NEW LOGARITHMIC SOBOLEV INEQUALITIES AND AN ε-REGULARITY THEOREM FOR THE RICCI FLOW

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ABSTRACT. In this note we prove a new ε-regularity theorem for the Ricci flow. Let $(M^n, g(t))$ with $t \in [-T, 0]$ be a Ricci flow and H_{x_0} the conjugate heat kernel centered at a point $(x_0, 0)$ in the final time slice. Substituting H_{x_0} into Perelman's $\mathcal{W}$-functional produces a monotone function $\mathcal{W}_{x_0}(s)$ of $s \in [-T, 0]$, the pointed entropy, with $\mathcal{W}_{x_0}(s) \leq 0$, and $\mathcal{W}_{x_0}(s) = 0$ iff $(M, g(t))$ is isometric to the trivial flow on $\mathbb{R}^n$. Our main theorem asserts the following: There exists an $\varepsilon > 0$, depending only on T and on lower scalar curvature and μ-entropy bounds for $(M, g(-T))$, such that $\mathcal{W}_{x_0}(s) \geq -\varepsilon$ implies $|\mathrm{Rm}| \leq r^{-2}$ on $P_{\varepsilon r}(x_0, 0)$, where $r^2 = |s|$ and $P_\rho(x, t) \equiv B_\rho(x, t) \times (t - \rho^2, t]$.

The main technical challenge of the theorem is to prove an effective Lipschitz bound in x for the s-average of $\mathcal{W}_x(s)$. To accomplish this, we require a new log-Sobolev inequality. It is well known by Perelman that the metric measure spaces $(M, g(t), d\mathrm{vol}_{g(t)})$ satisfy a log-Sobolev; however we prove that this is also true for the conjugate heat kernel weighted spaces $(M, g(t), H_{x_0}(-, t)\, d\mathrm{vol}_{g(t)})$. Our log-Sobolev constants for these weighted spaces are in fact universal and sharp.

The weighted log-Sobolev has other consequences as well, including an average Gaussian upper bound on the conjugate heat kernel that only depends on a two-sided scalar curvature bound.

1. INTRODUCTION

Throughout this paper we will assume that the pair

$$(M^n, g(t)), \;\; t \in [-T, 0], \tag{1.1}$$

is a smooth Ricci flow. For simplicity we assume each time slice is complete of bounded geometry. It may also be convenient to assume M is compact when we rely on Perelman's monotonicity formula. However, our results will ultimately only depend on entropy and scalar curvature bounds.

Given a point $(x_0, 0) \in M \times [-T, 0]$ on the final time slice we write

$$H_{x_0}(y, s) = H(x_0, 0 \,|\, y, s) = (4\pi|s|)^{-\frac{n}{2}} \exp(-f_{x_0}(y, s)) \tag{1.2}$$

for the conjugate heat kernel based at $(x_0, 0)$, and

$$d\nu_{x_0}(y, s) = H_{x_0}(y, s)\, d\mathrm{vol}_{g(s)}(y) \tag{1.3}$$

for the associated probability measures on M; see also Definition 2.6.

Definition 1.4 (Perelman [18])**.** Let (M^n, g) be a Riemannian manifold. Given an $f \in C^\infty(M)$ and $\tau > 0$ such that $(4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mathrm{vol}$ has unit mass, we define associated entropy functionals

$$\mathcal{W}(g, f, \tau) \equiv \int [\tau(|\nabla f|^2 + R) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mathrm{vol}, \tag{1.5}$$

$$\mu(g, \tau) \equiv \inf \left\{ \mathcal{W}(g, f, \tau) : \int (4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mathrm{vol} = 1 \right\}. \tag{1.6}$$

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Perelman [18] discovered that $\mathcal{W}$ is nondecreasing in $t = t_0 - \tau$ ($t_0 \in \mathbb{R}$) if $(4\pi\tau)^{-\frac{n}{2}}e^{-f}$ evolves by the conjugate heat equation coupled to the Ricci flow in the time variable $t < t_0$; see Theorem 2.14. As a consequence, the quantity $\mu(g(t), t_0 - t)$ is nondecreasing in $t < t_0$ along any Ricci flow.

1.1. **Pointed entropy and ε-regularity.** Our primary concern in this paper will be a localized version of the $\mathcal{W}$-entropy. Namely, given a Ricci flow as in (1.1), there exists for each point $(x_0, 0)$ in the 0-time slice and $s \in [-T, 0)$ the canonical metric probability space $(M^n, g(s), d\nu_{x_0}(s))$, where $d\nu_{x_0}(s) \equiv H_{x_0}(-, s)\, d\mathrm{vol}_{g(s)}$ is the conjugate heat kernel measure as in (1.3).

Definition 1.7. The pointed entropy at scale $\sqrt{|s|}$ based at x_0 is defined by

$$\mathcal{W}_{x_0}(s) \equiv \mathcal{W}(g(s), f_{x_0}(s), |s|). \tag{1.8}$$

It is a consequence of Perelman's gradient formula, see Proposition 2.23, that

$$\lim_{s\to 0} \mathcal{W}_{x_0}(s) = 0, \;\; \frac{d}{ds}\mathcal{W}_{x_0}(s) \geq 0. \tag{1.9}$$

Moreover, the pointed entropy at a given point and scale vanishes if and only if the flow is isometric to the trivial flow on Euclidean space. One can view this as an example of a rigidity theorem.

What is often useful in such situations is an "almost rigidity" statement: If the pointed entropy *at a single point* x_0 is close to 0, then the Ricci flow ought to be smoothly close to flat $\mathbb{R}^n$ *near* x_0. There are many examples of such statements in the literature. For instance, in the case of Einstein manifolds one has an ε-regularity theorem based on the volume ratio $\mathcal{V}_x(r) \equiv \log(\mathrm{Vol}(B_r(x))/\omega_n r^n)$ going back to Anderson [1], which is at the backbone of the regularity theory in [6].

To make all of this precise we introduce the *regularity scale* of a point. Given $(x,t) \in M \times [-T, 0]$ a bound on the curvature $|\mathrm{Rm}|(x,t)$ at this point gives remarkably little information. On the other hand, a bound on the curvature in a spacetime neighborhood of (x,t) tells us everything about the local geometry of the Ricci flow. Thus we make the following definition:

Definition 1.10. Given a Ricci flow $(M^n, g(t))$ as in (1.1) we define the following.

(1) Given $(x,t) \in M \times [-T, 0]$ and $r > 0$ with $-T \leq t - r^2$ we define the parabolic ball

$$P_r(x,t) \equiv B_r(x,t) \times (t - r^2, t]. \tag{1.11}$$

(2) Given $(x,t) \in M \times [-T, 0]$ we define the regularity scale

$$r_{|\mathrm{Rm}|}(x,t) \equiv \sup\{r > 0 : \sup_{P_r(x,t)} |\mathrm{Rm}| \leq r^{-2}\}. \tag{1.12}$$

Remark 1.13. The regularity scale is defined to be scale invariant: If $r_{|\mathrm{Rm}|}(x,0) = r$ and if we rescale the Ricci flow by r^{-1}, so that $P_r(x,0)$ is mapped to $P_1(\tilde{x},0)$, then we have $|\mathrm{Rm}| \leq 1$ on $P_1(\tilde{x},0)$.

Remark 1.14. The regularity scale controls not only the curvature but also its derivatives. Namely, by standard parabolic estimates there exists for each $k \in \mathbb{N}$ a dimensional constant $C(n,k)$ such that if $r_{|\mathrm{Rm}|}(x,t) \equiv r$ then we have the estimates

$$\sup_{P_{\frac{r}{2}}(x,t)} |\nabla^k \mathrm{Rm}| \leq C(n,k) r^{-2-k}. \tag{1.15}$$

Our main theorem then takes the following form: *There exists an $\varepsilon > 0$ such that $\mathcal{W}_{x_0}(s) \geq -\varepsilon$ implies* $r_{|\mathrm{Rm}|}(x_0,0)^2 \geq \varepsilon|s|$. Of course the use of such an estimate is only as good as what ε depends on, and we are able to bound ε below purely in terms of T and lower scalar curvature and μ-entropy bounds at the initial time $-T$, which is really the most one might hope for.

Theorem 1.16. *For each $C > 0$ there exists an $\varepsilon = \varepsilon(n, C) > 0$ such that the following holds. Let $(M^n, g(t))$ be any Ricci flow as in* (1.1) *such that*

$$R[g(s)] \geq -\frac{C}{|s|}, \quad \inf_{\tau \in (0, 2|s|)} \mu(g(s), \tau) \geq -C, \tag{1.17}$$

for some $s \in [-T, 0)$. If the pointed entropy satisfies

$$\mathcal{W}_{x_0}(s) \geq -\varepsilon \tag{1.18}$$

for some point x_0 in the 0*-time slice, then we have*

$$r_{|\mathrm{Rm}|}(x_0, 0)^2 \geq \varepsilon |s|. \tag{1.19}$$

Remark 1.20. One typically takes the point of view that $(M, g(-T))$ is a fixed Riemannian manifold but that the runtime T of the flow is variable. One would then like to *derive* (1.17) from information about the initial time slice $(M, g(-T))$. The appropriate conditions to impose are

$$R[g(-T)] \geq -\frac{C}{|s|}, \quad \inf_{\tau \in (T-|s|, T+|s|)} \mu(g(-T), \tau) \geq -C, \tag{1.21}$$

because both $\inf R[g(t)]$ and $\mu(g(t), t_0 - t)$ for any fixed $t_0 > t$ are nondecreasing in t. Even if one then only cares about fixed values of s, (1.21) still exhibits an explicit T-dependence in the entropy condition which may degenerate as $T \to \infty$. This issue is familiar from [18].

The known ε-regularity theorems for the Ricci flow either allow ε to depend on a type I sectional curvature bound [8], or require a smallness condition such as (1.18) to hold *at every x in a definite neighborhood of* x_0 [17], which clearly follows from (1.18) under a type I condition.

In order to understand the technical difficulties encountered in the proof of Theorem 1.16, let us give a brief outline of the argument. We would like to follow Anderson's proof [1] of his ε-regularity theorem for Einstein manifolds as much as possible. Namely, assume the theorem fails. Then for all $\varepsilon > 0$ we can find Ricci flows $(M_\varepsilon, g_\varepsilon(t))$ and points $(x_\varepsilon, 0)$ such that (1.17) and (1.18) are satisfied but (1.19) fails. We can assume $s_\varepsilon = 1$ by rescaling. After a careful point picking we would then like to say that x_ε more or less minimizes the function $r_{|\mathrm{Rm}|}(x, 0)$ in a small but definite ball *while still satisfying* $\mathcal{W}_{x_\varepsilon}(1) \geq -2\varepsilon$. Then, after rescaling so that $r_{|\mathrm{Rm}|}(\tilde{x}_\varepsilon, 0) = 1$, we can take a subsequence converging smoothly to a complete pointed Ricci flow $(\tilde{M}_\infty, \tilde{g}_\infty(t), \tilde{x}_\infty)$ such that $r_{|\mathrm{Rm}|}(\tilde{x}_\infty, 0) = 1$, yet $\mathcal{W}_{\tilde{x}_\infty}(s) = 0$ for all s and hence $(\tilde{M}_\infty, \tilde{g}_\infty(t)) \cong \mathbb{R}^n$, which is the desired contradiction.

However, in order for this to go through we require $\mathcal{W}_x(s)$ to depend on x in a Lipschitz manner (for the volume ratio $\mathcal{V}_x(r)$ in the Einstein case this is clear by volume comparison). We are in fact unable to prove this assuming only (1.17). What does turn out to be possible under (1.17), though, is to obtain Lipschitz control on the *s-average of $\mathcal{W}_s(x)$*, which is still enough for our purposes. To accomplish this we introduce new log-Sobolev estimates for heat kernel measures. In addition, this s-average turns out to be an interesting monotone quantity in its own right; we will put this to some use in obtaining new integral bounds for the conjugate heat kernel, see Theorem 4.5.

1.2. **Log-Sobolev inequalities for the conjugate heat kernel measure.** Consider a smooth metric probability space $(M, g, d\nu)$, where $d\nu = e^{-f} d\mathrm{vol}_g$. If the Bakry-Émery condition

$$\mathrm{Ric} + \nabla^2 f \geq \frac{1}{2} g \tag{1.22}$$

is satisfied, then a celebrated classical theorem from [2] asserts that $(M, g, d\nu)$ satisfies a log-Sobolev inequality. That is, for every smooth function ϕ with compact support on M we have

$$\int \phi^2 \, d\nu = 1 \implies \int \phi^2 \log \phi^2 \, d\nu \leq 4 \int |\nabla \phi|^2 \, d\nu. \tag{1.23}$$

The case of flat $\mathbb{R}^n$ equipped with the Gaussian measure $d\nu = (4\pi)^{-\frac{n}{2}} \exp(-\frac{1}{4}|x|^2)dx$ is already very interesting; this case is equivalent to the original log-Sobolev inequality proved by Gross [10].

The log-Sobolev (1.23) can be used to prove that the operator $\Delta_f \phi \equiv \Delta\phi - \langle \nabla f, \nabla \phi \rangle$, which is $d\nu$-selfadjoint, has discrete spectrum. Linearizing (1.23) around $\phi \equiv 1$ yields a Poincaré for $d\nu$ that tells us that the smallest positive eigenvalue of Δ_f is at least $\frac{1}{2}$, and it follows from [2] that equality holds if and only if M splits off a line, in which case the $\frac{1}{2}$-eigenfunction is linear.

Remark 1.24. If M is compact, an improvement depending on $\operatorname{diam} M$ for the spectral gap $\frac{1}{2}$ was recently proved in [9]. Intriguingly, on every nontrivial normalized gradient shrinking Ricci soliton, $\mathrm{Ric} + \nabla^2 f = \frac{1}{2} g$, the soliton function f itself is always a 1-eigenfunction of Δ_f.

In the context of a Ricci flow $(M^n, g(t))$ with its conjugate heat kernel measures (1.3), Perelman's monotonicity formula (2.16) would suggest that the metric probability spaces $(M, g(s), d\nu_{x_0}(s))$ are typically well approximated by shrinking solitons. Thus the Bakry-Émery inequality (1.23) may at least help to motivate (if not prove) the following result, which is our main technical tool.

Theorem 1.25. *Let* $(M^n, g(t))$ *be a Ricci flow as in* (1.1). *Fix* $x_0 \in M$ *and* $s \in [-T, 0)$.

(1) *For all* $\phi \in C_0^\infty(M)$ *with* $\int \phi \, d\nu_{x_0}(s) = 0$,

$$\int \phi^2 \, d\nu_{x_0}(s) \leq 2|s| \int |\nabla \phi|^2_{g(s)} \, d\nu_{x_0}(s). \tag{1.26}$$

Equality holds if and only if either $\phi \equiv 0$, *or* $(M, g(t)) = (M', g'(t)) \times (\mathbb{R}, dz^2)$ *isometrically for all* $t \in [s, 0]$ *with* $z(x_0) = 0$ *and* $\phi = \lambda z$ *for some constant* $\lambda \in \mathbb{R}^*$.

(2) *For all* $\phi \in C_0^\infty(M)$ *with* $\int \phi^2 \, d\nu_{x_0}(s) = 1$,

$$\int \phi^2 \log \phi^2 \, d\nu_{x_0}(s) \leq 4|s| \int |\nabla \phi|^2_{g(s)} \, d\nu_{x_0}(s). \tag{1.27}$$

Equality holds if and only if either $\phi \equiv 1$, *or* $(M, g(t)) = (M', g'(t)) \times (\mathbb{R}, dz^2)$ *isometrically for all* $t \in [s, 0]$ *with* $z(x_0) = 0$ *and* $\phi = \exp(\lambda z - 2\lambda^2 |s|)$ *for some constant* $\lambda \in \mathbb{R}^*$.

Remark 1.28. Perelman's monotonicity formula implies an unweighted log-Sobolev inequality (2.19) whose optimal constant depends on T and on various bounds for the geometry of $(M, g(-T))$. The weighted inequalities in Theorem 1.25 on the other hand are sharp and completely universal. As far as we can tell, however, the applications of (2.19) and (1.27) are essentially disjoint.

Remark 1.29. A static version of Theorem 1.25 for the heat kernel measure on complete Riemannian manifolds with $\mathrm{Ric} \geq 0$ was proved independently by Bakry-Ledoux [3] and (with a spurious extra factor of n) Bueler [4]. These papers inspired our proof of Theorem 1.25.

1.3. **Integral and pointwise bounds for the conjugate heat kernel.** Before returning to our discussion of the ε-regularity result, Theorem 1.16, we wish to explain an interesting consequence of Theorem 1.25. The basic idea is to test (1.27) with functions ϕ that are well adapted to the metric geometry; this is the so-called "Herbst argument" in metric measure theory [14].

The following appears to be the sharpest possible result such methods can yield.

Theorem 1.30. *Let* $(M^n, g(t))$ *be a Ricci flow as in* (1.1) *and let* $d\nu = d\nu_{x_0}(s)$ *be a conjugate heat kernel measure as in* (1.3). *Then the Gaussian concentration inequality*

$$\nu(A)\nu(B) \leq \exp\left(-\frac{1}{8|s|}\mathrm{dist}_{g(s)}(A,B)^2\right) \tag{1.31}$$

holds for all $A, B \subseteq M$. *Here* dist *refers to the usual set distance, not the Hausdorff distance.*

Let us observe the following natural consequence. Given $x_1, x_2 \in M$ in the s-time slice, we can apply (1.31) to $d\nu \equiv d\nu_{x_2}$, choosing A, B to be the metric balls $B_r(x_1, s), B_r(x_2, s)$ with $r^2 \equiv |s|$. If $B_r(x_1, s)$ is in addition noncollapsed with a uniform constant, which is of course a fair assumption after Perelman [18], then we obtain the following average Gaussian upper bound:

$$⨍_{B_r(x_1,s)} H_{x_2}(s)\,d\mathrm{vol}_{g(s)} \leq \frac{C|s|^{-\frac{n}{2}}}{\nu_{x_2}(B_r(x_2,s))}\exp\left(-\frac{1}{C|s|}d_{g(s)}(x_1,x_2)^2\right). \tag{1.32}$$

The primary concern with this estimate is a lack of effective lower bound on $\nu_{x_2}(B_r(x_2, s))$. We can fix this to some extent by bringing in a pointwise Gaussian lower bound for H_{x_2} from [21], see Theorem 2.37, however this forces us to work with time-0 balls, not time-s balls.

Corollary 1.33. *For each* $C > 0$ *there exists a* $C' = C'(n, C) > 0$ *such that the following holds. Let* $(M^n, g(t))$ *be any Ricci flow as in* (1.1) *such that, for some* $s \in [-T, 0)$,

$$\sup_{t\in[s,0]} \|R[g(t)]\|_\infty \leq \frac{C}{|s|}, \quad \inf_{\tau\in(0,2|s|)} \mu(g(s),\tau) \geq -C. \tag{1.34}$$

Let $x_1, x_2 \in M$ *and put* $r^2 \equiv |s|$. *Then we have an average Gaussian upper bound*

$$⨍_{B_r(x_1,0)} H_{x_2}(s)\,d\mathrm{vol}_{g(s)} \leq C'|s|^{-\frac{n}{2}}\exp\left(-\frac{1}{C'|s|}\mathrm{dist}_{g(s)}(B_r(x_1,0),B_r(x_2,0))^2\right). \tag{1.35}$$

As a consequence, we get the following distance distortion type estimate:

$$\mathrm{dist}_{g(s)}(B_r(x_1,0),B_r(x_2,0)) \leq C' d_{g(0)}(x_1,x_2). \tag{1.36}$$

Remark 1.37. Both (1.32) and (1.35) fall short of what one might hope to have. For example, there is a method of proving *pointwise* Gaussian upper bounds for heat kernels on static manifolds relying on nothing more than a log-Sobolev [7]. Unfortunately this approach seems to break down for the Ricci flow for lack of control on the distance distortion between different time slices.

1.4. **Lipschitz continuity of the pointed Nash entropy.** We now return to the main flow of the argument and explain how the Poincaré inequality of Theorem 1.25(1) helps us to complete the proof of Theorem 1.16. As we said at the end of Section 1.1, what we require is a Lipschitz bound in x for $\mathcal{W}_x(s)$, or at least for a weaker quantity than $\mathcal{W}_x(s)$ that still controls the soliton behavior of our Ricci flow near x. It turns out that we can work with the *time average* of $\mathcal{W}_x(s)$:

Definition 1.38. Given $x_0 \in M$ and $s \in [-T, 0)$, we define the pointed Nash entropy by

$$\mathcal{N}_{x_0}(s) \equiv \frac{1}{|s|}\int_s^0 \mathcal{W}_{x_0}(r)\,dr = \int_M f_{x_0}(s)\,d\nu_{x_0}(s) - \frac{n}{2}. \tag{1.39}$$

See Proposition 2.26 for the equality in (1.39) and other basic properties. Thus, $\mathcal{N}_{x_0}(s)$ is closely related to the quantity used by Nash [16] in proving Hölder continuity of weak solutions.

Theorem 1.40. *For each $C > 0$ there exists a $C' = C'(n, C) > 0$ such that the following holds. Let $(M^n, g(t))$ be any Ricci flow as in* (1.1) *such that*

$$R[g(s)] \geq -\frac{C}{|s|}, \quad \inf_{\tau \in (0, 2|s|)} \mu(g(s), \tau) \geq -C, \tag{1.41}$$

for some $s \in [-T, 0)$. Then the mapping

$$(M, g(0)) \ni x \mapsto f_x(s) H_x(s) \in L^1(M, d\mathrm{vol}_{g(s)}) \tag{1.42}$$

is globally $C'|s|^{-\frac{1}{2}}$-Lipschitz. In particular this implies that

$$|\mathcal{N}_{x_1}(s) - \mathcal{N}_{x_2}(s)| \leq C'|s|^{-\frac{1}{2}} d_{g(0)}(x_1, x_2). \tag{1.43}$$

The idea here, and the reason for introducing $\mathcal{N}_x$ in the first place, is that $\mathcal{N}_x$, unlike $\mathcal{W}_x$, does not depend on derivatives of H. Thus, $\nabla_x \mathcal{N}_x$ can be bounded by the gradient estimates of [19, 20]. The Poincaré (1.26) comes in as a crucial tool to control the L^2-norm of $f_x(s)$ in the process.

Finally, with Theorem 1.40 in hand it is not difficult to complete the proof of Theorem 1.16 (or in fact of a slightly strengthened version in which $\mathcal{W}_{x_0}$ gets replaced by $\mathcal{N}_{x_0}$) because the contradiction argument outlined in Section 1.1 works the same way with $\mathcal{N}_{x_0}$ in place of $\mathcal{W}_{x_0}$.

2. Background material

2.1. The heat operator and its conjugate.

Let $(M^n, g(t))$ be a Ricci flow as in (1.1).

Definition 2.1. The heat operator and its conjugate along the flow are defined by

$$\Box \equiv \partial_t - \Delta, \tag{2.2}$$

$$\Box^* \equiv -\partial_t - \Delta + R. \tag{2.3}$$

This is a sensible definition because of the following identity.

Lemma 2.4. *Let Ω be a smooth bounded domain in M and $[t_1, t_2] \subseteq [-T, 0]$. Then*

$$\int_{t_1}^{t_2} \int_\Omega (\Box u)v - (\Box^* v)u = \int_\Omega uv \,\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} v - \frac{\partial v}{\partial n} u \right) \tag{2.5}$$

for all smooth functions $u, v : M \times [t_1, t_2] \to \mathbb{R}$.

Definition 2.6. For $x, y \in M$ and $s < t$ in $[-T, 0]$, we let $H(x, t \,|\, y, s)$ denote the conjugate heat kernel based at (x, t), i.e. the unique minimal positive solution to the equations

$$\Box^*_{y,s} H(x, t \,|\, y, s) = (-\partial_s - \Delta_{y, g(s)} + R(y, s)) H(x, t \,|\, y, s) = 0, \tag{2.7}$$

$$\lim_{s \to t} H(x, t \,|\, y, s) = \delta_x(y). \tag{2.8}$$

If we wish to fix a point $x \in M$ we may write $H_x(y, s) \equiv H(x, 0 \,|\, y, s) \equiv (4\pi|s|)^{-\frac{n}{2}} e^{-f_x(y,s)}$.

Lemma 2.9. *The conjugate heat kernel satisfies the following properties.*

1. $\int H(x, t \,|\, y, s) \, d\mathrm{vol}_{g(s)}(y) = 1$.
2. $H(x, t \,|\, y, s)$ *is also the fundamental solution of $\Box_{x,t} = \partial_t - \Delta_{x, g(t)}$ with pole at (y, s).*
3. $\int H(x, t \,|\, y, s) \, d\mathrm{vol}_{g(t)}(x) \leq \exp(\rho(t - s))$, *where $\rho \equiv \|R[g(-T)]^-\|_\infty$.*

Proof. (1) is simply the mass conserving property of the conjugate heat equation.

(2) requires some more care, and indeed reflects a very general fact concerning the fundamental solutions of parabolic operators and their formal adjoints. Let us write

$$u \equiv H_{\Box}(-,-\,|\,y,s), \tag{2.10}$$

$$v \equiv H_{\Box^*}(x,t\,|\,-,-), \tag{2.11}$$

for the fundamental solutions of $\Box$ and $\Box^*$ with poles at (y,s) and (x,t) respectively. We then apply (2.5) to u and v on the following three domains in the limit as $\varepsilon \to 0$:

$$[s, s+\varepsilon^2] \times (M \setminus B_\varepsilon(y,s)), \;\; [s+\varepsilon^2, t-\varepsilon^2] \times M, \;\; [t-\varepsilon^2, t] \times (M \setminus B_\varepsilon(x,t)).$$

Using standard local asymptotics for u and v at their poles, this yields $u(x,t) = v(y,s)$.

(3) follows by differentiating $\int H(x,t\,|\,y,s)\,d\mathrm{vol}_{g(t)}(x)$ by t, substituting $\Box_{x,t}H = 0$ from (2), and using that the minimum of the scalar curvature is nondecreasing along the Ricci flow. $\Box$

Finally, we recall a key tool that has already been put to good use in [15, 19].

Lemma 2.12. *The following parabolic Bochner formula holds for all spacetime functions u:*

$$\Box \frac{1}{2}|\nabla u|^2 = -|\nabla^2 u|^2 + \langle \nabla u, \nabla \Box u\rangle. \tag{2.13}$$

Proof. Using $\frac{\partial}{\partial t}|\nabla u|^2 = 2\langle \nabla \frac{\partial}{\partial t}u, \nabla u\rangle + 2\mathrm{Ric}(\nabla u, \nabla u)$, this reduces to the usual computation. The Ricci term here cancels with the Ricci term from the standard elliptic Bochner formula. $\Box$

2.2. **Properties of the entropy functionals.** We reviewed the definitions of Perelman's entropy functionals $\mathcal{W}(g,f,\tau)$ and $\mu(g,\tau)$ in Section 1. In addition we introduced localized versions $\mathcal{W}_{x_0}(s)$ and $\mathcal{N}_{x_0}(s)$. In this section we collect basic properties and applications of these functionals.

We begin with Perelman's foundational monotonicity formula and two corollaries [13, 18].

Theorem 2.14. *Fix a smooth probability measure dv on M and let $f(t)$, $g(t)$ be families of functions and metrics on M parametrized by $t \in [-T, 0]$. Fix any $t_0 \in \mathbb{R}$ and put $\tau(t) \equiv t_0 - t$. If*

$$\frac{\partial g}{\partial t} = -2(\mathrm{Ric} + \nabla^2 f), \;\; (4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mathrm{vol} = dv, \tag{2.15}$$

for all $t < t_0$, then we have

$$\frac{d}{dt}\mathcal{W}(g,f,\tau) = 2\tau \int \left|\mathrm{Ric} + \nabla^2 f - \frac{g}{2\tau}\right|^2 dv. \tag{2.16}$$

Up to correcting by the diffeomorphisms generated by ∇f, (2.15) is equivalent to g evolving by Ricci flow and $(4\pi\tau)^{-\frac{n}{2}}e^{-f}$ evolving by the conjugate heat equation associated with this Ricci flow. For the rest of the present section, $(M^n, g(t))$ will denote a Ricci flow as in (1.1).

Corollary 2.17. *For all $t_0 \in \mathbb{R}$, the quantity $\mu(g(t), t_0 - t)$ is nondecreasing in $t < t_0$. The quantity is constant if and only if the flow is isometric to a gradient shrinking soliton with singular time t_0 and soliton function $f(t)$, where $f(t)$ denotes any minimizer in the definition of $\mu(g(t), t_0 - t)$.*

Corollary 2.18. *For a fixed $t_0 \in \mathbb{R}$, define $\mu_0 \equiv \mu(g(-T), t_0 + T)$ and write $\tau \equiv t_0 - t$. Then*

$$\int \phi^2 \log \phi^2 \, d\mathrm{vol} \leq \tau \int (4|\nabla\phi|^2 + R\phi^2)\, d\mathrm{vol} - \frac{n}{2}\log 4\pi\tau - n - \mu_0 \tag{2.19}$$

holds for all $\phi \in C_0^\infty(M)$ with $\int \phi^2 \, d\mathrm{vol} = 1$ as long as $t < t_0$.

We will then also need to recall how (2.19) implies Perelman's no local collapsing theorem [18]. We state an improved version of this result that only requires an upper scalar curvature bound [13]. As reported in [13], this is due to Perelman as well. The way we organize the proof may be slightly simpler than the version in [13] and was inspired by an argument in [5]; see also [20].

Theorem 2.20. *Fix any $t \in [-T, 0]$, $x \in M$, and $r > 0$, and suppose that we have*

$$\inf_{\rho\in(0,r)} \mu(g(-T), t+T+\rho^2) \geq -C, \quad \sup_{B_r(x,t)} R[g(t)] \leq Cr^{-2}. \tag{2.21}$$

Defining $\kappa \equiv \exp(-(2^{n+4}+2C))$ it then follows that

$$|B_r(x,t)| \geq \kappa r^n. \tag{2.22}$$

Proof. We work in the t-time slice. For $\rho \in (0, r]$ define a Lipschitz function ψ to be $\equiv 1$ on $B(x, \frac{\rho}{2})$, $\equiv 0$ off $B(x,\rho)$, and linear in $d(x,-)$ in between, and apply (2.19) to $\phi \equiv \psi/\|\psi\|_2$. Using Jensen's inequality with respect to $\frac{d\text{vol}}{|B(x,\rho)|}$ on $B(x,\rho)$ to bound the left-hand side from below, we obtain

$$\log \frac{1}{|B(x,\rho)|} \leq \frac{16\tau}{\rho^2}\left(\frac{|B(x,\rho)|}{|B(x,\frac{\rho}{2})|} - 1\right) + \frac{C\tau}{\rho^2} - \frac{n}{2}\log(4\pi\tau) - n - \mu_0.$$

This holds for any given $\tau > 0$, with μ_0 depending on τ by definition. We now make $\tau \equiv \rho^2$. Using our definition of κ, it is then easy to prove the following implication:

$$\left|B\left(x, \frac{\rho}{2}\right)\right| \geq \kappa\left(\frac{\rho}{2}\right)^n \implies |B(x,\rho)| \geq \kappa\rho^n.$$

Since κ is smaller than the volume of $B_{\mathbb{R}^n}(1)$, the claim follows from this by iteration. □

Finally, we summarize some basic properties of the localized entropies $\mathcal{W}_x(s)$ and $\mathcal{N}_x(s)$ that we introduced in Definitions 1.7 and 1.38. The first couple of facts are clear from Theorem 2.14.

Proposition 2.23. *The following hold for all $x \in M$ and all $s \in [-T, 0)$.*

(1) $\lim_{s\to 0} \mathcal{W}_x(s) = 0$.
(2) $\mu(g(-T), T) \leq \mathcal{W}_x(s) \leq 0$.
(3) $\mathcal{W}_x(s) = -\int_s^0 2|r| \int |\text{Ric} + \nabla^2 f_x - \frac{g}{2|r|}|^2 \, d\nu_x(r)\, dr$.

The following straightforward computation explains the use of the Nash entropy.

Lemma 2.24. *Let u be a smooth positive solution to the conjugate heat equation, of rapid decay and of unit mass. Fix $t_0 \in \mathbb{R}$, put $\tau \equiv t_0 - t$ for $t < t_0$, and write $u \equiv (4\pi\tau)^{-\frac{n}{2}}e^{-f}$. Then*

$$\frac{d}{dt}\left(\tau \int u \log u \, d\text{vol}\right) = \mathcal{W}(g, f, \tau) + n + \frac{n}{2}\log 4\pi\tau. \tag{2.25}$$

Let us then summarize what we learn from this together with Proposition 2.23.

Proposition 2.26. *The following hold for all $x \in M$ and $s \in [-T, 0)$.*

(1) $\mathcal{W}_x(s) \leq \mathcal{N}_x(s) \leq 0$.
(2) $\frac{d}{ds}\mathcal{N}_x(s) = \frac{1}{|s|}(\mathcal{N}_x(s) - \mathcal{W}_x(s)) \geq 0$.
(3) $\mathcal{N}_x(s) = -\int \log H_x(s)\, d\nu_x(s) - \frac{n}{2}(1 + \log 4\pi|s|) = \int f_x(s)\, d\nu_x(s) - \frac{n}{2}$.
(4) $\mathcal{N}_x(s) = -\int_s^0 2|r|(1 - \frac{r}{s}) \int |\text{Ric} + \nabla^2 f_x - \frac{g}{2|r|}|^2 \, d\nu_x(r)\, dr$.

2.3. **Heat kernel estimates.** Here we provide careful statements and applications of some useful estimates due to Zhang [19, 20, 21]. As usual, we let $(M^n, g(t))$ denote a Ricci flow as in (1.1).

In [20], Davies's method [7] for deriving L^∞ heat kernel estimates from log-Sobolev inequalities is applied to the Ricci flow, based on Corollary 2.18. During the proof, the scalar curvature term in (2.19) gets compensated by the evolution of the Riemannian measure, so that the final result holds without any upper scalar curvature assumptions.

Theorem 2.27. *Define two auxiliary quantities*

$$\rho \equiv \|R[g(-T)]^-\|_\infty, \;\; \mu \equiv \inf_{\tau\in(0,2T)} \mu(g(-T),\tau). \tag{2.28}$$

Let $u : M \times [t_1, t_2] \to \mathbb{R}^+$ with $[t_1, t_2] \subseteq [-T, 0]$ be a smooth positive solution to

$$\frac{\partial u}{\partial t} = \Delta_{g(t)} u. \tag{2.29}$$

For each $t \in [t_1, t_2]$ we then have

$$\|u(t)\|_\infty \leq (4\pi(t-t_1))^{-\frac{n}{2}} e^{\rho(t-t_1)-\mu} \|u(t_1)\|_1. \tag{2.30}$$

Hamilton [11] proved a Harnack inequality for positive solutions to the heat equation on manifolds with lower Ricci bounds that only involves the gradient in the space variables. In [19] this idea was applied to the Ricci flow, using the Bochner formula of Lemma 2.12 as the key tool.

Theorem 2.31. *Let $u : M \times [t_1, t_2] \to \mathbb{R}^+$ with $[t_1, t_2] \subseteq [-T, 0]$ be a smooth positive solution to*

$$\frac{\partial u}{\partial t} = \Delta_{g(t)} u. \tag{2.32}$$

Then we have a spatial Harnack estimate

$$\left|\nabla \sqrt{\log \frac{\sup u}{u}}\right| \leq \frac{1}{\sqrt{t-t_1}}. \tag{2.33}$$

The following corollary was not stated in [19] but will be useful for us in Section 4.

Corollary 2.34. *For each $C > 0$ there exists a $C' = C'(n, C) > 0$ such that if we have*

$$R[g(-s)] \geq -\frac{C}{|s|}, \;\; \inf_{\tau\in(0,2|s|)} \mu(g(s),\tau) \geq -C, \tag{2.35}$$

and if we write $H(x, 0 \,|\, y, s) = (4\pi|s|)^{-\frac{n}{2}} \exp(-f_x(y, s))$ as before, then

$$|\nabla_x f_x|^2 \leq \frac{C'}{|s|}(C' + f_x). \tag{2.36}$$

Proof. Fix y, s and let $u(x, t) \equiv H(x, t \,|\, y, s)$, so that u solves the heat equation by Lemma 2.9. We can apply Theorem 2.27 with $T = |s|$ and $[t_1, t_2] = [s + \varepsilon, 0]$ for $\varepsilon \to 0$ to conclude that

$$\bar{u} \equiv \sup_{[\frac{s}{2},0]\times M} u \leq C'|s|^{-\frac{n}{2}}.$$

On the other hand, Theorem 2.31 with $[t_1, t_2] = [\frac{s}{2}, 0]$ yields $|\nabla_x f_x|^2 \leq \frac{C'}{|s|} \log \frac{\bar{u}}{u}$ at $(x, 0)$. □

Finally, in [21], Perelman's Harnack inequality from [18] is used to prove a Gaussian lower bound for H in terms of distance in the final time slice, by bringing in Theorems 2.31 and 2.27.

Theorem 2.37. *Define* ρ, μ *as in* (2.28) *and write* $\tau \equiv t - s$ *for* $s < t$ *in* $[-T, 0]$. *Then*

$$H(x,t\,|\,y,s) \geq (8\pi\tau)^{-\frac{n}{2}} \exp\left(-\frac{4}{\tau}d_{g(t)}(x,y)^2 - \frac{1}{\sqrt{\tau}}\int_0^\tau \sqrt{\sigma}R(y,t-\sigma)\,d\sigma - \rho\tau + \mu\right). \tag{2.38}$$

Notice that (2.38) involves the $g(t)$-distance because Theorem 2.31 bounds the x-gradient.

3. Log-Sobolev and Gaussian concentration

Section 3.1 proves Theorem 1.25 using methods in the spirit of the papers [3, 4], which deal with the analogous problem for static Riemannian manifolds with Ric ≥ 0. In fact, this proof is not far removed from the usual proof of a log-Sobolev under the Bakry-Émery condition (1.22). Section 3.2 deduces the Gaussian concentration (Theorem 1.30), using a standard argument from the theory of log-Sobolev inequalities. Corollary 1.33 is then deduced as a consequence.

3.1. **Proof of the Poincaré and log-Sobolev inequalities.** The starting point is to rewrite our two inequalities in a more convenient way. Writing $d\nu = d\nu_{x_0}(s)$ and passing to square roots in the log-Sobolev, we need to prove that for all $u \in C_0^\infty(M)$, with $u \geq 0$ in the second case,

$$\int u^2\,d\nu - \left(\int u\,d\nu\right)^2 \leq 2|s|\int |\nabla u|^2\,d\nu, \tag{3.1}$$

$$\int u\log u\,d\nu - \left(\int u\,d\nu\right)\log\left(\int u\,d\nu\right) \leq |s|\int \frac{|\nabla u|^2}{u}\,d\nu. \tag{3.2}$$

Note that (3.2) in fact implies (3.1) by linearizing around $u \equiv 1$, but information about the equality case is lost in this way. However, we will prove (3.1) and (3.2) completely in parallel, with almost identical discussions of the respective equality cases.

The key insight, not unlike [2], is that the heat kernel provides a homotopy between the two terms that are being subtracted on the left-hand sides of (3.1), (3.2). The proof then reduces to deriving a gradient estimate for the forward heat equation via the Bochner formula (Lemma 2.12).

In [4], this is done (in the static case) by bounding the Hodge heat kernel on 1-forms in terms of the heat kernel on scalars using a Kato type inequality; unfortunately, see [12], this method loses a factor of $n = \operatorname{rank} T^*M$, which is not accounted for in [4]. While the same idea works for the Ricci flow, we will therefore instead follow in spirit the approach of [3], where a precise gradient estimate is obtained (in the static case) by applying the heat kernel homotopy principle *once again*.

For $s \leq t$ in $[-T, 0]$, we write $P_{st}u$ for the evolution of $u \in C_0^\infty(M)$ from time s to time t under the forward heat equation coupled to the Ricci flow. In other words, by Lemma 2.9,

$$(P_{st}u)(x) = \int u(y)H(x,t\,|\,y,s)\,d\mathrm{vol}_{g(s)}(y). \tag{3.3}$$

Given this, the following lemma records the key homotopy principle.

Lemma 3.4. (1) *For any family of smooth functions* U_t *parametrized by* $t \in [-T, 0]$,

$$\frac{d}{dt}P_{t0}U_t = P_{t0}\Box_t U_t. \tag{3.5}$$

(2) *Let* $u \in C_0^\infty(M)$ *and put* $u_t = P_{st}u$, *so that* $\Box_t u_t = 0$. *Fix* $\phi, \psi : \mathbb{R} \to \mathbb{R}$. *Then we have*

$$U_t \equiv \phi(u_t) \Longrightarrow \Box U = -\phi''(u)|\nabla u|^2, \tag{3.6}$$

$$U_t \equiv \psi(u_t)|\nabla u_t|^2_{g(t)} \Longrightarrow \Box U = -2\psi(u)|\nabla^2 u|^2 - 4\psi'(u)\langle\nabla^2 u, du\otimes du\rangle - \psi''(u)|du\otimes du|^2, \tag{3.7}$$

omitting all subscripts t *and* $g(t)$ *on the right-hand side.*

Proof. (1) The representation formula (3.3) yields

$$\frac{d}{dt}P_{t0}U_t(x_0) = \frac{d}{dt}\int U_t(x)H(x_0,0\,|\,x,t)\,d\mathrm{vol}_{g(t)}(x).$$

Now use that $H(x_0,0\,|\,x,t)$ solves the conjugate heat equation in (x,t) and $\frac{\partial}{\partial t}d\mathrm{vol} = -R\,d\mathrm{vol}$.

(2) The first claim is straightforward. The key to the second claim is Lemma 2.12. □

Combining (3.5), (3.6), we can now rewrite the left-hand sides of (3.1), (3.2) as follows:

$$\int \phi(u)\,d\nu - \phi\left(\int u\,d\nu\right) = -\int_s^0 \frac{d}{dt}P_{t0}(\phi(P_{st}u))(x_0)\,dt$$
$$= \int_s^0 P_{t0}(\phi''(P_{st}u)|\nabla P_{st}u|^2_{g(t)})(x_0)\,dt, \tag{3.8}$$

with $\phi(x) = x^2$ and $\phi(x) = x\log x$, respectively. We next estimate the integrand by combining (3.5), (3.7) with $\psi = \phi''$, replacing $t, 0$ by r, t for a new variable $r \in [s,t]$. Thus, in the x^2 case,

$$|\nabla P_{st}u|^2_{g(t)} = P_{st}(|\nabla u|^2_{g(s)}) - 2\int_s^t P_{rt}|\nabla^2 P_{sr}u|^2_{g(r)}\,dr. \tag{3.9}$$

Substituting this into (3.8) immediately yields the Poincaré inequality of Theorem 1.25(1). In the $x\log x$ case, (3.7) simplifies quite drastically and we obtain

$$\frac{|\nabla P_{st}u|^2_{g(t)}}{P_{st}u} = P_{st}\left(\frac{|\nabla u|^2_{g(s)}}{u}\right) - 2\int_s^t P_{rt}((P_{sr}u)|\nabla^2 \log P_{sr}u|^2_{g(r)})\,dr\,. \tag{3.10}$$

Again we substitute this into (3.8) to prove the required log-Sobolev inequality. We can then finish the proof of Theorem 1.25 by observing that equality can occur if and only if either $u = const$, or the flow lines of ∇u or $\nabla \log u$ split off as isometric $\mathbb{R}$-factors.

3.2. **Gaussian concentration.** Given the inequalities in Theorem 1.25, proving the concentration estimate claimed in Theorem 1.30 is by now a standard exercise in abstract metric measure theory: log-Sobolev inequalities yield Gaussian concentration (Herbst), and Poincaré inequalities, which are weaker, still yield exponential concentration (Gromov-Milman). We refer to Ledoux [14] for a good exposition; the following proof merely recalls the relevant points from [14].

Proof of Theorem 1.30. We proceed from the log-Sobolev in the version (1.27). The first idea is that this allows us to bound, in a specific fashion, the Laplace transform of 1-Lipschitz functions with zero average. The second idea is to apply this bound to the distance function from a set.

As for the Laplace transform bound, fix $F \in C^\infty(M)$ with

$$\int F\,d\nu = 0,\ \ |\nabla F| \leq 1.$$

Let us define a Laplace type transform, or moment generating function, by

$$U(\lambda) \equiv \frac{1}{\lambda}\log\int e^{\lambda F}\,d\nu.$$

This is $O(\lambda)$ as $\lambda \to 0$. Moreover, (1.27) applied to $\phi^2 = e^{\lambda F}/\int e^{\lambda F}\,d\nu$ easily yields

$$\frac{dU}{d\lambda} \leq |s|$$

for all $\lambda > 0$. Thus, altogether,

$$\int e^{\lambda F}\,d\nu \leq e^{|s|\lambda^2}.$$

We now apply the preceding inequality to the two test functions

$$F \equiv \pm(G - \int G\, d\nu), \;\; G(y) \equiv \mathrm{dist}(y, B).$$

Then we immediately obtain that

$$e^{\lambda \mathrm{dist}(A,B)}\nu(A)\nu(B) \leq \int_A \int_B e^{\lambda(F(y_1)-F(y_2))}\, d\nu(y_1)\, d\nu(y_2) \leq e^{2|s|\lambda^2},$$

and the theorem follows from this by optimizing in λ. □

Proof of Corollary 1.33. We begin by applying Theorem 1.30 with x_0 replaced by x_2 and

$$A = B_r(x_1, 0), \;\; B = B_r(x_2, 0).$$

In order to derive (1.35) we must bound the factor $\nu(B)$ on the left-hand side of (1.31) from below. This can be done using two ingredients. First, using Zhang's Theorem 2.37,

$$\inf_{B_r(x_2,0)} H_{x_2}(s) \geq \frac{1}{C'}|s|^{-\frac{n}{2}}.$$

Second, the evolution of the volume form under Ricci flow and Theorem 2.20 tell us that

$$\mathrm{Vol}_{g(s)}(B_r(x_2, 0)) \geq \frac{1}{C'}\mathrm{Vol}_{g(0)}(B_r(x_2, 0)) \geq \frac{1}{C'}r^n.$$

Together these show that $\nu(B) \geq \frac{1}{C'}$. Then (1.35) follows by dividing through by $\mathrm{Vol}_{g(s)}(B_r(x_1, 0))$, which we can bound from below by the same argument we just used for $\mathrm{Vol}_{g(s)}(B_r(x_2, 0))$.

Finally, Theorem 2.37 also tells us that

$$\inf_{B_r(x_1,0)} H_{x_2}(s) \geq \frac{1}{C'}|s|^{-\frac{n}{2}} \exp\left(-\frac{C'}{|s|}d_{g(0)}(x_1, x_2)^2\right),$$

which allows us to deduce (1.36) from (1.35). □

4. Lipschitz continuity of the pointed Nash entropy

We now prove Theorem 1.40 as a consequence of Corollary 2.34 and our weighted Poincaré (1.26). The standing assumption throughout this section is that $(M^n, g(t))$ is a Ricci flow parametrized by $t \in [-T, 0]$ satisfying (1.41). In order to prove that the mapping $x \mapsto f_x(s)H_x(s)$ from $(M, g(0))$ to $L^1(M, d\mathrm{vol}_{g(s)})$ is Lipschitz, clearly all we need to do is estimate the integral

$$\mathcal{I} \equiv \int |\nabla_x(f_x(y, s)H_x(y, s))|\, d\mathrm{vol}_{g(s)}(y). \tag{4.1}$$

Inserting the expression for H_x in terms of f_x, and writing $d\nu \equiv d\nu_x(s)$, we find that

$$\mathcal{I} = \int |\nabla_x f_x - f_x \nabla_x f_x|\, d\nu \leq \|\nabla_x f_x\|_2(1 + \|f_x\|_2), \tag{4.2}$$

where the subscript 2 indicates the L^2-norm with respect to the probability measure ν. Now using Corollary 2.34 we have the pointwise estimate

$$|\nabla_x f_x|^2 \leq \frac{C'}{|s|}(C' + f_x). \tag{4.3}$$

If we substitute this into (4.2) we therefore get

$$\mathcal{I} \leq C'|s|^{-\frac{1}{2}}\left(1 + \int |f_x|^2\, d\nu\right). \tag{4.4}$$

To deal with this term we need the Poincaré inequality (1.26) and some simple observations based on monotonicity of the Nash entropy. Precisely, the following lets us complete the proof.

Theorem 4.5. *Under* (1.41) *the following hold.*

(1) $\int f_x \, d\nu \in [\frac{n}{2} - C, \frac{n}{2}]$.
(2) $\int |\nabla f_x|^2 \, d\nu \leq (\frac{n}{2} + C)\frac{1}{|s|}$.
(3) $\int |f_x|^2 \, d\nu \leq 2(n + 2C)^2$.

Notice carefully that ∇f_x means $\nabla_y f_x$ here, not $\nabla_x f_x$ as in (4.2).

Proof of Theorem 4.5. The first statement follows from Propositions 2.23 and 2.26. For the second statement, notice that the inequality $\mathcal{W}_x(s) \leq \mathcal{N}_x(s)$ is equivalent to

$$\int (|\nabla f_x|^2 + R) \, d\nu \leq \frac{n}{2|s|} .$$

Finally, the Poincaré inequality of Theorem 1.25 gives us

$$\int |f_x|^2 \, d\nu \leq 2|s| \int |\nabla f_x|^2 \, d\nu + \left(\int f_x \, d\nu \right)^2 , \tag{4.6}$$

and the two terms on the right-hand side are bounded by the first two statements. □

5. Proof of the ε-regularity theorem

Throughout this section we are considering a Ricci flow $(M^n, g(t))$ as in (1.1) that satisfies (1.17). We begin by proving Proposition 5.2, which is a more restrictive version of Theorem 1.16, and then use the continuity statement of Theorem 1.40 to complete the proof of Theorem 1.16 in full.

For a point $y \in M$ on the 0-time slice we define the normalized time-scale

$$t(y) \equiv -\min\{T, r_{|\mathrm{Rm}|}(y, 0)^2\}. \tag{5.1}$$

Proposition 5.2. *There exists an $\varepsilon = \varepsilon(n, C) > 0$ such that if we have*

$$\forall y \in B_\delta(x, 0) : \mathcal{N}_{t(y)}(y) \geq -\varepsilon \tag{5.3}$$

for some $x \in M$ and $0 < \delta \leq \sqrt{T}$, then we also have

$$\forall y \in B_\delta(x, 0) : r_{|\mathrm{Rm}|}(y, 0) \geq \varepsilon \cdot d_{g(0)}(y, \partial B_\delta(x, 0)). \tag{5.4}$$

Proof. (1) By rescaling there is no harm in assuming $\delta = 1 \leq T$. Let us assume for some $n, C > 0$ that the lemma fails. In this case we have for all $i \in \mathbb{N}$ that there exists a complete Ricci flow

$$(M_i^n, g_i(t), (x_i, 0))$$

with $t \in [-1, 0]$ and $x_i \in M_i$ such that for each $y \in B_1(x_i, 0)$ we have

$$\mathcal{N}_{t(y)}(y) \geq -\frac{1}{i}, \tag{5.5}$$

whereas any point $y_i \in B_1(x_i, 0)$ minimizing the quantity

$$w(y) \equiv \frac{r_{|\mathrm{Rm}|}(y, 0)}{d_{g_i(0)}(y, \partial B_1(x_i, 0))} \tag{5.6}$$

must necessarily satisfy

$$0 < w(y_i) \leq \frac{1}{i}. \tag{5.7}$$

(2) Choose any such y_i, define $r_i \equiv r_{|\mathrm{Rm}|}(y_i, 0)$, and consider the rescaled Ricci flows

$$(\tilde{M}_i, \tilde{g}_i(t), (\tilde{y}_i, 0)),\ \ \tilde{g}_i(t) \equiv \frac{1}{r_i^2} g_i(r_i^2 t),\ \ t \in [-\frac{1}{r_i^2}, 0].$$

Certainly $r_{|\mathrm{Rm}|}(\tilde{y}_i, 0) = 1$, and $\tilde{y}_i$ moves away from the boundary in that

$$d_i \equiv \frac{1}{2} d_{\tilde{g}_i(0)}(\tilde{y}_i, \partial B_{\frac{1}{r_i}}(\tilde{x}_i, 0)) \geq \frac{i}{2}. \tag{5.8}$$

On the other hand, since y_i minimizes w, we find that

$$\tilde{y} \in B_{d_i}(\tilde{y}_i, 0) \implies r_{|\mathrm{Rm}|}(\tilde{y}, 0) \geq \frac{1}{2}. \tag{5.9}$$

Moreover, by Perelman's no local collapsing, see Theorem 2.20, we have that

$$\mathrm{Vol}_{\tilde{g}_i(0)}(B_1(\tilde{y}, 0)) \geq \kappa(n). \tag{5.10}$$

Thus we have uniform smooth bounds on $P_{1/4}(\tilde{y}, 0)$ for all $\tilde{y} \in B_{d_i}(\tilde{y}_i, 0)$, and uniform noncollapsing at scale 1 on $B_{d_i}(\tilde{y}_i, 0)$. This allows us to pass to a subsequence to derive a pointed C^∞ limit

$$(\tilde{M}_i, \tilde{g}_i(t), (\tilde{y}_i, 0)) \to (\tilde{M}_\infty, \tilde{g}_\infty(t), (\tilde{y}_\infty, 0)).$$

The limit exists for $t \in [-\frac{1}{16}, 0]$ and is complete with bounded curvature while satisfying

$$r_{|\mathrm{Rm}|}(\tilde{y}_\infty, 0) = 1. \tag{5.11}$$

(3) To contradict this with (5.5) we need a few estimates for the heat kernel. Namely,

$$(4\pi|t|)^{-\frac{n}{2}} \exp(-l^{|t|}_{\tilde{y}_i}(\tilde{y})) \leq H(\tilde{y}_i, 0 \,|\, \tilde{y}, t) \leq C'(n, C)|t|^{-\frac{n}{2}} \tag{5.12}$$

for all $t \in [-1, 0)$ and $\tilde{y} \in \tilde{M}_i$ in the rescaled flows $(\tilde{M}_i, \tilde{g}_i(t), (\tilde{y}_i, 0))$, where l denotes the reduced length function of Perelman. The lower bound follows from Perelman's Harnack inequality [18], and the upper bound follows from Zhang's Theorem 2.27, originally proved in [20].

To clarify this point, note that in principle the required heat kernel estimates already follow from (5.9) on $B_{d_i}(\tilde{y}_i, 0)$ and neither depend on any special structure of the Ricci flow nor on the assumed value of C. We used some results from [18, 20] instead just for the sake of simplicity.

(4) As usual, let us write

$$H(\tilde{y}_i, 0 \,|\, \tilde{y}, t) \equiv (4\pi|t|)^{-\frac{n}{2}} e^{-f_i(\tilde{y}, t)}. \tag{5.13}$$

Then the two bounds in (5.12) together with the local regularity (5.9) tell us that

$$f_i(\tilde{y}, t) \to f_\infty(\tilde{y}, t) \tag{5.14}$$

smoothly on compact subsets because the heat kernels $H_{\tilde{y}_i}$ satisfy uniform derivative bounds. Now the entropy smallness (5.5) together with Proposition 2.26 shows that

$$\int_{-\frac{1}{16}}^{0} 2|t|(1 - 16|t|) \int_{\tilde{M}_i} \left| \mathrm{Ric}[\tilde{g}_i] + \nabla^2 f_i - \frac{\tilde{g}_i}{2|t|} \right|^2 d\nu_{\tilde{y}_i}(t)\, dt \leq \frac{1}{i}. \tag{5.15}$$

Thus, by Fatou's lemma, the function f_∞ constructed in (5.14) is a soliton potential for the limiting Ricci flow with singular time $t = 0$. Hence the t-time slice of the limiting flow is isometric to the $16|t|$-rescaling of the $(-\frac{1}{16})$-time slice. But then the only way for the curvature to stay bounded as $t \to 0$ is if $(\tilde{M}_\infty, \tilde{g}_\infty(t))$ is flat for all t. This contradicts (5.11) and thus proves the lemma. □

We are now in good shape to in fact prove a strengthening of our main theorem.

Theorem 5.16. *Theorem 1.16 is true and we can even replace* (1.18) *by* $\mathcal{N}_{x_0}(s) \geq -\varepsilon$.

Proof. Define $\delta \equiv \min\{1, \frac{1}{2C'_{1.40}}\varepsilon_{5.2}\}$ and $\varepsilon \equiv \frac{\delta}{2}\varepsilon_{5.2}$. We can assume $-s = T = 1 \geq \delta$. Then

$$\forall x \in B_\delta(x_0, 0) : \mathcal{N}_x(1) \geq -\varepsilon_{5.2}$$

by Theorem 1.40. Thus Proposition 5.2 tells us that $r_{|\mathrm{Rm}|}(x_0, 0) \geq \varepsilon_{5.2}\delta \geq \varepsilon$. □

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