "[A] variable is a letter or a string of letters used to stand for a number. . . . At any particular time, a variable will stand for one particular number, called the *value* of the variable, which may change from time to time. . . . The value of a variable may change millions of times. . . . [W]e will associate with each variable a *window box*. The associated variable is engraved on the top of each box, and inside is a strip of paper with the *present value* of the variable written on it. The variable is a name for the number that currently appears inside."

9. "A variable is a named entity possessing a value that may change during execution of the program. A variable is associated with a specific memory location and the variable's value is the content of that memory location."

10. "[A]ny symbol whose meaning is not determinate is called a *variable*, and the various determinations of which its meaning is susceptible are called *values* of the variable. The values may be any set of entities, propositions, functions, classes or relations, according to circumstances. If a statement is made about 'Mr. A and Mr. B,' 'Mr. A' and 'Mr. B' are variables whose values are confined to men. A variable may either have a conventionally-assigned range of values, or may (in the absence of any indication of the range of values) have as the range of its values all determinations which render the statement in which it occurs significant."

Here are some points to consider in reviewing this list.

a. Can you condense each of the preceding definitions to just one word so as to expand the list we generated in part 1?

b. Contrast the "full" definition you produced in part 1 with each of the definitions given in the preceding list.

Our multiple uses of the term *variable* makes it hard for students to understand.

c. Suggest criteria by which the definitions might be classified into categories. Then try to find the commonalities within categories, and see whether a common core appears for the concept of variable across your suggested classification. Of course, the discipline from which the definition comes is one natural classification. However, other bases come to mind; for example, you might look at the definitions to find their "didactical intentions."

What follows are some of the observations that emerged in our discussion. Regarding (*a*), we note that our list of single-word characterizations of variable was hardly complete. Additional terms that can be extracted from the preceding set of definitions include the following: change, quantity, motion, entity (general purpose term), window box. Of course, our expanded list hardly makes life any easier; with all these varied meanings, the concept seems even more elusive than it did before!

Some definitions (including the oldest, which dates back to 1710 and thus captures much of the original intention behind the definition) give the reader a sense of "how it works"; others (e.g., descriptions of motion, of an "empty space," or of a "window box") give us the feeling of what a variable is (or behaves like) in pictorial, concrete terms. Others led naturally to discussions of related terms—for example, the modern notion of variable depends on the notion of domain. Indeed, can you understand one without understanding the other? Is that part of the difficulty in making the definitions?

More generally, the difficulty might be that to understand what something is, you need to understand how it's used (how it works). Or, to borrow from Aristotle, "to know what a thing is, is the same as knowing why it is." Note here that definition 7 addresses some of the "why." "Variables are useful in two ways: they make it easy to state laws, and by mere substitution the solution of a problem expressed in terms of variables yields the result for arbitrarily many individual cases without new calculations." That is, variables are a formal tool in the service of generalization. Variables are used for making general statements, characterizing general procedures, investigating the generality of mathematical issues, and handling finitely or infinitely many cases at once. Indeed, the idea of variable came to life as a notational tool for making generalizations. In the words of Alfred North Whitehead (1911), "by relieving the brain of all unnecessary work a good notation sets it free to concentrate on more advanced problems, and, in effect, increases the mental power of the race." Developments that followed from the use of this tool—developments that clearly stretched humankind's mental power—include algebra, analytic geometry, calculus, and so on. (For a historical discussion see, e.g., Boyer [1985]).

It turns out, however, that "generalization" is too broad a generalization to capture what we want. Consider, for example, the following two mathematical statements:

$$a + 3 = 3 + a$$

and

$$\varepsilon \text{ approaches } 0.$$

The idea of generalization is present in both. But now look through our aforementioned list of definitions and identify the sentences that best characterize the roles that the variables play in each of these statements. Our choice is given in table 1. The headings in table 1, "polyvalent names" and "variable objects," are taken from Freudenthal; he makes a clear distinction between these two types of statements (1983, 491):

> Polyvalent names are a means to formulate general statements, that is statements that hold for all objects they name. . . . The mathematical habit of calling polyvalent names variables is of a rather recent date. Originally "variable" meant something that really varies.

TABLE 1

Two Different Conceptions of Variable

	Polyvalent Names	Variable Objects
Example	$a + 3 = 3 + a$	ε approaches 0
Description	A quantity that may assume one of a specified set of values (def. 2)	Supposed to be continually increasing or decreasing ... motion of increase or decrease (def. 3)
	Symbol that can be replaced by any element of some designated set (def. 3)	Capable of varying in value (def. 4)

This dual usage of the same term *variable* raises some interesting mathematical and pedagogical issues. Freudenthal decries the blending of the two ideas. He stresses the importance of the original, kinematic sense of the notion of variable, especially when one comes to study functions and expressions similar to "ε approaches 0." From his point of view, the attempt on the part of mathematical purists to outlaw such phraseology—using a single name for both polyvalent names and variable objects, as "things which stripped from mere frill boil down to the same" (Freudenthal 1983, 493)—was misguided, for it hides the underlying intuitions. Yet such distillation is one of the fundamental aspects of mathematical activity, no less an adversary than Alfred North Whitehead lines up on the other side of the fence from Freudenthal. Whitehead (1911) describes this habit "of using the same symbol in different but allied senses" as "puzzling to those engaged in tracing out meanings, but . . . very convenient in practice. . . . The one essential requisite for a symbol in . . . [the mathematician's] eyes is that whatever its possible varieties of meaning, the formal laws for its use shall always be the same."

Students need more experience in observing patterns and in making generalizations before they use variables.

This practice may lead to good mathematics, but it is not helpful for most students. It means that the mathematical meaning of a statement is determined by its context rather than by the formal rules that apply to it. Students have to study the formal rules but then learn to interpret statements from their contexts. Here are three examples to illustrate the point.

1. Consider something as simple as the equals sign. We all know that "equals" is an equivalence relationship: if $a = b$, then $b = a$. But consider these two statements:

   a. For every real number x,
$$\frac{1}{x-1} - \frac{1}{x+1} = \frac{2}{x^2-1}.$$

   b. For every real number x,
$$\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}.$$

In a formal sense, these two statements are equivalent. But in reality, (*a*) is a simple statement about subtracting algebraic fractions, whereas (*b*) represents a partial fractions decomposition of the fraction $2/(x^2 - 1)$. (For a more elementary example, consider the statements "$2 \times 3 = 6$" and "$6 = 3 \times 2$." The first statement is about multiplication, the second about factoring.) That is, in context, the equals sign is not read as "is formally equivalent to"; it is read as "yields." For example, (*b*) is interpreted to mean "the fraction
$$\frac{2}{x^2-1}$$
yields
$$\frac{1}{x-1} - \frac{1}{x+1}$$
when it is decomposed."

2. "Consider the expression $a/(a + 1)$, where a is a positive integer." When you read that statement do you imagine—

   a. one particular value (perhaps 5/6, where $a = 5$);

   b. an algebraic expression to be manipulated (as in an inductive proof, perhaps);

   c. a static set of values including the fractions 1/2, 2/3, and so on; or

   d. the motion of a "running" over **N** and therefore representing a fraction which approaches 1 in value as a increases without bound?

Your interpretation determines how easily you can work with the statement.

3. Finally, suppose a problem begins with the following statement: "Prove that for any number x. . . ." Some students, of course, will simply select a particular number ("Let $x = 7$"); they fail to understand that the mathematician seeks to derive the general result from the consideration of a specific, but generic case. Even so, however, the generalization may be understood in two ways:

   a. We proved the result about an unspecified number x precisely because such an unspecified number has no distinguishing characteristics. Hence, we can conclude that it is a representative of a whole class, and thus the proof is general.

   b. We proved the result about all numbers at once, since any number could have been used in the computation instead of x. That is, x was an unspecified placeholder representing all numbers at the same time.

   These subtle differences can result in different interpretations when students are asked to engage in more complex proofs and mathematical arguments.

In sum, *the meaning of variable is variable;* using the term differently in different contexts can make it hard for students to understand. As teachers, we should be sensitive to this multiple usage.

A Diversion

At the beginning of part 2 we gave ten definitions of the term *variable*. Those definitions were taken from the sources listed below. Can you guess which definition came from which source? Answers are given at the end of the article.

a. Murray, James A. H. *A New English Dictionary on Historical Principles* 10. Part 2. Oxford: Clarendon Press, 1928.

b. Klein, Ernest. *A Comprehensive Etymological Dictionary of the English Language*. Amsterdam, N.Y.: Elsevier Scientific Publishing Co., 1971.

c. Whitehead, Alfred N., and Bertrand Russell. *Principia Mathematica*. 2d ed. Cambridge: Cambridge University Press, 1925.

d. *Academic American Encyclopedia*. Danbury, Conn.: Grolier, 1983.

e. Forsythe, Alexandra I., Thomas A. Keenan, Elliot I. Organick, and Warren Stenberg. *Computer Science*. New York: John Wiley & Sons, 1969.

f. Carberry, M. Sandra, A. Toni Cohen, and Hatem Khalil. *Principles of Computer Science: Concepts, Algorithms, Data Structures, and Applications*. Rockville, Md.: Computer Science Press, 1986.

g. Harris, John. *Lexicon Technicum, or, an Universal English Dictionary of Arts and Sciences: Explaining Not Only the Terms of Art, but the Arts Themselves*. London: D. Brown, 1710.

h. Gellert, W., H. Küstner, M. Hellwich, and H. Kästner, eds. *The VNR Concise Encyclopedia of Mathematics*. New York: Van Nostrand Reinhold, 1975.

i. *Webster's Third New International Dictionary*. Springfield, Mass.: G. & C. Merriam Co., 1966.

j. Glenn, John A., and Graham H. Littler, eds. *Dictionary of Mathematics*. Totowa, N.J.: Barnes & Noble Books, 1984.

On the Learnable Meaning of Variable

As we have seen, the concept of variable is both variable and elusive. It is hard to describe and a good deal harder to learn. In discussing the issues raised in parts 1 and 2, our group generated a fair amount of heat—and perhaps just a bit of light. Here are three suggestions related to teaching the concept of variable that emerged from our discussions.

1. Variables are tools for expressing mathematical generalizations. It should help, therefore, if students had the habit of verbalizing such generalizations before they were asked to formalize those generalizations using the language of mathematics. It might be useful to make a habit of asking students to summarize some of their observations about arithmetic in their own

words, in short exercises. ("What happens when you add three times a number to two times that number?" or "Can you say whether the sum of two odd numbers will necessarily be even or odd?") By asking students to observe patterns and to summarize them verbally, we may help make the transition from arithmetic to algebra. When algebraic language is introduced, it may then serve students as a convenient shorthand for expressing ideas with which they have already grappled.

2. The dynamic aspects of the variable concept should be stressed whenever it is appropriate and feasible. At the simplest level, students can be led to observe that the cost (variable) that appears on a gasoline pump is a (dynamic, linear) function of the amount of gasoline that emerges from the pump. Other dynamic relationships include time-dependent phenomena, for example, the temperature in a room at time t, the distance an object falls in t seconds, and so on. We note that the availability of computers and computer-based tools now makes it much easier to capture the dynamic aspects of such phenomena. Real-time probes attached to microcomputers can record and graph the temperatures of various objects (e.g., ice-and-water mixtures) automatically (see, e.g., Nachmias and Linn [1987]); they will also produce the distance and velocity graphs as a person moves forward and backward from a sensor attached to the computer (Brasell 1987). And, of course, computer programming itself makes use of variables in lots of different ways. With languages such as Logo available for elementary school students to explore (Papert 1980), graphing games such as Green Globs (Dugdale 1984), and "dynamic graphers" (Schoenfeld, in press) of various sorts on the horizon, opportunities to make the notion of variable more meaningful will continue to expand.

3. Much of the algebra curriculum consists of solutions of equations independent of the context from which they were drawn. Many applications of algebra, for example, the standard word problems, are so neatly packaged, and taught in separate units, that students miss the point—that the language of algebra is a powerful means of capturing the mathematical essence of a wide variety of situations and not just the ones they've been taught to solve. We recommend that students be exposed to a wide range of problems for which algebra, especially in its role of a tool for mathematical generalization, can be useful. Here are two such problems.

Problem 1

A friend of mine observed that

$$39 \times 62 = 93 \times 26;$$

that is, when you reverse the numbers 39 and 62, you find that the product of the reversed numbers is the same as the product of the two original numbers. Do any other pairs of two-digit numbers have this property?

Problem 2

In England a 15-percent "value added tax" must be paid on all purchases. Suppose you go to a discount house, where all items are marked 20 percent off. Would you rather get the discount first and then pay the tax on the discounted price, or would you rather add in the value-added tax and then get the 20-percent discount on the total?

What about a 30-percent-off sale in a state that has a 7-percent sales tax? What about other discounts and other tax rates? (This problem is borrowed from Mason, Burton, and Stacey's [1982] *Mathematical Thinking*. The reader might enjoy their discussion of the problem and others as well.)

Conclusion

Mathematical notation is a wonderful and powerful tool. It is also subtle and difficult to learn. When we have mastered it, we often forget just how hard it was and just what went into the development of our understanding. Look at how hard it is to learn to ride a bicycle and how easy it seems once you know how. This article has been an exercise in trying to recapture some of the subtlety, and difficulty, of the idea of variable: to bring out into the open some of the concept's properties that might make it hard to learn. We hope it has been useful, and we welcome your comments.

Answers

1(*b*), 2(*i*), 3(*g*), 4(*a*), 5(*d*), 6(*j*), 7(*h*), 8(*e*), 9(*f*), 10(*c*)

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