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EXPLICIT CONSTANTS FOR GAUSSIAN UPPER BOUNDS ON HEAT KERNELS

By E. B. Davies

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1. Introduction. We use logarithmic Sobolev inequalities to obtain Gaussian upper bounds on the heat kernels associated with various second order elliptic or hypoelliptic operators. These bounds are valid without any continuity conditions on the coefficients, and under very weak curvature conditions in the case of Laplace-Beltrami operators. Moreover the constants are completely explicit and the bounds are close to optimal.

If $K(t, x, y)$ is the heat kernel for a second order elliptic operator $H \geqslant 0$ on a manifold M of dimension N, we say that e^{-Ht} is ultracontractive if there is a uniform bound of the type

$$0 \leqslant K(t, x, y) \leqslant c(t) \tag{1.1}$$

where $c(t) < \infty$ for all $t > 0$. There are a great variety of different ways of proving such bounds, and for the case

$$c(t) = a_1 t^{-\mu/2} \tag{1.2}$$

which is the only one we shall treat here, it is known that (1.1) is equivalent to a Sobolev inequality

$$\|f\|^2_{2\mu/(\mu-2)} \leqslant a_2 \langle Hf, f \rangle \tag{1.3}$$

and also to a logarithmic Sobolev inequality

$$\int f^2 \log f \leqslant \epsilon \langle Hf, f \rangle + \beta(\epsilon) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2 \tag{1.4}$$

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for all $0 \leqslant f \in C_c^\infty$ and all $0 < \epsilon < \infty$, where

$$\beta(\epsilon) = a_3 - \frac{\mu}{4} \log \epsilon. \tag{1.5}$$

We refer to [2, 8, 16, 17] for explicit expression relating the various constants.

If H is a Laplace-Beltrami operator then (1.1) and (1.2) hold with $\mu = N$ if the isoperimetric constant is nonzero [3, p. 101] or if the Ricci curvature is bounded below and the injectivity radius is nonzero [3, p. 127, 198]. A number of other cases are treated in [6].

Although we shall only treat cases for which (1.4) and (1.5) hold for all $0 < \epsilon < \infty$, we mention other situations in which our method may be applied. We first remark that (1.1) and (1.2) hold for small $t > 0$ if (1.4) and (1.5) hold for $0 < \epsilon < 1$. This enables us to deal with Example 15 Case A of [6], where $\mu > N$ and M is a compact manifold with a power cusp. It is also possible to replace (1.5) by

$$\beta(\epsilon) = \beta_0 + \beta_1 \epsilon^{-\sigma}.$$

However the integral $M(t)$ computed in Theorem 3 can only be finite for the given choice of $\epsilon(p)$ if $\lambda > 1$ and $0 < \sigma < 1$. Thus we can obtain Gaussian upper bounds for the heat kernel of the Laplace-Beltrami operator of a compact manifold with an exponential cusp, as discussed in Example 15, Case B of [6].

Although the bound (1.1) is nontrivial it is only really efficient when $x = y$, and away from the diagonal one expects a bound of the form

$$0 \leqslant K(t, x, y) \leqslant a_\delta t^{-\mu/2} \exp\left[-\frac{d(x, y)^2}{4(1 + \delta)t} \right] \tag{1.6}$$

in the case of Laplace-Beltrami operators, where d is the metric and $\delta > 0$ is arbitrary. Such results have indeed been proved under a variety of conditions [3, 4, 12], but our point here is to show that (1.6) is a simple consequence of (1.1), and that no further hypotheses are needed.

In the majority of this paper we consider Laplace-Beltrami operators, but our technique is also applicable to second order elliptic operators on $L^2(\mathbf{R}^N)$ and even to certain hypoelliptic operators. The passage from (1.1) to (1.6) then requires boundedness of the coefficients but no continuity hypotheses. In spite of the extensive work on this problem [1, 13], we present this explicitly in Section 3 because of its simplicity and the explicit constants obtained.

If M is a Riemannian manifold of dimension N with volume element dv and metric d, the Laplace-Beltrami operator H is associated with the quadratic form

$$Q(f) = \int |\nabla f|^2 dv.$$

This form is closable on C_c^∞ and its closure is associated in a standard way with a self-adjoint operator $H \geqslant 0$, which coincides on C_c^∞ with $-\Delta$. Since Q is a Dirichlet form we have [9]

$$\|e^{-Ht}f\|_p \leqslant \|f\|_p \tag{1.7}$$

for all $t \geqslant 0$ and $1 \leqslant p \leqslant \infty$. Moreover e^{-Ht} has a C^∞ kernel $K(t, x, y)$ and the bound (1.1) may be rewritten as an operator bound

$$\|e^{-Ht}f\|_\infty \leqslant c(t)\|f\|_1. \tag{1.8}$$

One may interpolate between (1.7) and (1.8) to obtain other L^p to L^q bounds, and ultimately to get (1.3) or (1.4).

We shall obtain our off-diagonal heat kernel bounds by proving ultracontractivity for the weighted semigroup $\phi^{-1}e^{-Ht}\phi$, where ϕ is chosen appropriately. This is done by proving an L^p logarithmic Sobolev inequality for $\phi^{-1}H\phi$ for all $2 \leqslant p < \infty$, and then integrating it in the standard manner [8, 10]. The use of such operators $\phi^{-1}H\phi$ has been commonplace for several years [5, 15] but we believe that its combination with logarithmic Sobolev inequalities is new.

Throughout the paper we assume that H satisfies (1.4) and (1.5) with given constants a_3 and μ. We make fundamental use of the fact [8, 10] that these imply

$$\int f^p \log f dv \leqslant \epsilon \langle Hf, f^{p-1}\rangle + \frac{2\beta(\epsilon)}{p}\|f\|_p^p + \|f\|_p^p \log\|f\|_p \tag{1.9}$$

for all $0 \leqslant f \in C_c^\infty$ and all $2 \leqslant p < \infty$.

2. Weighted ultracontractive bounds. We put $\phi = e^{\alpha\psi}$ where $\alpha \in \mathbf{R}$ and $\psi: M \to \mathbf{R}$ satisfies $|\nabla \psi| \leqslant 1$ everywhere. In order to avoid technical problems we shall initially assume that ψ is C^∞ and bounded, but this condition will be removed later by a limiting argument.

LEMMA 1. *If* $0 \leqslant f \in C_c^\infty$ *and* $2 < p < \infty$ *then*

$$\langle Hf, f^{p-1}\rangle \leqslant 2\langle \phi^{-1}H\phi f, f^{p-1}\rangle + \alpha^2 p\|f\|_p^p. \tag{2.1}$$

Proof. Elementary calculus yields

$$\langle \phi^{-1}H\phi f, f^{p-1}\rangle = \int \nabla(\phi f)\cdot\nabla(\phi^{-1}f^{p-1})dv$$

$$= \int (f\nabla\phi + \phi\nabla f)\cdot(-\phi^{-2}f^{p-1}\nabla\phi + \phi^{-1}(p-1)f^{p-2}\nabla f)dv$$

$$= \int(-\alpha^2 f^p(\nabla\psi)^2 + \alpha(p-1)f^{p-1}\nabla f\cdot\nabla\psi.$$

$$-\alpha f^{p-1}\nabla f\cdot\nabla\psi + (p-1)f^{p-2}(\nabla f)^2)dv$$

$$\geqslant -\alpha^2\int f^p dv - |\alpha|(p-2)\int f^{p-1}|\nabla f|dv + \int \nabla f\cdot\nabla(f^{p-1})dv.$$

Using the bound

$$2\int f^{p-1}|\nabla f|dv \leqslant s\int (f^{p/2-1}|\nabla f|)^2 dv + s^{-1}\int f^p dv$$

$$= \frac{s}{p-1}\langle Hf, f^{p-1}\rangle + s^{-1}\|f\|_p^p \tag{2.2}$$

we obtain

$$\langle \phi^{-1}H\phi f, f^{p-1}\rangle \geqslant \left(1 - \frac{|\alpha|(p-2)s}{2(p-1)}\right)\langle Hf, f^{p-1}\rangle$$

$$-\left(\alpha^2 + \frac{|\alpha|(p-2)}{2s}\right)\|f\|_p^p.$$

The estimate now follows upon putting

$$s = \frac{p-1}{(p-2)|\alpha|}.$$

LEMMA 2. *If* $2 < p < \infty$ *and* $0 \leqslant f \in C_c^\infty$ *then*

$$\int f^p \log f dv \leqslant \epsilon \langle \phi^{-1} H \phi f, f^{p-1} \rangle$$

$$+ \gamma(\epsilon, p) \|f\|_p^p + \|f\|_p^p \log \|f\|_p \tag{2.3}$$

where

$$\gamma(\epsilon, p) = \frac{2}{p}\left(a_3 - \frac{\mu}{4} \log \frac{\epsilon}{2}\right) + \frac{\epsilon \alpha^2 p}{2}.$$

Proof. By combining (1.9) and (2.1) we obtain

$$\int f^p \log f dv \leqslant 2\epsilon \langle \phi^{-1} H \phi f, f^{p-1} \rangle$$

$$+ \epsilon \alpha^2 p \|f\|_p^p + \frac{2\beta(\epsilon)}{p} \|f\|_p^p + \|f\|_p^p \log \|f\|_p$$

from which the result follows upon replacing ϵ by $\epsilon/2$.

THEOREM 3. *We have*

$$\|\phi^{-1} e^{-Ht} \phi f\|_\infty \leqslant a_\delta t^{-\mu/4} e^{(1+\delta)\alpha^2 t} \|f\|_2 \tag{2.4}$$

for all $\delta > 0$, $t > 0$ and $f \in L^2$.

Proof. We follow [8, 10] closely. We put

$$\epsilon(p) = \lambda 2^\lambda t p^{-\lambda}$$

where $\lambda > 1$, and define $p(s)$ for $0 \leqslant s < t$ by the implicit formula

$$s = \int_2^p \frac{\epsilon(q)}{q} dq$$

so that $p(s) \to \infty$ as $s \to t$. If we also put

$$N(s) = \int_2^{p(s)} \frac{\gamma(\epsilon(q), q)}{q} \, dq$$

then an explicit calculation using the formulae

$$\frac{dp}{ds} = \frac{p}{\epsilon(p)}, \qquad \frac{dN}{ds} = \frac{\gamma(\epsilon(p), p)}{p} \frac{dp}{ds}$$

yields the inequality

$$\text{(2.5)} \qquad \frac{d}{ds}\{ \| \phi^{-1} e^{-Hs} \phi f \|_{p(s)} e^{-N(s)} \} \leqslant 0$$

for all $0 < s < t$. This implies that

$$\| \phi^{-1} e^{-Hs} \phi f \|_{p(s)} \leqslant e^{N(s)} \| f \|_2.$$

In the limit $s \to t$ putting

$$M(t) = \int_2^{\infty} \frac{\gamma(\epsilon(p), p)}{p} \, dp$$

we obtain

$$\| \phi^{-1} e^{-Ht} \phi f \|_{\infty} \leqslant e^{M(t)} \| f \|_2.$$

The validity of (2.5) depends upon the choice of a suitable domain $\mathfrak{D}$ on which the formal calculation may be justified. This is a technical problem which we defer to Section 4.

It remains to calculate $M(t)$. We have

$$M(t) = \int_2^{\infty} \gamma(\lambda 2^{\lambda} t p^{-\lambda}, p) p^{-1} dp$$

$$= \int_2^{\infty} \frac{2}{p^2} (a_3 - \frac{\mu}{4} \log(\lambda 2^{\lambda - 1} t p^{-\lambda}) dp$$

$$+ \int_2^\infty \frac{\alpha^2}{2} \lambda 2^\lambda t p^{-\lambda} d\lambda$$

$$= -\frac{\mu}{4} \log t + \alpha^2 t \frac{\lambda}{\lambda - 1} + \text{const.}$$

We now choose λ so that

$$\frac{\lambda}{\lambda - 1} = 1 + \delta.$$

COROLLARY 4. *We have*

$$0 \leqslant K(t, x, y) \leqslant a_\delta t^{-\mu/2} \exp[(1 + \delta)\alpha^2 t + \alpha(\psi(x) - \psi(y))]$$

for all $t > 0$, $x, y \in M$ *and* $\alpha \in \mathbf{R}$.

Proof. Replacing ϕ by ϕ^{-1} and then taking adjoints of (2.4) we obtain

$$\| \phi^{-1} e^{-Ht} \phi f \|_2 \leqslant g(t) \| f \|_1$$

where

$$g(t) = a_\delta t^{-\mu/4} e^{(1+\delta)\alpha^2 t}$$

Composing this estimate with (2.4) yields

$$\| \phi^{-1} e^{-Ht} \phi f \|_\infty \leqslant g(t/2)^2 \| f \|_1$$

or equivalently

$$0 \leqslant \phi(x)^{-1} K(t, x, y) \phi(y) \leqslant g(t/2)^2$$

THEOREM 5. *We have*

$$0 \leqslant K(t, x, y) \leqslant a_\delta t^{-\mu/2} \exp\left[-\frac{(\psi(x) - \psi(y))^2}{4(1 + \delta)t} \right] \tag{2.6}$$

for all $t > 0$ *and* $x, y \in M$.

Proof. This depends upon making the choice

$$\alpha = \frac{\psi(y) - \psi(x)}{2(1+\delta)t}.$$

COROLLARY 6. *We have*

$$0 \leqslant K(t, x, y) \leqslant a_\delta t^{-\mu/2} \exp\left[-\frac{d(x, y)^2}{4(1+\delta)t}\right]$$

for all $t > 0$ *and* $x, y \in M$.

Proof. If we put

$$\psi(s) = \min\{d(s, x),\ d(y, x)\}$$

into Theorem 5 we obtain the required bound. Now ψ is bounded, constant outside the ball with centre x and radius $d(y, x)$. Moreover ψ satisfies $|\nabla \psi| \leqslant 1$ in the weak sense that

$$|\psi(s_0) - \psi(s_1)| \leqslant d(s_0, s_1).$$

However ψ is not C^∞. We therefore construct a sequence ψ_n of C^∞ functions which converge uniformly to ψ, equal $d(x, y)$ outside the stated ball, and satisfy $|\nabla \psi_n| \leqslant 1$. This yields the required bound in the limit.

3. Other second order operators. We now consider a very general second order operator on $L^2(\mathbf{R}^N)$ starting with a quadratic form Q defined on C_c^∞ by

$$Q(f) = \int \sum_{i,j} a_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial \bar{f}}{\partial x_j}\, d^N x.$$

We assume that $0 \leqslant a(x) \in L^1_{\mathrm{loc}}$ so that Q is well-defined and nonnegative. We assume that the form Q is closable so that the closure is associated with a self-adjoint operator $H \geqslant 0$ on $L^2(\mathbf{R}^N)$ whose associated semigroup e^{-Ht} is positively preserving and satisfies

$$\|e^{-Ht} f\|_p \leqslant \|f\|_p$$

for all $t \geqslant 0$ and $1 \leqslant p \leqslant \infty$. The condition that Q is closable holds [9] if $a(x) \in C^1$ or $a^{-1}(x) \in L^\infty_{\text{loc}}$.

We next assume that e^{-Ht} has a kernel $K(t, x, y)$ which satisfies

$$0 \leqslant K(t, x, y) \leqslant at^{-\mu/2}$$

for all $t > 0$ and $x, y \in \mathbf{R}^N$. This holds with $\mu = N$ if $0 < \beta \leqslant a(x)$ for some $\beta > 0$ and all $x \in \mathbf{R}^N$, because the relevant logarithmic Sobolev inequality holds by a simple comparison of H with $-\beta\Delta$. It also holds for $0 < t < 1$ with some $\mu \geqslant N$ for certain hypoelliptic operators with C^∞ coefficients [11, 14].

Finally let $\alpha \in \mathbf{R}$ and let $\psi \colon \mathbf{R}^N \to \mathbf{R}$ be a C^∞ bounded function such that

$$\sum_{i,j} a_{ij}(x) \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} \leqslant 1 \tag{3.1}$$

almost everywhere. All the estimates in Section 2 remain valid, and we are able to deduce the bound (2.6) of Theorem 5 in our new situation. This leads to an explicit bound for uniformly elliptic operators. Although we do not obtain lower bounds or bounds on the derivatives of the heat kernel as in [1, 13] our upper bound is close to optimal.

THEOREM 7. *If $a(x)$ is measurable and*

$$0 < \beta \leqslant a(x) \leqslant \gamma < \infty \tag{3.2}$$

for all $x \in \mathbf{R}^N$, then the kernel $K(t, x, y)$ of e^{-Ht} satisfies

$$0 \leqslant K(t, x, y) \leqslant a_1 t^{-N/2} \exp\left[-\frac{|x-y|^2}{4\gamma(1+\delta)t}\right]$$

for all $\delta > 0$, $t > 0$ and $x, y \in \mathbf{R}^N$.

Proof. If we put

$$\psi(s) = \gamma^{-1/2} \min(|s-y|, |x-y|)$$

then ψ is bounded and satisfies (3.1) but is not $\mathbf{C}^\infty$. By a simple regularisation procedure we obtain a sequence ψ_n of C^∞ bounded functions which converge uniformly to ψ and all satisfy (3.1). We deduce that ψ still satisfies (2.6) and obtain the stated bound.

If we let $\mathcal{G}$ be the set of C^∞ bounded functions ψ on $\mathbf{R}^N$ which satisfy (3.1) then we may define a metric d_a on $\mathbf{R}^N$ by

$$d_a(x, y) = \sup\{\psi(x) - \psi(y) : \psi \in \mathcal{G}\}$$

We conjecture that this metric coincides with the metric d_L defined in [14] for hypoelliptic operators. If so the next theorem would yield stronger heat kernel bounds than those of [14].

THEOREM 8. *If* $\delta > 0$ *then there is a bound*

$$0 \leqslant K(t, x, y) \leqslant a_\delta t^{-\mu/2} \exp\left[-\frac{d_a(x, y)^2}{4(1 + \delta)t}\right]$$

valid for all $0 < t < 1$ *and* $x, y \in \mathbf{R}^N$.

Returning to the situation where the coefficients $a_{ij}(x)$ satisfy (3.2) so that H is uniformly elliptic, we now use Theorem 7 to deduce a similar bound for the heat kernel $K_\lambda(t, x, y)$ associated with

$$H_\lambda = H + \lambda V$$

under the assumption that the potential V satisfies

$$|V| \leqslant \epsilon H + \gamma_0 + \gamma_1 \epsilon^{-\gamma_2} \tag{3.3}$$

for all $\epsilon > 0$ and some $\gamma_i < \infty$. This condition is slightly stronger than the condition $V \in K_N$ unless we restrict to a region $\Omega \subseteq \mathbf{R}^N$, in which case it allows strong singularities of V at the boundary of Ω by [7]. The point of (3.3) is that it implies a bound

$$0 \leqslant K_\lambda(t, x, y) \leqslant a(\lambda) t^{-N/2} e^{c(\lambda)t}$$

by [7]. Our next result follows Theorem B.7.1. of [15].

THEOREM 9. *If a satisfies* (3.2) *and V satisfies* (3.3) *then for any* $\delta >$ 0 *we have*

$$0 \leqslant K_\lambda(t, x, y) \leqslant at^{-N/2} \exp\left[-\frac{|x-y|^2}{4\gamma(1+\delta)t} + ct\right]$$

where a and c depend upon $\delta, \beta, \gamma, \lambda, \gamma_0, \gamma_1, \gamma_2, N$.

Proof. It is a standard observation [15], following from functional integration or the Trotter product formula, that

$$\lambda \to \log K_\lambda(t, x, y)$$

is convex. Therefore if $0 < s < 1$ we have

$$0 \leqslant K_\lambda(t, x, y) \leqslant K_0(t, x, y)^{1-s} K_{\lambda/s}(t, x, y)^s$$

$$\leqslant a_1^{1-s} a_{\lambda/s}^s t^{-N/2} \exp\left[-\frac{(1-s)|x-y|^2}{4\gamma(1+\delta/2)t} + sc(\lambda/s)t\right]$$

We now choose s so that

$$\frac{1-s}{1+\delta/2} = \frac{1}{1+\delta}.$$

4. Some technical problems. We describe how to justify the formal calculation (2.5) of Theorem 3. We put

$$\mathfrak{D}_1 = \bigcup_{t>0} e^{-Ht}(L^1 \cap L^\infty)$$

and

$$\mathfrak{D} = \phi^{-1}\mathfrak{D}_1.$$

Since $\phi^{\pm 1}$ are bounded it is clear that $\mathfrak{D}$ is dense in L^p for all $2 \leqslant p < \infty$ and that $\mathfrak{D}$ is invariant under the semigroup

$$e^{-Kt} = \phi^{-1} e^{-Ht} \phi.$$

Since $H \geqslant 0$ is self-adjoint one sees by interpolation that e^{-Ht} is holomorphic on L^p for all $2 < p < \infty$, so the same holds for e^{-Kt}. In particular

$\mathfrak{D}$ lies in the domain of K_p^n for all n and $2 < p < \infty$, where K_p is the generator of e^{-Kt} on L^p. Although K_p is given formally by

$$K_p f = \phi^{-1} H \phi f$$

we cannot assume that $C_c^\infty \subseteq \mathrm{Dom}(K_p)$, particularly in Section 3. There is now no difficulty in justifying the differentiation in (2.5) but we have to prove that (2.3) holds for all $0 \leqslant f \in \mathfrak{D}$, or equivalently

$$\int (\phi^{-1}g)^p \log(\phi^{-1}g)dv \leqslant \epsilon \langle Hg, \phi^{-p}g^{p-1}\rangle$$

$$+ \gamma(\epsilon, p) \|\phi^{-1}g\|_p^p + \|\phi^{-1}g\|_p^p \log \|\phi^{-1}g\|_p \tag{4.1}$$

for all $0 \leqslant g \in \mathfrak{D}_1$, starting from the fact that this holds for all $g \in C_c^\infty$ by Lemma 2.

We now note that $\mathfrak{D}_1 \subseteq \mathfrak{D}_2$ where

$$\mathfrak{D}_2 = \mathrm{Quad}(H) \cap L^1 \cap L^\infty \subseteq L^2.$$

We shall use the inner product

$$\langle f, g\rangle' = \langle f, g\rangle + \langle H^{1/2}f, H^{1/2}g\rangle$$

on Quad(H) and note that Quad(H) is complete for the norm

$$|||f||| = \{\langle f, f\rangle'\}^{1/2}$$

We say $f_n \overset{w}{\to} f$ in Quad(H) if

$$\langle f_n, g\rangle' \to \langle f, g\rangle'$$

for all $g \in \mathrm{Quad}(H)$. A sufficient condition for this is that $|||f_n|||$ is uniformly bounded and $\|f_n - f\| \to 0$.

Lemma 10. *If* $0 \leqslant g \in \mathfrak{D}_2$ *then* $\phi^{\pm 1}g \in \mathfrak{D}_2$ *and* $g^s \in \mathfrak{D}_2$ *for any* $1 < s < \infty$. *Therefore* (4.1) *is implied by the condition that*

$$\int (\phi^{-1}g)^p \log(\phi^{-1}g)dv \leqslant \epsilon\langle H^{1/2}g,\, H^{1/2}(\phi^{-p}g^{p-1})\rangle$$

$$+ \gamma(\epsilon, p)\,\|\phi^{-1}g\|_p^p + \|\phi^{-1}g\|_p^p \log\|\phi^{-1}g\|_p \tag{4.2}$$

for all $0 \leqslant g \in \mathfrak{D}_2$ *and* $2 < p < \infty$.

Proof. It is evident that L^∞ is invariant under $\phi^{\pm 1}$. The invariance of $\operatorname{Quad}(H)$ depends upon an easy computation which shows that $|||f|||$ and $|||\phi^{\pm 1}f|||$ are equivalent norms on the dense linear subspace C_c^∞ of $\operatorname{Quad}(H)$. If $g \in \mathfrak{D}_2$ then

$$g^s = F(g)$$

where F has bounded derivative and vanishes as 0. Since Q is a Dirichlet form one concludes that

$$Q(F(g)) \leqslant cQ(g) < \infty.$$

The passage from (4.1) to (4.2) depends only on the fact that

$$\langle Hu, v\rangle = \langle H^{1/2}u,\, H^{1/2}v\rangle$$

if $u \in \mathfrak{D}_1 \subseteq \operatorname{Dom}(H)$ and $v \in \operatorname{Quad}(H)$.

PROPOSITION 11. *If* $2 < p < \infty$ and (4.2) holds for all $0 \leqslant g \in C_c^\infty$ *then it also holds for all* $0 \leqslant g \in \mathfrak{D}_2$.

Proof. If $0 \leqslant g \in \mathfrak{D}_2$ then there exists a sequence $h_n \in C_c^\infty$ such that $|||h_n - g||| \to 0$. If $\|g\|_\infty = k$ then there exist $F_n \in C^\infty$ with $0 \leqslant F_n \leqslant 2k$, $0 \leqslant F_n' \leqslant 1$ and $F_n(0) = 0$ such that $g_n = F_n(h_n)$ satisfy $g_n \in C_c^\infty$, $0 \leqslant g_n \leqslant 2k$, $\|g_n - g\| \to 0$ and $Q(g_n) \leqslant Q(g)$. Since these bounds imply $g_n \xrightarrow{w} g$ and

$$\limsup |||g_n||| \leqslant |||g|||$$

we deduce that

$$|||g_n - g||| \to 0 \tag{4.3}$$

Next $0 \leqslant g_n \leqslant 2k$ implies $g_n^{p-1} = G(g_n)$ where G has bounded derivative and $G(0) = 0$. Since Q is a Dirichlet form it follows that

$$Q(g_n^{p-1}) \leqslant cQ(g_n) \leqslant cQ(g).$$

Using the fact that multiplication by ϕ^{-p} is bounded operator on Quad(H) we deduce that

$$|||\phi^{-p} g_n^{p-1}||| \leqslant d < \infty$$

for all n. This implies

$$\phi^{-p} g_n^{p-1} \xrightarrow{w} \phi^{-p} g^{p-1} \tag{4.4}$$

Finally the equations (4.3) and (4.4) imply that

$$\langle g_n, \phi^{-p} g_n^{p-1} \rangle' \to \langle g, \phi^{-p} g^{p-1} \rangle'$$

The convergence to the left-hand side of (4.2) as $n \to \infty$ is proved by applying Fatou's lemma to the positive and negative parts of the integrand separately and using the facts that

$$\int (\phi^{-1} g_n)^2 dx \to \int (\phi^{-1} g)^2 dx$$

$$\int (\phi^{-1} g_n)^{p+1} dx \to \int (\phi^{-1} g)^{p+1} dx$$

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