ARGUMENTATION AND PROOF IN EXAMPLES TAKEN FROM FRENCH AND GERMAN TEXTBOOKS

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A study of French and German curricula of secondary schools have shown that arguments of plausibility and arguments of necessity are both encouraged in mathematic teaching. We show that the mathematics textbooks used in these curricula give examples of argumentations and proofs involving both kinds of arguments. These examples illustrate the theoretical frame presented in this paper and chosen to explain the use and the combination of arguments which can be understood through the concept of the function of validation.

We have shown in [Cabassut 2003, 2005] that plausible reasoning, pragmatic proofs, visual perception or induction are present in validations used in the mathematical curricula of secondary schools in France and Baden-Wurtemberg. We expose here the theoretical frame chosen to compare the validation of mathematic statements in France and in Germany.

1 The theoretical frame

1.1 Toulmin's theory of arguments

From [Toulmin 1958] we consider an argument as a three-part structure (data, warrant, claim). We apply a warrant to data to produce a claim. We call arguments of plausibility the "arguments in which the warrant entitles us to draw our conclusions only tentatively (qualifying it with a 'probably') subject to possible exceptions ('presumably') or conditionally ('provided that …')" [Toulmin 1958, 148]. Toulmin calls them 'probable arguments'. As probability has a strong mathematical connotation, we prefer the term 'argument of plausibility'. It's also a reference to the plausible reasoning of [Polya 1954] described under the name of abduction by [Peirce 1960, 5.189]: "The surprising fact C is observed, but if A were true, C would be a matter of course; hence, there is reason to suspect that A is true". We call arguments of necessity "the arguments in which the warrant entitles us to argue unequivocally to the conclusion" [Toulmin 1958, 148]. The 'modus ponens' is an example of argument of necessity: A is observed, and 'if A then C' is true, then C is necessarily true. We call 'validation' a reasoning that intends to assert, necessarily or plausibly, the truth of a statement. A 'proof' is a validation using only arguments of necessity and an 'argumentation' is a validation using arguments of plausibility and maybe arguments of necessity.

[Perelman, Olbrechts-Tyteca 1969] distinguishes different audiences to which the validation is addressed. The self as audience and a particular audience are subject to persuasion which is grounded on the specificity of the character of the audience. A universal audience is subject to conviction which is based on rationality. We can consider that these audiences or their rationalities are attached to the institutions in which the validation is established.

For [Balacheff 1991, 188-189] "argumentation and mathematical proof are not of the same nature: The aim of argumentation is to obtain the agreement of the partner of the interaction, but not in the first place to establish the truth of some statement. As a social behaviour it is an open process, in other words it allows the use of any kind of means; whereas, for mathematical proofs, we have to fit the requirement for the use of knowledge taken from a body of knowledge on which people (mathematician) agree". We can consider that in a mathematical institution where a statement is proved (for example in a journal of mathematics, or in a mathematical seminar), the people (the mathematicians) agree with the mathematical knowledge that only admits arguments of necessity. The arguments of necessity in a mathematical proof of a statement establish the necessity of the truth of this statement. When a mathematician uses an argument of plausibility to sustain a conjecture, or as a heuristic to look for the proof of an assertion, he doesn't make his assertion more true or more plausible from the point of view of mathematical theory. This mathematician can obtain the agreement from other mathematicians on the plausibility of his conjecture, but this agreement will be based on intuition, experimental methods or everyday mathematical practice using inferences taken from outside of mathematical theory. For example, plausible reasoning is not admitted in mathematical theory. In an other institutions, for example in biology classes, or in maths classes during the heuristic phase of problem-solving, or in a pupils' group discussing the proof of a statement, an argumentation that uses arguments of plausibility will establish the plausibility of the statement if the arguments of plausibility employed are part of the body of knowledge of this institution. In this case all the members of this institution agree with the argumentation. In mathematical institutions or other institutions, the partners' agreement means that arguments employed are in the body of knowledge shared by all the members of the institution.

Mathematical institutions have formal theories to prove statements. Other institutions often have informal theory to argue for statements. "Demonstrative reasoning has rigid standards, codified and clarified by logic (formal or demonstrative logic), which is the theory of demonstrative reasoning. The standards of plausible reasoning are fluid, and there is no theory of such reasoning that could be compared to demonstrative logic in clarity or would command comparable consensus" [Polya 1954].

To explain the combination of arguments of necessity and arguments of plausibility in a same validation we need to identify the different functions of a validation.

1.2 Functions of a validation

De Villiers proposed different functions of the proof in mathematics : "verification (concerned with the truth of a statement), explanation (providing insight into why it is true), systematisation (the organisation of various results into a deductive system of axioms, major concepts and theorems), discovery (the discovery or invention of new results), communication (the transmission of mathematical knowledge)" [De Villiers 1990, 18]. We extend these functions of the proof to the validation in the teaching of mathematics. For the function of verification we distinguish two functions: the function of plausibility verifies the plausibility of the truth of an assertion by means of arguments of plausibility; the function of proof verifies the necessity of the truth of an assertion by means of arguments of necessity.

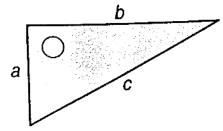
2 Examples from textbooks

2.1 mathematical properties available but not used

We consider a proof of Pythagora's theorem. We use Clarke's methodology looking at the differences for similar proofs. The first proof is found in a French textbook[1].

A. Découpage et constructions

1. Construire et découper, dans du carton, un triangle rectangle dont les côtés perpendiculaires mesurent par exemple :
a = 4 cm et b = 7,5 cm.
Ce triangle est une équerre.



2. À l'aide de cette équerre, réaliser les deux figures suivantes.

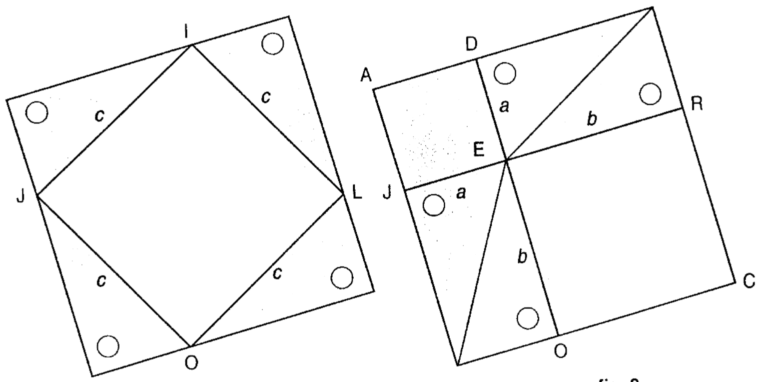


fig. 1 fig. 2

B. Observations, calculs et conclusion

1. En observant les figures 1 et 2, expliquer pourquoi :
aire(JOLI) = aire(JADE) + aire(OCRE).

2. Exprimer, en fonction de a, b ou c, les aires des carrés JOLI, JADE et OCRE.

3. Compléter par les lettres a, b ou c l'égalité suivante (qui résulte de ce qui précède) :

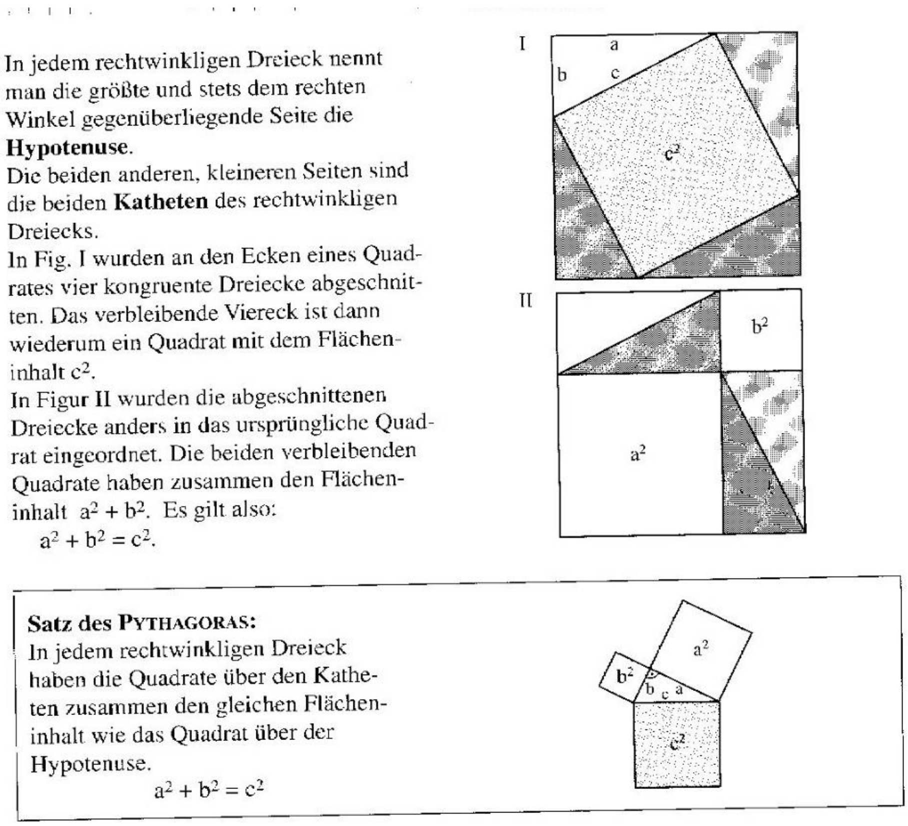
$[\ldots]^2 = [\ldots]^2 + [\ldots]^2$

The proof is based on a puzzle technic[2] (pragmatic argument) where it is asserted without mathematical argument that the inner quadrangle JILO is a square. At this class level, it could be proved that JILO is a lozenge and that ∠JIL = 90° by consideration of angles. We assume that this fact is not proved because the main function of this proof is explanation. Pythagoras' theorem is explained as a theorem on the equality of areas: the area of the inner square equals the area of the two other squares which appears clearly in the puzzle technic. To prove that JILO is a square

[1] The French textbook is : Le nouveau Pythagore class 4ème , 1998, Hatier, 165.

[2] [Knipping 2003, 84] observes classes where this proof is done. She mentions that the proof was known from the old Hindus.

would distract from the main explanation in terms of areas. We have found in Baden-Wurtemberg a textbook[3] from a similar grade level with the same kind of proof where it is not justified that the inner quadrangle is a square.



In jedem rechtwinkligen Dreieck nennt man die größte und stets dem rechten Winkel gegenüberliegende Seite die **Hypotenuse**.
Die beiden anderen, kleineren Seiten sind die beiden **Katheten** des rechtwinkligen Dreiecks.
In Fig. I wurden an den Ecken eines Quadrates vier kongruente Dreiecke abgeschnitten. Das verbleibende Viereck ist dann wiederum ein Quadrat mit dem Flächeninhalt c^2.
In Figur II wurden die abgeschnittenen Dreiecke anders in das ursprüngliche Quadrat eingeordnet. Die beiden verbleibenden Quadrate haben zusammen den Flächeninhalt $a^2 + b^2$. Es gilt also:

$$a^2 + b^2 = c^2.$$

Satz des Pythagoras:
In jedem rechtwinkligen Dreieck haben die Quadrate über den Katheten zusammen den gleichen Flächeninhalt wie das Quadrat über der Hypotenuse.

$$a^2 + b^2 = c^2$$

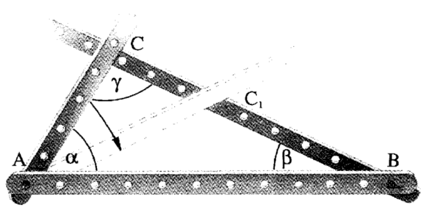
Both textbooks prefer a visual argument for which the inner quadrangle looks very plausibly like a square. The function of verification of the plausibility and the function of explanation make it so that a visual argument is preferred to a mathematical argument on angles. The German textbook used explicitly the isometric triangle property which is available; the French textbook used implicitly this property which is not available.

We have found other textbooks[4] where it is proven[5] that the inner quadrangle is a square with rigor and where the function of verification of the proof is stronger than in the previous textbooks.

[3] The German textbook is : Lambacher Schweizer class 9, Klett Verlag, 1997, 70.
[4] For example : Math 4ème, Bordas, 1998, 203.
[5] Using properties of angles.

2.2 mathematical property not available

1 a) Welche Winkel des Dreiecks ABC ändern sich, wenn man den Stab AC nach rechts schwenkt und der Stab BC seine Richtung beibehält? Welcher Winkel wird größer, welcher kleiner? Vergleiche die Zunahme des einen Winkels und die Abnahme des anderen mit dem Winkelmesser.
b) Was lässt sich über die Summe $\alpha + \beta + \gamma$ sagen, wenn sich AC und BC mehr und mehr der Lage von AB nähern?



In Fig. 1 sind g und h parallel. In diesem Fall ist $\alpha + \delta = 180°$.
Ist – wie in Fig. 2 – h nicht parallel zu g, so entsteht ein Dreieck ABC. Der Winkel γ bei C hat gegenüber Fig. 1 um eine Winkelweite β abgenommen; gleichzeitig ist bei B ein neuer Winkel derselben Weite β entstanden (Wechselwinkel an Parallelen).
Jetzt gilt daher: $\alpha + \beta + \gamma = 180°$.

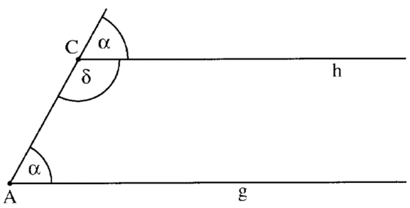


Fig.1

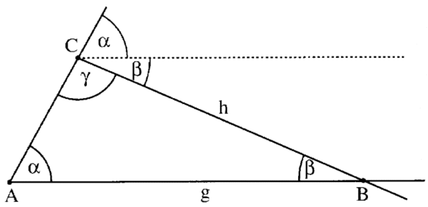


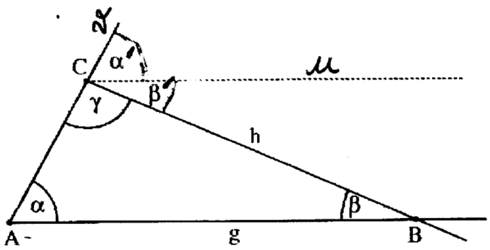
Fig.2

The proof of the property of the sum of angles in a triangle is illustrated in two textbooks. In both textbooks, the validation begins with an argumentation using pragmatic arguments (in the German textbook[6]: action with a Meccano game and measurement, in the French textbook[7]: cutting out, measurement), visual argument and mathematical arguments (calculation of the sum of angles). This previous validation develops two functions: the function of discovery of the property of the sum of angles in a triangle, and the function of the verification of the plausibility of the result. With the German Meccano technic, explanation of the proof is central: when the angle α decreases, the angle γ increases in a compensating way. The last German proof tries to develop the function of explanation, by determining the compensation β between α and γ, and the function of verification of the proof, by

[6] The German textbook is: Lambacher Schweizer class 7 (12-13 years old), Klett Verlag, 1994, 105.

[7] The French textbook is: “Nouveau transmath class 4ème” , 1997, Nathan, 225, class 5ème (12-13 years old).

using mathematical arguments of necessity. The use of mathematical argument of necessity seems to avoid the use of pragmatic arguments. In fact, there is a visual argument in figure 2 to assert that the angle β in C is equal to the alternate-interior angle β in B. Holland [2001, 56-57] shows that a formal proof without visual argument needs the technology of orientated angles: this technology is not available at this class level[8]. The proof with orientated angles without visual argument is the following:



step	data of the step	warrant of the step	claim of the step
1	hypothesis		ABC triangle with α=([AB),[AC)), β=([BC), [BA)), γ=([CA), [CB))
2	hypothesis		u is half straight line with C as origin, parallel to g, and with opposite direction to the half straight line [BA) with B as origin. A
3	hypothesis		v is the other half straight line of origin C completing the half straight line [CA)
4	hypothesis		α'= angle(u,v)
5	hypothesis		β'=angle([CB),u)
6	1, 2, 3, 4	definition of corresponding angles	α et α' are corresponding angles
7	1, 2, 5	definition of alternate-interior angles	β et β' are alternate-interior angles
8	2, 6	theorem of corresponding angles in case of parallelism	α=α'
9	2, 7	theorem of alternate-interior angles in case of parallelism	β=β'
10	3	definition of straight angle	([CA),v) = 180°
11	1, 2, 3, 4, 5	property of angles	([CA),v)= ([CA), [CB))+ ([CB),u) +(u,v) = α'+β'+γ
12	8, 9, 11	calculus	α+β+γ=180°

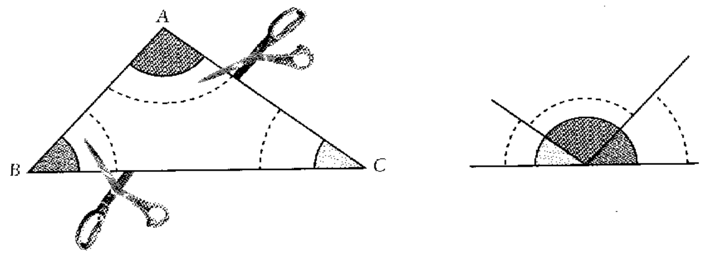
We can ask why the German textbook proposes another proof with a non mathematical argument (the visual argument revealed by Holland) following the

[8] class 7 for Germany and 5ème for France.

validations with Meccano or measurements proof which used also non mathematical arguments. On one hand, using visual arguments helps to develop the function of explanation and the function of verification of the proof, even if this last function is not completely accomplished in a formal way because of the visual argument; it could be considered as accomplished in a mathematics class where a visual argument could be, sometimes, used as an argument of necessity, for example in the cases in which orientation is involved. On the other hand, the function of systematisation is developed in the last proof because previous mathematical results[9] are used to establish, in a deductive way, a new result.

① En découpant

a. Trace sur papier blanc un triangle *ABC* comme celui dessiné ci-dessous.
b. Découpe chacun de ses angles, puis « regroupe »-les comme l'indique le dessin ci-dessous.



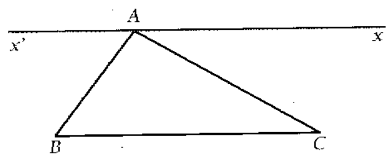
c. Quelle semble être la valeur de la somme des angles du triangle ?

② En mesurant avec ton rapporteur

a. Trace trois triangles.
b. Mesure les angles de chacun de ces triangles à l'aide d'un rapporteur, puis calcule la somme des angles de chaque triangle.
c. Quelle semble être la valeur de cette somme ?

③ Une démonstration à présent

La droite $(x'x)$ est parallèle à la droite (BC) et passe par A.



a. Compare les angles $\widehat{ABC}$ et $\widehat{BAx'}$, puis $\widehat{ACB}$ et $\widehat{CAx}$.
b. Explique alors pourquoi la somme des trois angles du triangle *ABC* est égale à 180°.

[9] Calculation rules, definition and properties of angles: corresponding, alternate-interior, supplementary.

3 Conclusion

We have seen that arguments of plausibility can be used to develop the functions of explanation and plausibility, or to replace a missing mathematical argument to help to develop the function of systematisation. We have seen that mathematical arguments can be used to develop the functions of proof, explanation or systematisation. In these examples both kind of arguments are combined in a same validation. In [Cabassut 2002, 2003, 2005] we show other examples[10] of these combinations. On one hand, there are non mathematical validations in others institutions[11]. Mathematical arguments can join non mathematical arguments to make a validation[12] in the institution that teaches mathematics: this new hybrid validation is called a didactic validation, and is created by transposing a validation used in a non-mathematical institution onto a validation used by institutions that teach mathematics. In this case, special functions are developed (function of explanation in the previous examples, function of discovery to prepare the limits in [Cabassut 2002]). On the other hand, the replacement[13] of some mathematical arguments (available or not) by non-mathematical arguments is a transposition of a mathematical proof onto a didactic validation where the functions of explanation, plausibility or proof, and systematisation can be developed. In this sense didactic validation is the double transposition of mathematical proof and of the validation of non-mathematical institutions. The combination of arguments of plausibility and mathematical arguments are generally not allowed in a context of assessment where the functions of communication, systematisation and proof are developed as shown in [Cabassut 2005]. The pupils have to understand the change in the didactical contract depending on what functions of the validation are developed and on what combinations of arguments are allowed. The main difficulty is that these two kinds of arguments refer to two different conceptions of truth.

[10] like formula of the circumference and area of a circle.

[11] like daily life or experimental sciences classes where pragmatic arguments can be sufficient.

[12] for example for the sum of angles of a triangle: measurement, use of Meccano or cutting.

[13] for example in the proof of Pythagora's theorem or in the last German and French proofs of the sum of angles of a triangle.

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