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Heat Kernel and Analysis on Manifolds

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Preface

The development of Mathematics in the past few decades has witnessed an unprecedented rise in the usage of the notion of heat kernel in the diverse and seemingly remote sections of Mathematics. In the paper [**217**], titled "*The ubiquitous heat kernel*", Jay Jorgenson and Serge Lang called the heat kernel "*... a universal gadget which is a dominant factor practically everywhere in mathematics, also in physics, and has very simple and powerful properties.*"

Already in a first Analysis course, one sees a special role of the exponential function $t \mapsto e^{at}$. No wonder that a far reaching generalization of the exponential function – the *heat semigroup* $\left\{e^{-tA}\right\}_{t\geq 0}$, where A is a positive definite linear operator, plays similarly an indispensable role in Mathematics and Physics, not the least because it solves the associated *heat equation* $\dot{u} + Au = 0$. If the operator A acts in a function space then frequently the action of the semigroup e^{-tA} is given by an integral operator, whose kernel is called then *the heat kernel* of A.

Needless to say that any knowledge of the heat kernel, for example, upper and/or lower estimates, can help in solving various problems related to the operator A and its spectrum, the solutions to the heat equation, as well as to the properties of the underlying space. If in addition the operator A is Markovian, that is, generates a Markov process (for example, this is the case when A is a second order elliptic differential operator), then one can use information about the heat kernel to answer questions concerning the process itself.

This book is devoted to the study of the heat equation and the heat kernel of the Laplace operator on Riemannian manifolds. Over 140 years ago, in 1867, Eugenio Beltrami [**29**] introduced the Laplace operator for a Riemannian metric, which is also referred to as the Laplace-Beltrami operator. The next key step towards analysis of this operator was made in 1954 by Matthew Gaffney [**126**], who showed that on geodesically complete manifolds the Laplace operator is essentially self-adjoint in L^2. Gaffney also proved in [**127**] the first non-trivial sufficient condition for the stochastic completeness of the heat semigroup, that is, for the preservation of the L^1-norm by this semigroup. Nearly at the same time S. Minakshisundaram [**275**] constructed the heat kernel on compact Riemannian manifolds using the parametrix method.

However, it was not until the mid-1970s when the geometric analysis of the Laplace operator and the heat equation was revolutionized in the groundbreaking work of Shing-Tung Yau, which completely reshaped the area. The culmination of this work was the proof by Li and Yau [**258**] in 1986 of the parabolic Harnack inequality and the heat kernel two-sided estimates on complete manifolds of non-negative Ricci curvature, which stimulated further research on heat kernel estimates by many authors. Apart from the general wide influence on geometric analysis, the gradient estimates of Li and Yau motivated Richard Hamilton in his program on Ricci flow that eventually lead to the resolution of the Poincaré conjecture by Grigory Perel'man, which can be viewed as a most spectacular application of heat kernels in geometry[1].

Another direction in heat kernel research was developed by Brian Davies [**96**] and Nick Varopoulos [**353**], [**355**], who used primarily function-analytic methods to relate heat kernel estimates to certain functional inequalities.

The purpose of this book is to provide an accessible for graduate students introduction to the geometric analysis of the Laplace operator and the heat equation, which would bridge the gap between the foundations of the subject and the current research. The book focuses on the following aspects of these notions, which form separate chapters or groups of chapters.

I. *Local geometric background.* A detailed introduction to Riemannian geometry is given, with emphasis on construction of the Riemannian measure and the Riemannian Laplace operator as an elliptic differential operator of second order, whose coefficients are determined by the Riemannian metric tensor.

II. *Spectral-theoretic properties.* It is a crucial observation that the Laplace operator can be extended to a self-adjoint operator in L^2 space, which enables one to invoke the spectral theory and functional calculus of self-adjoint operator and, hence, to construct the associated heat semigroup. To treat properly the domains of the self-adjoint Laplacian and that of the associated energy form, one needs the Sobolev function spaces on manifolds. A detailed introduction to the theory of distributions and Sobolev spaces is given in the setting of $\mathbb{R}^n$ and Riemannian manifolds.

III. *Markovian properties and maximum principles.* The above spectral-theoretic aspect of the Laplace operator exploits its ellipticity and symmetry. The fact that its order is 2 leads to the so-called Markovian properties, that is, to maximum and minimum principles for solutions to the Laplace equation and the heat equation. Various versions of maximum/minimum principles are presented in different parts of the book, in the weak, normal, and strong forms. The Markovian properties are tightly related to the diffusion Markov process associated with the Laplacian, where is reflected in

[1]Another striking application of heat kernels is the heat equation approach to the Atiyah-Singer index theorem – see [**12**], [**132**], [**317**].

the terminology. However, we do not treat stochastic processes here, leaving this topic for a prospective second volume.

IV. *Smoothness properties.* As it is well-known, elliptic and parabolic equations feature an added regularity phenomenon, when the degree of smoothness of solutions is higher than a priori necessary. A detailed account of the local regularity theory in $\mathbb{R}^n$ (and consequently on manifolds) is given for elliptic and parabolic operators with smooth coefficients. This includes the study of the smoothness of solutions in the scale of Sobolev spaces of positive and negative orders, as well as the embedding theorems of Sobolev spaces into C^k. The local estimates of solutions are used, in particular, to prove the existence of the heat kernel on an arbitrary manifold.

V. *Global geometric aspects.* These are those properties of solutions which depend on the geometry of the manifold in the large, such as the essential self-adjointness of the Laplace operator (that is, the uniqueness of the self-adjoint extension), the stochastic completeness of the heat kernel, the uniqueness in the bounded Cauchy problem for the heat equation, and the quantitative estimates of solutions, in particular, of the heat kernel. A special attention is given to upper bounds of the heat kernel, especially the on-diagonal upper bounds with the long-time dependence, and the Gaussian upper bounds reflecting the long-distance behavior. The lower bounds as well as the related uniform Harnack inequalities and gradient estimates are omitted and will be included in the second volume.

The prerequisites for reading of this books are Analysis in $\mathbb{R}^n$ and the basics of Functional Analysis, including Measure Theory, Hilbert spaces, and Spectral Theorem for self-adjoint operators (the necessary material from Functional Analysis is briefly surveyed in Appendix). The book can be used as a source for a number of graduate lecture courses on the following topics: Riemannian Geometry, Analysis on Manifolds, Sobolev Spaces, Partial Differential Equations, Heat Semigroups, Heat Kernel Estimates, and others. In fact, it grew up from a graduate course "Analysis on Manifolds" that was taught by the author in 1995-2005 at Imperial College London and in 2002, 2005 at Chinese University of Hong Kong.

The book is equipped with over 400 exercises whose level of difficulty ranges from "general nonsense" to quite involved. The exercises extend and illustrate the main text, some of them are used in the main text as lemmas. The detailed solutions of the exercises (about 200 pages) as well as their LaTeX sources are available on the web page of the AMS

http : //www.ams.org/bookpages/amsip-47

where also additional material on the subject of the book will be posted.

The book has little intersection with the existing monographs on the subject. The above mentioned upper bounds of heat kernels, which were obtained mostly by the author in 1990s, are presented for the first time in a book format. However, the background material is also significantly different from the previous accounts. The main distinctive feature of the foundation part of this book is a new method of construction of the heat kernel on an arbitrary Riemannian manifold. Since the above mentioned work by Minakshisundaram, the traditional method of constructing the heat kernel was by using the parametrix method (see, for example, [**36**], [**37**], [**51**], [**317**], [**326**]). However, a recent development of analysis on metric spaces, including fractals (see [**22**], [**186**], [**187**], [**224**]), has lead to emergence of other methods that are not linked so much to the local Euclidean structure of the underlying space.

Although singular spaces are not treated here, we still employ whenever possible those methods that could be applied also on such spaces. This desire has resulted in the abandonment of the parametrix method as well as the tools using smooth hypersurfaces such as the coarea formula and the boundary regularity of solutions, sometimes at expense of more technical arguments. Consequently, many proofs in this book are entirely new, even for the old well-known properties of the heat kernel and the Green function. A number of key theorems are presented with more than one proof, which should provide enough flexibility for building lecture courses for audiences with diverse background.

The material of Chapters 1 - 10, the first part of Chapter 11, and Chapter 13, belongs to the foundation of the subject. The rest of the book – the second part of Chapter 11, Chapters 12 and 14 - 16, contains more advanced results, obtained in the 1980s -1990s.

Let us briefly describe the contents of the individual chapters.

Chapters 1, 2, 6 contain the necessary material on the analysis in $\mathbb{R}^n$ and the regularity theory of elliptic and parabolic equations in $\mathbb{R}^n$. They do not depend on the other chapters and can be either read independently or used as a reference source on the subject.

Chapter 3 contains a rather elementary introduction to Riemannian geometry, which focuses on the Laplace-Beltrami operator and the Green formula.

Chapter 4 introduces the Dirichlet Laplace operator as a self-adjoint operator in L^2, which allows then to define the associated heat semigroup and to prove its basic properties. The spectral theorem is the main tool in this part.

Chapter 5 treats the Markovian properties of the heat semigroup, which amounts to the chain rule for the weak gradient, and the weak maximum principle for elliptic and parabolic problems. The account here does not use the smoothness of solutions; hence, the main tools are the Sobolev spaces.

Chapter 7 introduces the heat kernel on an arbitrary manifold as the integral kernel of the heat semigroup. The main tool is the regularity theory of Chapter 6, transplanted to manifolds. The existence of the heat kernel is derived from a local $L^2 \to L^\infty$ estimate of the heat semigroup, which in turn is a consequence of the Sobolev embedding theorem and the regularity theory. The latter implies also the smoothness of the heat kernel.

Chapter 8 deals with a number of issues related to the positivity or boundedness of solutions to the heat equation, which can be regarded as an extension of Chapter 5 using the smoothness of the solutions. It contains the results on the minimality of the heat semigroup and resolvent, the strong minimum principle for positive supersolutions, and some basic criteria for the stochastic completeness.

Chapter 9 treats the heat kernel as a fundamental solution. Based on that, some useful tools are introduced for verifying that a given function is the heat kernel, and some examples of heat kernels are given.

Chapter 10 deals with basic spectral properties of the Dirichlet Laplacian. It contains the variational principle for the bottom of the spectrum λ_1, the positivity of the bottom eigenfunction, the discreteness of the spectrum and the positivity of λ_1 in relatively compact domains, and the characterization of the long time behavior of the heat kernel in terms of λ_1.

Chapter 11 contains the material related to the use of the geodesic distance. It starts with the properties of Lipschitz functions, in particular, their weak differentiability, which allows then to use Lipschitz functions as test functions in various proofs. The following results are proved using the distance function: the essential self-adjointness of the Dirichlet Laplacian on geodesically complete manifolds, the volume tests for the stochastic completeness and parabolicity, and the estimates of the bottom of the spectrum.

Chapter 12 is the first of the four chapters dealing with upper bounds of the heat kernel. It contains the results on the integrated Gaussian estimates that are valid on an arbitrary manifold: the integrated maximum principle, the Davies-Gaffney inequality, Takeda's inequality, and some consequences. The proofs use the carefully chosen test functions based on the geodesic distance.

Chapter 13 is devoted to the Green function of the Laplace operator, which is constructed by integrating the heat kernel in time. Using the Green function together with the strong minimum principle allows to prove the local Harnack inequality for α-harmonic functions and its consequences – convergence theorems. As an example of application, the existence of the ground state on an arbitrary manifold is proved. Logically this Chapter belongs to the foundations of the subject and should have been placed much earlier in the sequence of the chapters. However, the proof of the local Harnack inequality requires one of the results of Chapter 12, which has necessitated the present order.

Chapter 14 deals with the on-diagonal upper bounds of the heat kernel, which requires additional hypothesis on the manifold in question. Normally such hypotheses are stated in terms of some isoperimetric or functional inequalities. We use here the approach based on the Faber-Krahn inequality for the bottom eigenvalue, which creates useful links with the spectral properties. The main result is that, to a certain extent, the on-diagonal upper bounds of the heat kernel are equivalent to the Faber-Krahn inequalities.

Chapter 15 continues the topic of the Gaussian estimates. The main technical result is Moser's mean-value inequality for solutions of the heat equation, which together with the integrated maximum principle allows to obtain pointwise Gaussian upper bounds of the heat kernel. We consider such estimates in the following three settings: arbitrary manifolds, the manifolds with the global Faber-Krahn inequality, and the manifolds with the relative Faber-Krahn inequality that leads to the Li-Yau estimates of the heat kernel.

Chapter 16 introduces alternative tools to deal with the Gaussian estimates. The main point is that the Gaussian upper bounds can be deduced directly from the on-diagonal upper bounds, although in a quite elaborate manner. As an application of these techniques, some on-diagonal lower estimates are proved.

Finally, Appendix A contains some reference material as was already mentioned above.

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Writing about maximum principle brings up memories of my teacher Eugene Landis. His masterful use of maximum principles has never been surpassed. My entire education in Analysis was hugely influenced by Landis, which has left an imprint on the style and choice of the material for this book.

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Alexander Grigor'yan
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CHAPTER 1

Laplace operator and the heat equation in $\mathbb{R}^n$

The *Laplace operator* in $\mathbb{R}^n$ is a partial differential operator defined by

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2},$$

where $x_1, ..., x_n$ are the Cartesian coordinates in $\mathbb{R}^n$. This operator plays a crucial role in many areas of mathematics and physics. In this Chapter we present some basic facts about the Laplace operator and the associated heat equation to motivate a similar study on manifolds.

1.1. Historical background

The Laplace operator came to Mathematics from Physics.

Laplace equation. Pierre-Simon Laplace discovered in 1784-85 that a gravitational field can be represented as the gradient of a potential function $U(x)$, and that this function satisfies in a free space the equation $\Delta U = 0$. This equation is referred to as the *Laplace equation.* The gravitational potential of a particle placed at the origin $o \in \mathbb{R}^3$ is given by $U(x) = -\frac{m}{|x|}$ where m is the mass of the particle. It is easy to verify that $\Delta \frac{1}{|x|} = 0$ in $\mathbb{R}^3 \setminus \{o\}$ whence $\Delta U = 0$ follows. The potential of a body located in an open set $\Omega \subset \mathbb{R}^3$ is given by

$$U(x) = -\int_{\Omega} \frac{\rho(y)\, dy}{|x-y|},$$

where ρ is the mass density of the body. Then it follows that $\Delta U(x) = 0$ outside $\overline{\Omega}$.

Heat equation. Fourier's law of heat conductivity ("Théorie analytique de la chaleur"[1], 1822) implies that the temperature $u(t,x)$ at time t and point $x \in \mathbb{R}^3$ satisfies the *heat equation*

$$\frac{\partial u}{\partial t} = k\Delta u,$$

in any region that is free of sources and sinks of the heat (here $k > 0$ is the coefficient of heat conductivity).

[1] "The analytic theory of heat"

Wave equation. It follows from Maxwell's equations ("Treatise on Electricity and Magnetism", 1873), that each component $u = u(t, x)$ of an electromagnetic field satisfies the *wave equation*,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u,$$

in any region that is free of charges and currents (here c is the speed of light). The wave equation appears also in other physical phenomena related to wave propagation.

Diffusion equation. Albert Einstein suggested a mathematical explanation of the Brownian motion in his paper "Über die von der molekulakinetischen Theorie der Wärme gefoderte Bewegung von in ruhenden flüssigkeite suspendierten Teilchen"[2] published in *Annalen der Physik*, 1905. He showed that the density $u(t, x)$ of the probability that the particle started at the origin $o \in \mathbb{R}^3$ reaches the point x in time t satisfies the *diffusion equation*

$$\frac{\partial u}{\partial t} = D\Delta u$$

(here $D > 0$ is the diffusion coefficient). Using this equation, Einstein predicted that the mean displacement of the particle after time t was $\sqrt{4Dt}$. The latter was verified experimentally by Jean Perrin in 1908, for which he was honored with the 1926 Nobel Prize for Physics. That work was a strong argument in favor of the molecular-kinetic theory and thereby confirmed the atomic structure of matter.

Schrödinger equation. In 1926, Erwin Schrödinger developed a new approach for describing motion of elementary particles in Quantum Mechanics. Developing further the idea of Louis de Broglie that the motion of a particle is governed by the wave function $\psi(t, x)$, Schrödinger formulated the following equation describing the dynamic of the wave function of a spin-less particle:

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + U\psi,$$

where m is the mass of the particle, U is the potential field, $\hbar$ is the Planck constant, and $i = \sqrt{-1}$. He then applied this equation to the hydrogen atom and predicted many of its properties with remarkable accuracy. Erwin Schrödinger shared the 1933 Nobel Prize for Physics with Paul Dirac.

1.2. The Green formula

The Laplace operator appears in many applications (including all the physical laws) through the *Green formula*, which is a consequence of the *divergence theorem*. Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth

[2] "On the motion of small particles suspended in liquids at rest required by the molecular-kinetic theory of heat"

boundary. Then the divergence theorem says that, for any vector field F that is C^1 in Ω and continuous in $\overline{\Omega}$,

$$\int_{\partial\Omega} F \cdot \nu \, d\sigma = \int_{\Omega} \operatorname{div} F \, dx, \tag{1.1}$$

where σ is the boundary area on $\partial\Omega$ and ν is the outward normal unit vector field on $\partial\Omega$.

For any continuous function f defined in an open subset Ω of $\mathbb{R}^n$, define its *support* by

$$\operatorname{supp} f = \overline{\{x \in \Omega : f(x) \neq 0\}}, \tag{1.2}$$

where the closure is taken in Ω. If $u, v \in C^1(\Omega)$ and one of the supports of u and v is compact then the following *integration-by-parts formula* takes place:

$$\int_{\Omega} \frac{\partial u}{\partial x_k} v \, dx = -\int_{\Omega} u \frac{\partial v}{\partial x_k} dx, \tag{1.3}$$

which follows from (1.1) for $F = uve_k$, where e_k is the unit vector in the direction of the axis x_k.

If $u, v \in C^2(\Omega)$ and one of the supports of u and v is compact then the following *Green formula* takes place:

$$\int_{\Omega} u \Delta v \, dx = -\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} \Delta u \, v \, dx, \tag{1.4}$$

which follows from (1.1) for $F = u\nabla v$ and $F = v\nabla u$. Alternatively, (1.4) follows easily from (1.3):

$$\int_{\Omega} u \Delta v \, dx = \sum_{k=1}^{n} \int_{\Omega} u \frac{\partial^2 v}{\partial x_k^2} dx = -\sum_{k=1}^{n} \int_{\Omega} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x_k} dx = -\int_{\Omega} \nabla u \cdot \nabla v \, dx.$$

Exercises.

1.1. Denote by $S_r(x)$ the sphere of radius $r > 0$ centered at the point $x \in \mathbb{R}^n$, that is

$$S_r(x) = \{y \in \mathbb{R}^n : |x - y| = r\}.$$

Let σ be the $(n-1)$-volume on $S_r(x)$, and note that $\sigma(S_r(x)) = \omega_n r^{n-1}$ where ω_n is the area of the unit $(n-1)$-sphere in $\mathbb{R}^n$. Prove that, for any $f \in C^2(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$,

$$\frac{1}{\omega_n r^{n-1}} \left(\int_{\mathbb{S}_r(x)} f d\sigma \right) - f(x) = \Delta f(x) \frac{r^2}{2n} + \bar{o}(r^2) \quad \text{as } r \to 0. \tag{1.5}$$

1.2. Denote a round ball in $\mathbb{R}^n$ by

$$B_R(x) = \{y \in \mathbb{R}^n : |x - y| < R\}$$

and note that its volume is equal to $c_n R^n$ where c_n is the volume of the unit ball in $\mathbb{R}^n$. Prove that, for any $f \in C^2(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$,

$$\frac{1}{c_n R^n} \left(\int_{B_R(x)} f(y) \, dy \right) - f(x) = \Delta f(x) \frac{R^2}{2(n+2)} + \bar{o}(R^2) \quad \text{as } R \to 0. \tag{1.6}$$

1.3. The heat equation

Our main subject will be the heat equation

$$\frac{\partial u}{\partial t} = \Delta u,$$

where $u = u(t, x)$, t varies in an interval of $\mathbb{R}$, and $x \in \mathbb{R}^n$. According to the classification theory of partial differential equations, the Laplace operator belongs to the family of *elliptic* operators, whereas the heat operator $\frac{\partial}{\partial t} - \Delta$ belongs to the family of *parabolic* operators. The difference between these families manifests in many properties of the equations, in particular, which boundary and initial value problems are well-posed.

One of the most interesting problems associated with the heat equation is the *Cauchy problem* (known also as the initial value problem): given a function $f(x)$ on $\mathbb{R}^n$, find $u(t, x)$ such that

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \\ u|_{t=0} = f, \end{cases} \tag{1.7}$$

where the function u is sought in the class $C^2(\mathbb{R}_+ \times \mathbb{R}^n)$ so that the heat equation makes sense. The exact meaning of the initial data $u|_{t=0} = f$ depends on the degree of smoothness of the function f. In this section, we consider only continuous functions f, and in this case $u|_{t=0} = f$ means, by definition, that $u(t, x) \to f(x)$ as $t \to 0$ locally uniformly in x. Equivalently, this means that the function $u(t, x)$, extended to $t = 0$ by setting $u(0, x) = f(x)$, is continuous in $[0, \infty) \times \mathbb{R}^n$.

We investigate here the existence and uniqueness in the Cauchy problem in the class of bounded solutions.

1.3.1. Heat kernel and existence in the Cauchy problem.

The following function plays the main role in the existence problem:

$$p_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right), \tag{1.8}$$

where $t > 0$ and $x \in \mathbb{R}^n$. The function $p_t(x)$ is called the *Gauss-Weierstrass* function or the *heat kernel* (see Fig. 1.1 and 1.2).

The main properties of the heat kernel are stated in the following lemma.

LEMMA 1.1. *The function $p_t(x)$ is C^∞ smooth in $\mathbb{R}_+ \times \mathbb{R}^n$, positive, satisfies the heat equation*

$$\frac{\partial p_t}{\partial t} = \Delta p_t, \tag{1.9}$$

the identity

$$\int_{\mathbb{R}^n} p_t(x)\,dx \equiv 1, \tag{1.10}$$

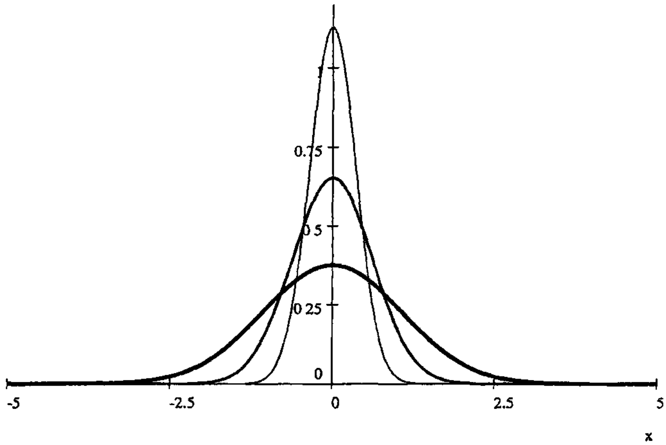

FIGURE 1.1. The graphs of the function $x \mapsto p_t(x)$ in $\mathbb{R}$ for $t = 1$ (thin), $t = 3$ (medium), and $t = 9$ (thick).

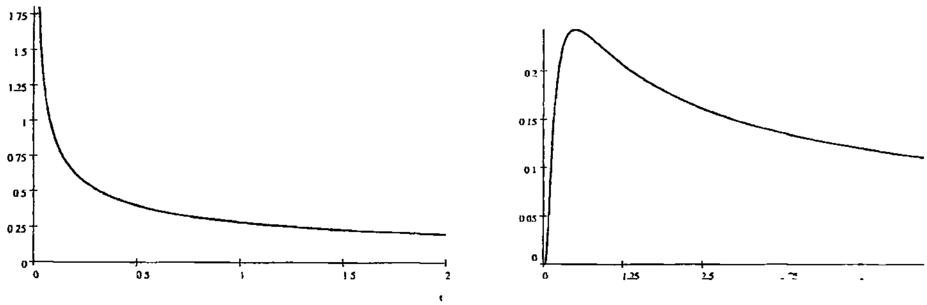

FIGURE 1.2. The graphs of the function $t \mapsto p_t(x)$ in $\mathbb{R}$ for $x = 0$ (left) and $x = 1$ (right)

and, for any $r > 0$,

$$\int_{\{|x|>r\}} p_t(x)dx \to 0 \quad as\ t \to 0.$$

PROOF. The smoothness and positivity of $p_t(x)$ are obvious. It is easier to verify the equation (1.9) using the function

$$u(t,x) := \log p_t(x) = -\frac{n}{2}\log t - \frac{|x|^2}{4t} - \text{const}.$$

Differentiating the identity $p_t = e^u$, we obtain

$$\frac{\partial p_t}{\partial t} = \frac{\partial u}{\partial t}e^u \quad \text{and} \quad \frac{\partial^2 p_t}{\partial x_k^2} = \left(\frac{\partial^2 u}{\partial x_k^2} + \left(\frac{\partial u}{\partial x_k}\right)^2\right)e^u.$$

Denoting by $\nabla u$the gradient of u, that is

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right),$$

we see that the equation (1.9) is equivalent to

$$\frac{\partial u}{\partial t} = \Delta u + |\nabla u|^2. \tag{1.12}$$

Computing the derivatives of u,

$$\frac{\partial u}{\partial t} = -\frac{n}{2t} + \frac{|x|^2}{4t^2}$$

and

$$\Delta u = -\frac{n}{2t}, \quad \nabla u = -\frac{1}{2t}(x_1, ..., x_n), \quad |\nabla u|^2 = \frac{|x|^2}{4t^2},$$

we obtain (1.12).

To prove (1.10), let us use the identity

$$\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi} \tag{1.13}$$

(cf. Example A.1), which implies by a change in the integral that

$$\int_{-\infty}^{\infty} e^{-s^2/4t} ds = \sqrt{4\pi t}.$$

Reducing the integration in $\mathbb{R}^n$ to repeated integrals, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} p_t(x)dx &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{x_1^2 + \cdots + x_n^2}{4t}\right) dx_1 \cdots dx_n \\ &= \frac{1}{(4\pi t)^{n/2}} \prod_{k=1}^{n} \int_{\mathbb{R}} \exp\left(-\frac{x_k^2}{4t}\right) dx_k \\ &= \frac{1}{(4\pi t)^{n/2}} \left(\sqrt{4\pi t}\right)^n \\ &= 1.\end{aligned}$$

Finally, to verify (1.11), let us make the change $y = t^{-1/2}x$ in the integral (1.11). Since $dy = t^{-n/2}dx$, the factor $t^{-n/2}$ cancels and we obtain

$$\int_{\{|x|>r\}} p_t(x)dx = \frac{1}{(4\pi)^{n/2}} \int_{\{|y|>t^{-1/2}r\}} e^{-|y|^2/4} dy. \tag{1.14}$$

Since the integral in the right hand side is convergent and $t^{-1/2}r \to \infty$ as $t \to \infty$, we obtain that it tends to 0 as $t \to \infty$, which was to be proved. $\square$

REMARK 1.2. It is obvious from (1.14) that, in fact,

$$\int_{\{|x|>r\}} p_t(x)dx \le \text{const}\exp\left(-\frac{r^2}{5t}\right),$$

so that the integral tends to 0 as $t \to 0$ faster than any power of t.

For any two continuous functions f, g in $\mathbb{R}^n$, their *convolution* $f * g$ is defined by

$$f * g\,(x) = \int_{\mathbb{R}^n} f\,(x-y)\,g\,(y)\,dy,$$

provided the integral converges for all $x \in \mathbb{R}^n$. It turns out that the Cauchy problem can be solved by taking the convolution of the heat kernel with the initial function f.

THEOREM 1.3. *If f is a bounded continuous function in $\mathbb{R}^n$ then the following function*

$$u\,(t,x) = p_t * f\,(x) \tag{1.15}$$

is C^∞ smooth in $\mathbb{R}_+ \times \mathbb{R}^n$, satisfies the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

and the initial data $u|_{t=0} = f$ in the sense that

$$u\,(t,x) \to f\,(x) \quad \textit{as } t \to 0 \tag{1.16}$$

locally uniformly in x. Moreover, the function u is bounded and, for all $t > 0$ and $x \in \mathbb{R}^n$,

$$\inf f \leq u\,(t,x) \leq \sup f. \tag{1.17}$$

PROOF. By the definition of the convolution, we have

$$u\,(t,x) = \int_{\mathbb{R}^n} p_t\,(x-y)\,f\,(y)\,dy = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) f\,(y)\,dy. \tag{1.18}$$

The function $(t,x) \mapsto p_t\,(x-y)$ is infinitely smooth in $\mathbb{R}_+ \times \mathbb{R}^n$ whence the same property of u follows from the fact that we can interchange the order of differentiation in t and x and integration in (1.18) (note that the integral in (1.18) converges locally uniformly in (t,x) and so does any integral obtained by differentiation of the integrand in t and x, thanks to the boundedness of f). In particular, using (1.9) we obtain

$$\frac{\partial u}{\partial t} - \Delta u = \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial t} - \Delta\right) p_t(x-y) f(y) dy = 0.$$

Let us verify (1.16). Using the identity (1.10), we can write

$$\begin{aligned} u(t,x) - f(x) &= \int_{\mathbb{R}^n} p_t(x-y) f(y) dy - \int_{\mathbb{R}^n} p_t(x-y) f(x) dy \\ &= \int_{\mathbb{R}^n} p_t(x-y)(f(y) - f(x)) dy. \end{aligned}$$

Since f is continuous at x, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|y - x| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Furthermore, since f is locally uniformly continuous, the same δ can be chosen for all x varying in a bounded set. Then we have

$$\left|\int_{\{|y-x|<\delta\}} p_t(x-y)(f(y)-f(x))dy\right| \le \varepsilon \int_{\mathbb{R}^n} p_t(x-y)\,dy = \varepsilon$$

and, changing $z = x - y$,

$$\left|\int_{\{|y-x|\ge\delta\}} p_t(x-y)(f(y)-f(x))dy\right| \le 2\sup|f| \int_{\{|z|\ge\delta\}} p_t(z)dz.$$

The right hand side in the last integral tends to 0 as $t \to 0$ by (1.11). Hence, we conclude that

$$\int_{\mathbb{R}^n} p_t(x-y)(f(y)-f(x))dy \to 0 \text{ as } t \to 0,$$

and the convergence is local uniform in x, which proves (1.16).

Finally, the positivity of the heat kernel and (1.10) imply that

$$u(x) \le \sup f \int_{\mathbb{R}^n} p_t(x-y)dy = \sup f$$

and in the same way $u \ge \inf f$. □

REMARK 1.4. It is clear from the proof that if $f(x)$ is uniformly continuous in $\mathbb{R}^n$ then $u(t,x) \to f(x)$ uniformly in $x \in \mathbb{R}^n$.

Exercises.

The next two questions provide a step-by-step guide for alternative proofs of Lemma 1.1 and (a version of) Theorem 1.3, using the Fourier transform. Recall that, for any function $u \in L^1(\mathbb{R}^n)$, its Fourier transform $\widehat{u}(\xi)$ is defined by

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} u(x)\,dx.$$

Using the Plancherel identity, the Fourier transform extends to all $u \in L^2(\mathbb{R}^n)$.

In all questions here, $p_t(x)$ is the heat kernel in $\mathbb{R}^n$ defined by (1.8).

1.3. Prove the following properties of the heat kernel.

(a) For all $t > 0$ and $\xi \in \mathbb{R}^n$,

$$\widehat{p_t}(\xi) = e^{-t|\xi|^2}. \tag{1.19}$$

(b) $\int_{\mathbb{R}^n} p_t(x)\,dx = 1$.

(c) For all $t, s > 0$, $p_t * p_s = p_{t+s}$.

(c) $\frac{\partial p_t}{\partial t} = \Delta p_t$.

1.4. Fix a function $f \in L^2(\mathbb{R}^n)$ and set $u_t = p_t * f$ for any $t > 0$. Prove the following properties of the function u_t.

(a) $\widehat{u_t}(\xi) = e^{-t|\xi|^2}\widehat{f}(\xi)$.

(b) $u_t(x)$ is smooth and satisfies the heat equation in $\mathbb{R}_+ \times \mathbb{R}^n$.

(c) $\|u_t\|_{L^2} \le \|f\|_{L^2}$ for all $t > 0$.

(d) $u(t,x) \to f(x)$ as $t \to 0$ in the norm of $L^2(\mathbb{R}^n)$.

(e) If $\widehat{f} \in L^1(\mathbb{R}^n)$ then $u(t,x) \to f(x)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^n$.

1.5. Prove the following properties of the heat kernel.

(a) For any $\varepsilon > 0$, $p_t(x) \to 0$ as $t \to 0$ uniformly in $\{x : |x| > \varepsilon\}$.

(b) $p_t(x) \to 0$ as $x \to \infty$ uniformly in $t \in (0, +\infty)$.
(c) For any $\varepsilon > 0$, $p_t(x)$ is continuous in $\{x : |x| > \varepsilon\}$ uniformly in $t \in (0, +\infty)$.

1.3.2. Maximum principle and uniqueness in the Cauchy problem. The uniqueness in the bounded Cauchy problem will follow from the *maximum principle*, which is of its own interest. Let $U \subset \mathbb{R}^n$ be a bounded open set. Fix some positive real T and consider the cylinder $\Gamma = (0,T) \times U$ as a subset in $\mathbb{R}^{n+1}$. The boundary $\partial\Gamma$ can be split into three parts: the top $\{T\} \times U$, the bottom $\{0\} \times U$ and the lateral boundary $[0,T] \times \partial U$ (where ∂U is the boundary of U in $\mathbb{R}^n$). Define the *parabolic* boundary $\partial_p\Gamma$ of the cylinder Γ as the union of its bottom and the lateral boundary, that is

$$\partial_p\Gamma := (\{0\} \times U) \cup ([0,T] \times \partial U)$$

(see Fig. 1.3). Note that $\partial_p\Gamma$ is a closed subset of $\mathbb{R}^{n+1}$.

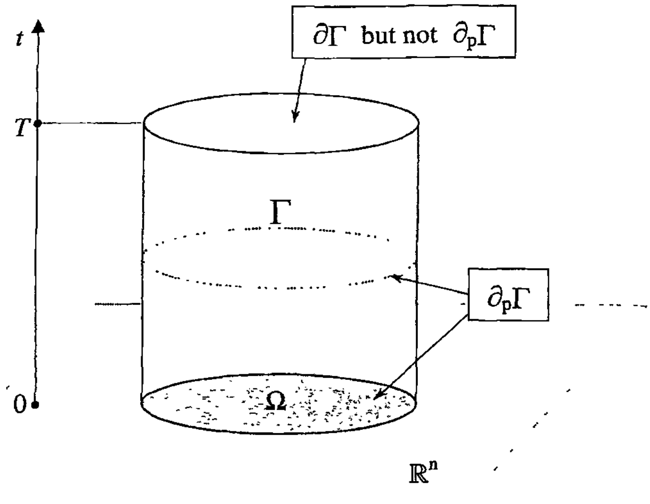

FIGURE 1.3. The parabolic boundary $\partial_p\Gamma$ contains the bottom and the lateral surface of the cylinder Γ, but does not include the top.

LEMMA 1.5. (Parabolic maximum principle) *If $u \in C^2(\Gamma) \cap C(\overline{\Gamma})$ and*

$$\frac{\partial u}{\partial t} - \Delta u \leq 0 \text{ in } \Gamma$$

then

$$\sup_{\Gamma} u = \sup_{\partial_p\Gamma} u. \tag{1.20}$$

In particular, if $u \leq 0$ on $\partial_p\Gamma$ then $u \leq 0$ in Γ.

By changing u to $-u$, we obtain the *minimum principle:* if

$$\frac{\partial u}{\partial t} - \Delta u \geq 0 \text{ in } \Gamma$$

then

$$\inf_{\Gamma} u = \inf_{\partial_p \Gamma} u.$$

In particular, if u solves the heat equation in Γ then the maximum and minimum of u in $\overline{\Gamma}$ are attained also in $\partial_p \Gamma$.

PROOF. Assume first that u satisfies a *strict* inequality in Γ:

$$\frac{\partial u}{\partial t} - \Delta u < 0. \tag{1.21}$$

By slightly reducing T, we can assume that (1.21) holds also at the top of Γ. Let (t_0, x_0) be a point of maximum of function u in $\overline{\Gamma}$. Let us show that $(t_0, x_0) \in \partial_p \Gamma$, which will imply (1.20). If $(t_0, x_0) \notin \partial_p \Gamma$ then (t_0, x_0) lies either inside Γ or at the top of Γ. In the both cases, $x_0 \in \Gamma$ and $t_0 > 0$. Since the function $x \mapsto u(t_0, x)$ in $\overline{U}$ attains the maximum at $x = x_0$, we have

$$\frac{\partial^2 u}{\partial x_j^2}(t_0, x_0) \leq 0 \text{ for all } j = 1, ..., n$$

whence $\Delta u(t_0, x_0) \leq 0$.

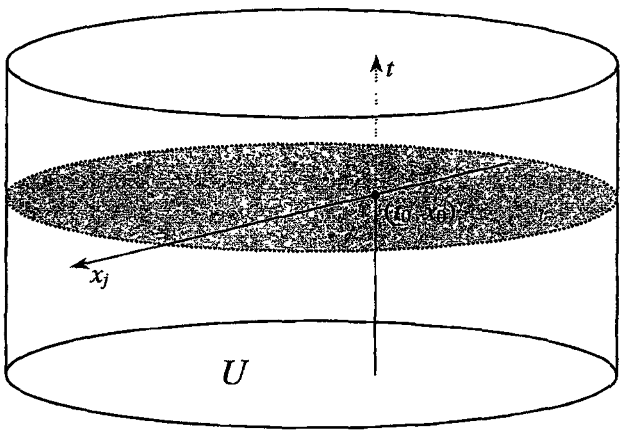

FIGURE 1.4. The restriction of $u(t, x)$ to the lines in the direction x_j and in the direction of t (downwards) attains the maximum at (t_0, x_0).

On the other hand, the function $t \mapsto u(t, x_0)$ in $(0, t_0]$ attains its maximum at $t = t_0$ whence

$$\frac{\partial u}{\partial t}(t_0, x_0) \geq 0$$

(if $t_0 < T$ then, in fact, $\frac{\partial u}{\partial t}(t_0, x_0) = 0$). Hence, we conclude that

$$\left(\frac{\partial u}{\partial t} - \Delta u\right)(t_0, x_0) \geq 0,$$

contradicting (1.21).

Consider now the general case. Set $u_\varepsilon = u - \varepsilon t$ where ε is a positive parameter. Clearly, we have

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \left(\frac{\partial u}{\partial t} - \Delta u\right) - \varepsilon < 0.$$

Hence, the previous case applies to the function u_ε, and we conclude that

$$\sup_{\Gamma}(u - \varepsilon t) = \sup_{\partial_p \Gamma}(u - \varepsilon t).$$

Letting $\varepsilon \to 0$ we obtain (1.20). $\square$

REMARK 1.6. The statement remains true for a more general operator

$$\frac{\partial}{\partial t} - \Delta - \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j},$$

where b_j are arbitrary functions in Γ. Indeed, the first order terms vanish at the point (t_0, x_0) because $\frac{\partial u}{\partial x_j}(t_0, x_0) = 0$, and the proof goes through unchanged.

Now we can prove the uniqueness theorem.

THEOREM 1.7. *For any continuous function $f(x)$, the Cauchy problem* (1.7) *has at most one bounded solution $u(t, x)$.*

PROOF. It suffices to prove that if u is a bounded solution to the Cauchy problem with $f = 0$ then $u \equiv 0$. Compare u to the function

$$v(t, x) = |x|^2 + 2nt,$$

which is non-negative and obviously satisfies the heat equation

$$\frac{\partial v}{\partial t} - \Delta v = 0.$$

Fix $\varepsilon > 0$ and compare u and εv in a cylinder $\Gamma = (0, T) \times B_R$. At the bottom of the cylinder (that is, at $t = 0$) we have $u = 0 \leq \varepsilon v$. At the lateral boundary of the cylinder (that is, at $|x| = R$) we have $u(x) \leq C$ where $C := \sup |u|$, and $\varepsilon v(x) \geq \varepsilon R^2$. Choosing R so big that $\varepsilon R^2 \geq C$, we obtain that $u \leq \varepsilon v$ on the lateral boundary of Γ.

Hence, the function $u - \varepsilon v$ satisfies the heat equation in Γ and $u - \varepsilon v \leq 0$ on the parabolic boundary $\partial_p \Gamma$. By Lemma 1.5, we conclude that $u - \varepsilon v \leq 0$ in Γ. Letting $R \to \infty$ and $T \to \infty$ we obtain $u - \varepsilon v \leq 0$ in $\mathbb{R}_+ \times \mathbb{R}^n$. Letting $\varepsilon \to 0$, we obtain $u \leq 0$. In the same way $u \geq 0$, whence $u \equiv 0$. $\square$

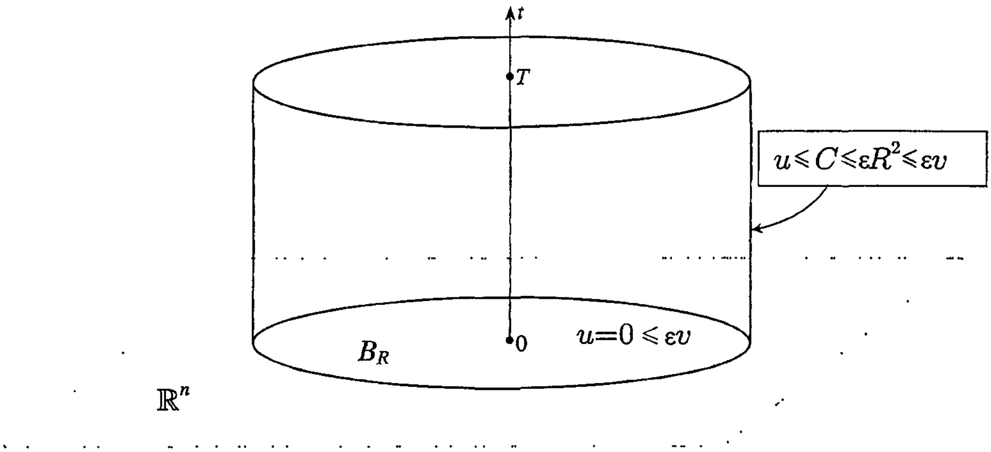

FIGURE 1.5. Comparison of functions u and εv on $\partial_p \Gamma$

REMARK 1.8. In fact, the uniqueness class for solutions to the Cauchy problem is much wider than the set of bounded functions. For example, the Tikhonov theorem says that if $u(t,x)$ solves the Cauchy problem with $f = 0$ and

$$|u(t,x)| \leq C \exp\left(C |x|^2\right)$$

for some constant C and all $t > 0$, $x \in \mathbb{R}^n$, then $u \equiv 0$. We do not prove this theorem here because it will easily follow from a much more general result of Chapter 11 (see Corollary 11.10).

Theorems 1.3 and 1.7 imply that, for any bounded continuous function f, the Cauchy problem has a unique bounded solution, given by $p_t * f$. Let us show an amusing example of application of this result to the heat kernel.

EXAMPLE 1.9. Let us prove that, for all $0 < s < t$,

$$p_{t-s} * p_s = p_t. \tag{1.22}$$

(cf. Exercise 1.3). Let f be continuous function in $\mathbb{R}^n$ with compact support. By Theorem 1.3, the function $u_t = p_t * f$ solves the bounded Cauchy problem with the initial function f. Consider now the Cauchy problem with the initial function u_s. Obviously, the function u_t gives the bounded solution to this problem at time $t - s$. On the other hand, the solution at time $t - s$ is given by $p_{t-s} * u_s$. Hence, we obtain the identity

$$u_t = p_{t-s} * u_s,$$

that is

$$p_t * f = p_{t-s} * (p_s * f).$$

By the associative law of convolution (which follows just by changing the order of integration), we have

$$p_{t-s} * (p_s * f) = (p_{t-s} * p_s) * f,$$

whence

$$p_t * f = (p_{t-s} * p_s) * f.$$

Since this is true for all functions f as above, we conclude that $p_t = p_{t-s} * p_s$.

Naturally, this identity can be proved by a direct computation, but such a computation is not very simple.

Exercises.

1.6. *(Elliptic maximum principle)* Let Ω be a bounded open set in $\mathbb{R}^n$, and consider the following differential operator in Ω

$$L = \Delta + \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j},$$

where b_j are smooth bounded functions in Ω.

(a) Show that there exists a function $v \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $Lv > 0$ in Ω.

(b) Prove that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $Lu \geq 0$ in Ω then

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

1.7. Evaluate the bounded solution $u(t, x)$ of the Cauchy problem with the initial function $f(x) = \exp(-|x|^2)$.

Notes

The material of this Chapter is standard and can be found in many textbooks on partial differential equations – see for example, **[38]**, **[118]**, **[121]**, **[130]**, **[243]**.

CHAPTER 2

Function spaces in $\mathbb{R}^n$

We have collected in this Chapter some properties of *distributions* and *Sobolev spaces* in $\mathbb{R}^n$ mostly related to the techniques of *mollifiers*. The knowledge of the Lebesgue measure, Lebesgue integration, and Hilbert spaces is assumed here. The reader is referred to Appendix A for the necessary background.

The full strength of the results of this Chapter will be used only in Chapter 6 in the regularity theory of elliptic and parabolic equations.

For the next Chapter 3, we will need only the material of Section 2.2 (the cutoff functions and partition of unity). In Chapter 4, we will introduce distributions and Sobolev spaces on manifolds, where the understanding of similar notions in $\mathbb{R}^n$ will be an advantage. At technical level, we will need there only the material of Section 2.3 (in fact, only Corollary 2.5). Chapter 5 does not use any results from the present Chapter.

Sections 2.1–2.6 are self-contained. Section 2.7 is somewhat away from the mainstream of this Chapter (although it depends on the results of the preceding sections) and can be considered as a continuation of Chapter 1. Also, it provides a certain motivation for the L^2-Cauchy problem on manifolds, which will be considered in Section 4.3. Technically, the results of Section 2.7 are used to prove the embedding theorems in Chapter 6, although alternative proofs are available as well.

2.1. Spaces C^k and L^p

Let $x^1, ..., x^n$ be the Cartesian coordinates in $\mathbb{R}^n$. We use the following short notation for partial derivatives:

$$\partial_i \equiv \frac{\partial}{\partial x^i}$$

and, for any *multiindex* $\alpha = (\alpha_1, ..., \alpha_n)$,

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} (\partial x^2)^{\alpha_2} ... (\partial x^n)^{\alpha_n}} = \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n}, \tag{2.1}$$

where $|\alpha| := \alpha_1 + ... + \alpha_n$ is the *order* of the multiindex. In particular, if $\alpha = 0$ then $\partial^\alpha u = u$.

For any open set $\Omega \subset \mathbb{R}^n$, $C(\Omega)$ denotes the class of all continuous functions in Ω, and $C^k(\Omega)$ denotes the class of all functions f from $C(\Omega)$ such $\partial^\alpha f \in C(\Omega)$ for all $|\alpha| \leq k$ (here k is a non-negative integer). Let

$C^\infty(\Omega)$ be the intersection of all $C^k(\Omega)$, and $C_0^\infty(\Omega)$ be the subspace of $C^\infty(\Omega)$, which consists of functions with compact support in Ω.

The sup-norm of any function $u \in C(\Omega)$ is defined by

$$\|u\|_{C(\Omega)} := \sup_\Omega |u|,$$

and the C^k-norm of $u \in C^k(\Omega)$ is defined by

$$\|u\|_{C^k(\Omega)} := \max_{|\alpha|\le k} \sup_\Omega |\partial^\alpha u|.$$

Despite the terminology, $\|u\|_{C^k(\Omega)}$ is *not* a norm in $C^k(\Omega)$ because it may take the value ∞. In fact, the topology of the space $C^k(\Omega)$ is defined by the family of *seminorms* $\|u\|_{C^k(\Omega')}$ where Ω' is any open subset of Ω such that $\Omega' \Subset \Omega$. The relation $E \Subset \Omega$ (*compact inclusion*) means that the closure $\overline{E}$ of the set E is compact and $\overline{E} \subset \Omega$.

Denote by μ the Lebesgue measure in $\mathbb{R}^n$. By "a measurable function" we always mean a function measurable with respect to measure μ. For any open set $\Omega \subset \mathbb{R}^n$, $L^p(\Omega)$ stands for the Lebesgue space $L^p(\Omega,\mu)$, $1 \le p \le \infty$ (see Section A.4.5). The *local* Lebesgue space $L^p_{loc}(\Omega)$ is the set of all measurable functions f in Ω such that $f \in L^p(\Omega')$ for any open set $\Omega' \Subset \Omega$. Clearly, $L^p_{loc}(\Omega)$ is a linear space, and it has a natural topology, defined by the family of seminorms $\|f\|_{L^p(\Omega')}$ where Ω' runs over all open sets compactly contained in Ω.

If V and W are two linear topological spaces then an *embedding* of V to W is a linear continuous injection $V \to W$. We will apply this notion when V, W are spaces of functions on the same set and a natural embedding of V to W is obtained by identifying functions from V as elements from W. In this case, we denote the embedding by $V \hookrightarrow W$ and will normally consider V as a subspace of W (although, in general, the topology of V is stronger than that of W).

Obviously, we have the embeddings

$$C^k(\Omega) \hookrightarrow C(\Omega) \hookrightarrow L^\infty_{loc}(\Omega).$$

For another example, let Ω' be an open subset of Ω. Any function from $f \in L^p(\Omega')$ can be identified as a function from $L^p(\Omega)$ just by setting $f = 0$ in $\Omega \setminus \Omega'$. Since this mapping from $L^p(\Omega')$ to $L^p(\Omega)$ is injective and bounded (in fact, norm preserving), we obtain a natural embedding $L^p(\Omega') \hookrightarrow L^p(\Omega)$. One can, of course, define also a mapping from $L^p(\Omega)$ to $L^p(\Omega')$ just by restricting a function on Ω to Ω'. Although this mapping is bounded, it is not injective and, hence, is not an embedding.

CLAIM. *$L^p_{loc}(\Omega) \hookrightarrow L^1_{loc}(\Omega)$ for any $p \in [1, +\infty]$.*

PROOF. Indeed, for all $f \in L^p_{loc}(\Omega)$ and $\Omega' \Subset \Omega$, we have by the Hölder inequality

$$\|f\|_{L^1(\Omega')} = \int_{\Omega'} 1 \cdot |f| \, d\mu \leq \mu(\Omega')^{1-1/p} \left(\int_{\Omega'} |f|^p \, d\mu \right)^{1/p} = C\|f\|_{L^p(\Omega')}, \tag{2.2}$$

where $C := \mu(\Omega')^{1-1/p} < \infty$ (strictly speaking, the above computations is valid only if $p < \infty$, but the case $p = \infty$ is trivial – cf. Exercise 2.1). Therefore, any function from $L^p_{loc}(\Omega)$ belongs also to $L^1_{loc}(\Omega)$, which defines a natural linear injection from $L^p_{loc}(\Omega)$ to $L^1_{loc}(\Omega)$, and this injection is continuous by (2.2). $\square$

It follows that all the function spaces considered above embed into $L^1_{loc}(\mathbb{R}^n)$.

Exercises.

2.1. Prove that $L^q_{loc}(\Omega) \hookrightarrow L^p_{loc}(\Omega)$ for all $1 \leq p < q \leq +\infty$.

2.2. Let $\{f_k\}$ be a sequence of functions from $L^p(\Omega)$ that converges to a function f in L^p norm, $1 \leq p \leq \infty$. Prove that if $f_k \geq 0$ a.e. then also $f \geq 0$ a.e..

2.2. Convolution and partition of unity

The purpose on this section is to approximate functions from L^1 and L^1_{loc} by smooth functions. The main technical tool for that is the notion of convolution. Recall that, for any two measurable functions f, g on $\mathbb{R}^n$, their convolution $f * g$ is defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(x - y) \, g(y) \, d\mu(y), \tag{2.3}$$

provided the integral converges in the Lebesgue sense. Note that the function $f(x-y)\,g(y)$ is measurable as a function of x, y and, by Fubini's theorem, if the above integral converges then it defines a measurable function of x.

Denote by $B_r(x)$ the ball of radius r centered at x, that is,

$$B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}.$$

LEMMA 2.1. *If $f \in L^1_{loc}(\mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ then the convolution $f * \varphi$ belongs to $C^\infty(\mathbb{R}^n)$ and, for any multiindex α,*

$$\partial^\alpha (f * \varphi) = f * \partial^\alpha \varphi. \tag{2.4}$$

Also, if $\operatorname{supp}\varphi \subset B_r(0)$ *then* $\operatorname{supp}(f * \varphi)$ *is contained in the r-neighborhood of* $\operatorname{supp} f$.

PROOF. Assuming that $\operatorname{supp}\varphi \subset B_r(0)$ and changing $z = x - y$ in (2.3), we obtain

$$f * \varphi(x) = \int_{\mathbb{R}^n} f(z) \, \varphi(x - z) \, dz = \int_{B_r(x)} f(z) \, \varphi(x - z) \, dz. \tag{2.5}$$

Since $f \in L^1(B_r(x))$ and φ is bounded, we see that the integral in the right hand side converges for all x. If x is away from the r-neighborhood of $\operatorname{supp} f$ then $B_r(x)$ and $\operatorname{supp} f$ are disjoint, whence it follows that $f * \varphi(x) = 0$, which proves the second claim of Lemma 2.1.

Let us show that $f * \varphi$ is continuous. If x' is close enough to x, namely, $|x - x'| < r$ then we have

$$f * \varphi(x') = \int_{B_r(x')} f(z) \varphi(x' - z)\, dz = \int_{B_{2r}(x)} f(z) \varphi(x' - z)\, dz.$$

Since f is integrable in $B_{2r}(x)$ and $\varphi(x' - z) \rightrightarrows \varphi(x - z)$ as $x' \to x$, we can pass to the limit in the above integral and obtain that $f * \varphi(x') \to f * \varphi(x)$.

Let us show that the derivative $\partial_j (f * \varphi)$ exists and is equal to $f * \partial_j \varphi$. If h is a non-zero vector in the direction x^j then we have

$$\frac{f * \varphi(x + h) - f * \varphi(x)}{|h|} = \int_{\mathbb{R}^n} f(z) \frac{\varphi(x + h - z) - \varphi(x - z)}{|h|} dz.$$

Again, if $|h|$ is small enough then the integration can be restricted to $z \in B_{2r}(x)$. Since f is integrable in this ball and

$$\frac{\varphi(x + h - z) - \varphi(x - z)}{|h|} \rightrightarrows \partial_j \varphi(x - z)$$

as $h \to 0$, we can pass to the limit under the integral and conclude that

$$\begin{aligned} \partial_j (f * \varphi)(x) &= \lim_{h \to 0} \frac{f * \varphi(x + h) - f * \varphi(x)}{|h|} \\ &= \int_{\mathbb{R}^n} f(z) \partial_j \varphi(x - z)\, dx \\ &= f * \partial_j \varphi(x). \end{aligned}$$

Applying the same argument to $f * \partial_j \varphi$ and continuing by induction, we obtain (2.4) for an arbitrary α and $f * \varphi \in C^\infty(\mathbb{R}^n)$. $\square$

We say that a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ is a *mollifier* if $\operatorname{supp} \varphi \subset B_1(0)$, $\varphi \geq 0$, and

$$\int_{\mathbb{R}^n} \varphi d\mu = 1. \tag{2.6}$$

For example, the following function

$$\varphi(x) = \begin{cases} c \exp\left(-\frac{1}{(|x|^2 - 1/4)^2}\right) & |x| < 1/2 \\ 0, & |x| \geq 1/2 \end{cases} \tag{2.7}$$

is a mollifier, for a suitable normalizing constant $c > 0$ (see Fig. 2.1).

If φ is a mollifier then, for any $0 < \varepsilon < 1$, the function

$$\varphi_\varepsilon := \varepsilon^{-n} \varphi\left(\frac{x}{\varepsilon}\right)$$

is also a mollifier, and $\operatorname{supp} \varphi_\varepsilon \subset B_\varepsilon(0)$.

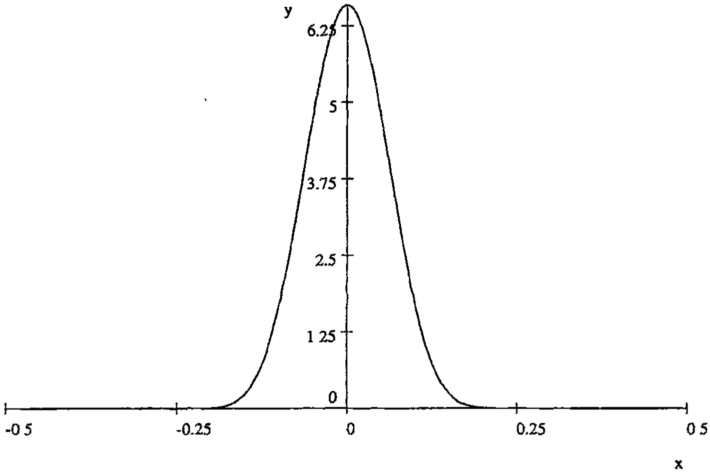

FIGURE 2.1. The mollifier (2.7) in $\mathbb{R}$.

THEOREM 2.2. (Partition of unity) *Let K be a compact subset of $\mathbb{R}^n$ and $\{U_j\}_{j=1}^k$ be a finite family of open sets covering K. Then there exist non-negative functions $\varphi_j \in C_0^\infty(U_j)$ such that $\sum_j \varphi_j \equiv 1$ in an open neighbourhood of K and $\sum_j \varphi_j \leq 1$ in $\mathbb{R}^n$.*

Such a family of functions φ_j is called a *partition of unity* at K subordinate to the covering $\{U_j\}$.

PROOF. Consider first the case when the family $\{U_j\}$ consists of a single set U. Then we will construct a function $\psi \in C_0^\infty(U)$ such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in an open neighbourhood of K. Such a function ψ is called a *cutoff function* of K in U.

Let V be an open neighborhood of K such that $V \Subset U$, and set $f = 1_V$. Fix a mollifier φ. Since $f \in L^1(\mathbb{R}^n)$, by Lemma 2.1 we have $f * \varphi_\varepsilon \in C^\infty(\mathbb{R}^n)$. If ε is small enough then $f * \varphi_\varepsilon$ is supported in U so that $f * \varphi_\varepsilon \in C_0^\infty(U)$. Clearly, $f * \varphi_\varepsilon \geq 0$ and

$$f * \varphi_\varepsilon(x) \leq \sup|f| \int_{\mathbb{R}^n} \varphi_\varepsilon(y)\, dy = \sup|f| = 1.$$

Finally, if ε is small enough then, for any $x \in K$, we have $B_\varepsilon(x) \subset V$, whence $f|_{B_\varepsilon(x)} = 1$ and

$$f * \varphi_\varepsilon(x) = \int_{B_\varepsilon(x)} f(z)\,\varphi_\varepsilon(x-z)\, dz = \int_{B_\varepsilon(x)} \varphi_\varepsilon(x-z)\, dz = 1.$$

Hence, the function $\psi = f * \varphi_\varepsilon$ satisfies all the requirements, provided ε is small enough.

Consider now the general case of an arbitrary finite family $\{U_j\}$. Any point $x \in K$ belongs to a set U_j. Hence, there is a ball B_x centered at x and such that $B_x \Subset U_j$. The family of balls $\{B_x\}_{x \in K}$ obviously covers K.

Select a finite subfamily with the same property, say $\{B_{x_i}\}$, and denote by V_j the union of those balls B_{x_i} for which $B_{x_i} \Subset U_j$ (see Fig. 2.2).

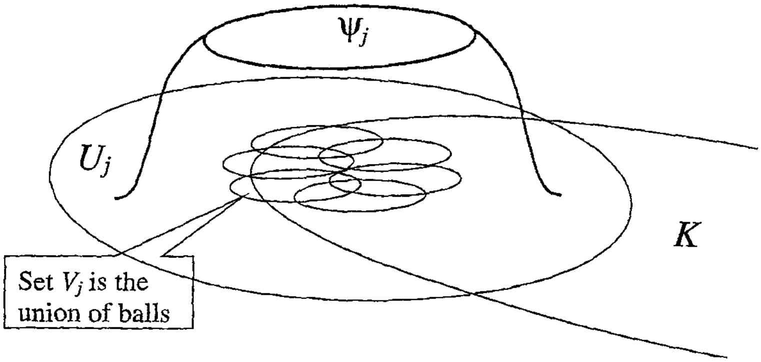

FIGURE 2.2. Function ψ_j is a cutoff function of V_j in U_j.

By construction, the set V_j is open, $V_j \Subset U_j$, and the union of all sets V_j covers K. Let ψ_j be a cutoff function of V_j in U_j, and set

$$\varphi_1 = \psi_1,\ \varphi_2 = \psi_2 (1 - \psi_1),\ \ldots,\ \varphi_k = \psi_k (1 - \psi_1) \ldots (1 - \psi_{k-1}).$$

Obviously, $\varphi_j \in C_0^\infty (U_j)$ and $\varphi_j \geq 0$. It is easy to check the identity

$$1 - \sum_j \varphi_j = (1 - \psi_1) \ldots (1 - \psi_k), \tag{2.8}$$

which, in particular, implies $\sum \varphi_j \leq 1$. Since $1 - \psi_j = 0$ on V_j, (2.8) implies also that $\sum_j \varphi_j \equiv 1$ on the union of sets V_j and, in particular, on K. $\square$

2.3. Approximation of integrable functions by smooth ones

THEOREM 2.3. *For any $1 \leq p < \infty$ and for any open set $\Omega \subset \mathbb{R}^n$, $C_0^\infty (\Omega)$ is dense in $L^p (\Omega)$, and the space $L^p (\Omega)$ is separable.*

PROOF. We need to show that any function $f \in L^p (\Omega)$ can be approximated in L^p norm by a sequence of functions from $C_0^\infty (\Omega)$. Recall that a simple function in Ω is a linear combination of the indicator functions 1_E where $E \subset \Omega$ is a measurable set with finite measure. Since the class of simple functions in dense in $L^p (\Omega)$ (see Section A.4.3), it suffices to prove the above claim in the case $f = 1_E$. By the regularity of the Lebesgue measure (see Section A.4.1), for any $\varepsilon > 0$ there exist a compact set K and an open set $U \subset \Omega$ such that

$$K \subset E \subset U,$$

and

$$\mu (U) \leq \mu (K) + \varepsilon.$$

Let $\psi \in C_0^\infty(U)$ be a cutoff function of K in U. Then $\psi = 1 = 1_E$ on K, $\psi = 1_E = 0$ outside U, whereas in $U \setminus K$ we have $|1_E - \psi| \leq 1$. Therefore,

$$\|1_E - \psi\|_{L^p}^p = \int_\Omega |1_E - \psi|^p \, d\mu \leq \mu(U \setminus K) \leq \varepsilon,$$

which settles the first claim.

To prove the separability of $L^p(\Omega)$, consider the following functions in $L^p(\Omega)$:

$$\sum_{i=1}^k \alpha_i 1_{Q_i} \tag{2.9}$$

where k is a positive integer, α_i are rationals, and Q_i are disjoint open boxes in Ω with rationals coordinates of the corners[1]. Clearly, the family of all such functions is countable. Let us show that this family is dense in $L^p(\Omega)$, which will prove the separability. As in the first part, it suffices to prove that, for any measurable set $E \subset \Omega$ of finite measure, the indicator function 1_E can be approximated in L^p norm by functions (2.9).

Let ε, K, U be as in the first part of the proof. Fix a rational $\delta > 0$ and consider the lattice $\delta\mathbb{Z}^n$, which induces the splitting of $\mathbb{R}^n$ into the cubes of the size δ. Let $Q_1, ..., Q_k$ be those (open) cubes that are contained in Ω. If δ is small enough then the closed cubes $\overline{Q}_1, ..., \overline{Q}_k$ cover the compact set K. Hence,

$$\mu(K) \leq \sum_{i=1}^k \mu(\overline{Q}_i) = \sum_{i=1}^k \mu(Q_i) \leq \mu(U),$$

whence it follows that

$$\|1_E - \sum_{i=1}^k 1_{Q_i}\|_{L^p}^p \leq \varepsilon,$$

which finishes the proof. $\square$

Mollifiers allow to construct smooth approximations to integrable function with additional properties. The following lemma has numerous extensions to other functional classes (cf. Lemma 2.10, Theorems 2.11, 2.13, 2.16, and Exercise 2.18).

LEMMA 2.4. *Let φ be a mollifier.*

(i) If f is a uniformly continuous function on $\mathbb{R}^n$ then $f * \varphi_\varepsilon \rightrightarrows f$ as $\varepsilon \to 0$. If $f \in C(\mathbb{R}^n)$ then $f * \varphi_\varepsilon \to f$ as $\varepsilon \to 0$ locally uniformly.

(ii) If $f \in L^1(\mathbb{R}^n)$ then also $f * \varphi \in L^1(\mathbb{R}^n)$ and

$$f * \varphi_\varepsilon \overset{L^1(\mathbb{R}^n)}{\longrightarrow} f \quad \text{as } \varepsilon \to 0. \tag{2.10}$$

---

[1]Here a box in $\mathbb{R}^n$ is a set of the form $I_1 \times ... \times I_n$ where each I_k is a bounded open interval $(a_k, b_k) \subset \mathbb{R}$. We choose the boxes with rational a_k, b_k.

(iii) If $f \in L^1_{loc}(\mathbb{R}^n)$ *then*

$$f * \varphi_\varepsilon \overset{L^1_{loc}(\mathbb{R}^n)}{\longrightarrow} f \quad \text{as } \varepsilon \to 0.$$

PROOF. (*i*) We have

$$\begin{aligned} f * \varphi_\varepsilon(x) - f(x) &= \int_{B_\varepsilon(0)} f(x-y)\varphi_\varepsilon(y)\,dy - f(x)\int_{B_\varepsilon(0)} \varphi_\varepsilon(y)\,dy \\ &= \int_{B_\varepsilon(0)} (f(x-y) - f(x))\varphi_\varepsilon(y)\,dy. \end{aligned}$$

The uniform continuity of f yields

$$\sup_{x\in\mathbb{R}^n, |y|<\varepsilon} |f(x-y) - f(x)| \to 0 \quad \text{as } \varepsilon \to 0,$$

which implies $f * \varphi_\varepsilon \rightrightarrows f$.

If $f \in C(\mathbb{R}^n)$ then $f(x)$ is uniformly continuous on compact sets, and the same argument works when x varies in a compact set rather than in $\mathbb{R}^n$.

(*ii*) Using Fubini's theorem and (2.6), we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |f * \varphi(x)|\,dx &\le \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |f|(x-y)\varphi(y)\,dy\right)dx \\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |f|(x-y)\,dx\right)\varphi(y)\,dy \\ &= \|f\|_{L^1}, \end{aligned}$$

that is,

$$\|f * \varphi\|_{L^1} \le \|f\|_{L^1}. \tag{2.11}$$

By Theorem 2.3, $C_0^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$. Hence, for a given $\delta > 0$, there exists $g \in C_0^\infty(\mathbb{R}^n)$ such that $\|f - g\|_{L^1} < \delta$ (in fact, we need only that g is a continuous function with compact support). Then we have

$$\|f * \varphi_\varepsilon - f\|_{L^1} \le \|f * \varphi_\varepsilon - g * \varphi_\varepsilon\|_{L^1} + \|g * \varphi_\varepsilon - g\|_{L^1} + \|g - f\|_{L^1}.$$

Using (2.11), we obtain

$$\|f * \varphi_\varepsilon - g * \varphi_\varepsilon\|_{L^1} = \|(f-g) * \varphi_\varepsilon\|_{L^1} \le \|f - g\|_{L^1} < \delta,$$

whence

$$\|f * \varphi_\varepsilon - f\|_{L^1} \le \|g * \varphi_\varepsilon - g\|_{L^1} + 2\delta. \tag{2.12}$$

By part (*i*), we have $g * \varphi_\varepsilon \rightrightarrows g$ as $\varepsilon \to 0$. Obviously, $\operatorname{supp}(g * \varphi_\varepsilon)$ is contained in the ε-neighborhood of $\operatorname{supp} g$, which implies

$$g * \varphi_\varepsilon \overset{L^1}{\longrightarrow} g.$$

Hence, (2.12) yields

$$\limsup_{\varepsilon\to 0} \|f * \varphi_\varepsilon - f\|_{L^1} \le 2\delta,$$

and, since $\delta > 0$ is arbitrary, we obtain (2.10).

(*iii*) It suffices to prove that, for any bounded open set $\Omega \subset \mathbb{R}^n$, $f * \varphi_\varepsilon \to f$ in $L^1(\Omega)$. Let Ω_1 be the 1-neighborhood of Ω and set $g = 1_{\Omega_1} f$. Then

$g \in L^1(\mathbb{R}^n)$ and, by part (ii), we have $g * \varphi_\varepsilon \to g$ in $L^1(\mathbb{R}^n)$. Since $f = g$ in Ω_1 and $\operatorname{supp}\varphi_\varepsilon \subset B_1(0)$, we obtain that $f * \varphi_\varepsilon = g * \varphi_\varepsilon$ in Ω. Therefore, we conclude that $f * \varphi_\varepsilon \to f$ in $L^1(\Omega)$. □

COROLLARY 2.5. *For any open set $\Omega \subset \mathbb{R}^n$, if $f \in L^1_{loc}(\Omega)$ and*

$$\int_\Omega f\psi d\mu = 0 \text{ for any } \psi \in C_0^\infty(\Omega), \tag{2.13}$$

then $f = 0$ a.e. in Ω.

PROOF. Let φ be a mollifier and fix an open set $\Omega' \Subset \Omega$. If $\varepsilon > 0$ is small enough then, for any $x \in \Omega'$, the function $\varphi_\varepsilon(x - \cdot)$ is supported in $B_\varepsilon(x) \subset \Omega$, which implies by (2.13)

$$f * \varphi_\varepsilon(x) = \int_\Omega f(z)\varphi_\varepsilon(x-z)\,dz = 0.$$

By Lemma 2.4, $f * \varphi_\varepsilon \to f$ in $L^1_{loc}(\mathbb{R}^n)$, whence it follows that $f = 0$ a.e. in Ω'. Since Ω' was arbitrary, we conclude $f = 0$ a.e.in Ω, which was to be proved. □

Exercises.

2.3. Prove that if $f \in L^\infty(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ then $f * g \in L^\infty(\mathbb{R}^n)$ and

$$\|f * g\|_{L^\infty} \le \|f\|_{L^\infty}\|g\|_{L^1}.$$

2.4. Prove that if $f, g \in L^1(\mathbb{R}^n)$ then $f * g \in L^1(\mathbb{R}^n)$ and

$$\|f * g\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}.$$

2.5. Prove that if $f, g, h \in L^1(\mathbb{R}^n)$ then $f * g = g * f$ and

$$(f * g) * h = f * (g * h).$$

2.6. Prove that if $C^k(\mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ then, for any multiindex α with $|\alpha| \le k$,

$$\partial^\alpha(f * \varphi) = (\partial^\alpha f) * \varphi.$$

2.7. Prove that if $f \in C^k(\mathbb{R}^n)$ and φ is a mollifier in $\mathbb{R}^n$ then $f * \varphi_\varepsilon \to f$ as $\varepsilon \to 0$ in the topology of $C^k(\mathbb{R}^n)$.

2.8. Let $f \in L^1_{loc}(\Omega)$. Prove that $f \ge 0$ a.e. if and only if

$$\int_\Omega f\psi d\mu \ge 0,$$

for all non-negative function $\psi \in C_0^\infty(\Omega)$.

2.4. Distributions

For any open set $\Omega \subset \mathbb{R}^n$, define the space of *test functions* $\mathcal{D}(\Omega)$ as follows. As a set, $\mathcal{D}(\Omega)$ is identical to $C_0^\infty(\Omega)$ but, in addition, $\mathcal{D}(\Omega)$ is endowed with the following *convergence*: a sequence $\{\varphi_k\}$ converges to φ in $\mathcal{D}(\Omega)$ if

(1) $\partial^\alpha\varphi_k \rightrightarrows \partial^\alpha\varphi$ for any multiindex α;
(2) all supports $\operatorname{supp}\varphi_k$ are contained in some compact set $K \subset \Omega$.

If these two conditions are satisfied then we will write $\varphi_k \xrightarrow{\mathcal{D}(\Omega)} \varphi$ or $\varphi_k \xrightarrow{\mathcal{D}} \varphi$. It is possible to prove that this convergence comes from a certain topology, which makes $\mathcal{D}(\Omega)$ into a linear topological space[2]. It is easy to see that if $\varphi_k \xrightarrow{\mathcal{D}} \varphi$ then also $\partial^\alpha \varphi_k \xrightarrow{\mathcal{D}} \partial^\alpha \varphi$ for any multiindex α.

Denote by $\mathcal{D}'(\Omega)$ the dual space to $\mathcal{D}(\Omega)$, that is, the space of all linear continuous functionals on $\mathcal{D}(\Omega)$. The elements of $\mathcal{D}'(\Omega)$ are called *distributions* in Ω. If $u \in \mathcal{D}'(\Omega)$ then the action of u at a test function $\varphi \in \mathcal{D}(\Omega)$ is denoted by (u, φ). The bracket (u, φ) is also referred to as the *pairing* of a distribution and a test function. The continuity of u means that $(u, \varphi_k) \to (u, \varphi)$ whenever $\varphi_k \xrightarrow{\mathcal{D}} \varphi$.

Obviously, $\mathcal{D}'(\Omega)$ is a linear space. We will use the following convergence in $\mathcal{D}'(\Omega)$: $u_k \xrightarrow{\mathcal{D}'} u$ if $(u_k, \varphi) \to (u, \varphi)$ for any $\varphi \in \mathcal{D}(\Omega)$ (this convergence is associated with the *weak* topology of $\mathcal{D}'(\Omega)$)

Any function $u \in L^1_{loc}(\Omega)$ can be identified as a distribution by the following rule[3]

$$(u, \varphi) = \int_\Omega u\varphi \, d\mu \text{ for any } \varphi \in \mathcal{D}(\Omega), \tag{2.14}$$

where μ is the Lebesgue measure. Clearly, $\varphi_k \xrightarrow{\mathcal{D}} \varphi$ implies $(u, \varphi_k) \to (u, \varphi)$ so that (2.14), indeed, defines a distribution. If $u \in L^1_{loc}(\Omega)$ defines by (2.14) the zero distribution then Corollary 2.5 yields that $u = 0$ as an element of $L^1_{loc}(\Omega)$. If a sequence u_k converges to u in $L^1_{loc}(\Omega)$ then obviously $(u_k, \varphi) \to (u, \varphi)$ for any $\varphi \in \mathcal{D}(\Omega)$, that is, $u_k \xrightarrow{\mathcal{D}'} u$. Therefore, the relation (4.1) defines an embedding

$$L^1_{loc}(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

From now on, we will regard $L^1_{loc}(\Omega)$ as a subspace of $\mathcal{D}'(\Omega)$. Hence, all other function spaces $C^k(\Omega)$, $L^p(\Omega)$, and $L^p_{loc}(\Omega)$ also become subspaces of $\mathcal{D}'(\Omega)$.

Another example of a distribution is the *delta function* δ_z: for any fixed point $z \in \Omega$, δ_z is defined by

$$(\delta_z, \varphi) = \varphi(z) \quad \text{for any } \varphi \in \mathcal{D}(\Omega).$$

---

[2]Any topology determines a convergence, which in this context is called a topological convergence. However, not every convergence is topological. For example, convergence almost everywhere is not a topological one. Although the convergence in $\mathcal{D}$ is topological, we never actually need the topology in $\mathcal{D}$ and will work only with the convergence.

[3]The notation (u, φ) is consistent with the usage of the brackets to denote the inner product in L^2. Hence, if $u \in L^2$ then (u, φ) means both the inner product of u and φ and the pairing of u and φ in the sense of distributions. If it is still necessary to distinguish these notions then we will use $(\cdot, \cdot)_{L^2}$ to denote the inner product in L^2. For example, the difference occurs when one considers complex valued functions (which we normally do not). In this case, $(u, \varphi)_{L^2} = (u, \overline{\varphi})$ where $\overline{\varphi}$ is the complex conjugate of φ.

This example shows that there are distributions that are not obtained from functions by the rule (2.14). The delta function belongs to a class of distributions that arise from measures. Indeed, any Radon (signed) measure ν in Ω determines a distribution by

$$(\nu, \varphi) = \int_\Omega \varphi d\nu.$$

Using the integration by parts formula, we see that, for all $u \in C^1(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega (\partial_j u)\, \varphi\, d\mu = -\int_\Omega u\, \partial_j \varphi\, d\mu.$$

This suggests the following extension of the operator ∂_j to the space $\mathcal{D}'(\Omega)$: for any distribution $u \in \mathcal{D}'(\Omega)$, define its derivative $\partial_j u \in \mathcal{D}'(\Omega)$ by the rule

$$(\partial_j u, \varphi) = -(u, \partial_j \varphi)\,, \ \text{ for all } \varphi \in \mathcal{D}(\Omega)\,. \tag{2.15}$$

Obviously, the right hand side of (2.15) is, indeed, a continuous linear functional on $\mathcal{D}(\Omega)$ and, hence, $\partial_j u$ is defined as an element of $\mathcal{D}'(\Omega)$. Now we can define $\partial^\alpha u$ for any multiindex α either inductively, using (2.1), or directly by

$$(\partial^\alpha u, \varphi) = (-1)^{|\alpha|}(u, \partial^\alpha \varphi)\,, \ \text{ for all } \varphi \in \mathcal{D}(\Omega)\,. \tag{2.16}$$

It is worth mentioning that ∂^α is a continuous operator in $\mathcal{D}'(\Omega)$ (cf. Exercise 2.13). Clearly, we have $\partial^\alpha \partial^\beta u = \partial^{\alpha+\beta} u$ for any $u \in \mathcal{D}'(\Omega)$ and for all multiindices α, β.

It is a consequence of the definition that all distributions are differentiable infinitely many times. In particular, any function $u \in L^1_{loc}(\Omega)$ has all partial derivatives $\partial^\alpha u$ as distributions. However, a function can be differentiated also in the *classical* sense, when $\partial_j u$ is defined pointwise as the limit of the difference quotient. We will distinguish the two kinds of derivatives by referring to them as *distributional* versus *classical* derivatives. It is clear from the above definition that if $u \in C^k(\Omega)$ then all the classical derivatives $\partial^\alpha u$ of the order $|\alpha| \le k$ coincide with their distributional counterparts.

Let us define one more operation on distributions: multiplication by a smooth function. If $u \in L^2_{loc}(\Omega)$ and $f \in C^\infty(\Omega)$ then we have obviously the identity

$$\int_\Omega (fu)\, \varphi d\mu = \int_\Omega u\, (f\varphi)\, d\mu \ \text{ for any } \varphi \in \mathcal{D}(\Omega)\,.$$

Hence, for a distribution $u \in \mathcal{D}'(\Omega)$ and a function $f \in C^\infty(\Omega)$, define a distribution fu by the identity

$$(fu, \varphi) = (u, f\varphi) \ \text{ for all } \varphi \in \mathcal{D}(\Omega)\,.$$

We say that a distribution $u \in \mathcal{D}'(\Omega)$ *vanishes* in an open set $U \subset \Omega$ if $(u, \varphi) = 0$ for any $\varphi \in \mathcal{D}(U)$. It is possible to prove that if u vanishes in a family of open sets then it vanishes also in their union (cf. Exercise 2.10). Hence, there is a maximal open set in Ω where u vanishes. Its complement in Ω is called the *support* of u and is denoted by $\operatorname{supp} u$. Clearly, $\operatorname{supp} u$ is a closed subset of Ω. For any function $u \in L^1_{loc}(\Omega)$, its support $\operatorname{supp} u$ is defined as the support of the associated distribution u. If $u \in C(\Omega)$ then this definition of the support is consistent with (1.2) (cf. Exercise 2.11).

Let us state for the record the following properties of distributions (the proofs are straightforward and are omitted).

CLAIM. *Let $u \in \mathcal{D}'(\Omega)$.*

(i) *For any derivative ∂^α, we have* $\operatorname{supp} \partial^\alpha u \subset \operatorname{supp} u$.
(ii) *If $\varphi_1, \varphi_2 \in \mathcal{D}(\Omega)$ and $\varphi_1 = \varphi_2$ in a neighborhood of* $\operatorname{supp} u$ *then* $(u, \varphi_1) = (u, \varphi_2)$.

If $\operatorname{supp} u$ is a compact subset of Ω then u can be canonically extended to a distribution in $\mathbb{R}^n$ as follows. Let $\psi \in \mathcal{D}(\Omega)$ be a cutoff function of a neighborhood of $\operatorname{supp} u$ in Ω (see Theorem 2.2). Then, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, the function $\psi\varphi$ belongs to $\mathcal{D}(\Omega)$, which allows to define (u, φ) by

$$(u, \varphi) := (u, \psi\varphi). \tag{2.17}$$

Note that if $\varphi \in \mathcal{D}(\Omega)$ then $\psi\varphi = \varphi$ in a neighborhood of $\operatorname{supp} u$ and, hence, $(u, \psi\varphi) = (u, \varphi)$. Therefore, the above extension of u is consistent with the action of u in $\mathcal{D}(\Omega)$. Also, this extension is independent of the choice of ψ because if ψ' is another cut-off function then $\psi = \psi'$ in a neighborhood of $\operatorname{supp} u$, which implies $(u, \psi\varphi) = (u, \psi'\varphi)$.

LEMMA 2.6. *Let u be a distribution in Ω with compact support and let $v = \partial^\alpha u$. Let u' and v' be the canonical extensions of u and v to $\mathbb{R}^n$ as described above. Then $v' = \partial^\alpha u'$ in $\mathbb{R}^n$.*

PROOF. In other words, this statement says that the extension operator commutes with ∂^α. It suffices to show that for the first order derivative. Hence, let us prove that, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\left(u', \partial_j \varphi\right) = -\left(v', \varphi\right),$$

which, in the view of (2.17), amounts to

$$(u, \psi \partial_j \varphi) = -\left(\partial_j u, \psi\varphi\right).$$

We have

$$-\left(\partial_j u, \psi\varphi\right) = \left(u, \partial_j (\psi\varphi)\right).$$

and

$$\partial_j (\psi\varphi) = (\partial_j \psi)\varphi + \psi \partial_j \varphi.$$

Since $\partial_j \psi \equiv 0$ in a neighborhood of $\operatorname{supp} u$ and hence, $(u, (\partial_j \psi)\varphi) = 0$, we obtain

$$\left(u, \partial_j (\psi\varphi)\right) = (u, \psi_j \partial\varphi),$$

which finishes the proof. □

LEMMA 2.7. *For any distribution $u \in \mathcal{D}'(\Omega)$ and for any open set $U \Subset \Omega$, there exist a positive integer N and a real $C > 0$ such that, for any $\varphi \in \mathcal{D}(U)$,*

$$|(u, \varphi)| \leq C \max_{|\alpha| \leq N} \sup_U |\partial^\alpha \varphi| . \tag{2.18}$$

PROOF. Assume that (2.18) does not hold for any C and N. Then, for any positive integer k, there exists $\varphi_k \in \mathcal{D}(U)$ such that

$$(u, \varphi_k) \geq k \max_{|\alpha| \leq k} \sup_U |\partial^\alpha \varphi_k| .$$

Multiplying φ_k by a constant, we can assume that $(u, \varphi_k) = 1$, which implies

$$\max_{|\alpha| \leq k} \sup_U |\partial^\alpha \varphi_k| \leq \frac{1}{k}.$$

It follows that, for any α, $\partial^\alpha \varphi_k$ converges to 0 uniformly on U. Since all $\operatorname{supp} \varphi_k$ are contained in U, we conclude that $\varphi_k \xrightarrow{\mathcal{D}} 0$. By the continuity of u, this should imply $(u, \varphi_k) \to 0$, which contradicts $(u, \varphi_k) = 1$. $\square$

Exercises.

2.9. For a function f on $\mathbb{R}$, denote by f'_{dist} its distributional derivative, reserving f' for the classical derivative.

(a) Prove that if $f \in C^1(\mathbb{R})$ then $f'_{dist} = f'$.
(b) Prove that the same is true if f is continuous and piecewise continuously differentiable.
(c) Evaluate f'_{dist} for $f(x) = |x|$.
(d) Let $f = 1_{[0,+\infty)}$. Prove that $f'_{dist} = \delta$, where δ is the Dirac delta-function at 0.

2.10. Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that two distributions $u, v \in \mathcal{D}'(\Omega)$ are equal on an open subset $U \subset \Omega$ if $(u, \varphi) = (v, \varphi)$ for all $\varphi \in \mathcal{D}(U)$.

(a) Let $\{\Omega_\alpha\}$ be a family of open subsets of Ω. Prove that if u and v are equal on each of the sets Ω_α then they are equal on their union $\cup_\alpha \Omega_\alpha$.
(b) Prove that for any $u \in \mathcal{D}'(\Omega)$ there exists the maximal open set $U \subset \Omega$ such that $u = 0$ in U.

REMARK. The closed set $\Omega \setminus U$ is called the support of the distribution u and is denoted by $\operatorname{supp} u$.

2.11. For any function $u(x)$, defined pointwise in Ω, set

$$S(u) = \overline{\{x \in \Omega : u(x) \neq 0\}},$$

where the bar means the closure in Ω.

(a) Prove that if $u \in C(\Omega)$ then its support $\operatorname{supp} u$ in the distributional sense coincides with $S(u)$.
(b) If $u \in L^1_{loc}(\Omega)$ then its support $\operatorname{supp} u$ in the distributional sense can be identified by

$$\operatorname{supp} u = \bigcap_{v = u \text{ a. e.}} S(v),$$

where the intersection is taken over all functions v in Ω, defined pointwise, which are equal to u almost everywhere.

2.12. Prove the product rule: if $u \in \mathcal{D}'(\Omega)$ and $f \in C^\infty(\Omega)$ then

$$\partial^\alpha (fu) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f \, \partial^\beta u, \tag{2.19}$$

where

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$$

is the product of the binomial coefficients, and $\beta \le \alpha$ means that $\beta_i \le \alpha_i$ for all $i = 1, ..., n$.

2.13. Let $\{u_k\}$ be a sequence of distributions in Ω such that $u_k \xrightarrow{\mathcal{D}'(\Omega)} u$.

(a) Prove that $\partial^\alpha u_k \xrightarrow{\mathcal{D}'} \partial^\alpha u$ for any multiindex α.

(b) Prove that $f u_k \xrightarrow{\mathcal{D}'} fu$ for any $f \in C^\infty(\Omega)$.

2.14. Let X be a topological space. Prove that a sequence $\{x_k\} \subset X$ converges to $x \in X$ (in the topology of X) if and only if any subsequence of $\{x_k\}$ contains a sub-subsequence that converges to x.

2.15. Prove that the convergence "almost everywhere" is not topological, that is, it is not determined by any topology.

2.16. Prove that the convergence in the space $\mathcal{D}(\Omega)$ is topological.

2.5. Approximation of distributions by smooth functions

For any distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ and a function $\varphi \in \mathcal{D}(\mathbb{R}^n)$, define the *convolution* $u * \varphi$ as a function in $\mathbb{R}^n$ by

$$(u * \varphi)(x) = (u, \varphi(x - \cdot)).$$

If $u \in L^1_{loc}(\mathbb{R}^n)$ then this definition obviously matches the one from Section 2.2 (cf. (2.5)).

LEMMA 2.8. *For all $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, the function $u * \varphi$ is continuous and, for any $\psi \in \mathcal{D}(\mathbb{R}^n)$,*

$$(u * \varphi, \psi) = (u, \varphi' * \psi) \tag{2.20}$$

where $\varphi'(x) = \varphi(-x)$.

PROOF. Let us show that $u * \varphi$ is a continuous function. Indeed, if $y \to x$ then obviously

$$\varphi(y - \cdot) \xrightarrow{\mathcal{D}} \varphi(x - \cdot)$$

whence we conclude that

$$(u * \varphi)(y) = (u, \varphi(y - \cdot)) \longrightarrow (u, \varphi(x - \cdot)) = (u * \varphi)(y).$$

In particular, function $u * \varphi$ can be considered as a distribution, which validates the left hand side of (2.20).

To prove (2.20), transform the left hand side of (2.20) as follows:

$$(u * \varphi, \psi) = \int_{\mathbb{R}^n} (u, \varphi(x - \cdot)) \, \psi(x) \, dx = \int_{\mathbb{R}^n} (u, \Phi(x, \cdot)) \, dx,$$

where

$$\Phi(x,y) := \varphi(x-y)\,\psi(x).$$

CLAIM. *For any function $\Phi(x,y) \in \mathcal{D}\left(\mathbb{R}^{2n}\right)$, we have*

$$\int_{\mathbb{R}^n} (u, \Phi(x,\cdot))\,dx = \left(u, \int_{\mathbb{R}^n} \Phi(x,\cdot)\,dx\right). \tag{2.21}$$

Using (2.21), the proof of (2.20) is finished by the observation that

$$\int_{\mathbb{R}^n} \Phi(x,y)\,dx = \int_{\mathbb{R}^n} \varphi'(y-x)\,\psi(x)\,dx = \left(\varphi' * \psi\right)(y).$$

To prove (2.21), let us approximate the integral of Φ by the Riemann sums, as follows:

$$\int_{\mathbb{R}^n} \Phi(x,y)\,dx = \lim_{\varepsilon\to 0} \sum_{k\in\mathbb{Z}^n} \Phi(\varepsilon k, y)\,\varepsilon^n. \tag{2.22}$$

Since the sum here is, in fact, finite, the both sides of (2.22) belong to $\mathcal{D}(\mathbb{R}^n)$ as functions of y, and the support of the right hand side is uniformly bounded for all $\varepsilon > 0$. Since $|\nabla\Phi|$ is uniformly bounded, the limit in (2.22) is uniform with respect to y. Applying the same argument for any derivative $\partial_y^\alpha \Phi$, we obtain that the limit in (2.22) can be understood in the sense of the convergence in $\mathcal{D}(\mathbb{R}^n)$. Therefore, (2.22) implies

$$\begin{aligned}
\left(u, \int_{\mathbb{R}^n} \Phi(x,\cdot)\,dx\right) &= \lim_{\varepsilon\to 0}\left(u, \sum_{k\in\mathbb{Z}^n} \Phi(\varepsilon k,\cdot)\,\varepsilon^n\right) \\
&= \lim_{\varepsilon\to 0} \sum_{k\in\mathbb{Z}^n} (u, \Phi(\varepsilon k,\cdot))\,\varepsilon^n \\
&= \int_{\mathbb{R}^n} (u, \Phi(x,\cdot))\,dx,
\end{aligned}$$

which finishes the proof. □

The following statement extends Lemma 2.1 to distributions.

LEMMA 2.9. *If $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ then $u * \varphi \in C^\infty(\mathbb{R}^n)$ and, for any multiindex α,*

$$\partial^\alpha(u*\varphi) = (\partial^\alpha u) * \varphi = u * (\partial^\alpha \varphi). \tag{2.23}$$

If $\operatorname{supp}\varphi \subset B_r(0)$ *then* $\operatorname{supp}(u*\varphi)$ *is contained in the r-neighborhood of* $\operatorname{supp} u$.

PROOF. Since

$$\operatorname{supp}\varphi(x-\cdot) \subset B_r(x),$$

if x is away from the r-neighborhood of $\operatorname{supp} u$ then $\operatorname{supp}\varphi(x-\cdot)$ and $\operatorname{supp} u$ are disjoint whence $u*\varphi(x) = 0$, which proves the second claim.

The second equality in (2.23) is easily proved as follows:

$$\begin{aligned}(\partial^\alpha u) * \varphi(x) &= (\partial^\alpha u, \varphi(x-\cdot)) = (-1)^{|\alpha|}(u, \partial^\alpha[\varphi(x-\cdot)]) \\ &= (u, (\partial^\alpha \varphi)(x-\cdot)) = u * (\partial^\alpha \varphi)(x).\end{aligned}$$

Before we prove the first equality in (2.23) and the smoothness of $u * \varphi$, recall that, by Lemma 2.8, $u * \varphi \in C(\mathbb{R}^n)$. For any $\psi \in \mathcal{D}(\mathbb{R}^n)$, we have by Lemma 2.8

$$(\partial^\alpha(u * \varphi), \psi) = (-1)^\alpha (u * \varphi, \partial^\alpha \psi) = (-1)^\alpha \left(u, \varphi' * \partial^\alpha \psi\right), \tag{2.24}$$

where the derivative $\partial^\alpha(u * \varphi)$ is understood in the distributional sense. By Lemma 2.1, we have

$$\varphi' * \partial^\alpha \psi = \partial^\alpha \left(\varphi' * \psi\right),$$

whence

$$(-1)^\alpha \left(u, \varphi' * \partial^\alpha \psi\right) = (-1)^\alpha \left(u, \partial^\alpha\left(\varphi' * \psi\right)\right) = \left(\partial^\alpha u, \varphi' * \psi\right) = ((\partial^\alpha u) * \varphi, \psi).$$

Together with (2.24), this proves the first equality in (2.23).

We still need to prove that $u * \varphi \in C^\infty(\mathbb{R}^n)$. What we already know is that $u * \varphi$ is continuous and all its distributional derivatives $\partial^\alpha(u * \varphi)$ are continuous as well. The proof will be completed if we prove the following fact (here ∂^{dist} and ∂^{clas} stand for the distributional and classical derivatives, respectively).

CLAIM. *If $f \in C(\mathbb{R}^n)$ and $\partial_j^{dist} f \in C(\mathbb{R}^n)$ then $\partial_j^{clas} f$ exists at any point and is equal to $\partial_j^{dist} f$.*

Let φ be a mollifier. By Lemma 2.1, the function $f * \varphi_\varepsilon$ is C^∞-smooth. Setting $g = \partial_j^{dist} f$, using the identity (2.4) of Lemma 2.1 and the identity (2.23), we obtain

$$\partial_j^{clas}(f * \varphi_\varepsilon) = \partial_j^{dist}(f * \varphi_\varepsilon) = (\partial_j^{dist} f) * \varphi_\varepsilon = g * \varphi_\varepsilon.$$

By Lemma 2.4, we obtain $f * \varphi_\varepsilon \to f$ as $\varepsilon \to 0$ and

$$\partial_j^{clas}(f * \varphi_\varepsilon) = g * \varphi_\varepsilon \to g,$$

where the convergence is locally uniform. This implies that $\partial_j^{clas} f$ exists at any point and is equal to g. □

In the rest of this section, we extend Lemma 2.4 to the spaces $\mathcal{D}'$ and L^2.

LEMMA 2.10. *Let φ be a mollifier in $\mathbb{R}^n$.*

(i) If $u \in \mathcal{D}(\mathbb{R}^n)$ then $u * \varphi_\varepsilon \xrightarrow{\mathcal{D}} u$.

(ii) If $u \in \mathcal{D}'(\mathbb{R}^n)$ then $u * \varphi_\varepsilon \xrightarrow{\mathcal{D}'} u$.

PROOF. (*i*) By Lemma 2.4, we have $u * \varphi_\varepsilon \rightrightarrows u$. Using Lemma 2.1 or 2.9, we obtain, for any multiindex α,

$$\partial^\alpha(u * \varphi_\varepsilon) = (\partial^\alpha u) * \varphi_\varepsilon \rightrightarrows \partial^\alpha u.$$

Finally, since all the supports of $u * \varphi_\varepsilon$ are uniformly bounded when $\varepsilon \to 0$, we obtain that $u * \varphi_\varepsilon \xrightarrow{\mathcal{D}} u$.

(*ii*) By Lemma 2.8, for any $\psi \in \mathcal{D}(\mathbb{R}^n)$,

$$(u * \varphi_\varepsilon, \psi) = \left(u, \varphi'_\varepsilon * \psi\right).$$

Since $\varphi'_\varepsilon * \psi \xrightarrow{\mathcal{D}} \psi$ by part (*i*), we conclude that

$$(u * \varphi_\varepsilon, \psi) \to (u, \psi)$$

which implies $u * \varphi_\varepsilon \xrightarrow{\mathcal{D}'} u$. □

THEOREM 2.11. *Let φ be a mollifier in $\mathbb{R}^n$.*

(*i*) *If $u \in L^2(\mathbb{R}^n)$ then $u * \varphi$ is also in $L^2(\mathbb{R}^n)$ and*

$$\|u * \varphi\|_{L^2} \le \|u\|_{L^2}. \tag{2.25}$$

Moreover, we have

$$u * \varphi_\varepsilon \xrightarrow{L^2} u \ \text{ as } \varepsilon \to 0.$$

(*ii*) *If $u \in L^2_{loc}(\mathbb{R}^n)$ then*

$$u * \varphi_\varepsilon \xrightarrow{L^2_{loc}} u \ \text{ as } \varepsilon \to 0.$$

(*iii*) *If $u \in \mathcal{D}'(\mathbb{R}^n)$ and*

$$\liminf_{\varepsilon \to 0} \|u * \varphi_\varepsilon\|_{L^2} < \infty$$

then $u \in L^2(\mathbb{R}^n)$ and

$$\|u\|_{L^2} \le \liminf_{\varepsilon \to 0} \|u * \varphi_\varepsilon\|_{L^2}. \tag{2.26}$$

PROOF. $(i) + (ii)$ Applying the Cauchy-Schwarz inequality and using

$$\int_{\mathbb{R}^n} \varphi(y)\, dy = 1,$$

we obtain

$$\begin{aligned}
|u * \varphi(x)|^2 &= \left(\int_{\mathbb{R}^n} \varphi(y)\, u(x-y)\, dy\right)^2 \\
&= \left(\int_{\mathbb{R}^n} \varphi(y)^{1/2} \varphi(y)^{1/2} u(x-y)\, dy\right)^2 \\
&\le \int_{\mathbb{R}^n} \varphi(y)\, dy \int_{\mathbb{R}^n} \varphi(y)\, u^2(x-y)\, dy \\
&= \int_{\mathbb{R}^n} \varphi(y)\, u^2(x-y)\, dy,
\end{aligned}$$

whence

$$\|u * \varphi\|^2_{L^2} \le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(y)\, u^2(x-y)\, dx dy = \|u\|^2_{L^2}.$$

Alternatively, (2.25) follows from Exercise 2.20 with $q(x, y) = \varphi(x-y)$ and $K = 1$.

After we have proved using (2.25), the convergencies in L^2 and L^2_{loc} are treated in the same way as those in L^1 and L^1_{loc} in Lemma 2.4.

(*iii*) Let $\varepsilon_k \to 0$ be a sequence such that

$$\liminf_{\varepsilon\to 0} \|u * \varphi_\varepsilon\|_{L^2} = \lim_{k\to\infty} \|u * \varphi_{\varepsilon_k}\|_{L^2}.$$

Set $u_k = u * \varphi_{\varepsilon_k}$. Since the sequence $\{u_k\}$ is bounded in L^2, by the weak compactness of a ball in L^2, there exists a subsequence $\{u_{k_i}\}$ that converges weakly in L^2, say to $v \in L^2$. The weak convergence in L^2 obviously implies the convergence in $\mathcal{D}'$, whence $u_{k_i} \xrightarrow{\mathcal{D}'} v$. By Lemma 2.10, we have $u_k \xrightarrow{\mathcal{D}'} u$, which implies $u = v$ and, hence, $u \in L^2$.

We are left to verify (2.26). The fact that u_k converges to u weakly in L^2 implies, in particular, that

$$(u_k, u)_{L^2} \to (u, u)_{L^2}.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\|u\|^2_{L^2} \le \lim_{k\to\infty} \|u_k\|_{L^2} \|u\|_{L^2},$$

whence (2.26) follows. □

REMARK 2.12. It is useful to observe that the proof of inequality (2.25) works for a more general class of functions φ, in particular, if φ is a non-negative integrable function on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} \varphi(y)\, dy \le 1$$

(cf. Exercise 2.19).

Exercises.

2.17. Prove that if $u, v \in L^2(\mathbb{R}^n)$ and $\partial_i u, \partial_i v \in L^2(\mathbb{R}^n)$ for some index i, then

$$(\partial_i u, v)_{L^2} = -(v, \partial_i v)_{L^2}. \tag{2.27}$$

2.18. Let $1 < p < \infty$, $u \in L^p(\mathbb{R}^n)$, and φ be a mollifier in $\mathbb{R}^n$.

(*a*) Prove that $u * \varphi \in L^p$ and

$$\|u * \varphi\|_{L^p} \le \|u\|_{L^p}.$$

(*b*) Prove that

$$u * \varphi_\varepsilon \xrightarrow{L^p} u \text{ as } \varepsilon \to 0.$$

2.19. Prove that if $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, and $g \in L^1(\mathbb{R}^n)$ then $f * g$ exists, belongs to $L^p(\mathbb{R}^n)$, and

$$\|f * g\|_p \le \|f\|_p \|g\|_1.$$

2.20. *(Lemma of Schur)* Let (M, μ) be a measure space with a σ-finite measure μ. Let $q(x, y)$ be a non-negative measurable function $M \times M$ such that, for a constant K,

$$\int_M q(x, y)\, d\mu(y) \le K \text{ for almost all } x \tag{2.28}$$

and

$$\int_M q(x, y)\, d\mu(x) \le K \text{ for almost all } y. \tag{2.29}$$

Prove that, for any $f \in L^r(M,\mu)$, $1 \le r \le \infty$, the function

$$Qf(x) := \int_M q(x,y) f(y)\, d\mu(y)$$

belongs to $L^r(M,\mu)$ and

$$\|Qf\|_{L^r} \le K \|f\|_{L^r}. \tag{2.30}$$

2.21. Under the condition of Exercise 2.20, assume in addition that, for some constant C,

$$q(x,y) \le C,$$

for almost all $x, y \in M$. Prove that, for any $f \in L^r(M,\mu)$, $1 \le r \le +\infty$, the function Qf belongs to $L^s(M,\mu)$ for any $s \in (r, +\infty]$ and

$$\|Qf\|_{L^s} \le C^{1/r-1/s} K^{1/r'+1/s} \|f\|_{L^r}, \tag{2.31}$$

where r' is the Hölder conjugate to r.

2.22. A function f on a set $S \subset \mathbb{R}^n$ is called Lipschitz if, for some constant L, called the Lipschitz constant, the following holds:

$$|f(x) - f(y)| \le L|x-y| \quad \text{for all } x, y \in S.$$

Let U be an open subset of $\mathbb{R}^n$ and let f be a Lipschitz function in U with the Lipschitz constant L. For any $\varepsilon > 0$, set

$$U_\varepsilon = \left\{ x \in U : \overline{B_\varepsilon(x)} \subset U \right\}.$$

Let φ be a mollifier in $\mathbb{R}^n$.

(a) Show that U_ε is an open set and

$$U = \bigcup_{k=1}^{\infty} U_{1/k}. \tag{2.32}$$

Extend f to $\mathbb{R}^n$ by setting $f = 0$ outside U. Prove that $f * \varphi_\varepsilon$ is Lipschitz in U_ε with the same Lipschitz constant L.

(b) Prove that, for any $\delta > 0$, $f * \varphi_\varepsilon \rightrightarrows f$ in U_δ as $\varepsilon \to 0$.

2.23. Prove that if f is a Lipschitz function in an open set $U \subset \mathbb{R}^n$ then all the distributional partial derivatives $\partial_j f$ belong to $L^\infty(U)$ and $|\nabla f| \le L$ a.e. where

$$|\nabla f| := \left(\sum_{j=1}^{n} (\partial_j f)^2 \right)^{1/2}$$

and L is the Lipschitz constant of f.

2.24. Prove that if f and g are two bounded Lipschitz functions in an open set $U \subset \mathbb{R}^n$ then fg is also Lipschitz. Prove the product rule for the distributional derivatives:

$$\partial_j (fg) = (\partial_j f) g + f (\partial_j g).$$

2.25. Let $f(x)$ be a Lipschitz function on an interval $[a,b] \subset \mathbb{R}$. Prove that if f' is its distributional derivative then

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

Prove that if g is another Lipschitz function on $[a,b]$ then

$$\int_a^b f' g\, dx = [fg]_a^b - \int_a^b f g'\, dx. \tag{2.33}$$

2.6. Weak derivatives and Sobolev spaces

2.6.1. Spaces of positive order. If $u \in L^2_{loc}(\Omega)$ and its distributional derivative $\partial^\alpha u$ happens to belong to $L^2_{loc}(\Omega)$ then we say that $\partial^\alpha u$ is a *weak derivative*[4] of u.

For any non-negative integer k, consider the following space:

$$W^k(\Omega) = \left\{u \in L^2(\Omega) : \partial^\alpha u \in L^2(\Omega) \text{ for all } \alpha \text{ with } |\alpha| \le k\right\},$$

which is a linear space with the following inner product:

$$(u,v)_{W^k(\Omega)} := \sum_{|\alpha|\le k} \int_\Omega \partial^\alpha u\, \partial^\alpha v\, d\mu = \sum_{|\alpha|\le k} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)}.$$

The associated norm is given by

$$\|u\|^2_{W^k(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega |\partial^\alpha u|^2\, d\mu = \sum_{|\alpha|\le k} \|\partial^\alpha u\|^2_{L^2(\Omega)}$$

In fact, $W^k(\Omega)$ is a Hilbert space (cf. Exercise 2.28).

The spaces $W^k(\Omega)$ are called the *Sobolev spaces*. For example, $W^0(\Omega) \equiv L^2(\Omega)$,

$$W^1(\Omega) = \left\{u \in L^2(\Omega) : \partial_j u \in L^2(\Omega),\ j = 1,...,n\right\},$$

and

$$(u,v)_{W^1} = (u,v)_{L^2} + \sum_{j=1}^n (\partial_j u, \partial_j v)_{L^2}.$$

Obviously, we have

$$W^{k+1}(\Omega) \hookrightarrow W^k(\Omega),$$

for any $k \ge 0$.

Let us mention the following simple properties of the Sobolev spaces.

CLAIM. *(a) If $u \in W^k$ and $|\alpha| \le k$ then $\partial^\alpha u \in W^{k-|\alpha|}$.*

(b) If $\partial^\alpha u \in W^k$ for all α with $|\alpha| \le m$ then $u \in W^{k+m}$.

PROOF. The first property is obvious. To prove the second one, observe that any multiindex β with $|\beta| \le k+m$ can be presented in the form $\beta = \alpha + \alpha'$ where $|\alpha| \le m$ and $|\alpha'| \le k$. Hence, $\partial^\beta u = \partial^{\alpha'}(\partial^\alpha u) \in W^{k-|\alpha'|} \subset L^2$, whence the claim follows. $\square$

Let Ω' be an open subset of Ω. For any $u \in W^k(\Omega)$, the restriction of u to Ω' belongs to $W^k(\Omega')$ and

$$\|u\|_{W^k(\Omega')} \le \|u\|_{W^k(\Omega)}.$$

Define the *local* Sobolev space $W^k_{loc}(\Omega)$ as the class of all distributions $u \in \mathcal{D}'(\Omega)$ such that $u \in W^k(\Omega')$ for any open set $\Omega' \Subset \Omega$. The topology in

[4]The weak derivative $\partial^\alpha u$ can be equivalenlty defined as a function from $L^2_{loc}(\Omega)$ that satisfies the identity (2.16).

$W^k_{loc}(\Omega)$ is defined by the family of the seminorms $\|u\|_{W^k(\Omega')}$. Let us mention also that

$$W^k_{loc}(\Omega) = \left\{u \in L^2_{loc}(\Omega) : \partial^\alpha u \in L^2_{loc}(\Omega) \text{ for all } \alpha \text{ with } |\alpha| \le k\right\}.$$

The scale of spaces $W^k_{loc}(\Omega)$ is in some sense analogous to that of $C^k(\Omega)$, although, for the spaces W^k_{loc}, we use *weak* derivatives, whereas the spaces C^k are associated with *continuous* derivatives. If $u \in C^k$ then all the classical derivatives of u are also weak derivatives and, for any open set $\Omega' \Subset \Omega$,

$$\|u\|_{W^k(\Omega')} \le C\|u\|_{C^k(\Omega')}.$$

Hence, we have an embedding

$$C^k(\Omega) \hookrightarrow W^k_{loc}(\Omega).$$

The next statement extends Theorem 2.11 to the spaces W^k.

THEOREM 2.13. *Let φ be a mollifier in $\mathbb{R}^n$ and k be a non-negative integer.*

(*i*) *If $u \in W^k(\mathbb{R}^n)$ then $u * \varphi$ is also in $W^k(\mathbb{R}^n)$ and*

$$\|u * \varphi\|_{W^k} \le \|u\|_{W^k}. \tag{2.34}$$

Moreover, we have

$$u * \varphi_\varepsilon \xrightarrow{W^k} u \text{ as } \varepsilon \to 0. \tag{2.35}$$

(*ii*) *If $u \in \mathcal{D}'(\mathbb{R}^n)$ and*

$$\liminf_{\varepsilon \to 0} \|u * \varphi_\varepsilon\|_{W^k} < \infty, \tag{2.36}$$

then $u \in W^k(\mathbb{R}^n)$ and

$$\|u\|_{W^k} \le \liminf_{\varepsilon \to 0} \|u * \varphi_\varepsilon\|_{W^k}.$$

PROOF. (*i*) By Lemma 2.9, we have

$$\partial^\alpha(u * \varphi) = (\partial^\alpha u) * \varphi. \tag{2.37}$$

Applying Theorem 2.11 to $\partial^\alpha u$, where $|\alpha| \le k$, we obtain

$$\|\partial^\alpha(u * \varphi)\|_{L^2} \le \|\partial^\alpha u\|_{L^2}$$

and

$$\partial^\alpha(u * \varphi) \xrightarrow{L^2} \partial^\alpha \varphi,$$

whence (2.34) and (2.35) follow.

(*ii*) For any multiindex α with $|\alpha| \le k$, we have by (2.36) and (2.37) that

$$\liminf_{\varepsilon \to 0} \|(\partial^\alpha u) * \varphi_\varepsilon\|_{L^2} < \infty.$$

By Theorem 2.11, we conclude that $\partial^\alpha u \in L^2$ and

$$\|\partial^\alpha u\|_{L^2} \le \liminf_{\varepsilon \to 0} \|\partial^\alpha(u * \varphi_\varepsilon)\|_{L^2},$$

whence the both claims follows. $\square$

Exercises.

2.26. Let $f \in C^k(\Omega)$, where k is a non-negative integer.

(a) Prove that if

$$\|f\|_{C^k(\Omega)} < \infty$$

then, for any $u \in W^k(\Omega)$, also $fu \in W^k(\Omega)$ and

$$\|fu\|_{W^k(\Omega)} \leq C\|f\|_{C^k(\Omega)}\|u\|_{W^k(\Omega)}, \tag{2.38}$$

where the constant C depends only on k, n.

(b) Prove that if $u \in W^k_{loc}(\Omega)$ then $fu \in W^k_{loc}(\Omega)$.

2.27. Assume that $f_k \to f$ in W^k and $\partial^\alpha f \to g$ in W^k, for some multiindex α such that $|\alpha| \leq k$. Prove that $g = \partial^\alpha f$.

2.28. Prove that, for any open set $\Omega \subset \mathbb{R}^n$, the space $W^k(\Omega)$ is complete.

2.29. Denote by $W^k_c(\Omega)$ the subset of $W^k(\Omega)$, which consists of functions with compact support in Ω. Prove that $\mathcal{D}(\Omega)$ is dense in $W^k_c(\Omega)$.

2.30. Prove that $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^k(\mathbb{R}^n)$, for any non-negative integer k. Warning: for an arbitrary open set $\Omega \subset \mathbb{R}^n$, $\mathcal{D}(\Omega)$ may not be dense in $W^k(\Omega)$.

2.31. Denote by $W^1_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^1(\Omega)$. Prove that, for any $u \in W^1(\Omega)$ and $v \in W^1_0(\Omega)$,

$$(\partial_i u, v)_{L^2} = -(u, \partial_i v)_{L^2}. \tag{2.39}$$

2.32. Let $u \in L^2(\mathbb{R}^n)$ and $\partial^\alpha u \in L^2(\mathbb{R}^n)$ for some multiindex α.

(a) Prove that

$$\widehat{\partial^\alpha u} = (i\xi)^\alpha\,\widehat{u}(\xi), \tag{2.40}$$

where $\widehat{u}$ is the Fourier transform of u and $\xi^\alpha \equiv \xi_1^{\alpha_1}...\xi_n^{\alpha_n}$, $i^\alpha \equiv i^{|\alpha|}$.

(b) Prove the following identity

$$\|\partial^\alpha u\|^2_{L^2} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\widehat{u}(\xi)|^2|\xi^\alpha|^2\,d\xi. \tag{2.41}$$

2.33. Let $u \in L^2(\mathbb{R}^n)$. Prove that if the right hand side of (2.41) is finite then $\partial^\alpha u$ belongs to $L^2(\mathbb{R}^n)$ and, hence, the identity (2.41) holds.

2.34. Prove that the space $W^k(\mathbb{R}^n)$ (where k is a positive integer) can be characterized in terms of the Fourier transform as follows: a function $u \in L^2(\mathbb{R}^n)$ belongs to $W^k(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n}|\widehat{u}(\xi)|^2\left(1+|\xi|^2\right)^k d\xi < \infty.$$

Moreover, the following relation holds:

$$\|u\|^2_{W^k} \simeq \int_{\mathbb{R}^n}|\widehat{u}(\xi)|^2\left(1+|\xi|^2\right)^k d\xi, \tag{2.42}$$

where the sign $\simeq$ means that the ratio of the both sides is bounded from above and below by positive constants.

2.6.2. Spaces of negative order. In the previous section, the Sobolev space W^k of order k was defined for any non-negative integer k. Our next goal is to define the Sobolev spaces of negative orders.

Fix an open set $\Omega \subset \mathbb{R}^n$ and, for any positive integer k and a distribution $u \in \mathcal{D}'(\Omega)$, set

$$\|u\|_{W^{-k}} := \sup_{\varphi \in \mathcal{D}(\Omega)\setminus\{0\}} \frac{(u,\varphi)}{\|\varphi\|_{W^k}}. \tag{2.43}$$

Then the space $W^{-k}(\Omega)$ is defined by

$$W^{-k}(\Omega) := \left\{u \in \mathcal{D}'(\Omega) : \|u\|_{W^{-k}} < \infty\right\}.$$

It follows directly from the definition (2.43) that

$$|(u,\varphi)| \leq \|u\|_{W^{-k}(\Omega)} \|\varphi\|_{W^k(\Omega)},$$

for all $u \in W^{-k}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$.

Here are some simple properties of the spaces $W^k(\Omega)$ for all $k \in \mathbb{Z}$.

Claim. *If $k < m$ then*

$$\|u\|_{W^k} \leq \|u\|_{W^m} \tag{2.44}$$

and, consequently, $W^m \hookrightarrow W^k$. In particular, if $k < 0$ then $L^2 \hookrightarrow W^k$.

Proof. If $k \geq 0$ then this property is already known, so assume $k < 0$. If $m > 0$ then we can replace it by $m = 0$. Hence, we can assume $k < m \leq 0$.

Observe that the definition (2.43) is valid also for $k = 0$, that is, for the L^2-norm, which follows from the fact that $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$ (see Theorem 2.3). Since $|k| > |m|$, we have $\|\varphi\|_{W^{|k|}} \geq \|\varphi\|_{W^{|m|}}$, and (2.44) follows from (2.43). □

Claim. *If $k \in \mathbb{Z}$ and $u \in W^k(\Omega)$ then $\partial_j u \in W^{k-1}(\Omega)$ and*

$$\|\partial_j u\|_{W^{k-1}} \leq \|u\|_{W^k}. \tag{2.45}$$

Proof. If $k \geq 1$ then this is already known, so assume $k \leq 0$. For any $\varphi \in \mathcal{D}$, we have

$$(\partial_j u, \varphi) = -(u, \partial_j \varphi),$$

whence

$$(\partial_j u, \varphi) \leq \|u\|_{W^k} \|\partial_j \varphi\|_{W^{|k|}} \leq \|u\|_{W^k} \|\varphi\|_{W^{|k|+1}},$$

and (2.45) follows. □

In particular, we obtain that if $u \in L^2$ then $\partial^\alpha u \in W^{-|\alpha|}$, which gives many examples of distributions from W^k with negative k.

Claim. *If $\Omega' \subset \Omega$ then, for any $k \in \mathbb{Z}$,*

$$\|u\|_{W^k(\Omega')} \leq \|u\|_{W^k(\Omega)}.$$

PROOF. For $k \geq 0$ this is already known. For $k < 0$ it follows from (2.43) and the fact that $\mathcal{D}(\Omega') \subset \mathcal{D}(\Omega)$. $\square$

The function space $W^k_{loc}(\Omega)$ for $k < 0$ is defined in the same way as that for $k \geq 0$ in the previous section. Namely, a distribution $u \in \mathcal{D}'(\Omega)$ belongs to $W^k_{loc}(\Omega)$ if $u \in W^k(\Omega')$ for any open set $\Omega' \Subset \Omega$. The topology in $W^k_{loc}(\Omega)$ is defined by the family of the seminorms $\|u\|_{W^k(\Omega')}$. It follows from the above statements that $W^k_{loc}(\Omega)$ increases when Ω expands and when k decreases.

It is interesting to mention that any function from $L^1_{loc}(\Omega)$ belongs to $W^{-k}_{loc}(\Omega)$ if $k > n/2$ – see Example 6.2 below.

LEMMA 2.14. *Let $k \in \mathbb{Z}$.*

(a) If $f \in \mathcal{D}(\Omega)$ and $u \in W^k_{loc}(\Omega)$ then $fu \in W^k(\Omega)$ and

$$\|fu\|_{W^k(\Omega)} \leq C\|f\|_{C^{|k|}(\Omega)}\|u\|_{W^k(\Omega')}, \tag{2.46}$$

where Ω' is an open set containing $\operatorname{supp} f$ *and the constant C depends on k, n.*

(b) If $f \in C^\infty(\Omega)$ and $u \in W^k_{loc}(\Omega)$ then $fu \in W^k_{loc}(\Omega)$.

PROOF. (a) If $k \geq 0$ then we obtain by Exercise 2.26 $fu \in W^k(\Omega')$ and

$$\|fu\|_{W^k(\Omega')} \leq C\|f\|_{C^k(\Omega')}\|u\|_{W^k(\Omega')},$$

whence the claim follows.

Let now $k < 0$. Assuming that φ ranges in $\mathcal{D}(\Omega)$ and $\|\varphi\|_{W^{|k|}(\Omega)} = 1$, we have

$$\|fu\|_{W^k(\Omega)} = \sup_\varphi (fu, \varphi) = \sup_\varphi (u, f\varphi) \leq \sup_\varphi \|f\varphi\|_{W^{|k|}(\Omega')}\|u\|_{W^k(\Omega')},$$

where the last inequality holds because $f\varphi \in \mathcal{D}(\Omega')$. We are left to notice that

$$\|f\varphi\|_{W^{|k|}(\Omega')} \leq C\|f\|_{C^{|k|}}\|\varphi\|_{W^{|k|}} = C\|f\|_{C^{|k|}},$$

whence the claim follows.

(b) Let us show that $fu \in W^k(\Omega')$ for any open set $\Omega' \Subset \Omega$. Fix a function $\varphi \in \mathcal{D}(\Omega)$ such that $\varphi \equiv 1$ in Ω'. Then $\varphi f \in \mathcal{D}(\Omega)$ and, by the previous part, $\varphi fu \in W^k(\Omega)$. It follows that $\varphi fu \in W^k(\Omega')$ and, hence, $fu \in W^k(\Omega')$. $\square$

LEMMA 2.15. *Let k be a positive integer. For any $u \in W^{-k}(\mathbb{R}^n)$, there exists a unique function $v \in W^k(\mathbb{R}^n)$ such that*

$$u = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \partial^{2\alpha} v. \tag{2.47}$$

Moreover, we have the identity

$$\|u\|_{W^{-k}} = \|v\|_{W^k}. \tag{2.48}$$

Note that, for any $v \in W^k(\mathbb{R}^n)$, the equation (2.47) defines $u \in W^{-k}(\mathbb{R}^n)$. Hence, we obtain a norm preserving bijection between $W^k(\mathbb{R}^n)$ and $W^{-k}(\mathbb{R}^n)$.

PROOF. By definition, $u \in W^k(\mathbb{R}^n)$ means that

$$\|u\|_{W^{-k}} \equiv \sup_{\varphi \in \mathcal{D}(\mathbb{R}^n)\setminus\{0\}} \frac{|(u,\varphi)|}{\|\varphi\|_{W^k}} < \infty.$$

Hence, u can be considered as a linear functional on $\mathcal{D}(\mathbb{R}^n)$ bounded in the norm W^k. Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^k(\mathbb{R}^n)$ (see Exercise 2.30), this functional uniquely extends to a bounded functional on $W^k(\mathbb{R}^n)$, with the same norm. Denote it by $F_u(\varphi)$.

However, W^k is Hilbert space and, by the Riesz representation theorem, there exists a unique function $v \in W^k(\mathbb{R}^n)$ such that

$$F_u(\varphi) = (v,\varphi)_{W^k} \quad \text{for all } \varphi \in W^k(\mathbb{R}^n).$$

In particular, this means, that for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\begin{aligned}(u,\varphi) &= (v,\varphi)_{W^k} = \sum_{|\alpha|\le k} (\partial^\alpha v, \partial^\alpha \varphi)_{L^2} \\ &= \sum_{|\alpha|\le k} (\partial^\alpha v, \partial^\alpha \varphi) = \sum_{|\alpha|\le k} (-1)^{|\alpha|} \left(\partial^{2\alpha} v, \varphi\right),\end{aligned}$$

which proves the first claim.

The functional $\varphi \mapsto (v,\varphi)_{W^k}$ on $W^k(\mathbb{R}^n)$ has the norm $\|v\|_{W^k}$, whence it follows that $\|F_u\| = \|v\|_{W^k}$ where $\|F_u\|$ is the norm of the function F_u on $W^k(\mathbb{R}^n)$. By the first part of the proof, $\|F_u\| = \|u\|_{W^{-k}}$, whence (2.48) follows. ☐

The following statement extends Theorem 2.13 to the Sobolev spaces of negative order.

THEOREM 2.16. *Let k be a positive integer. If $u \in W^{-k}(\mathbb{R}^n)$ and φ is a mollifier in $\mathbb{R}^n$ then*

$$u * \varphi_\varepsilon \xrightarrow{W^{-k}} u \text{ as } \varepsilon \to 0.$$

PROOF. Consider the following differential operator

$$D = \sum_{|\alpha|\le k} (-1)^{|\alpha|} \partial^{2\alpha},$$

which maps W^k into W^{-k}. By Lemma 2.15, for any $u \in W^{-k}(\mathbb{R}^n)$, there exists a unique $v \in W^k(\mathbb{R}^n)$ such that $u = Dv$, and also

$$\|Dv\|_{W^{-k}} = \|v\|_{W^k}.$$

Using Lemma 2.9, we obtain

$$u * \varphi_\varepsilon - u = (Dv) * \varphi_\varepsilon - Dv = D(v * \varphi_\varepsilon - v),$$

whence

$$\|u * \varphi_\varepsilon - u\|_{W^{-k}} = \|D(v * \varphi_\varepsilon - v)\|_{W^{-k}} = \|v * \varphi_\varepsilon - v\|_{W^k}.$$

Since, by Theorem 2.13, the right hand side here tends to 0, we obtain that $u * \varphi_\varepsilon \to u$ in W^{-k}, which was to be proved. $\square$

Exercises.

2.35. Let k be a positive integer. Prove that if $u \in W^{-k}(\mathbb{R}^n)$ and φ is a mollifier in $\mathbb{R}^n$ then

$$\|u * \varphi\|_{W^{-k}} \leq \|u\|_{W^{-k}}. \tag{2.49}$$

2.36. Prove that, for any positive integer k, the space W^{-k} with the norm $\|\cdot\|_{W^{-k}}$ is a Hilbert space.

2.7. Heat semigroup in $\mathbb{R}^n$

Let Δ be the Laplace operator in $\mathbb{R}^n$ and

$$p_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) \tag{2.50}$$

be the heat kernel in $\mathbb{R}^n$ (cf. Section 1.3). For any $t \geq 0$, denote by P_t the following operator on functions

$$P_t f = \begin{cases} p_t * f, & t > 0, \\ f, & t = 0, \end{cases}$$

whenever the convolution $p_t * f$ makes sense. Denote by $C_b(\mathbb{R}^n)$ the class of bounded continuous functions in $\mathbb{R}^n$. By Theorem 1.3, if $f \in C_b(\mathbb{R}^n)$ then the function $P_t f(x)$ is C^∞-smooth in $\mathbb{R}_+ \times \mathbb{R}^n$ and solves in $\mathbb{R}_+ \times \mathbb{R}^n$ the heat equation

$$\partial_t(P_t f) = \Delta(P_t f). \tag{2.51}$$

Besides, $P_t f(x)$ is bounded and continuous in $[0, +\infty) \times \mathbb{R}^n$.

In particular, for any fixed $t \geq 0$, we can consider P_t as an operator from $C_b(\mathbb{R}^n)$ to $C_b(\mathbb{R}^n)$ such that $P_t f \to f$ as $t \to 0$ locally uniformly. The identity

$$p_t * p_s = p_{t+s} \tag{2.52}$$

(see Example 1.9) implies

$$P_t P_s = P_{t+s},$$

for all $t, s \geq 0$. Hence, the family $\{P_t\}_{t \geq 0}$ is a *semigroup*. It is called the *heat semigroup* of the Laplace operator in $\mathbb{R}^n$.

Here we consider some properties of the heat semigroup, which extend Theorem 1.3 to the class L^2. These properties are closely related to the properties of mollifiers considered in the previous sections, which is not surprising because the heat kernel as a function of x in many respects looks like a mollifier although with non-compact support (compare, for example, Fig. 1.1 and 2.1).

In Chapters 4 and 7, the heat semigroup will be considered on an arbitrary weighted manifold, and most of these properties will be retained, although from a difference perspective.

We use the notation $\partial_t \equiv \frac{\partial}{\partial t}$ and $\partial_j \equiv \frac{\partial}{\partial x^j}$ for $j = 1, ..., n$. Denote by $C_b^k(\mathbb{R}^n)$ the subspace of $C^k(\mathbb{R}^n)$ that consists of functions u whose all partial derivatives up to the order k are bounded functions.

LEMMA 2.17. *If $f \in C_0^\infty(\mathbb{R}^n)$ then $P_t f \in C_b^\infty([0, +\infty) \times \mathbb{R}^n)$. Moreover, the following identities hold in $[0, +\infty) \times \mathbb{R}^n$:*

$$\partial_j (P_t f) = P_t (\partial_j f) \tag{2.53}$$

and

$$\partial_t (P_t f) = P_t (\Delta f). \tag{2.54}$$

PROOF. The function $P_t f$ is bounded and C^∞-smooth in $(0, +\infty) \times \mathbb{R}^n$ by Theorem 1.3. The identity (2.53) for $t > 0$ follows from Lemma 2.1 because

$$\partial_j (P_t f) = \partial_j (p_t * f) = p_t * \partial_j f = P_t (\partial_j f). \tag{2.55}$$

This proves also (2.54), because using the heat equation (2.51) and iterating (2.53), we obtain

$$\partial_t (P_t f) = \Delta (P_t f) = P_t (\Delta f).$$

To extend all this to $t = 0$, observe that the right hand sides of (2.53) and (2.54) are continuos functions up to $t = 0$. Therefore, the derivatives in the left hand side exist and satisfy these identities also up to $t = 0$. In particular, we obtain that $P_t f$ is C^1-smooth up to $t = 0$. Since $\partial_j f$ and Δf are bounded functions, the identities (2.53) and (2.54) imply that $\partial_j (P_t f)$ and $\partial_t (P_t f)$ are bounded in $[0, +\infty) \times \mathbb{R}^n$, that is,

$$P_t f \in C_b^1([0, +\infty) \times \mathbb{R}^n). \tag{2.56}$$

Since $\partial_j f$ and Δf belong to $C_0^\infty(\mathbb{R}^n)$, we obtain by (2.53), (2.54) and (2.56) that $\partial_j (P_t f)$ and $\partial_t (P_t f)$ are also in the class $C_b^1([0, +\infty) \times \mathbb{R}^n)$, which implies that

$$P_t f \in C_b^2([0, +\infty) \times \mathbb{R}^n).$$

Continuing by induction, we conclude the proof. □

The following statement is similar to Theorem 2.11 but a mollifier is replaced by the heat kernel.

LEMMA 2.18. *If $f \in L^2(\mathbb{R}^n)$ then $P_t f \in L^2(\mathbb{R}^n)$ for any $t \geq 0$, and*

$$\|P_t f\|_{L^2} \leq \|f\|_{L^2}. \tag{2.57}$$

Moreover, we have

$$P_t f \xrightarrow{L^2} f \text{ as } t \to 0. \tag{2.58}$$

PROOF. For $t = 0$ the claim is trivial. If $t > 0$ then we have by Lemma 1.1

$$\int_{\mathbb{R}^n} p_t(x)\, dx \equiv 1. \tag{2.59}$$

Since $P_t f = f * p_t$, the first claim follows from an extension of Theorem 2.11 by Remark 2.12.

Thanks to (2.57) and the fact that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ (see Theorem 2.3), it suffices to prove (2.58) for $f \in C_0^\infty(\mathbb{R}^n)$. Assuming that, we have by Theorem 1.3,

$$\sup_{\mathbb{R}^n} |P_t f - f| \to 0 \text{ as } t \to 0,$$

because the function f is uniformly continuos (cf. Remark 1.4). Since (2.59) implies

$$\|P_t f\|_{L^1} \le \|f\|_{L^1}$$

(cf. Lemma 2.4), we obtain

$$\|P_t f - f\|_{L^2}^2 \le \sup_{\mathbb{R}^n} |P_t f - f| \, \|P_t f - f\|_{L^1} \le 2 \sup_{\mathbb{R}^n} |P_t f - f| \, \|f\|_{L^1} \to 0.$$

$\square$

See also Exercise 1.4 for an alternative proof of (2.57) and (2.58).

Hence, P_t can be now considered as a bounded operator from L^2 to L^2. The semigroup property

$$P_t P_s = P_{t+s} \tag{2.60}$$

obviously extends to L^2 because C_0^∞ is dense in L^2. A new feature of P_t which comes with L^2 spaces, is the *symmetry*, in the following sense:

$$(P_t f, g)_{L^2} = (f, P_t g)_{L^2}, \tag{2.61}$$

for all $f, g \in L^2$. Indeed, if $f, g \in C_0^\infty$ then this trivially follows from

$$\begin{aligned} (P_t f, g) &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} p_t(x-y) f(y)\, dy \right) g(x)\, dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p_t(x-y) f(y) g(x)\, dy dx \end{aligned}$$

and from a similar expression for $(f, P_t g)$, because $p_t(x-y) = p_t(y-x)$; then the extension to L^2 is obvious.

In the next statement, we will use the notion of convexity. Recall that a function $\varphi(t)$ on $[0, +\infty)$ is called convex if, for all $t, s \ge 0$ and $\varepsilon \in (0,1)$,

$$\varphi(\varepsilon t + (1-\varepsilon) s) \le \varepsilon \varphi(t) + (1-\varepsilon)\varphi(s). \tag{2.62}$$

If φ is continuous then it suffices to have this property for $\varepsilon = 1/2$, that is,

$$\varphi\left(\frac{t+s}{2}\right) \le \frac{\varphi(t) + \varphi(s)}{2}. \tag{2.63}$$

Indeed, by iterating (2.63), one obtains (2.62) for all binary fractions ε, and then for all real ε by continuity. A non-negative function φ is called log-convex if $\log\varphi$ is convex. The latter obviously amounts to

$$\varphi\left(\frac{t+s}{2}\right) \leq \sqrt{\varphi(t)\,\varphi(s)}. \tag{2.64}$$

Comparing (2.63) and (2.64) we see that the log-convexity implies the convexity.

The following *convexity lemma* is frequently useful.

LEMMA 2.19. *For any $f \in L^2(\mathbb{R}^n)$, the function*

$$\varphi(t) := (P_t f, f)_{L^2}$$

on $t \in [0, +\infty)$ is non-negative, decreasing, continuous, and log-*convex.*

PROOF. The proof is based only on the properties (2.57), (2.58), (2.60), (2.61) of the semigroup P_t and, hence, the statement of Lemma 2.19 remains true in any other setting where these properties can be verified. In particular, this will be the case for the heat semigroup on an arbitrary manifold – see Section 4.3.

We start with the observation that, by (2.60) and (2.61),

$$\varphi(t) = \left(P_{t/2}P_{t/2}f, f\right) = \left(P_{t/2}f, P_{t/2}f\right) = \|P_{t/2}f\|^2, \tag{2.65}$$

which implies $\varphi(t) \geq 0$. Using (2.60) and (2.57), we obtain, for all $t, s \geq 0$,

$$\|P_{t+s}f\| = \|P_s(P_t f)\| \leq \|P_t f\|,$$

that is, the function $t \mapsto \|P_t f\|$ is decreasing, which implies by (2.65) that $\varphi(t)$ is also decreasing. The triangle inequality and (2.58) yield

$$\|P_t f\| - \|P_{t+s}f\| \leq \|P_t f - P_{t+s}f\| = \|P_t(f - P_s f)\| \leq \|f - P_s f\| \to 0$$

as $s \to 0+$ (and the same holds if $s \to 0-$), which implies that the function $t \mapsto \|P_t f\|$ is continuous and, hence, so is $\varphi(t)$.

Finally, we have by the Cauchy-Schwarz inequality

$$\varphi(2t+2s) = (P_{t+s}f, f) = (P_s f, P_t f) \leq \|P_s f\|\|P_t f\| = \sqrt{\varphi(2s)\,\varphi(2t)},$$

which proves the log-convexity of φ. □

In the next statement, we show that the rate of convergence $P_t f \to f$ as $t \to 0$ depends on the regularity of f. If $f \in W^1$ then denote by ∇f the "vector" $(\partial_1 f, ..., \partial_n f)$ of its first order partial derivatives, and set

$$|\nabla f|^2 := \sum_{j=1}^n |\partial_j f|^2$$

and $\|\nabla f\|_{L^2} \equiv \|\,|\nabla f|\,\|_{L^2}$ so that

$$\|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2 = \|f\|_{W^1}^2.$$

If $f \in W^2$ then its distributional Laplacian Δf belongs to L^2 and

$$\|\Delta f\|_{L^2} \leq \sum_{j=1}^{n} \|\partial_j^2 f\|_{L^2} \leq n\|f\|_{W^2}.$$

LEMMA 2.20. *If $f \in W^1(\mathbb{R}^n)$ then, for any $t > 0$,*

$$\|P_t f - f\|_{L^2} \leq \sqrt{t}\|\nabla f\|_{L^2}. \tag{2.66}$$

If $f \in W^2(\mathbb{R}^n)$ then, for any $t > 0$,

$$\|P_t f - f\|_{L^2} \leq t\|\Delta f\|_{L^2}. \tag{2.67}$$

PROOF. It suffices to prove the both claims for $f \in \mathcal{D}(\mathbb{R}^n)$ because $\mathcal{D}(\mathbb{R}^n)$ is dense both in $W^1(\mathbb{R}^n)$ and $W^2(\mathbb{R}^n)$ (cf. Exercise 2.30) and the expressions in (2.66) and (2.67) are continuous in W^1 and W^2, respectively. Using the notation $\varphi(t) = (P_t f, f)$ as in Lemma 2.19, we have

$$\|P_t f - f\|^2 = (P_t f, P_t f) - 2(P_t f, f) + (f, f) = \varphi(2t) - 2\varphi(t) + \varphi(0). \tag{2.68}$$

Since $\varphi(2t) \leq \varphi(t)$, this implies

$$\|P_t f - f\|^2 \leq \varphi(0) - \varphi(t). \tag{2.69}$$

Using Lemma 2.17 and the Green formula, we can compute the derivative $\varphi'(0)$ as follows:

$$\varphi'(0) = (\partial_t (P_t f)|_{t=0}, f) = (\Delta f, f) = -\int_{\mathbb{R}^n} |\nabla f|^2 dx = -\|\nabla f\|_{L^2}^2, \tag{2.70}$$

which together with the convexity of φ yields

$$\varphi(t) - \varphi(0) = t\varphi'(\xi) \geq t\varphi'(0) = -t\|\nabla f\|^2, \tag{2.71}$$

where $\xi \in (0, t)$. Combining (2.69) and (2.71), we obtain (2.66).

To prove (2.67), we need the second derivative of φ, which is computed as follows using Lemma 2.17:

$$\varphi'(t) = (\partial_t (P_t f), f) = (\Delta(P_t f), f) = (P_t f, \Delta f),$$

and

$$\varphi''(t) = (\partial_t P_t f, \Delta f) = (P_t(\Delta f), \Delta f). \tag{2.72}$$

By Lemma 2.19, $(P_t(\Delta f), \Delta f)$ is non-increasing in t; hence, $\varphi''(t)$ is non-increasing. Using (2.68) and (2.72) for $t = 0$, we obtain

$$\|P_t f - f\|^2 = \varphi(2t) - 2\varphi(t) + \varphi(0) = \varphi''(\xi)t^2 \leq \varphi''(0)t^2 = \|\Delta f\|^2 t^2,$$

which finishes the proof. □

See Exercise 2.39 for a Fourier transform proof of Lemma 2.20, and Exercises 4.39, 4.40 for an extension of Lemma 2.20 to a general setting of manifolds.

DEFINITION 2.21. Let B be a Banach space and I be an interval in $\mathbb{R}$. A path $u : I \to B$ is said to be *strong differentiable* at $t \in I$, if the limit

$$\lim_{\varepsilon \to 0} \frac{u(t+\varepsilon) - u(t)}{\varepsilon} \tag{2.73}$$

exists in the norm of B. The value of the limit is called the *strong derivative* of u at t and is denoted by $u'(t)$ or $\frac{du}{dt}$.

The word "strong" refers to the fact that the limit in (2.73) is understood in the strong topology of B, that is, the norm topology. If the limit is understood in the weak topology of B then one obtains the weak derivative.

In the next statement, we consider the function $t \mapsto P_t f$ as a path in L^2.

THEOREM 2.22. *If $f \in W^2(\mathbb{R}^n)$ then the path $t \to P_t f$ is strongly differentiable in $L^2(\mathbb{R}^n)$ for all $t \in [0, +\infty)$, and*

$$\frac{d}{dt}(P_t f) = \Delta(P_t f). \tag{2.74}$$

Combining with Lemma 2.18, we see that the path $u(t) = P_t f$ solves the Cauchy problem in the L^2 sense: it satisfies the heat equation and the initial data

$$\begin{cases} \frac{du}{dt} = \Delta u, \\ u|_{t=0} = f, \end{cases}$$

where the limits in the both conditions are understood in the L^2-norm.

PROOF. Let us prove first that, when $t \to 0$,

$$Q_t f := \frac{P_t f - f}{t} \longrightarrow \Delta f \quad \text{in } L^2(\mathbb{R}^n). \tag{2.75}$$

Assume that $f \in \mathcal{D}(\mathbb{R}^n)$. Then, by Lemma 2.17, the function $P_t f$ is smooth in $[0, +\infty) \times \mathbb{R}^n$, bounded, and all its derivatives are bounded. Therefore, by (2.54),

$$\frac{P_t f(x) - f(x)}{t} \rightrightarrows \partial_t P_t f(x)|_{t=0} = \Delta f(x).$$

It follows that, for any bounded open set $\Omega \subset \mathbb{R}^n$, $Q_t f \to \Delta f$ in $L^2(\Omega)$. Choose Ω to contain $K := \operatorname{supp} f$, and prove that also

$$Q_t f \to \Delta f \quad \text{in } L^2(\Omega^c), \tag{2.76}$$

which will imply (2.75). Since in Ω^c we have $f = \Delta f = 0$ and $Q_t f = \frac{1}{t} P_t f$, (2.76) amounts to

$$\|P_t f\|_{L^2(\Omega^c)} = o(t) \quad \text{as } t \to \infty.$$

Since function $P_t f(x)$ is bounded, it suffices to prove that

$$\|P_t f\|_{L^1(\Omega^c)} = o(t) \quad \text{as } t \to \infty. \tag{2.77}$$

Denoting by ε the distance from K to the boundary of Ω, we obtain

$$\begin{aligned}\|P_t f\|_{L^1(\Omega^c)} &\leq \int_{\Omega^c}\left(\int_K p_t(x-y)\,|f(y)|\,dy\right)dx \\ &= \int_K\left(\int_{\Omega^c} p_t(x-y)\,dx\right)|f(y)|\,dy \\ &\leq \int_K\left(\int_{\{|x-y|>\varepsilon\}} p_t(x-y)\,dx\right)|f(y)|\,dy \\ &= \int_K |f(y)|\,dy \int_{\{|z|>\varepsilon\}} p_t(z)\,dz.\end{aligned}$$

By Remark 1.2, the last integral decays as $t \to 0$ faster than any power of t, which proves (2.77).

Let us prove (2.75) for $f \in W^2(\mathbb{R}^n)$. By Exercise 2.30, there exists a sequence $\{f_k\} \subset \mathcal{D}(\mathbb{R}^n)$ such that $f_k \to f$ in $W^2(\mathbb{R}^n)$. Observing that, by Lemma 2.20,

$$\|Q_t(f-f_k)\|_{L^2} \leq \|\Delta(f-f_k)\|_{L^2},$$

we obtain

$$\begin{aligned}\|Q_t f - \Delta f\|_{L^2} &\leq \|Q_t f - Q_t f_k\|_{L^2} + \|Q_t f_k - \Delta f_k\|_{L^2} + \|\Delta f_k - \Delta f\|_{L^2} \\ &\leq \|Q_t f_k - \Delta f_k\|_{L^2} + 2\,\|\Delta f_k - \Delta f\|_{L^2}.\end{aligned}$$

Letting $t \to 0$ and then $k \to \infty$, we obtain (2.75).

Note that (2.75) is a particular case of (2.74) for $t = 0$. Let us prove (2.74) for all $t > 0$. First show that, for any multiindex α of order ≤ 2,

$$\partial^\alpha(P_t f) = P_t(\partial^\alpha f), \tag{2.78}$$

which will imply $P_t f \in W^2$. Indeed, for any test function $\psi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\begin{aligned}(P_t(\partial^\alpha f), \psi) &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \partial^\alpha f(x-y)\,p_t(y)\,dy\right)\psi(x)\,dx \\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \partial^\alpha f(x-y)\,\psi(x)\,dx\right)p_t(y)\,dy \\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(x-y)\,\partial^\alpha\psi(x)\,dx\right)p_t(y)\,dy \\ &= (-1)^{|\alpha|}(P_t f, \partial^\alpha\psi) = (\partial^\alpha P_t f, \psi),\end{aligned}$$

whence (2.78) follows. Applying (2.75) to function $P_t f \in W^2$, we obtain

$$\frac{P_{t+s}f - P_t f}{s} = \frac{P_s(P_t f) - P_t f}{s} \xrightarrow{L^2} \Delta(P_t f) \quad \text{as } s \to 0,$$

which finishes the proof. $\square$

Exercises.

2.37. Evaluate function $\varphi(t)$ from Lemma 2.19 for $f(x) = \exp\left(-|x|^2\right)$.

2.38. Show that Lemma 2.17 remains true for $f \in C_b^\infty(\mathbb{R}^n)$.

2.39. Give an alternative proof of Lemma 2.20 using the Fourier transform and Exercises 1.4, 2.32.

Notes

This Chapter contains a standard material on distributions and mollifiers in $\mathbb{R}^n$ – see, for example, [**207**], [**327**], [**356**].

CHAPTER 3

Laplace operator on a Riemannian manifold

We introduce in this Chapter the notions of smooth and Riemannian manifolds, Riemannian measure, and the Riemannian Laplace operator. From the previous Chapters, we use here only the material of Section 2.2. However, acquaintance with measure theory and integration is required (see Appendix A).

The core of this Chapter is the material of Sections 3.1–3.6, which is needed in the rest of the book except for Chapter 6. The material of Sections 3.7–3.10 is mostly used for constructing examples of manifolds. In Section 3.11, we introduce the geodesic distance, which will be seriously used only in Chapters 11 and 15.

3.1. Smooth manifolds

Let M be a topological space. Recall M is called *Hausdorff* if, for any two disjoint points $x, y \in M$, there exist two disjoint open sets $U, V \subset M$ containing x and y, respectively. We say that M has a *countable base* if there exists a countable family $\{B_j\}$ of open sets in M such that any other open set is a union of some sets B_j. The family $\{B_j\}$ is called a base of the topology of M.

DEFINITION 3.1. A n-dimensional *chart* on M is any couple (U, φ) where U is an open subset of M and φ is a homeomorphism of U onto an open subset of $\mathbb{R}^n$ (which is called the image of the chart).

DEFINITION 3.2. A *C-manifold* of dimension n is a Hausdorff topological space M with a countable base such that any point of M belongs to a n-dimensional chart.

Let M be a C-manifold of dimension n. For any chart (U, φ) on M, the *local coordinate system* $x^1, x^2, ..., x^n$ is defined in U by taking the φ-pullback of the Cartesian coordinate system in $\mathbb{R}^n$. Hence, loosely speaking, a chart is an open set $U \subset M$ with a local coordinate system. Normally, we will identify U with its image so that the coordinates $x^1, x^2, ..., x^n$ can be regarded as the Cartesian coordinates in a region in $\mathbb{R}^n$. However, there are some subtleties with this identification, which we would like to clarify before we proceed further.

If $U \subset M$ is an open set and $E \subset M$ then the relation $E \Subset U$ (*compact inclusion*) means that the closure $\overline{E}$ of E in M is compact and $\overline{E} \subset U$. The compact inclusion will be frequently used but it may become ambiguous if U is a chart on M because in this case $E \Subset U$ can be understood also in the sense of the topology of $\mathbb{R}^n$. Let us show that the two meanings of $E \Subset U$ are identical. Assume $E \subset U$ and denote temporarily by $\widetilde{E}$ the closure of E in $\mathbb{R}^n$. If $E \Subset U$ in the topology of $\mathbb{R}^n$ then $\widetilde{E}$ is compact in $\mathbb{R}^n$ and, hence, its pullback to M (also denoted by $\widetilde{E}$) is compact in M. The fact that M is Hausdorff implies that any compact subset of M is closed. Therefore, $\widetilde{E}$ is closed in M, which implies $\overline{E} \subset \widetilde{E}$ and, hence, the inclusion $E \Subset U$ holds also in M. The converse is proved in the same way.

If U and V are two charts on a C-manifold M then in the intersection $U \cap V$ two coordinate systems are defined, say $x^1, x^2, ..., x^n$ and $y^1, y^2, ..., y^n$. The change of the coordinates is given then by *continuous* functions $y^i = y^i(x^1, ..., x^n)$ and $x^i = x^i\left(y^1, ..., y^n\right)$. Indeed, if φ is the mapping of U to $\mathbb{R}^n$ and ψ is the mapping of V to $\mathbb{R}^n$ then the functions $y^i = y^i(x^1, ..., x^n)$ are the components of the mapping $\psi \circ \varphi^{-1}$ and the functions $x^i\left(y^1, ..., y^n\right)$ are the components of the mapping $\varphi \circ \psi^{-1}$ (see Fig. 3.1).

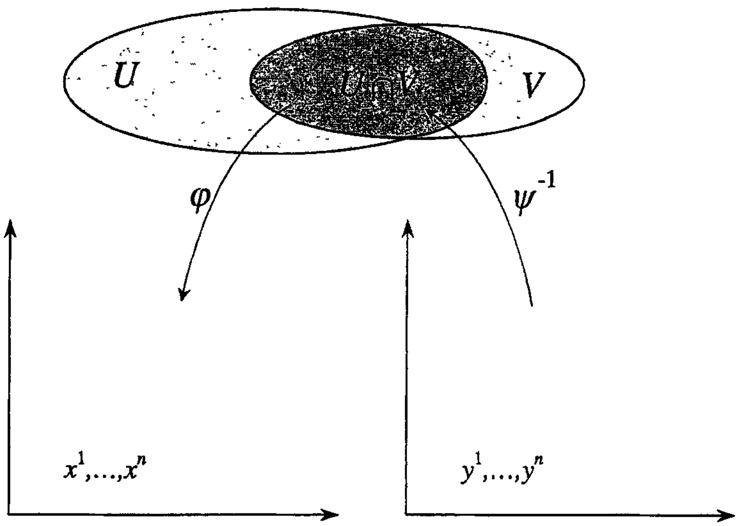

FIGURE 3.1. The mapping $\varphi \circ \psi^{-1}$

A family $\mathcal{A}$ of charts on a C-manifold is called a C^k-*atlas* (where k is a positive integer or $+\infty$) if the charts from $\mathcal{A}$ covers all M and the change of coordinates in the intersection of any two charts from $\mathcal{A}$ is given by C^k-functions. Two C^k-atlases are said to be compatible if their union is again a C^k-atlas. The union of all compatible C^k-atlases determines a C^k-*structure* on M.

DEFINITION 3.3. A smooth manifold is a C-manifold endowed with a C^∞-structure.

Alternatively, one can say that a smooth manifold is a couple $(M, \mathcal{A})$, where M is a C-manifold and $\mathcal{A}$ is a C^∞-atlas on M.

By a chart on a smooth manifold we will always mean a chart from its C^∞-structure, that is, any chart compatible with the defining atlas $\mathcal{A}$.

A trivial example of a smooth manifold is $\mathbb{R}^n$ with the C^∞-atlas consisting of a single chart $(\mathbb{R}^n, \mathrm{id})$. By default, the term "manifold" will be used as a synonymous of "smooth manifold".

If f is a (real valued) function on a manifold M and k is a non-negative integer or ∞ then we write $f \in C^k(M)$ (or $f \in C^k$) if the restriction of f to any chart is a C^k function of the local coordinates $x^1, x^2, ..., x^n$. The set $C^k(M)$ is a linear space with respect to the usual addition of functions and multiplication by constant.

For any function $f \in C(M)$, its support is defined by

$$\operatorname{supp} f = \overline{\{x \in M : f(x) \neq 0\}},$$

where the bar stands for the closure of the set in M. Denote by $C_0^k(M)$ the subspace of $C^k(M)$, which consists of functions whose support is compact. The fact that compact sets in M are closed implies that if f vanishes outside a compact set $K \subset M$ then $\operatorname{supp} f \subset K$.

If Ω is an open subset of M then Ω naturally inherits all the above structures of M and becomes a smooth manifold if M is so. Indeed, the open sets in Ω are defined as the intersections of open sets in M with Ω, and in the same way one defines charts and atlases in Ω.

The hypothesis of a countable base will be mostly used via the next simple lemma.

LEMMA 3.4. *For any manifold M, there is a countable family $\{U_i\}_{i=1}^\infty$ of relatively compact charts covering all M and such that the closure $\overline{U_i}$ is contained in a chart.*

PROOF. Any point $x \in M$ is contained in a chart, say V_x. Choose $U_x \Subset V_x$ to be a small open ball around x so that U_x is also a chart. By definition, manifold M has a countable base, say $\{B_j\}_{j=1}^\infty$. Let us mark each set B_j which is contained in some set U_x. Since U_x is open, it is a union of some marked sets B_j. It follows that all marked B_j cover M. Select for each marked B_j exactly *one* set U_x containing B_j. Thus, we obtain a countable family of sets U_x covering M, which finishes the proof. □

In particular, we see that a manifold M is a *locally compact* topological space.

The following statement extends Theorem 2.2 and provides a convenient vehicle for transporting the local properties of $\mathbb{R}^n$ to manifolds.

THEOREM 3.5. *Let K be a compact subset of a smooth manifold M and $\{U_j\}_{j=1}^k$ be a finite family of open sets covering K. Then there exist non-negative functions $\varphi_j \in C_0^\infty(U_j)$ such that $\sum_j \varphi_j \equiv 1$ in an open neighbourhood of K and $\sum_j \varphi_j \leq 1$ in M.*

A sequence of functions $\{\varphi_j\}$ as in Theorem 3.5 is called a *partition of unity* at K subordinate to the cover $\{U_j\}$.

A particular case of Theorem 3.5 with $k = 1$ says that, for any compact K and any open set $U \supset K$, there exists a function $\varphi \in C_0^\infty(U)$ such that $\varphi \equiv 1$ in a neighborhood of K and, besides, $0 \le \varphi \le 1$. Such a function φ is called a *cutoff function* of K in U.

PROOF. If each set U_j is a chart then the proof of Theorem 2.2 goes through unchanged. In the general case, for any point $x \in K$, there is a chart V_x containing x and such that $V_x \subset U_j$ for some j. Out of the family $\{V_x\}_{x\in K}$ covering K select a finite subfamily $\{V_i\}$ also covering K. Since each V_i is a chart, there exists a partition of unity $\{\psi_i\}$ at K subordinate to $\{V_i\}$. Now define φ_1 to be the sum of those functions ψ_i which are supported in U_1; φ_2 to be the sum of those functions ψ_i which are supported in U_2 but not supported in U_1; ... ; φ_k to be the sum of those functions ψ_i which are supported in U_k but not supported in $U_1, ..., U_{k-1}$. Clearly, each φ_j is non-negative and belongs to $C_0^\infty(U_j)$. Since V_i is covered by some U_j, each ψ_i is supported in some U_j and, hence, each ψ_i will be used in the above construction exactly once. This implies

$$\sum_i \varphi_i \equiv \sum_j \psi_j,$$

which finishes the proof. $\square$

COROLLARY 3.6. *Let $\{\Omega_\alpha\}$ be an arbitrary covering of M by open sets. Then, for any function $f \in C_0^\infty(M)$, there exists a finite sequence $\{f_i\}_{i=1}^k$ of functions from $C_0^\infty(M)$ such that each f_i is supported in one of the sets Ω_α and*

$$f \equiv f_1 + ... + f_k. \tag{3.1}$$

PROOF. Let $K = \operatorname{supp} f$ and let $\Omega_1, ..., \Omega_k$ be a finite subfamily of $\{\Omega_\alpha\}$ covering K. By Theorem 3.5, there exists a partition of unity $\{\varphi_i\}_{i=1}^k$ at K subordinate to $\{\Omega_i\}_{i=1}^k$. Set $f_i = f\varphi_i$ so that $f_i \in C_0^\infty(\Omega_i)$. Then

$$\sum_i f_i = f,$$

because on K we have $\sum_i \varphi_i \equiv 1$, and outside K all functions f, f_i vanish. $\square$

Exercises.

3.1. Prove that, on any C-manifold M, there exists a countable sequence $\{\Omega_k\}$ of relatively compact open sets such that $\Omega_k \Subset \Omega_{k+1}$ and the union of all Ω_k is M. Prove also that if M is connected then the sets Ω_k can also be taken connected.

REMARK. An increasing sequence $\{\Omega_k\}$ of open subsets of M whose union is M, is called an *exhaustion sequence*. If in addition $\Omega_k \Subset \Omega_{k+1}$ (that is, Ω_k is relatively compact and $\overline{\Omega}_k \subset \Omega_{k+1}$) then the sequence $\{\Omega_k\}$ is called a *compact exhaustion sequence*.

3.2. Prove that, on any C-manifold M, there is a countable locally finite family of relatively compact charts covering all M. (A family of sets is called locally finite if any compact set intersects at most finitely many sets from this family).

3.2. Tangent vectors

Let M be a smooth manifold.

DEFINITION 3.7. A mapping $\xi : C^\infty(M) \to \mathbb{R}$ is called an $\mathbb{R}$-*differentiation* at a point $x_0 \in M$ if

- ξ is linear;
- ξ satisfies the product rule in the following form:

$$\xi(fg) = \xi(f)\, g(x_0) + \xi(g)\, f(x_0),$$

for all $f, g \in C^\infty$.

The set of all $\mathbb{R}$-differentiations at x_0 is denoted by $T_{x_0}M$. For any $\xi, \eta \in T_{x_0}M$ one defines the sum $\xi + \eta$ as the sum of two functions on C^∞, and similarly one defined $\lambda\xi$ for any $\lambda \in \mathbb{R}$. It is easy to check that both $\xi + \eta$ and $\lambda\xi$ are again $\mathbb{R}$-differentiations, so that $T_{x_0}M$ is a linear space over $\mathbb{R}$. The linear space $T_{x_0}M$ is called the *tangent space* of M at x_0, and its elements (that is, $\mathbb{R}$-differentiations) are also called *tangent vectors* at x_0.

THEOREM 3.8. *If M is a smooth manifold of dimension n then the tangent space $T_{x_0}M$ is a linear space of the same dimension n.*

We will prove this after a series of claims.

CLAIM 1. *Let $U \subset M$ be an open set and $U_0 \Subset U$ be its open subset. Then, for any function $f \in C^\infty(U)$, there exists a function $F \in C^\infty(M)$ such that $f \equiv F$ in U_0.*

PROOF. Indeed, let ψ be a cutoff function of U_0 in U (see Theorem 3.5). Then define function F by

$$\begin{cases} F = \psi f & \text{in } U, \\ F = 0 & \text{in } M \setminus U, \end{cases}$$

which clearly satisfies all the requirements. □

CLAIM 2. *Let $f \in C^\infty(M)$ and let $f \equiv 0$ in an open neighbourhood U of a point $x_0 \in M$. Then $\xi(f) = 0$ for any $\xi \in T_{x_0}M$. Consequently, if F_1 and F_2 are smooth functions on M such that $F_1 \equiv F_2$ in an open neighbourhood of a point $x_0 \in M$ then $\xi(F_1) = \xi(F_2)$ for any $\xi \in T_{x_0}M$.*

PROOF. Let U_0 be a neighborhood of x_0 such that $U_0 \Subset U$ and let ψ be a cutoff function of U_0 in U. Then we have $f\psi \equiv 0$ on M, which implies the identity $f \equiv f(1-\psi)$. By the product rule, we obtain

$$\xi(f) = \xi(f(1-\psi)) = \xi(f)(1-\psi)(x_0) + \xi(1-\psi)\, f(x_0) = 0,$$

because $f(x_0) = (1-\psi)(x_0) = 0$. The second part follows from the first one applied to the function $f = F_1 - F_2$. □

REMARK 3.9. Originally a tangent vector $\xi \in T_{x_0}M$ is defined as a functional on $C^\infty(M)$. The results of Claims 1 and 2 imply that ξ can be regarded as a functional on $C^\infty(U)$ where U is any neighbourhood of x_0. Indeed, by Claim 1, for any $f \in C^\infty(U)$ there exists a function $F \in C^\infty(M)$ such that $f = F$ in a neighborhood of x_0; hence, set $\xi(f) := \xi(F)$. By Claim 2, this definition of $\xi(f)$ does not depend on the choice of F.

CLAIM 3. *Let f be a smooth function in a ball $B = B_R(o)$ in $\mathbb{R}^n$ where o is the origin. Then there exist smooth functions $h_1, h_2, ..., h_n$ in B such that, for any $x \in B$,*

$$f(x) = f(o) + x^i h_i(x), \tag{3.2}$$

where we assume summation over the repeated index i. Also, we have

$$h_i(o) = \frac{\partial f}{\partial x^i}(o). \tag{3.3}$$

PROOF. By the fundamental theorem of calculus applied to the function $t \mapsto f(tx)$ on the interval $t \in [0,1]$, we have

$$f(x) = f(o) + \int_0^1 \frac{d}{dt} f(tx)\, dt, \tag{3.4}$$

whence it follows

$$f(x) = f(o) + \int_0^1 x^i \frac{\partial f}{\partial x^i}(tx) dt.$$

Setting

$$h_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(tx) dt$$

we obtain (3.2). Clearly, $h_i \in C^\infty(B)$. The identity (3.3) follows from the line above by substitution $x = o$. □

CLAIM 4. *Under the hypothesis of* Claim 3, *there exist smooth functions h_{ij} in B, (where $i, j = 1, 2, ..., n$) such that, for any $x \in B$,*

$$f(x) = f(o) + x^i \frac{\partial f}{\partial x^i}(o) + x^i x^j h_{ij}(x). \tag{3.5}$$

PROOF. Applying (3.2) to the function h_j instead of f we obtain that there exist smooth functions h_{ij} in B, $i = 1, 2, ..., n$ such that

$$h_j(x) = h_j(o) + x^i h_{ij}(x).$$

Substituting this into the representation (3.2) for f and using $h_j(o) = \frac{\partial f}{\partial x^j}(o)$ we obtain

$$f(x) = f(o) + x^i h_i(x) = f(o) + x^i \frac{\partial f}{\partial x^i}(o) + x^i x^j h_{ij}(x).$$

□

Now we can prove Theorem 3.8.

PROOF OF THEOREM 3.8. Let $x^1, x^2, ..., x^n$ be local coordinates in a chart U containing x_0. All the partial derivatives $\frac{\partial}{\partial x^i}$ evaluated at x_0 are $\mathbb{R}$-differentiations at x_0, and they are linearly independent. We will prove that any tangent vector $\xi \in T_{x_0}M$ is represented in the form

$$\xi = \xi^i \frac{\partial}{\partial x^i} \quad \text{where} \quad \xi^i = \xi\left(x^i\right) \tag{3.6}$$

(note that, by Remark 3.9, the $\mathbb{R}$-differentiation ξ applies also to smooth functions defined in a neighborhood of x_0), which will imply that $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$ is a basis in the linear space $T_{x_0}M$ and hence $\dim T_{x_0}M = n$.

Without loss of generality, we can assume that x_0 is the origin of the chart U. For any smooth function f on M, we have by (3.5) the following representation in a ball $B \subset U$ centred at x_0:

$$f(x) = f(x_0) + x^i \frac{\partial f}{\partial x^i}(x_0) + x^i x^j h_{ij}(x),$$

where h_{ij} are some smooth functions in B. Using the linearity of ξ, we obtain

$$\xi(f) = \xi(1) f(x_0) + \xi\left(x^i\right) \frac{\partial f}{\partial x^i}(x_0) + \xi\left(x^i x^j h_{ij}\right). \tag{3.7}$$

By the product rule, we have

$$\xi(1) = \xi(1 \cdot 1) = \xi(1)1 + \xi(1)1 = 2\xi(1),$$

whence $\xi(1) = 0$. Set $u_i = x^j h_{ij}$. By the linearity and the product rule,

$$\xi\left(x^i u_i\right) = \xi\left(x^i\right) u_i(x_0) + \xi(u_i) x^i(x_0) = 0,$$

because x^i and hence u_i vanish at x_0. Hence, in the right hand side of (3.7), the first and the third term vanish, and we obtain

$$\xi(f) = \xi^i \frac{\partial f}{\partial x^i}, \tag{3.8}$$

which was to be proved. □

The numbers ξ^i are referred to as the *components* of the vector ξ in the coordinate system $x^1, ..., x^n$. One often uses the following alternative notation for $\xi(f)$:

$$\xi(f) \equiv \frac{\partial f}{\partial \xi}.$$

Then the identity (3.8) takes the form

$$\frac{\partial f}{\partial \xi} = \xi^i \frac{\partial f}{\partial x^i},$$

which allows to think of ξ as a *direction* at x_0 and to interpret $\frac{\partial f}{\partial \xi}$ as a directional derivative.

A *vector field* on a smooth manifold M is a family $\{v(x)\}_{x\in M}$ of tangent vectors such that $v(x) \in T_xM$ for any $x \in M$. In the local coordinates $x^1, ..., x^n$, it can be represented in the form

$$v(x) = v^i(x) \frac{\partial}{\partial x^i}.$$

The vector field $v(x)$ is called *smooth* if all the functions $v^i(x)$ are smooth in any chart.

Fix a point $x \in M$ and let f be a smooth function in a neighborhood of x. Define the notion of the *differential* df at x as follows: df is a linear functional on T_xM given by

$$\langle df, \xi\rangle = \xi(f) \text{ for any } \xi \in T_xM, \tag{3.9}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing of a linear functional on $T_xM$and a vector from T_xM. Hence, df is an element of the *dual space* T_x^*M, which is called a *cotangent space*. It is known from linear algebra that the dual space is also a linear space of the same dimension n. The elements of T_x^*M are called *covectors*.

Any basis $\{e_1, ..., e_n\}$in T_xM has a dual basis $\{e^1, ..., e^n\}$ in the dual space T_x^*M, which is defined by

$$\langle e^i, e_j\rangle = \delta_j^i := \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases}$$

For example, the basis $\left\{\frac{\partial}{\partial x^i}\right\}$ has dual $\{dx^i\}$ because

$$\langle dx^i, \frac{\partial}{\partial x^j}\rangle = \frac{\partial}{\partial x^j}x^i = \delta_j^i.$$

The covector df can be represented in the basis $\{dx^i\}$ as follows:

$$df = \frac{\partial f}{\partial x^i}dx^i, \tag{3.10}$$

that is, the partial derivatives $\frac{\partial f}{\partial x^i}$ are the components of the differential df in the basis $\{dx^i\}$. Indeed, for any $j = 1, ..., n$,

$$\langle \frac{\partial f}{\partial x^i}dx^i, \frac{\partial}{\partial x^j}\rangle = \frac{\partial f}{\partial x^i}\langle dx^i, \frac{\partial}{\partial x^j}\rangle = \frac{\partial f}{\partial x^i}\delta_j^i = \frac{\partial f}{\partial x^j} = \langle df, \frac{\partial}{\partial x^j}\rangle.$$

3.3. Riemannian metric

Let M be a smooth n-dimensional manifold. A *Riemannian metric* (or a Riemannian *metric tensor*) on M is a family[1] $\mathbf{g} = \{\mathbf{g}(x)\}_{x\in M}$ such that $\mathbf{g}(x)$ is a symmetric, positive definite, bilinear form on the tangent space T_xM, smoothly depending on $x \in M$.

[1]One can also say that $\mathbf{g}$ is a smooth $(0, 2)$-tensor field on M.

Using the metric tensor, one defines an inner product $\langle\cdot,\cdot\rangle_{\mathbf{g}}$ in any tangent space T_xM by[2]

$$\langle\xi,\eta\rangle_{\mathbf{g}} \equiv \mathbf{g}(x)(\xi,\eta),$$

for all tangent vectors $\xi,\eta \in T_xM$. Hence, T_xM becomes a Euclidean space. For any tangent vector $\xi \in T_xM$, its *length* is defined by

$$|\xi|_{\mathbf{g}} = \langle\xi,\xi\rangle_{\mathbf{g}}^{1/2}.$$

In the local coordinates $x^1,...,x^n$, the inner product in T_xM has the form

$$\langle\xi,\eta\rangle_{\mathbf{g}} = g_{ij}(x)\,\xi^i\eta^j, \tag{3.11}$$

where $(g_{ij}(x))_{i,j=1}^n$ is a symmetric positive definite $n\times n$ matrix. The functions $g_{ij}(x)$ are called the *components* of the tensor $\mathbf{g}$ in the coordinates $x^1,...,x^n$. The condition that $\mathbf{g}(x)$ smoothly depends on x means that all the components $g_{ij}(x)$ are C^∞-functions in the corresponding charts. It follows from (3.11) that

$$g_{ij} = \langle\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\rangle_{\mathbf{g}}. \tag{3.12}$$

DEFINITION 3.10. A Riemannian manifold is a couple $(M,\mathbf{g})$ where $\mathbf{g}$ is a Riemannian metric on a smooth manifold M.

A trivial example of a Riemannian manifold is $\mathbb{R}^n$ with the *canonical Euclidean metric* $\mathbf{g}_{\mathbb{R}^n}$ defined in the Cartesian coordinates $x^1,...,x^n$ by

$$\mathbf{g}_{\mathbb{R}^n} = \left(dx^1\right)^2 + ... + \left(dx^n\right)^2.$$

For this metric, we have $(g_{ij}) = \mathrm{id}$.

It is frequently convenient to write the metric tensor $\mathbf{g}$ in the form

$$\mathbf{g} = g_{ij}dx^idx^j, \tag{3.13}$$

where dx^idx^j stands for the *tensor product* of the covectors dx^i and dx^j, which is a bilinear functional on T_xM defined by

$$dx^idx^j(\xi,\eta) = \langle dx^i,\xi\rangle\langle dx^j,\eta\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the pairing of covectors and vectors. Indeed, since

$$\langle dx^i,\xi\rangle = \xi\left(x^i\right) = \xi^i,$$

(3.13) is equivalent to $\mathbf{g}(\xi,\eta) = g_{ij}\xi^i\eta^j$, which is just another form of (3.11).

Let $(M,\mathbf{g})$ be a Riemannian manifold. The metric tensor $\mathbf{g}$ provides a canonical way of identifying the tangent space T_xM with the cotangent space T_x^*M. Indeed, for any vector $\xi \in T_xM$, denote by $\mathbf{g}(x)\,\xi$ a covector that is defined by the identity

$$\langle\mathbf{g}(x)\,\xi,\eta\rangle = \langle\xi,\eta\rangle_{\mathbf{g}} \text{ for all } \eta\in T_xM. \tag{3.14}$$

[2]In the context when the metric tensor $\mathbf{g}$ is fixed, we will normally drop the subscript $\mathbf{g}$ from all notation.

Clearly, this makes $\mathbf{g}(x)$ into a linear mapping from T_xM to T_x^*M. In the local coordinates, we have

$$(\mathbf{g}(x)\xi)_j \eta^j = g_{ij}\xi^i\eta^j.$$

which implies

$$(\mathbf{g}(x)\xi)_j = g_{ij}\xi^i.$$

In particular, the components of the linear operator $\mathbf{g}(x)$ are g_{ij} – the same as the components of the metric tensor. With a slight abuse of notation, one writes $\xi_j \equiv (\mathbf{g}(x)\xi)_j$ so that the same letter is used to denote a vector and the corresponding covector. In this notation, we have $\xi_j = g_{ij}\xi^i$.

Observe that if $\xi \neq 0$ then $\mathbf{g}(x)\xi$ is also non-zero as covector, because $\langle \mathbf{g}(x)\xi, \xi\rangle > 0$.Therefore, the mapping

$$\mathbf{g}(x) : T_xM \to T_x^*M$$

is injective and, hence, also bijective. Consequently, it has the inverse mapping

$$\mathbf{g}^{-1}(x) : T_x^*M \to T_xM,$$

whose components are denoted by $\left(g^{ij}\right)$ so that

$$\left(g^{ij}\right) = (g_{ij})^{-1}.$$

Hence, for any covector $\omega \in T_x^*M$, $\mathbf{g}^{-1}(x)\omega$ is a vector whose components are given by

$$\omega^i := \left(\mathbf{g}^{-1}(x)\omega\right)^i = g^{ij}\omega_j. \tag{3.15}$$

Obviously, $\mathbf{g}^{-1}(x)$ can be considered as an inner product in T_x^*M: for all $v, \omega \in T_x^*M$, set

$$\langle v, \omega\rangle_{\mathbf{g}^{-1}} := \langle \mathbf{g}^{-1}(x)v, \mathbf{g}^{-1}(x)\omega\rangle_{\mathbf{g}} = \langle v, \mathbf{g}^{-1}(x)\omega\rangle.$$

It follows that, in the local coordinates,

$$\langle v, \omega\rangle_{\mathbf{g}^{-1}} = g^{ij}v_i\omega_j.$$

For any smooth function f on M, define its *gradient* $\nabla f(x)$ at a point $x \in M$ by

$$\nabla f(x) = \mathbf{g}^{-1}(x)\, df(x) \tag{3.16}$$

that is, $\nabla f(x)$ is a covector version of $df(x)$. Applying (3.14) with $\xi = \nabla f(x)$, we obtain, for any $\eta \in T_xM$,

$$\langle \nabla f, \eta\rangle_{\mathbf{g}} = \langle df, \eta\rangle = \frac{\partial f}{\partial \eta}, \tag{3.17}$$

which can be considered as an alternative definition of the gradient. In the local coordinates $x^1, ..., x^n$, we obtain by (3.15) and (3.16)

$$(\nabla f)^i = g^{ij}\frac{\partial f}{\partial x^j}. \tag{3.18}$$

If h is another smooth function on M then setting in (3.17) $\eta = \nabla h$ and using (3.18), we obtain

$$\langle \nabla f, \nabla h \rangle_{\mathrm{g}} = \langle df, \nabla h \rangle = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial h}{\partial x^j} = \langle df, dh \rangle_{\mathrm{g}^{-1}}. \tag{3.19}$$

Exercises.

3.3. Prove the product rule for d and ∇:

$$d(uv) = udv + vdu$$

and

$$\nabla(uv) = u\nabla v + v\nabla u, \tag{3.20}$$

where u and v are smooth function on M.

3.4. Prove the chain rule for d and ∇:

$$df(u) = f'(u)\,du$$

and

$$\nabla f(u) = f'(u)\,\nabla u$$

where u and f are smooth functions on M and $\mathbb{R}$, respectively.

3.4. Riemannian measure

Some basic knowledge of the measure theory is required in this section (see the reference material in Appendix A, Section A.4).

Let M be a smooth manifold of dimension n. Let $\mathcal{B}(M)$ be the smallest σ-algebra containing all open sets in M. The elements of $\mathcal{B}(M)$ are called *Borel sets*. We say that a set $E \subset M$ is *measurable* if, for any chart U, the intersection $E \cap U$ is a Lebesgue measurable set in U. Obviously, the family of all measurable sets in M forms a σ-algebra; denote it by $\Lambda(M)$. Since any open subset of M is measurable, it follows that also all Borel sets are measurable, that is, $\mathcal{B}(M) \subset \Lambda(M)$.

The purpose of this section is to show that any Riemannian manifold (M, g) features a canonical measure ν, defined on $\Lambda(M)$, which is called the *Riemannian measure (or volume)*. This measure is defined by means of the following theorem.

THEOREM 3.11. *For any Riemannian manifold (M, g), there exists a unique measure ν on $\Lambda(M)$ such that, in any chart U,*

$$d\nu = \sqrt{\det g}\, d\lambda, \tag{3.21}$$

where $g = (g_{ij})$ is the matrix of the Riemannian metric g *in U, and λ is the Lebesgue measure in U.*

Furthermore, the measure ν is complete, $\nu(K) < \infty$ for any compact set $K \subset M$, $\nu(\Omega) > 0$ for any non-empty open set $\Omega \subset M$, and ν is regular in the following sense: for any set $A \in \Lambda(M)$,

$$\nu(A) = \sup\{\nu(K) : K \subset A, \ K \text{ compact}\} \tag{3.22}$$

and

$$\nu(A) = \inf\{\nu(U) : A \subset \Omega,\ \Omega \text{ open}\}. \tag{3.23}$$

For the proof we use the following lemma.

LEMMA 3.12. *Let $x^1,...,x^n$ and $y^1,...,y^n$ be the local coordinate systems in open sets U and V, respectively. Denote by g^x and g^y the matrices of $\mathbf{g}$ in the coordinates $x^1,...,x^n$ and $y^1,...,y^n$, respectively. Let $J = \left(J_i^k\right)_{k,i=1}^n$ be the Jacobian matrix of the change $y = y(x)$ defined in $U \cap V$ by*

$$J_i^k = \frac{\partial y^k}{\partial x^i}, \tag{3.24}$$

where k is the row index and i is the column index. Then we have the following identity in $U \cap V$:

$$g^x = J^T g^y J, \tag{3.25}$$

where J^T is the transposed matrix.

PROOF. By the chain rule, we have

$$\frac{\partial}{\partial x^i} = \frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k} = J_i^k \frac{\partial}{\partial y^k},$$

whence by (3.12)

$$g_{ij}^x = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle_{\mathbf{g}} = J_i^k J_j^l \langle \frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^l} \rangle_{\mathbf{g}} = J_i^k g_{kl}^y J_j^l. \tag{3.26}$$

Noticing that

$$J_i^k g_{kl}^y J_j^l = \left(J^T g^y J\right)_{ij},$$

we obtain

$$g_{ij}^x = \left(J^T g^y J\right)_{ij}$$

whence (3.25) follows. □

PROOF OF THEOREM 3.11. The condition (3.21) means that, for any measurable set $A \subset U$,

$$\nu(A) = \int_A \sqrt{\det g}\, d\lambda. \tag{3.27}$$

By measure theory, the identity (3.27), indeed, defines a measure ν on the σ-algebra $\Lambda(U)$ of Lebesgue measurable sets in U (see Section A.4.3).

We will show that the measure ν defined by (3.27) in each chart, can be uniquely extended to $\Lambda(M)$. However, before that, we need to ensure that the measures in different charts agree on their intersection.

CLAIM. *If U and V are two charts on M and A is a measurable set in $U \cap V$ then $\nu(A)$ defined by (3.27) has the same values in both charts.*

Let $x^1,...,x^n$ and $y^1,...,y^n$ be the local coordinate systems in U and V, respectively. Denote by g^x and g^y the matrices of $\mathbf{g}$ in the coordinates

$x^1, ..., x^n$ and $y^1, ..., y^n$, respectively. We need to show that, for any measurable set $A \subset W := U \cap V$,

$$\int_A \sqrt{\det g^x} dx = \int_A \sqrt{\det g^y} dy,$$

where dx and dy stand for the Lebesgue measures in U and V, respectively.

By (3.25), we have

$$\det g^x = (\det J)^2 \det g^y. \tag{3.28}$$

Next, let us use the following formula for change of variables in multivariable integration: if f is a non-negative measurable function in W then

$$\int_W f dy = \int_W f \, |\det J| \, dx.$$

Applying this for $f = 1_A \sqrt{\det g^y}$, where $A \subset W$ is a measurable set, and using (3.28), we obtain

$$\int_A \sqrt{\det g^y} dy = \int_A \sqrt{\det g^y} \, |\det J| \, dx = \int_A \sqrt{\det g^x} dx,$$

which was to be proved.

Now let us prove the uniqueness of measure ν. By Lemma 3.4, there is a countable family $\{U_i\}_{i=1}^{\infty}$ of relatively compact charts covering M and such that $\overline{U}_i$ is contained in a chart. For any measurable set A on M, define the sequence of sets $A_i \subset U_i$ by

$$A_1 = A \cap U_1, \ A_2 = A \cap U_2 \setminus U_1, ..., \ A_i = A \cap U_i \setminus U_1 \setminus ... \setminus U_{i-1}, ... \tag{3.29}$$

(see Fig. 3.2).

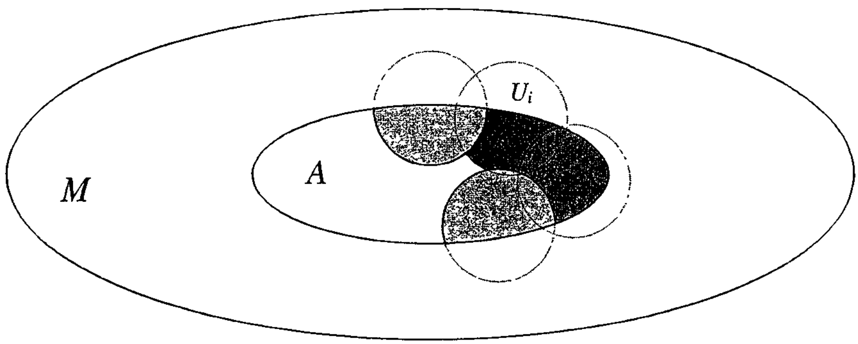

FIGURE 3.2. Splitting A into disjoint sets A_i.

Clearly, $A = \bigsqcup_i A_i$ where the sign $\sqcup$ means "disjoint union". Therefore, for any extension of ν, we should have

$$\nu(A) = \sum_i \nu(A_i). \tag{3.30}$$

However, the value $\nu(A_i)$ is uniquely defined because A_i is contained in the chart U_i. Hence, $\nu(A)$ is also uniquely defined, which was to be proved.

To prove the existence of ν, we use the same construction: for any measurable set A, define $\nu(A)$ by (3.27), using the fact that $\nu(A_i)$ is already defined. Let us show that ν is a measure, that is, ν is σ-additive. Let $\{A^k\}$ be a sequence of disjoint measurable sets in M and let

$$A = \bigsqcup_k A^k.$$

Defining the sets A_i^k similarly to (3.29), we obtain that

$$A_i = \bigsqcup_k A_i^k.$$

Since ν is σ-additive in each chart U_i, we obtain

$$\nu(A_i) = \sum_k \nu(A_i^k).$$

Adding up in i and interchanging the summation in i and k, we obtain

$$\nu(A) = \sum_k \nu\left(A^k\right),$$

which was to be proved.

Let us show that measure ν is complete, that is, any null set of ν is measurable. Let N be a null set of ν, that is, $N \subset A$ for some set A with $\nu(A) = 0$. Defining N_i similarly to A_i by (3.29), we obtain $N_i \subset A_i$. Since $\nu(A_i) = 0$, it follow from the formula (3.27) in U_i and $\sqrt{\det g} > 0$ that also $\lambda(A_i) = 0$. Thus, N_i is a null set for the Lebesgue measure λ in U_i. Since the Lebesgue measure is complete, we conclude that N_i is measurable and, hence, N is measurable.

Any compact set $K \subset M$ can covered by a finite number of charts U_i and, hence, K is a finite union of some sets $K_i = K \cap U_i$. Applying (3.27) in a chart containing $\overline{U}_i$ and noticing $\sqrt{\det g}$ is bounded on $\overline{U}_i$, we obtain $\nu(K_i) < \infty$, which implies $\nu(K) < \infty$.

Any non-empty open set $\Omega \subset M$ contains some chart U, whence it follows from (3.27) that

$$\nu(\Omega) \geq \nu(U) = \int_U \sqrt{\det g} d\lambda > 0.$$

Let us prove the inner regularity of ν, that is (3.22). Let A be a relatively compact measurable subset of M. Then there is a finite family $\{U_i\}_{i=1}^m$ of charts that cover $\overline{A}$. We can assume that each $\overline{U}_i$ is compact and is contained in another chart V_i. By the regularity of the Lebesgue measure, each set $A_i = A \cap U_i$ can be approximated by a compact set $K_i \subset A_i$ such that $\lambda_i(A_i \setminus K_i) < \varepsilon_i$ where λ_i is the Lebesgue measure in V_i and $\varepsilon_i > 0$ is any given number. Set $C_i = \sup_{U_i} \sqrt{\det g}$, $K = \bigcup_{i=1}^m K_i$, and observe that

$$\nu(A \setminus K) \le \sum_{i=1}^m \nu(A_i \setminus K_i) \le \sum_{i=1}^m C_i \varepsilon_i.$$

Since ε_i can be chosen arbitrarily small, the right hand side can be made arbitrarily small, which proves (3.22). If A is an arbitrary measurable subset of M, then take a compact exhaustion $\{\Omega_k\}$ of M and apply the previous argument to $A_k = A \cap \Omega_k$. Let K_k be a compact subset of A_k such that $\nu(A_k \setminus K_k) < \varepsilon_k$ where $\{\varepsilon_k\}$ is any sequence of positive numbers such that $\varepsilon_k \to 0$ as $k \to \infty$. Then we have

$$\lim_{k\to\infty} \nu(K_k) = \lim_{k\to\infty} \nu(A_k) = \nu(A),$$

which proves (3.22).

Finally, let us prove the outer regularity of ν, that is (3.23). Let now $\{U_i\}$ be a countable family of charts that cover M and such that each $\overline{U}_i$ is contained in another chart V_i, such that $\overline{V}_i$ is compact and is contained yet in another chart. By the regularity of the Lebesgue measure, the set $A_i = A \cap U_i$ can be approximated by an open set $\Omega_i \supset A_i$ so that $\Omega_i \subset V_i$ and $\lambda_i(\Omega_i \setminus A_i) < \varepsilon_i$. Setting $C_i = \sup_{V_i} \sqrt{\det g}$ and $\Omega = \bigcup_{i=1}^\infty \Omega_i$, we obtain as above

$$\nu(\Omega \setminus A) \le \sum_{i=1}^\infty C_i \varepsilon_i.$$

Since the right hand side can be made arbitrarily small by the choice of ε_i, we obtain (3.23). $\square$

The extension of measure ν from the charts to the whole manifold can also be done using the Carathéodory extension of measures. Consider the following family of subsets of M:

$$S = \{A \subset M : A \text{ is a relatively compact measurable set and } \overline{A} \text{ is contained in a chart}\}.$$

Observe that S is a semi-ring and, by the above Claim, ν is defined as a measure on S. Hence, the Carathéodory extension of ν exists and is a complete measure on M. It is not difficult to check that the domain of this measure is exactly $\Lambda(M)$. Since the union of sets U_i from Lemma 3.4 is M and $\nu(U_i) < \infty$, the measure ν on S is σ-finite and, hence, its extension to $\Lambda(M)$ is unique.

Since the Riemannian measure ν is finite on compact sets, any continuous function with compact support is integrable against ν. Let us record the following simple property of measure ν, which will be used in the next section.

LEMMA 3.13. *If $f \in C(M)$ and*

$$\int_M f\varphi d\nu = 0 \tag{3.31}$$

for all $\varphi \in C_0^\infty(M)$ then $f \equiv 0$.

PROOF. Assume that $f(x_0) \neq 0$ for some point $x_0 \in M$, say, $f(x_0) > 0$. Then, by the continuity of f, $f(x)$ is strictly positive in a open neighborhood Ω of x_0. Let φ be a cutoff function of $\{x_0\}$ in Ω. Then $\varphi \equiv 1$ in an open neighborhood U of x_0. Since $\nu(U) > 0$, it follows that

$$\int_M f\varphi d\nu = \int_\Omega f\varphi d\nu \geq \int_U f d\nu > 0,$$

which contradicts (3.31). □

Exercises.

3.5. Let $\mathbf{g}$, $\widetilde{\mathbf{g}}$ be two Riemannian metric tensors on a smooth manifold M and let g and $\widetilde{g}$ be the matrices of $\mathbf{g}$ and $\widetilde{\mathbf{g}}$ respectively in some coordinate system. Prove that the ratio

$$\frac{\det \widetilde{g}}{\det g}$$

does not depend on the choice of the coordinates (although separately $\det g$ and $\det \widetilde{g}$ do depend on the coordinate system).

3.6. Let $\mathbf{g}$, $\widetilde{\mathbf{g}}$ be two Riemannian metric tensors on a smooth manifold M such that

$$\frac{\widetilde{\mathbf{g}}}{\mathbf{g}} \leq C, \tag{3.32}$$

that is, for all $x \in M$ and $\xi \in T_xM$,

$$\widetilde{\mathbf{g}}(\xi, \xi) \leq C\mathbf{g}(\xi, \xi).$$

(a) Prove that if ν and $\widetilde{\nu}$ are the Riemannian volumes of $\mathbf{g}$ and $\widetilde{\mathbf{g}}$, respectively. then

$$\frac{d\widetilde{\nu}}{d\nu} \leq C^{n/2},$$

where $n = \dim M$.

(b) Prove that, for any smooth function f on M,

$$|\nabla f|^2_{\mathbf{g}} \leq C\,|\nabla f|^2_{\widetilde{\mathbf{g}}}.$$

3.5. Divergence theorem

For any smooth vector field $v(x)$ on a Riemannian manifold $(M, \mathbf{g})$, its *divergence* $\operatorname{div} v(x)$ is a smooth function on M, defined by means of the following statement.

THEOREM 3.14. (Divergence theorem) *For any C^∞-vector field $v(x)$ on a Riemannian manifold M, there exists a unique smooth function on M, denoted by* $\operatorname{div} v$, *such that the following identity holds*

$$\int_M (\operatorname{div} v)\, u\, d\nu = -\int_M \langle v, \nabla u\rangle_{\mathbf{g}} d\nu, \tag{3.33}$$

for all $u \in C_0^\infty(M)$.

PROOF. The uniqueness of $\operatorname{div} v$ is simple: if there are two candidates for $\operatorname{div} v$, say $(\operatorname{div} v)'$ and $(\operatorname{div} v)''$ then, for all $u \in C_0^\infty(M)$,

$$\int_M (\operatorname{div} v)' u \, d\nu = \int_M (\operatorname{div} v)'' u \, d\nu,$$

which implies $(\operatorname{div} v)' = (\operatorname{div} v)''$ by Lemma 3.13.

To prove the existence of $\operatorname{div} v$, let us first show that $\operatorname{div} v$ exists in any chart. Namely, if U is a chart on M with the coordinates $x^1, ..., x^n$ then, using (3.17), (3.21), and the integration-by-parts formula in U, we obtain, for any $u \in C_0^\infty(U)$,

$$\begin{aligned}\int_U \langle v, \nabla u\rangle_{\mathbf{g}} d\nu &= \int_U \langle v, du\rangle d\nu \\ &= \int_U v^k \frac{\partial u}{\partial x^k} \sqrt{\det g} d\lambda \\ &= -\int_U \frac{\partial}{\partial x^k}\left(v^k \sqrt{\det g}\right) u \, d\lambda \\ &= -\int_U \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^k}\left(v^k \sqrt{\det g}\right) u \, d\nu. \qquad (3.34)\end{aligned}$$

Comparing with (3.33) we see that the divergence in U can be defined by

$$\operatorname{div} v = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^k}\left(\sqrt{\det g} v^k\right). \qquad (3.35)$$

If U and V are two charts then (3.35) defines the divergences in U and in V, which agree in $U \cap V$ by the uniqueness statement. Hence, (3.35) defines $\operatorname{div} v$ as a function on the entire manifold M, and the divergence defined in this way satisfies the identity (3.33) for all test functions u compactly supported in one of the charts.

We are left to extend the identity (3.33) to all functions $u \in C_0^\infty(M)$. Let $\{\Omega_\alpha\}$ be any family of charts covering M. By Corollary 3.6, any function $u \in C_0^\infty(M)$ can be represented as a sum $u_1 + ... + u_k$, where each u_i is smooth and compactly supported in some Ω_α. Hence, (3.33) holds for each of the functions u_i, and adding up all such identities, we obtain (3.33) for function u. $\square$

It follows from (3.35) that

$$\operatorname{div} v = \frac{\partial v^k}{\partial x^k} + v^k \frac{\partial}{\partial x^k} \log \sqrt{\det g}.$$

In particular, if $\det g \equiv 1$ then we obtain the same formula as in $\mathbb{R}^n$: $\operatorname{div} v = \frac{\partial v^k}{\partial x^k}$.

COROLLARY 3.15. *The identity (3.33) holds also if $u(x)$ is any smooth function on M and $v(x)$ is a compactly supported smooth vector field on M.*

PROOF. Let $K = \operatorname{supp} v$. By Theorem 3.5, there exists a cutoff function of K, that is, a function $\varphi \in C_0^\infty(M)$ such that $\varphi \equiv 1$ in a neighbourhood of K. Then $u\varphi \in C_0^\infty(M)$, and we obtain by Theorem 3.14

$$\int_M \operatorname{div} v\, u\, d\nu = \int_M \operatorname{div} v\, (u\varphi)\, d\nu = -\int_M \langle v, \nabla (u\varphi)\rangle_{\mathbf{g}} d\nu = -\int_M \langle v, \nabla u\rangle_{\mathbf{g}} d\nu.$$

□

Alternative definition of divergence. Let us define the divergence $\operatorname{div} v$ in any chart by

$$\operatorname{div} v = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^k}\left(\sqrt{\det g}\, v^k\right), \tag{3.36}$$

and show by a direct computation that, in the intersection of any two charts, (3.36) defines the same function. This approach allows to avoid integration in the definition of divergence but it is more technically involved (besides, we need integration and Theorem 3.14 anyway).

We will use the following formula: if $a = (a_j^i)$ is a non-singular $n \times n$ matrix smoothly depending on a real parameter t and $(\widetilde{a}_j^i)$ is its inverse (where i is the row index and j is the column index) then

$$\frac{\partial}{\partial t} \log \det a = \widetilde{a}_k^l \frac{\partial a_l^k}{\partial t}. \tag{3.37}$$

In the common domain of two coordinate systems $x^1, ..., x^n$ and $y^1, ..., y^n$, set

$$J_i^k = \frac{\partial y^k}{\partial x^i} \text{ and } \widetilde{J}_k^i = \frac{\partial x^i}{\partial y^k}.$$

Let g be the matrix of the tensor $\mathbf{g}$ and v^i be the components of the vector v in coordinates $x^1, ..., x^n$, and let $\widetilde{g}$ be the matrix of $\mathbf{g}$ and $\widetilde{v}^k$ be the components of the vector v in coordinates $y^1, ..., y^n$. Then we have

$$v = v^i \frac{\partial}{\partial x^i} = v^i \frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k} = v^i J_i^k \frac{\partial}{\partial y^k}$$

so that

$$\widetilde{v}^k = v^i J_i^k.$$

Since by (3.28)

$$\sqrt{\det \widetilde{g}} = \sqrt{\det g}\, |\det J|^{-1},$$

where $J = \left(J_i^k\right)$, the divergence of v in coordinates $y^1, ..., y^n$ is given by

$$\begin{aligned}
\operatorname{div} v &= \frac{1}{\sqrt{\det \widetilde{g}}} \frac{\partial}{\partial y^k}\left(\sqrt{\det \widetilde{g}}\, \widetilde{v}^k\right) = \frac{\det J}{\sqrt{\det g}} \widetilde{J}_k^j \frac{\partial}{\partial x^j}\left(\sqrt{\det g}\, v^i (\det J)^{-1} J_i^k\right) \\
&= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^j}\left(\sqrt{\det g}\, v^i\right) \widetilde{J}_k^j J_i^k + v^i \widetilde{J}_k^j J_i^k \det J \frac{\partial}{\partial x^j} (\det J)^{-1} + v^i \widetilde{J}_k^j \frac{\partial J_i^k}{\partial x^j} \\
&= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, v^i\right) - v^i \frac{\partial}{\partial x^i} \log \det J + v^i \widetilde{J}_k^j \frac{\partial J_i^k}{\partial x^j},
\end{aligned}$$

where we have used the fact that the matrices (J_i^k) and $(\widetilde{J}_k^j)$ are mutually inverse. To finish the proof, it suffices to show that, for any index i,

$$-\frac{\partial}{\partial x^i} \log \det J + \widetilde{J}_k^j \frac{\partial J_i^k}{\partial x^j} = 0. \tag{3.38}$$

By (3.37), we have

$$\frac{\partial}{\partial x^i} \log \det J = \widetilde{J}_k^j \frac{\partial J_j^k}{\partial x^i}.$$

Noticing that

$$\frac{\partial J_j^k}{\partial x^i} = \frac{\partial^2 y^k}{\partial x^j \partial x^i} = \frac{\partial^2 y^k}{\partial x^i \partial x^j} = \frac{\partial J_i^k}{\partial x^j},$$

we obtain (3.38).

3.6. Laplace operator and weighted manifolds

Having defined gradient and divergence, we can now define the *Laplace operator* (called also the *Laplace-Beltrami operator*) on any Riemannian manifold $(M, \mathbf{g})$ as follows:

$$\Delta = \operatorname{div} \circ \nabla.$$

That is, for any smooth function f on M,

$$\Delta f = \operatorname{div}(\nabla f), \tag{3.39}$$

so that Δf is also a smooth function on M. In local coordinates, we have

$$\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x^j} \right), \tag{3.40}$$

where $g = (g_{ij})$.

THEOREM 3.16. (Green formula) *If u and v are smooth functions on a Riemannian manifold M and one of them has a compact support then*

$$\int_M u \Delta v \, d\nu = -\int_M \langle \nabla u, \nabla v \rangle_{\mathbf{g}} d\nu = \int_M v \Delta u \, d\nu. \tag{3.41}$$

PROOF. Consider the vector field ∇v. Clearly, $\operatorname{supp} \nabla v \subset \operatorname{supp} v$ so that either $\operatorname{supp} u$ or $\operatorname{supp} \nabla v$ is compact. By Theorem 3.14, Corollary 3.15, and (3.39), we obtain

$$\int_M u \Delta v \, d\nu = \int_M u \operatorname{div}(\nabla v) \, d\nu = -\int_M \langle \nabla u, \nabla v \rangle_{\mathbf{g}} d\nu.$$

The second identity in (3.41) is proved similarly. □

For example, if $(g_{ij}) \equiv \mathrm{id}$ then also $(g^{ij}) \equiv \mathrm{id}$, and (3.40) takes the form

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{(\partial x^i)^2}.$$

Any smooth positive function $\Upsilon(x)$ on a Riemannian manifold $(M, \mathbf{g})$ gives rise to a measure μ on M given by $d\mu = \Upsilon d\nu$. The function Υ is called the *density function* of the measure μ. For example, the density function of the Riemannian measure ν is 1.

DEFINITION 3.17. A triple $(M, \mathbf{g}, \mu)$ is called a weighted manifold, if $(M, \mathbf{g})$ is a Riemannian manifold and μ is a measure on M with a smooth positive density function.

The definition of gradient on a weighted manifold $(M, \mathbf{g}, \mu)$ is the same as on $(M, \mathbf{g})$, but the definition of divergence changes. For any smooth vector field v on M, define its *weighted divergence* $\operatorname{div}_\mu v$ by

$$\operatorname{div}_\mu v = \frac{1}{\Upsilon} \operatorname{div}(\Upsilon v).$$

It follows immediately from this definition and (3.33) that the following extension of Theorem 3.14 takes place: for all smooth vector fields v and functions u,

$$\int_M \operatorname{div}_\mu v \, u \, d\mu = -\int_M \langle v, \nabla u \rangle_{\mathbf{g}} d\mu, \tag{3.42}$$

provided v or u has a compact support.

Define the *weighted Laplace operator* Δ_μ by

$$\Delta_\mu = \operatorname{div}_\mu \circ \nabla.$$

The Green formulas remain true, that is, if u and v are smooth functions on M and one of them has a compact support then

$$\int_M u \Delta_\mu v \, d\mu = -\int_M \langle \nabla u, \nabla v \rangle_{\mathbf{g}} d\mu = \int_M v \Delta_\mu u \, d\mu. \tag{3.43}$$

In the local coordinates $x^1, ..., x^n$, we have

$$\operatorname{div}_\mu v = \frac{1}{\rho} \frac{\partial}{\partial x^i} \left(\rho v^i\right) \tag{3.44}$$

and

$$\Delta_\mu = \frac{1}{\rho} \frac{\partial}{\partial x^i} \left(\rho g^{ij} \frac{\partial}{\partial x^j}\right). \tag{3.45}$$

where $\rho = \Upsilon \sqrt{\det g}$. Note also that $d\mu = \rho d\lambda$, where λ is the Lebesgue measure in U.

Sometimes is it useful to know that the right hand side of (3.45) can be expanded as follows:

$$\Delta = g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \left(\frac{1}{\rho} \frac{\partial \rho}{\partial x^i} g^{ij} + \frac{\partial g^{ij}}{\partial x^i}\right) \frac{\partial}{\partial x^j}. \tag{3.46}$$

EXAMPLE 3.18. Consider the weighted manifold $(\mathbb{R}, \mathbf{g}, \mu)$ where $\mathbf{g}$ is the canonical Euclidean metric and $d\mu = \Upsilon dx$. Then by (3.45) or (3.46)

$$\Delta_\mu f = \frac{1}{\Upsilon} \frac{d}{dx} \left(\Upsilon \frac{df}{dx}\right) = f'' + \frac{\Upsilon'}{\Upsilon} f'.$$

For example, if $\Upsilon = e^{-x^2}$ then

$$\Delta_\mu f = f'' - 2x f'. \tag{3.47}$$

Exercises.

3.7. *(Product rule for divergence)* Prove that, for any smooth function u and any smooth vector field ω,

$$\operatorname{div}_\mu(u\omega) = \langle \nabla u, \omega\rangle + u \operatorname{div}_\mu \omega \tag{3.48}$$

3.8. *(Product rule for the Laplacian)* Prove that, for any two smooth functions u and v,

$$\Delta_\mu(uv) = u\Delta_\mu v + 2\langle \nabla u, \nabla v\rangle_{\mathbf{g}} + (\Delta_\mu u)\, v. \tag{3.49}$$

3.9. *(Chain rule for the Laplacian)* Prove that

$$\Delta_\mu f(u) = f''(u)\,|\nabla u|^2_{\mathbf{g}} + f'(u)\,\Delta_\mu u,$$

where u and f are smooth functions on M and $\mathbb{R}$, respectively.

3.10. The *Hermite polynomials* $h_k(x)$ are defined by

$$h_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2},$$

where $k = 0, 1, 2, \ldots$. Show that the Hermite polynomials are the eigenfunctions of the operator (3.47).

3.11. Let $a(x)$, $b(x)$ be smooth positive functions on a weighted manifold $(M, \mathbf{g}, \mu)$, and define new metric $\widetilde{\mathbf{g}}$ and measure $\widetilde{\mu}$ by

$$\widetilde{\mathbf{g}} = a\mathbf{g} \quad \text{and} \quad d\widetilde{\mu} = b\,d\mu.$$

Prove that the Laplace operator $\widetilde{\Delta}_{\widetilde{\mu}}$ of the weighted manifold $(M, \widetilde{\mathbf{g}}, \widetilde{\mu})$ is given by

$$\widetilde{\Delta}_{\widetilde{\mu}} = \frac{1}{b} \operatorname{div}_\mu\left(\frac{b}{a}\nabla\right).$$

In particular, if $a = b$ then

$$\widetilde{\Delta}_{\widetilde{\mu}} = \frac{1}{a}\Delta_\mu.$$

3.12. Consider the following operator L on a weighted manifold $(M, \mathbf{g}, \mu)$:

$$Lu = \frac{1}{b}\operatorname{div}_\mu(A\nabla u),$$

where $b = b(x)$ is a smooth positive function on M and $A = A(x)$ is a smooth field of positive definite symmetric operators on T_xM. Prove that L coincides with the Laplace operator $\widetilde{\Delta}_{\widetilde{\mu}}$ of the weighted manifold $(M, \widetilde{\mathbf{g}}, \widetilde{\mu})$ where

$$\widetilde{\mathbf{g}} = b\mathbf{g}A^{-1} \text{ and } d\widetilde{\mu} = b\,d\mu.$$

3.13. Consider the following operator L on a weighted manifold $(M, \mathbf{g}, \mu)$:

$$Lu = \Delta_\mu u + \langle \nabla v, \nabla u\rangle_{\mathbf{g}},$$

where v is a smooth function on M. Prove that $L = \Delta_{\widetilde{\mu}}$ for some measure $\widetilde{\mu}$, and determine this measure.

3.7. Submanifolds

If M is a smooth manifold then any open subset $\Omega \subset M$ trivially becomes a smooth manifold by restricting all charts to Ω. Also, if $\mathbf{g}$ is a Riemannian metric on M then $\mathbf{g}|_\Omega$ is a Riemannian metric on Ω. Hence, any open subset Ω of M can be considered as a (Riemannian) submanifold of a (Riemannian) manifold M of the same dimension.

Consider a more interesting notion of a submanifold of smaller dimension. Any subset S of a smooth manifold M can be regarded as a topological space with induced topology. It is easy to see that S inherits from M the properties of being Hausdorff and having a countable base.

A set $S \subset M$ is called an (embedded) *submanifold* of dimension m if, for any point $x_0 \in S$, there is a chart (U, φ) on M covering x_0 such that the intersection $S \cap U$ is given in U by the equations

$$x^{m+1} = x^{m+2} = ... = x^n = 0,$$

where $x^1, x^2, ..., x^n$ are the local coordinates in U (see Fig. 3.3).

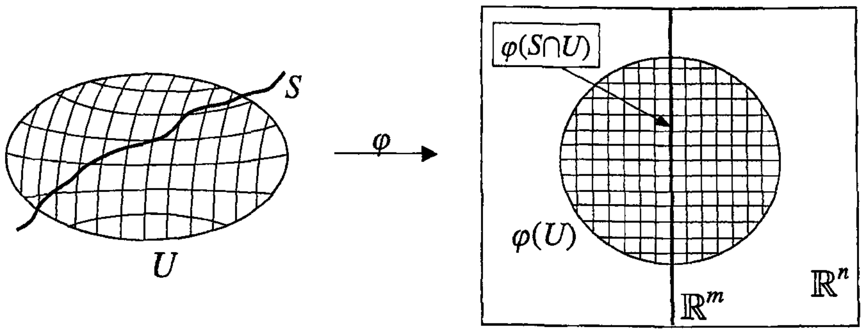

FIGURE 3.3. The image $\varphi(S \cap U)$ lies in $\mathbb{R}^m \subset \mathbb{R}^n$.

In particular, this means that the image $\varphi(U \cap S)$ is contained in the m-dimensional subspace of $\mathbb{R}^n$

$$\left\{x \in \mathbb{R}^n : x^{m+1} = x^{m+2} = ... = x^n = 0\right\},$$

which can be identified with $\mathbb{R}^m$, so that $\varphi|_{U\cap S}$ can be considered as a mapping from $U \cap S$ to $\mathbb{R}^m$. Hence, $(U \cap S, \varphi|_{U\cap S})$ is a m-dimensional chart on S (with the coordinates $x^1, x^2, ..., x^m$). With the atlas of all such charts, *the submanifold S is a smooth m-dimensional manifold.*

Let ξ be an $\mathbb{R}$-differentiation on S at a point $x_0 \in S$. For any smooth function f on M, its restriction $f|_S$ is a smooth function on S. Hence, setting

$$\xi(f) := \xi(f|_S), \tag{3.50}$$

we see that ξ can be extended to an $\mathbb{R}$-differentiation on M at the same point x_0. Therefore, (3.50) provides a natural identification of $T_{x_0}S$ as a subspace of $T_{x_0}M$.

Let $(M, \mathbf{g})$ be a Riemannian manifold. If $x_0 \in S$ then by restricting the tensor $\mathbf{g}$ in $T_{x_0}M$ to the subspace $T_{x_0}S$, we obtain the Riemannian metric $\mathbf{g}_S$ on S, which is called the *induced metric*.

Let $(M, \mathbf{g}, \mu)$ be a weighted manifold and Υ be the density function of measure μ. Define the *induced measure* μ_S on S by the condition that μ_S has the density function $\Upsilon|_S$ with respect to the Riemannian measure of $\mathbf{g}_S$. Hence, we obtain the weighted manifold $(S, \mathbf{g}_S, \mu_S)$. If $\dim S = n-1$ then the measure μ_S is also referred to as *area* as opposed to the n-dimensional measure μ, which in this context is called *volume*.

LEMMA 3.19. *Let M be a smooth manifold of dimension n and $F: M \to \mathbb{R}$ be a smooth function on M. Consider the null set of F, that is*

$$N = \{x \in M : F(x) = 0\}.$$

If

$$dF \neq 0 \text{ on } N \tag{3.51}$$

then N is a submanifold of dimension $n-1$.

PROOF. For any point $x_0 \in N$, there is a chart U on M containing x_0 and such that $dF \neq 0$ in U. This means that the row-vector $\left(\frac{\partial F}{\partial x^i}\right)$ does not vanish in U. By the implicit function theorem, there exists an open set $V \subset U$ containing x_0 and an index $i \in \{1, ..., n\}$ such that the equation $F(x) = 0$ in V can be resolved with respect to the coordinate x^i; that is, the equation $F(x) = 0$ is equivalent in V to

$$x^i = f(x^1, ... \overset{i}{\checkmark} ..., x^n),$$

where f is a smooth function and the sign $\overset{i}{\checkmark}$ means that the coordinate x^i is omitted from the list.

For simplicity of notation, set $i = n$ so that the equation of set N in V becomes

$$x^n = f\left(x^1, ..., x^{n-1}\right).$$

After the change of coordinates

$$\begin{aligned} y^1 &= x^1, \\ &\;\;\; ... \\ y^{n-1} &= x^{n-1}, \\ y^n &= x^n - f\left(x^1, ..., x^{n-1}\right), \end{aligned}$$

the equation of N in V becomes $y^n = 0$ and hence N is a $(n-1)$-dimensional submanifold. ☐

EXAMPLE 3.20. Consider in $\mathbb{R}^{n+1}$ the following equation

$$\left(x^1\right)^2 + ... + \left(x^{n+1}\right)^2 = 1,$$

which defines the unit sphere $\mathbb{S}^n$. Since $\mathbb{S}^n$ is the null set of the function

$$F(x) = \left(x^1\right)^2 + ... + \left(x^{n+1}\right)^2 - 1,$$

whose differential $dF = (2x^1, ..., 2x^{n+1})$ does not vanish on $\mathbb{S}^n$, we conclude that $\mathbb{S}^n$ is a submanifold of $\mathbb{R}^{n+1}$ of dimension n. Furthermore, considering $\mathbb{R}^{n+1}$ as a Riemannian manifold with the canonical Euclidean metric, we see that $\mathbb{S}^n$ can be regarded as Riemannian manifold with the induced metric, which is called the *canonical spherical metric* and is denoted by $\mathbf{g}_{\mathbb{S}^n}$.

Exercises.

3.14. Let M be a smooth manifold of dimension n and N be its submanifold of dimension $n-1$ given by the equation $F(x) = 0$ where F is a smooth function on M such that $dF \neq 0$ on N. Prove that, for any $x \in N$, the tangent space T_xN is determined as a subspace of T_xM by the equation

$$T_xN = \{\xi \in T_xM : \langle dF, \xi \rangle = 0\}. \tag{3.52}$$

In the case when $M = \mathbb{R}^n$, show that the tangent space T_xN can be naturally identified with the hyperplane in $\mathbb{R}^n$ that goes through x and has the normal

$$\nabla F = \left(\frac{\partial F}{\partial x^1}, ..., \frac{\partial F}{\partial x^n}\right).$$

In other words, the tangent space T_xN is identified with the tangent hyperplane to the hypersurface N at the point x.

3.8. Product manifolds

Let X, Y be smooth manifolds of dimensions n and m, respectively, and let $M = X \times Y$ be the direct product of X and Y as topological spaces. The space M consists of the couples (x, y) where $x \in X$ and $y \in Y$, and it can be naturally endowed with a structure of a smooth manifold. Indeed, if U and V are charts on X and Y respectively, with the coordinates $x^1, ..., x^n$ and $y^1, ..., y^m$ then $U \times V$ is a chart on M with the coordinates $x^1, ..., x^n, y^1, ..., y^m$. The atlas of all such charts makes M into a smooth manifold.

For any point $(x, y) \in M$, the tangent space $T_{(x,y)}M$ is naturally identified as the direct sum $T_xX \oplus T_yY$ of the linear spaces. Indeed, fix a point $(x, y) \in M$. Any $\mathbb{R}$-differentiation $\xi \in T_xX$ can be considered as an $\mathbb{R}$-differentiation on functions $f(x, y)$ on M by freezing the variable y, that is

$$\xi(f) = \xi(f(\cdot, y)).$$

This identifies T_xX as a subspace of $T_{(x,y)}M$, and the same applied to T_yY. Let us show that the intersection of T_xX and T_yY in $T_{(x,y)}M$ is $\{0\}$. Indeed, if $\xi \in T_xX \cap T_yY$ then, for some vectors $a \in T_xX$ and $b \in T_yY$ and all $f \in C^\infty(M)$,

$$\xi(f) = a(f(\cdot, y)) = b(f(x, \cdot)),$$

whence it follows that

$$a^i \frac{\partial f}{\partial x^i}(x, y) \equiv b^j \frac{\partial f}{\partial y^j}(x, y),$$

which is not possible for all f, unless all $a^i = b^j = 0$. Since $\dim T_x X = n$, $\dim T_y Y = m$, and $\dim T_{(x,y)} = n + m$, we conclude that

$$T_{(x,y)}M = T_x X \oplus T_y Y. \tag{3.53}$$

If $\mathbf{g}_X$ and $\mathbf{g}_Y$ are Riemannian metric tensors on X and Y, respectively, then define the metric tensor $\mathbf{g}$ on M as the *direct sum*

$$\mathbf{g} = \mathbf{g}_X + \mathbf{g}_Y. \tag{3.54}$$

Namely, for any $(x, y) \in M$, any vector $\xi \in T_{(x,y)}M$ uniquely splits into the sum

$$\xi = \xi_X + \xi_Y,$$

where $\xi_X \in T_x X$ and $\xi_Y \in T_y Y$; then set

$$(\mathbf{g}_X + \mathbf{g}_Y)(x, y)(\xi, \eta) = \mathbf{g}_X(x)(\xi_X, \eta_X) + \mathbf{g}_Y(y)(\xi_Y, \eta_Y).$$

In the local coordinates $x^1, ..., x^n, y^1, ..., y^m$, we have

$$\mathbf{g}_X + \mathbf{g}_Y = (g_X)_{ij}\, dx^i dx^j + (g_Y)_{kl}\, dy^k dy^l.$$

The manifold $(M, \mathbf{g})$ is called the *Riemannian* (or direct) *product* of $(X, \mathbf{g}_X)$ and $(Y, \mathbf{g}_Y)$.

Note that the matrix g of the metric tensor $\mathbf{g}$ has the block form

$$g = \begin{pmatrix} \boxed{g_X} & 0 \\ 0 & \boxed{g_Y} \end{pmatrix},$$

which implies a similar form for g^{-1} and

$$\det g = \det g_X \det g_Y.$$

If ν_X and ν_Y are the Riemannian measures on X and Y, respectively, then the Riemannian measure ν of M is given by

$$d\nu = \sqrt{\det g}\, dx^1 ... dx^n dy^1 ... dy^m = d\nu_X d\nu_Y.$$

Hence, ν is the product of measures ν_X and ν_Y, that is,

$$\nu = \nu_X \times \nu_Y$$

(see Section A.4.6 for the definition of products of measures).

Denoting by Δ_X and Δ_Y the Laplace operator on X and Y, respectively, and by $z^1, ..., z^{n+m}$ the coordinates $x^1, ..., x^n, y^1, ..., y^m$, we obtain the following expression of the Laplace operator Δ on M:

$$\begin{aligned}
\Delta &= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial z^i} \left(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial z^j} \right) \\
&= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g}\, g_X^{ij} \frac{\partial}{\partial x^j} \right) + \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial y^i} \left(\sqrt{\det g}\, g_Y^{ij} \frac{\partial}{\partial y^j} \right) \\
&= \frac{1}{\sqrt{\det g_X}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g_X}\, g_X^{ij} \frac{\partial}{\partial x^j} \right) + \frac{1}{\sqrt{\det g_Y}} \frac{\partial}{\partial y^i} \left(\sqrt{\det g_Y}\, g_Y^{ij} \frac{\partial}{\partial y^j} \right),
\end{aligned}$$

that is,

$$\Delta = \Delta_X + \Delta_Y.$$

Let $(X, \mathbf{g}_X, \mu_X)$ and $(Y, \mathbf{g}_Y, \mu_Y)$ be weighted manifold. Setting $M = X \times Y$, $\mathbf{g} = \mathbf{g}_X + \mathbf{g}_Y$ and $\mu = \mu_X \times \mu_Y$, we obtain a weighted manifold $(M, \mathbf{g}, \mu)$, which is called the direct product of weighted manifolds $(X, \mathbf{g}_X, \mu_X)$ and $(Y, \mathbf{g}_Y, \mu_Y)$. A computation similar to the above shows that

$$\Delta_\mu = \Delta_{\mu_X} + \Delta_{\mu_Y}.$$

There are other possibility to define a Riemannian tensor $\mathbf{g}$ on the product manifold $M = X \times Y$. For example, if $\psi(x)$ is a smooth positive function on X then consider the metric

$$\mathbf{g} = \mathbf{g}_X + \psi^2(x)\, \mathbf{g}_Y. \tag{3.55}$$

The Riemannian manifold $(M, \mathbf{g})$ with this metric is called a *warped product* of $(X, \mathbf{g}_X)$ and $(Y, \mathbf{g}_Y)$. In the local coordinates, we have

$$\mathbf{g} = (g_X)_{ij}\, dx^i dx^j + \psi^2(x)\, (g_Y)_{kl}\, dy^k dy^l.$$

Exercises.

3.15. Prove that the Riemannian measure ν of the metric (3.55) is given by

$$d\nu = \psi^m(x)\, d\nu_X d\nu_Y, \tag{3.56}$$

and the Laplace operator Δ of this metric is given by

$$\Delta f = \Delta_X f + m \langle \nabla_X \log \psi, \nabla_X f \rangle_{\mathbf{g}_X} + \frac{1}{\psi^2(x)} \Delta_Y f, \tag{3.57}$$

where ∇_X is gradient on X.

3.9. Polar coordinates in $\mathbb{R}^n, \mathbb{S}^n, \mathbb{H}^n$

Euclidean space. In $\mathbb{R}^n$, $n \geq 2$, every point $x \neq o$ can be represented in the *polar coordinates* as a couple (r, θ) where $r := |x| > 0$ is the *polar radius* and $\theta := \frac{x}{|x|} \in \mathbb{S}^{n-1}$ is the *polar angle.*

CLAIM. *The canonical Euclidean metric $\mathbf{g}_{\mathbb{R}^n}$ has the following representation in the polar coordinates:*

$$\mathbf{g}_{\mathbb{R}^n} = dr^2 + r^2 \mathbf{g}_{\mathbb{S}^{n-1}}, \tag{3.58}$$

where $\mathbf{g}_{\mathbb{S}^{n-1}}$ is the canonical spherical metric.

PROOF. Let $\theta^1, ..., \theta^{n-1}$ be local coordinates on $\mathbb{S}^{n-1}$ and let

$$\mathbf{g}_{\mathbb{S}^{n-1}} = \gamma_{ij} d\theta^i d\theta^j. \tag{3.59}$$

Then $r, \theta^1, ..., \theta^{n-1}$ are local coordinates on $\mathbb{R}^n$, and (3.58) means that

$$\mathbf{g}_{\mathbb{R}^n} = dr^2 + r^2 \gamma_{ij} d\theta^i d\theta^j. \tag{3.60}$$

We start with the identity $x = r\theta$, which implies that the Cartesian coordinates $x^1, ..., x^n$ can be expressed via the polar coordinates $r, \theta^1, ..., \theta^{n-1}$ as follows:

$$x^i = r f^i\left(\theta^1, ..., \theta^{n-1}\right), \tag{3.61}$$

where f^i is the x^i-coordinate in $\mathbb{R}^n$ of the point $\theta \in \mathbb{S}^{n-1}$. Clearly, $f^1, ..., f^n$ are smooth functions of $\theta^1, ..., \theta^{n-1}$ and

$$\left(f^1\right)^2 + ... + \left(f^n\right)^2 \equiv 1. \tag{3.62}$$

Applying differential d to x^i and using the product rule for d, we obtain

$$dx^i = f^i dr + r df^i,$$

whence, taking the tensor product,

$$\left(dx^i\right)^2 = \left(f^i\right)^2 dr^2 + (rdr)\left(f^i df^i\right) + \left(f^i df^i\right)(rdr) + r^2\left(df^i\right)^2.$$

Summing up these identities for all i and using (3.62) and its consequence

$$\sum_i f^i df^i = 0, \tag{3.63}$$

we obtain

$$\mathbf{g}_{\mathbb{R}^n} = \sum_i \left(dx^i\right)^2 = dr^2 + r^2 \sum_i \left(df^i\right)^2.$$

Clearly, we have

$$\left(df^i\right)^2 = \left(\frac{\partial f^i}{\partial \theta^j} d\theta^j\right)^2 = \frac{\partial f^i}{\partial \theta^j}\frac{\partial f^i}{\partial \theta^k} d\theta^j d\theta^k,$$

which implies that the sum $\sum_i \left(df^i\right)^2$ can be represented in the form

$$\sum_i \left(df^i\right)^2 = \gamma_{jk} d\theta^j d\theta^k, \tag{3.64}$$

where γ_{jk} are smooth functions of $\theta^1, ..., \theta^{n-1}$. Hence, we obtain the identity (3.60).

We are left to verify that $\gamma_{ij} d\theta^i d\theta^j$ is the canonical spherical metric. Indeed, the metric $\mathbf{g}_{\mathbb{S}^{n-1}}$ is obtained by restricting the metric $\mathbf{g}_{\mathbb{R}^n}$ to $\mathbb{S}^{n-1}$. On $\mathbb{S}^{n-1}$ we have the coordinates $\theta^1, ..., \theta^{n-1}$ while $r = 1$ and $dr = 0$. Indeed, for any $\xi \in T_x\mathbb{S}^{n-1}$, we have

$$\langle dr, \xi\rangle = \xi(r) = \xi\left(r|_{\mathbb{S}^{n-1}}\right) = \xi(1) = 0.$$

Therefore, substituting in (3.60) $r = 1$ and $dr = 0$, we obtain (3.59). □

Sphere. Consider now the polar coordinates on $\mathbb{S}^n$. Let p be the north pole of $\mathbb{S}^n$ and q be the south pole of $\mathbb{S}^n$ (that is, p is the point $(0, ..., 0, 1)$ in $\mathbb{R}^{n+1}$ and $q = -p$). For any point $x \in \mathbb{S}^n \setminus \{p, q\}$, define $r \in (0, \pi)$ and $\theta \in \mathbb{S}^{n-1}$ by

$$\cos r = x^{n+1} \text{ and } \theta = \frac{x'}{|x'|}, \tag{3.65}$$

where x' is the projection of x onto $\mathbb{R}^n = \left\{x \in \mathbb{R}^{n+1} : x^{n+1} = 0\right\}$. Clearly, the polar radius r is the angle between the position vectors of x and p, and r can be regarded as the latitude of the point x measured from the pole. The polar angle θ can be regarded as the longitude of the point x (see Fig. 3.4).

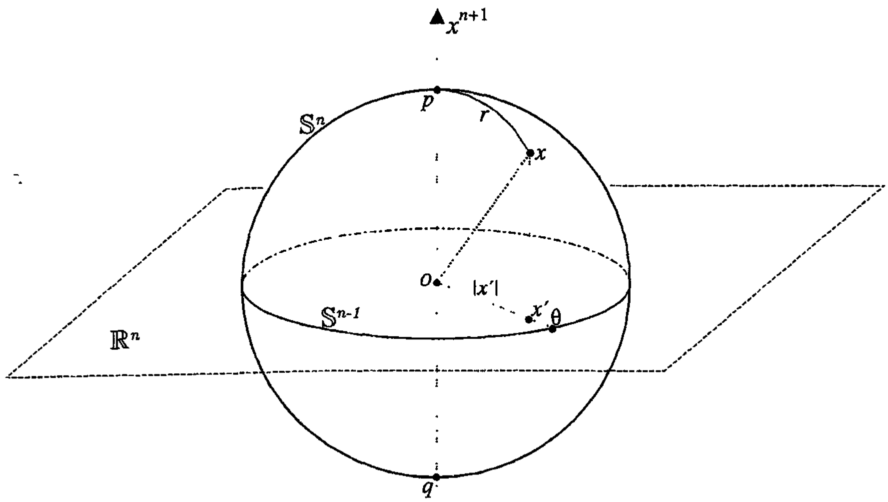

FIGURE 3.4. Polar coordinates on $\mathbb{S}^n$

CLAIM. *The canonical spherical metric* $\mathbf{g}_{\mathbb{S}^n}$ *has the following representation in the polar coordinates:*

$$\mathbf{g}_{\mathbb{S}^n} = dr^2 + \sin^2 r \mathbf{g}_{\mathbb{S}^{n-1}}. \tag{3.66}$$

PROOF. Let $\theta^1, ..., \theta^{n-1}$ are local coordinates on $\mathbb{S}^{n-1}$ and let us write down the metric $\mathbf{g}_{\mathbb{S}^n}$ in the local coordinates $r, \theta^1, ..., \theta^{n-1}$. Obviously, for any point $x \in \mathbb{S}^n \setminus \{p, q\}$, we have $|x'| = \sin r$ whence $x' = (\sin r)\, \theta$. Hence, the Cartesian coordinates $x^1, ..., x^{n+1}$ of the point x can be expressed as follows:

$$\begin{aligned} x^i &= \sin r f^i\left(\theta^1, ..., \theta^{n-1}\right),\ i = 1, ..., n, \\ x^{n+1} &= \cos r, \end{aligned}$$

where f^i are the same functions as in (3.61). Therefore, we obtain using (3.62), (3.63), and (3.64),

$$\begin{aligned}
&\left(dx^1\right)^2 + ... + (dx^n)^2 + \left(dx^{n+1}\right)^2 \\
= &\sum_{i=1}^n \left(f^i \cos r dr + \sin r df^i\right)^2 + \sin^2 r dr^2 \\
= &\sum_{i=1}^n \left(\left(f^i\right)^2 \cos^2 r dr^2 + \sin^2 r \left(df^i\right)^2\right) \\
&+ \sum_{i=1}^n (\sin r \cos r dr)\left(f^i df^i\right) + \sum_{i=1}^n \left(f^i df^i\right)(\sin r \cos r dr) \\
&+ \sin^2 r dr^2 \\
= &\cos^2 r dr^2 + \sin^2 r \sum_{i=1}^n \left(df^i\right)^2 + \sin^2 r dr^2 \\
= &dr^2 + \sin^2 r \gamma_{ij} d\theta^i d\theta^j .
\end{aligned}$$

Since we already know that $\gamma_{ij} d\theta^i d\theta^j$ is the canonical metric on $\mathbb{S}^{n-1}$, we obtain (3.66). $\square$

Hyperbolic space. The *hyperbolic space* $\mathbb{H}^n$, $n \geq 2$, is defined as follows. Consider in $\mathbb{R}^{n+1}$ a *hyperboloid* H given by the equation[3]

$$\left(x^{n+1}\right)^2 - \left(x'\right)^2 = 1, \tag{3.67}$$

where $x' = \left(x^1, ..., x^n\right) \in \mathbb{R}^n$ and $x^{n+1} > 0$. By Lemma 3.19, H is a submanifold of $\mathbb{R}^{n+1}$ of dimension n.

Consider in $\mathbb{R}^{n+1}$ the *Minkowski metric*

$$\mathbf{g}_{Mink} = \left(dx^1\right)^2 + ... + (dx^n)^2 - \left(dx^{n+1}\right)^2, \tag{3.68}$$

which is a bilinear symmetric form in any tangent space $T_x\mathbb{R}^{n+1}$ but not positive definite (so, $\mathbf{g}_{Mink}$ is not a Riemannian metric, but is a *pseudo-*Riemannian metric). Let $\mathbf{g}_H$ be the restriction of the tensor $\mathbf{g}_{Mink}$ to H. We will prove below that $\mathbf{g}_H$ is positive definite so that $(H, \mathbf{g}_H)$ is a Riemannian manifold. By definition, this manifold is called the *hyperbolic space* and is denoted by $\mathbb{H}^n$, and the metric $\mathbf{g}_H$ is called the *canonical hyperbolic metric* and is denoted also by $\mathbf{g}_{\mathbb{H}^n}$.

Our main purpose here is to introduce the polar coordinates in $\mathbb{H}^n$ and to represent $\mathbf{g}_{\mathbb{H}^n}$ in the polar coordinates. As a by-product, we will see that $\mathbf{g}_{\mathbb{H}^n}$ is positive definite.

Let p be the pole of $\mathbb{H}^n$, that is $p = (0, ..., 0, 1) \in \mathbb{R}^{n+1}$. For any point $x \in \mathbb{H}^n \setminus \{p\}$, define $r > 0$ and $\theta \in \mathbb{S}^{n-1}$ by

$$\cosh r = x^{n+1} \quad \text{and} \quad \theta = \frac{x'}{|x'|} \tag{3.69}$$

[3]For comparison, the equation of $\mathbb{S}^n$ can be written in the form $\left(x^{n+1}\right)^2 + \left(x'\right)^2 = 1$.

(see Fig. 3.5).

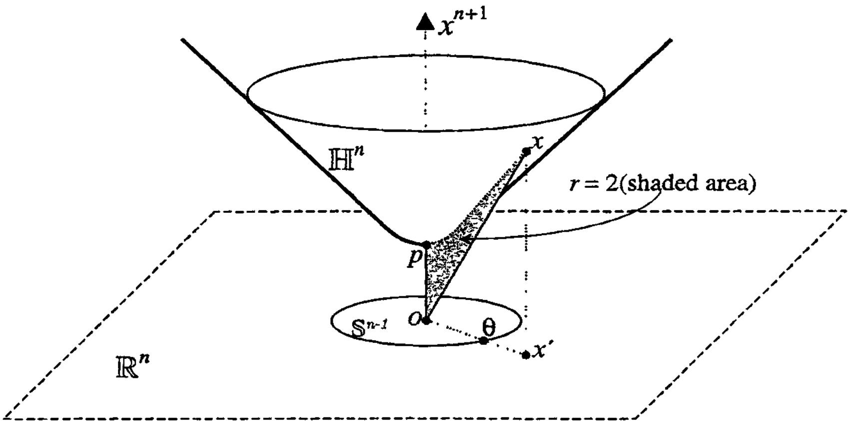

FIGURE 3.5. Polar coordinates on $\mathbb{H}^n$

CLAIM. *The canonical hyperbolic metric $\mathbf{g}_{\mathbb{H}^n}$ has the following representation in the polar coordinates:*

$$\mathbf{g}_{\mathbb{H}^n} = dr^2 + \sinh^2 r \mathbf{g}_{\mathbb{S}^{n-1}}. \tag{3.70}$$

In particular, we see from (3.70) that the tensor $\mathbf{g}_{\mathbb{H}^n}$ is positive definite on $T_x\mathbb{H}^n$ for any $x \in \mathbb{H}^n \setminus \{p\}$. The fact that $\mathbf{g}_{\mathbb{H}^n}$ is positive definite on $T_p\mathbb{H}^n$ follows directly from (3.68) because $dx^{n+1} = 0$ on $T_p\mathbb{H}^n$.

PROOF. Let $\theta^1, ..., \theta^{n-1}$ be local coordinates on $\mathbb{S}^{n-1}$ and let us write down the metric $\mathbf{g}_{\mathbb{H}^n}$ in the local coordinates $r, \theta^1, ..., \theta^{n-1}$. For any point $x \in \mathbb{H}^n \setminus \{p\}$, we have

$$|x'| = \sqrt{|x^{n+1}|^2 - 1} = \sqrt{\cosh^2 r - 1} = \sinh r,$$

whence $x' = (\sinh r)\,\theta$. Hence, the Cartesian coordinates $x^1, ..., x^{n+1}$ of the point x can be expressed as follows:

$$\begin{aligned} x^i &= \sinh r f^i\left(\theta^1, ..., \theta^{n-1}\right), \ i = 1, ..., n, \\ x^{n+1} &= \cosh r, \end{aligned}$$

where f^i are the same functions as in (3.61). It follows that

$$\begin{aligned}
& \left(dx^1\right)^2 + ... + (dx^n)^2 - \left(dx^{n+1}\right)^2 \\
= & \sum_{i=1}^{n} \left(f^i \cosh r dr + \sinh r df^i\right)^2 - \sinh^2 r dr^2 \\
= & \sum_{i=1}^{n} \left(\left(f^i\right)^2 \cosh^2 r dr^2 + \sinh^2 r \left(df^i\right)^2\right) \\
& + \sum_{i=1}^{n} (\sinh r \cosh r dr) \left(f^i df^i\right) + \sum_{i=1}^{n} \left(f^i df^i\right) (\sinh r \cosh r dr) \\
& - \sinh^2 r dr^2 \\
= & \cosh^2 r dr^2 + \sinh^2 r \sum_{i=1}^{n} \left(df^i\right)^2 - \sinh^2 r dr^2 \\
= & dr^2 + \sinh^2 r \gamma_{ij} d\theta^i d\theta^j .
\end{aligned}$$

Since $\gamma_{ij} d\theta^i d\theta^j$ is the canonical metric on $\mathbb{S}^{n-1}$, we obtain (3.70). □

Exercises.

3.16. Let q be the south pole of $\mathbb{S}^n$. For any point $x \in \mathbb{S}^n \setminus \{q\}$, its stereographic projection is the point y at the subspace

$$\mathbb{R}^n = \left\{z \in \mathbb{R}^{n+1} : z^{n+1} = 0\right\},$$

which belongs to the straight line through x and q. Show that the stereographic projection is a bijection $x \leftrightarrow y$ between $\mathbb{S}^n \setminus \{q\}$ and $\mathbb{R}^n$ given by

$$y = \frac{x'}{x^{n+1} + 1},$$

where $x = \left(x^1, ..., x^{n+1}\right)$ and $x' = \left(x^1, ..., x^n\right)$. Prove that, in the Cartesian coordinates $y^1, ..., y^n$, the canonical spherical metric has the form

$$\mathbf{g}_{\mathbb{S}^n} = \frac{4}{\left(1 + |y|^2\right)^2} \mathbf{g}_{\mathbb{R}^n},$$

where $|y|^2 = \sum \left(y^i\right)^2$ and $\mathbf{g}_{\mathbb{R}^n} = \left(dy^1\right)^2 + ... + (dy^n)^2$ is the canonical Euclidean metric.

3.17. Prove that the canonical hyperbolic metric $\mathbf{g}_{\mathbb{H}^n}$ is positive definite using directly the definition of $\mathbf{g}_{\mathbb{H}^n}$ as the restriction of the Minkowski metric to the hyperboloid.

3.18. Show that the equation

$$y = \frac{x'}{x^{n+1} + 1} \tag{3.71}$$

determines a bijection of the hyperboloid $\mathbb{H}^n$ onto the unit ball $\mathbb{B}^n = \{|y| < 1\}$ in $\mathbb{R}^n$. Prove that, in the Cartesian coordinates $y^1, ..., y^n$ in $\mathbb{B}^n$, the canonical hyperbolic metric has the form

$$\mathbf{g}_{\mathbb{H}^n} = \frac{4}{\left(1 - |y|^2\right)^2} \mathbf{g}_{\mathbb{R}^n}, \tag{3.72}$$

where $|y|^2 = \sum \left(y^i\right)^2$ and $\mathbf{g}_{\mathbb{R}^n} = \left(dy^1\right)^2 + ... + (dy^n)^2$ is the canonical Euclidean metric.

REMARK. The ball $\mathbb{B}^n$ with the metric (3.72) is called the *Poincaré model* of the hyperbolic space. Representation of the metric $\mathbf{g}_{\mathbb{H}^n}$ in this form gives yet another proof of its positive definiteness.

3.19. Prove that the relation between the polar coordinates (r,θ) in $\mathbb{H}^n$ and the coordinates $y^1,...,y^n$ in the Poincaré model of Exercise 3.18 are given by

$$\cosh r = \frac{1+|y|^2}{1-|y|^2} \text{ and } \theta = \frac{y}{|y|}.$$

3.10. Model manifolds

DEFINITION 3.21. An n-dimensional Riemannian manifold $(M,\mathbf{g})$ is called a *Riemannian model* if the following two conditions are satisfied:

(1) There is a chart on M that covers all M, and the image of this chart in $\mathbb{R}^n$ is a ball
$$B_{r_0} := \{x \in \mathbb{R}^n : |x| < r_0\}$$
of radius $r_0 \in (0,+\infty]$ (in particular, if $r_0 = \infty$ then $B_{r_0} = \mathbb{R}^n$).

(2) The metric $\mathbf{g}$ in the polar coordinates (r,θ) in the above chart has the form
$$\mathbf{g} = dr^2 + \psi^2(r)\,\mathbf{g}_{\mathbb{S}^{n-1}}, \tag{3.73}$$
where $\psi(r)$ is a smooth positive function on $(0,r_0)$.

The number r_0 is called the radius of the model M.

To simplify the terminology and notation, we usually identify a model M with the ball B_{r_0}. Then the polar coordinates (r,θ) are defined in $M\setminus\{o\}$ where o is the origin of $\mathbb{R}^n$. If $\theta^1,...,\theta^{n-1}$ are the local coordinates on $\mathbb{S}^{n-1}$ and
$$\mathbf{g}_{\mathbb{S}^{n-1}} = \gamma_{ij}d\theta^i d\theta^j,$$
then $r,\theta^1,...,\theta^{n-1}$ are local coordinates on $M\setminus\{o\}$, and (3.73) is equivalent to
$$\mathbf{g} = dr^2 + \psi^2(r)\,\gamma_{ij}d\theta^i d\theta^j. \tag{3.74}$$
Observe also that away from a neighborhood of o, $\psi(r)$ may be any smooth positive function. However, $\psi(r)$ should satisfy certain conditions near o to ensure that the metric (3.73) extends smoothly to o (see [**133**]).

In some cases, the polar coordinates on a Riemannian manifold can be used to identify this manifold or its part as a model. For example, the results of Section 3.9 imply the following:

- $\mathbb{R}^n$ is a model with the radius $r_0 = \infty$ and $\psi(r) = r$;
- $\mathbb{S}^n$ without a pole is a model with the radius $r_0 = \pi$ and $\psi(r) = \sin r$;
- $\mathbb{H}^n$ is a model with the radius $r_0 = \infty$ and $\psi(r) = \sinh r$.

The following statement is a particular case of Exercise 3.15.

LEMMA 3.22. *On a model manifold $(M,\mathbf{g})$ with metric (3.73), the Riemannian measure ν is given in the polar coordinates by*
$$d\nu = \psi(r)^{n-1}\,dr d\theta, \tag{3.75}$$

where $d\theta$ stands for the Riemannian measure on $\mathbb{S}^{n-1}$, and the Laplace operator on $(M, \mathbf{g})$ has the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \left(\frac{d}{dr}\log\psi^{n-1}\right)\frac{\partial}{\partial r} + \frac{1}{\psi^2(r)}\Delta_{\mathbb{S}^{n-1}}. \tag{3.76}$$

PROOF. Let $g = (g_{ij})$ be the matrix of the tensor $\mathbf{g}$ in coordinates $r, \theta^1, ..., \theta^{n-1}$. For simplicity of notation, set $\theta^0 = r$ and assume that the indices i, j vary from 0 to $n-1$. It follows from (3.74) that

$$g = \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & \psi^2(r)\gamma_{ij} \\ 0 & \end{array}\right) \tag{3.77}$$

and

$$(g^{ij}) = g^{-1} = \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & \psi^{-2}(r)\gamma_{ij} \\ 0 & \end{array}\right). \tag{3.78}$$

In particular, we have

$$\det g = \psi^{2(n-1)}\det\gamma, \tag{3.79}$$

where $\gamma = (\gamma_{ij})$, which implies (3.75).

Using representation (3.40) of Δ in local coordinates, that is,

$$\Delta = \frac{1}{\sqrt{\det g}}\sum_{i,j=0}^{n-1}\frac{\partial}{\partial\theta^i}\left(\sqrt{\det g}\,g^{ij}\frac{\partial}{\partial\theta^j}\right), \tag{3.80}$$

and that $g^{00} = 1$, $g^{0i} = 0$ for $i \geq 1$, we obtain

$$\Delta = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial r}\left(\sqrt{\det g}\frac{\partial}{\partial r}\right) + \sum_{i,j=1}^{n-1}\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial\theta^i}\left(\sqrt{\det g}\,g^{ij}\frac{\partial}{\partial\theta^j}\right). \tag{3.81}$$

Applying (3.78) and (3.79) and noticing that ψ depends only on r and γ_{ij} depend only on $\theta^1, ..., \theta^{n-1}$, we obtain

$$\begin{aligned}\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial r}\left(\sqrt{\det g}\frac{\partial}{\partial r}\right) &= \frac{\partial^2}{\partial r^2} + \left(\frac{\partial}{\partial r}\log\sqrt{\det g}\right)\frac{\partial}{\partial r} \\ &= \frac{\partial^2}{\partial r^2} + \left(\frac{d}{dr}\log\psi^{n-1}\right)\frac{\partial}{\partial r}\end{aligned}$$

and

$$\begin{aligned}\sum_{i,j=1}^{n-1}\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial\theta^i}\left(\sqrt{\det g}\,g^{ij}\frac{\partial}{\partial\theta^j}\right) &= \sum_{i,j=1}^{n-1}\frac{\psi^{-2}(r)}{\sqrt{\det\gamma}}\frac{\partial}{\partial\theta^i}\left(\sqrt{\det\gamma}\,\gamma^{ij}\frac{\partial}{\partial\theta^j}\right) \\ &= \frac{1}{\psi^2(r)}\Delta_{\mathbb{S}^{n-1}}.\end{aligned}$$

Substituting into (3.81), we obtain (3.76). $\square$

EXAMPLE 3.23. In $\mathbb{R}^n$, we have $\psi(r) = r$ and, hence,

$$d\nu = r^{n-1} dr d\theta \tag{3.82}$$

and

$$\Delta_{\mathbb{R}^n} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}}. \tag{3.83}$$

In $\mathbb{S}^n$, we have $\psi(r) = \sin r$ and, hence,

$$d\nu = \sin^{n-1} r \, dr d\theta$$

and

$$\Delta_{\mathbb{S}^n} = \frac{\partial^2}{\partial r^2} + (n-1)\cot r \frac{\partial}{\partial r} + \frac{1}{\sin^2 r}\Delta_{\mathbb{S}^{n-1}}. \tag{3.84}$$

In $\mathbb{H}^n$, we have $\psi(r) = \sinh r$ and, hence,

$$d\nu = \sinh^{n-1} r \, dr d\theta$$

and

$$\Delta_{\mathbb{H}^n} = \frac{\partial^2}{\partial r^2} + (n-1)\coth r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}}. \tag{3.85}$$

The formula (3.84) can be iterated in dimension to obtain a full expansion of $\Delta_{\mathbb{S}^n}$ in the polar coordinates (see Exercise 3.22).

Consider now a *weighted model* $(M, \mathbf{g}, \mu)$ where $(M, \mathbf{g})$ is a Riemannian model, and measure μ has the density function $\Upsilon(r)$, which depends only on r. Setting

$$\sigma(r) = \Upsilon(r)\psi^{n-1}(r),$$

we obtain from Lemma 3.22

$$d\mu = \sigma(r) \, dr d\theta \tag{3.86}$$

and

$$\Delta_\mu = \frac{\partial^2}{\partial r^2} + \frac{\sigma'}{\sigma}\frac{\partial}{\partial r} + \frac{1}{\psi^2(r)}\Delta_{\mathbb{S}^{n-1}}. \tag{3.87}$$

Let ω_n be the full Riemannian measure of $\mathbb{S}^{n-1}$, that is

$$\omega_n = \int_{\mathbb{S}^{n-1}} d\theta. \tag{3.88}$$

Then it follows from (3.86) that, for any $R \in (0, r_0)$,

$$\mu(B_R) = \omega_n \int_0^R \sigma(r) \, dr. \tag{3.89}$$

For example, in $\mathbb{R}^n$ we have $\sigma(r) = r^{n-1}$ and

$$\mu(B_R) = \frac{\omega_n}{n} R^n. \tag{3.90}$$

The function $R \mapsto \mu(B_R)$ is called the *volume function* of the model manifold. Define the *area function* $S(r)$ by

$$S(r) := \omega_n \sigma(r) = \omega_n \Upsilon(r)\psi^{n-1}(r). \tag{3.91}$$

It obviously follows from (3.89) and (3.87) that

$$\mu(B_R) = \int_0^R S(r)\,dr \tag{3.92}$$

and

$$\Delta_\mu = \frac{\partial^2}{\partial r^2} + \frac{S'(r)}{S(r)}\frac{\partial}{\partial r} + \frac{1}{\psi^2(r)}\Delta_{\mathbb{S}^{n-1}}\,. \tag{3.93}$$

To explain the terminology, consider the sphere

$$S_r = \{x \in \mathbb{R}^n : |x| = r\}$$

as a submanifold of M of dimension $n-1$ (cf. Example 3.20). It is easy to show that

$$S(r) = \mu_{S_r}(S_r),$$

where μ_{S_r} is the induced measure on S_r. Hence, $S(r)$ is the area of the sphere S_r, which explains the term "area function".

Exercises.

3.20. Let ω_n be defined by (3.88).

(a) Use (3.89) to obtain a recursive formula for ω_n.

(b) Evaluate ω_n for $n = 3, 4$ given $\omega_2 = 2\pi$. Evaluate the volume functions of $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{H}^n$ for $n = 2, 3, 4$.

3.21. Prove that, for any $n \geq 1$,

$$\omega_n = 2\frac{\pi^{n/2}}{\Gamma(n/2)}, \tag{3.94}$$

where Γ is the gamma function (cf. Section A.6).

3.22. Using (3.84), obtain a full expansion of $\Delta_{\mathbb{S}^n}$ in the polar coordinates for $n = 2, 3$. Hence, obtain a full expansion of $\Delta_{\mathbb{R}^n}$ and $\Delta_{\mathbb{H}^n}$ in the polar coordinates for $n = 2, 3$.

3.23. Consider in $\mathbb{H}^3$ a function u given in the polar coordinates by $u = \frac{r}{\sinh r}$.

(a) Prove that, in the domain of the polar coordinates, this function satisfies the equation

$$\Delta_{\mathbb{H}^3} u + u = 0. \tag{3.95}$$

(b) Prove that function u extends to a smooth function in the whole space $\mathbb{H}^3$ and, hence, satisfies (3.95) in $\mathbb{H}^3$.

HINT. Write function u in the coordinates of the Poincaré model (cf. Exercises 3.18 and 3.19).

3.24. Let M be a weighted model of radius r_0 and $u = u(r)$ be a smooth function on $M \setminus \{o\}$ depending only on the polar radius. Let $S(r)$ be its area function. Prove that u is harmonic, that is, $\Delta_\mu u = 0$, if and only if

$$u(r) = C\int_{r_1}^{r} \frac{dr}{S(r)} + C_1,$$

where C, C_1 arbitrary reals and $r_1 \in (0, r_0)$. Hence or otherwise, find all radial harmonic functions in $\mathbb{R}^n$, $\mathbb{S}^2$, $\mathbb{S}^3$, $\mathbb{H}^2$, $\mathbb{H}^3$.

3.25. Let M be a weighted model of radius r_0. Fix some $0 < a < b < r_0$ and consider the annulus

$$A = \{x \in M : a < |x| < b\}.$$

Prove the following Green formulas for any two function u, v of the class $C^2(A) \cap C^1(\overline{A})$:

$$\int_A (\Delta_\mu u)\, v\, d\mu = -\int_A \langle \nabla u, \nabla v \rangle d\mu + \int_{S_b} u_r v d\mu_{S_b} - \int_{S_a} u_r v d\mu_{S_a} \tag{3.96}$$

and

$$\begin{aligned} \int_A (\Delta_\mu u)\, v\, d\mu - \int_A (\Delta_\mu v)\, u\, d\mu &= \int_{S_b} (u_r v - v_r u)\, d\mu_{S_b} \\ &\quad - \int_{S_a} (u_r v - v_r u)\, d\mu_{S_a}, \end{aligned} \tag{3.97}$$

where $u_r = \frac{\partial u}{\partial r}$.

3.26. Let S be a surface of revolution in $\mathbb{R}^{n+1}$ given by the equation

$$|x'| = \Phi\left(x^{n+1}\right),$$

where Φ is a smooth positive function defined on an open interval.

(a) Prove that S is a submanifold of $\mathbb{R}^{n+1}$ of dimension n.

(b) Prove that the induced metric g_S of S is given in the coordinates $t = x^{n+1}$ and $\theta = \frac{x'}{|x'|} \in \mathbb{S}^{n-1}$ by

$$\mathrm{g}_S = \left(1 + \Phi'(t)^2\right) dt^2 + \Phi^2(t)\, \mathrm{g}_{\mathbb{S}^{n-1}}.$$

(c) Show that the change of the coordinate

$$\rho = \int \sqrt{1 + \Phi'(t)^2}\, dt$$

brings the metric g_S to the model form

$$\mathrm{g}_S = d\rho^2 + \Psi^2(\rho)\, \mathrm{g}_{\mathbb{S}^{n-1}}, \tag{3.98}$$

where Ψ is a smooth positive function.

3.27. Represent in the model form (3.98) the induced metric of the cylinder

$$Cyl = \left\{x \in \mathbb{R}^{n+1} : |x'| = 1\right\}$$

and that of the cone

$$Cone = \left\{x \in \mathbb{R}^{n+1} : x^{n+1} = |x'| > 0\right\}.$$

3.28. The pseudo-sphere PS is defined as follows

$$PS = \left\{ x \in \mathbb{R}^{n+1} : 0 < |x'| < 1,\ \ x^{n+1} = -\sqrt{1 - |x'|^2} + \log \frac{1 + \sqrt{1 - |x'|^2}}{|x'|} \right\}.$$

Show that the model form (3.98) of the induced metric of PS is

$$\mathrm{g}_{PS} = d\rho^2 + e^{-2\rho} \mathrm{g}_{\mathbb{S}^{n-1}}.$$

HINT. Use a variable s defined by $|x'| = \frac{1}{\cosh s}$.

3.29. For any two-dimensional Riemannian manifold (M, g), the Gauss curvature $K_{M,\mathrm{g}}(x)$ is defined in a certain way as a function on M. It is known that if the metric g has in coordinates x^1, x^2 the form

$$\mathrm{g} = \frac{\left(dx^1\right)^2 + \left(dx^2\right)^2}{f^2(x)}, \tag{3.99}$$

where f is a smooth positive function, then the Gauss curvature can be computed in this chart as follows

$$K_{M,\mathbf{g}} = f^2 \Delta \log f, \tag{3.100}$$

where $\Delta = \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2}$ is the Laplace operator of the metric $\left(dx^1\right)^2 + \left(dx^2\right)^2$.

(a) Using (3.100), evaluate the Gauss curvature of $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$.

(b) Consider in the half-plane $\mathbb{R}^2_+ := \left\{\left(x^1, x^2\right) \in \mathbb{R}^2 : x_2 > 0\right\}$ the metric

$$\mathbf{g} = \frac{\left(dx^1\right)^2 + \left(dx^2\right)^2}{(x^2)^2}.$$

Evaluate the Gauss curvature of this metric.

3.30. Let $\mathbf{g}$ be the metric (3.99) on a two-dimensional manifold M. Consider the metric $\widetilde{\mathbf{g}} = \frac{1}{h^2}\mathbf{g}$ where h is a smooth positive function on M. Prove that

$$K_{M,\widetilde{\mathbf{g}}} = \left(K_{M,\mathbf{g}} + \Delta_{\mathbf{g}} \log h\right) h^2,$$

where $\Delta_{\mathbf{g}}$ is the Laplace operator of the metric $\mathbf{g}$.

3.31. Let the metric $\mathbf{g}$ on a two-dimensional manifold M have in coordinates (r, θ) the form

$$\mathbf{g} = dr^2 + \psi^2(r)\, d\theta^2. \tag{3.101}$$

Prove that

$$K_{M,\mathbf{g}} = -\frac{\psi''(r)}{\psi(r)}. \tag{3.102}$$

3.32. Using (3.102), evaluate the Gauss curvature of the two-dimensional manifolds $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$, *Cyl*, *Cone*, *PS*.

3.33. Find all metrics $\mathbf{g}$ of the form (3.101) with constant Gauss curvature.

3.11. Length of paths and the geodesic distance

Let M be a smooth manifold. A *path* on M is any continuous mapping $\gamma : (a, b) \to M$ where $-\infty \le a < b \le +\infty$. In local coordinates $x^1, ..., x^n$, the path is given by its components $\gamma^i(t)$. If $\gamma^i(t)$ are smooth functions of t then the path γ is also called smooth.

For any smooth path $\gamma(t)$, its *velocity* $\dot\gamma(t)$ is an $\mathbb{R}$-differentiation at the point $\gamma(t)$ defined by

$$\dot\gamma(t)(f) = (f \circ \gamma)'(t) \quad \text{for all } f \in C^\infty(M), \tag{3.103}$$

where the dash $'$ means derivation in t. In the local coordinates, we have, using the notation $\dot\gamma^i \equiv \frac{d\gamma^i}{dt}$,

$$(f \circ \gamma)' = \dot\gamma^i \frac{\partial f}{\partial x^i},$$

whence it follows that

$$\dot\gamma = \dot\gamma^i \frac{\partial}{\partial x^i}.$$

This implies, in particular, that any tangent vector $\xi \in T_x M$ can be represented as the velocity of a path (for example, the path $\gamma^i(t) = x^i + t\xi^i$ will do).

Let now (M, g) be a Riemannian manifold. Recall that length of a tangent vector $\xi \in T_xM$ is defined by $|\xi| = \sqrt{\langle \xi, \xi \rangle_{\mathrm{g}}}$. For any smooth path $\gamma : (a, b) \to M$, its length $\ell(\gamma)$ is defined by

$$\ell(\gamma) = \int_a^b |\dot{\gamma}(t)|\, dt. \tag{3.104}$$

If the interval (a, b) is bounded and γ extends to a smooth mapping from the *closed* interval $[a, b]$ to M then $\ell(\gamma) < \infty$.

If the image of γ is contained in a chart U with coordinates $x^1, ..., x^n$ then

$$|\dot{\gamma}(t)| = \sqrt{g_{ij}(\gamma(t))\, \dot{\gamma}^i(t)\, \dot{\gamma}^j(t)}$$

and hence

$$\ell(\gamma) = \int_a^b \sqrt{g_{ij}\dot{\gamma}^i\dot{\gamma}^j}\, dt.$$

For example, if $(g_{ij}) \equiv \mathrm{id}$ then

$$\ell(\gamma) = \int_a^b \sqrt{(\dot{\gamma}^1)^2 + ... + (\dot{\gamma}^n)^2}\, dt.$$

Let us use the paths to define a distance function on the manifold (M, g). We say that a path $\gamma : [a, b] \to M$ connects points x and y if $\gamma(a) = x$ and $\gamma(b) = y$. The *geodesic distance* $d(x, y)$ between points $x, y \in M$ is defined by

$$d(x, y) = \inf_{\gamma} \ell(\gamma), \tag{3.105}$$

where the infimum is taken over all smooth paths connecting x and y. If the infimum in (3.105) is attained on a path γ then γ is called *a shortest* (or *a minimizing*) *geodesics* between x and y. If there is no path connecting x and y then, by definition, $d(x, y) = +\infty$.

Our purpose is to show that the geodesic distance is a metric[4] on M, and the topology of the metric space (M, d) coincides with the original topology of the smooth manifold M (see Corollary 3.26 below). We start with the following observation.

CLAIM. *The geodesic distance satisfies the following properties.*

(i) $d(x, y) \in [0, +\infty]$ *and* $d(x, x) = 0$.
(ii) *Symmetry:* $d(x, y) = d(y, x)$.
(iii) *The triangle inequality:* $d(x, y) \le d(x, z) + d(y, z)$.

[4]We allow a metric $d(x, y)$ to take value $+\infty$. It can always be replaced by a *finite* metric

$$\widetilde{d}(x, y) := \frac{d(x, y)}{1 + d(x, y)},$$

which determines the same topology as $d(x, y)$.

PROOF. Properties (i) and (ii) trivially follow from (3.105). To prove (iii), consider any smooth path γ_1 connecting x and z, and a smooth path γ_2 connecting z and y. Let γ be the path connecting x and y, which goes first from x to z along γ_1 and then from z to y along γ_2. Then we obtain from (3.105) that[5]

$$d(x,y) \leq \ell(\gamma) = \ell(\gamma_1) + \ell(\gamma_2),$$

whence the triangle inequality follows by minimizing in γ_1 and γ_2. $\square$

We still need to verify that $d(x,y) > 0$ for all distinct points x, y. A crucial step towards that is contained in the following lemma.

LEMMA 3.24. *For any point $p \in M$, there is a chart $U \ni p$ and $C \geq 1$ such that, for all $x, y \in U$,*

$$C^{-1}|x-y| \leq d(x,y) \leq C|x-y|. \tag{3.106}$$

PROOF. Fix a point $p \in M$ and a chart W around p with local coordinates $x^1, ..., x^n$. Let $V \Subset W$ be a Euclidean ball in W of (a small) radius r centered at p.

For any $x \in \overline{V}$ and any tangent vector $\xi \in T_xM$, its length $|\xi|_{\mathbf{g}}$ in the metric $\mathbf{g}$ is given by

$$|\xi|^2_{\mathbf{g}} = g_{ij}(x)\,\xi^i\xi^j,$$

whereas its length $|\xi|_{\mathbf{e}}$ in the Euclidean metric $\mathbf{e}$ is given by

$$|\xi|^2_{\mathbf{e}} = \sum_{i=1}^n \left(\xi^i\right)^2.$$

Since the matrix $(g_{ij}(x))$ is positive definite and continuously depends on x, there is a constant $C \geq 1$ such that

$$C^{-2}\sum_{i=1}^n \left(\xi^i\right)^2 \leq g_{ij}(x)\,\xi^i\xi^j \leq C^2 \sum_{i=1}^n \left(\xi^i\right)^2,$$

for all $x \in \overline{V}$ and $\xi \in T_xM$. Hence, we obtain

$$C^{-1}|\xi|_{\mathbf{e}} \leq |\xi|_{\mathbf{g}} \leq C|\xi|_{\mathbf{e}},$$

which implies that, for any smooth path γ in $\overline{V}$,

$$C^{-1}\ell_{\mathbf{e}}(\gamma) \leq \ell_{\mathbf{g}}(\gamma) \leq C\ell_{\mathbf{e}}(\gamma).$$

Connecting points $x, y \in V$ by a straight line segment γ and noticing that the image of γ is contained in V and $\ell_{\mathbf{e}}(\gamma) = |x-y|$ we obtain

$$d(x,y) \leq \ell_{\mathbf{g}}(\gamma) \leq C|x-y|.$$

[5]Some approximation argument is still needed to show $d(x,y) \leq \ell(\gamma)$ because the path γ is *piecewise* smooth rather than smooth.

Let U be the Euclidean ball in W of radius $\frac{1}{3}r$ centered at p. Let γ be any smooth path on M connecting points $x, y \in U$. If γ stays in V then $\ell_{\mathbf{e}}(\gamma) \geq |x-y|$, whence

$$\ell_{\mathbf{g}}(\gamma) \geq C^{-1}|x-y| \tag{3.107}$$

(cf. Exercise 3.37)

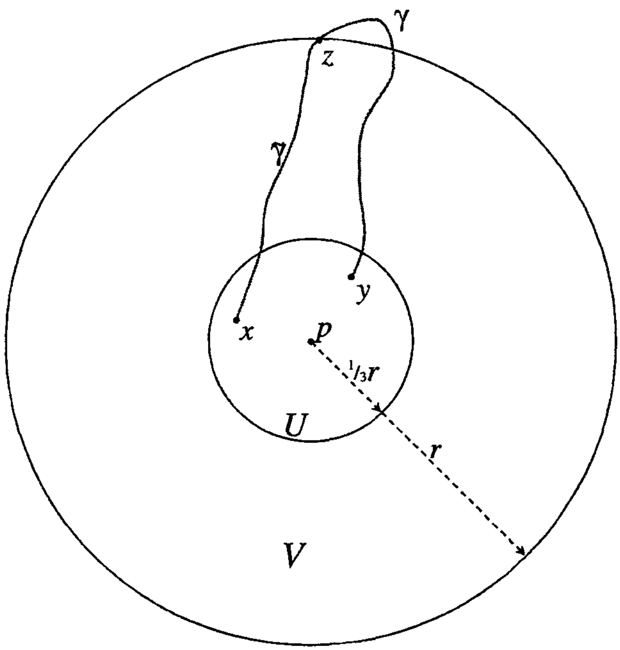

FIGURE 3.6. Path γ connecting the points x, y intersects ∂V at a point z.

If γ does not stay in V then it intersects the sphere ∂V (see Fig. 3.6). Denoting by $\widetilde{\gamma}$ be the part of γ that connects in $\overline{V}$ the point x to a point $z \in \partial V$, we obtain

$$\ell_{\mathbf{g}}(\gamma) \geq \ell_{\mathbf{g}}(\widetilde{\gamma}) \geq C^{-1}|x-z| \geq C^{-1}\frac{2}{3}r \geq C^{-1}|x-y|.$$

Hence, (3.107) holds for all paths γ connecting x and y, which implies

$$d(x,y) \geq C^{-1}|x-y|.$$

$\square$

COROLLARY 3.25. *We have $d(x,y) > 0$ for all distinct points $x, y \in M$. Consequently, the geodesic distance $d(x,y)$ satisfies the axioms of a metric and, hence, (M,d) is a metric space.*

PROOF. Fix a point $p \in M$ and let U be a chat as in Lemma 3.24. If $x \in U$ then $d(x,p) > 0$ by (3.106). We are left to treat the case $x \in M \setminus U$. Considering U as a part of $\mathbb{R}^n$, denote by $B_r(p)$ the Euclidean ball of radius $r > 0$ centered at p, that is,

$$B_r(p) = \{y \in \mathbb{R}^n : |y - p| < r\}.$$

Choose r small enough so that $B_r(p) \subset U$. Then any path from x to p must intersect the boundary of $B_r(p)$, which implies that the length of this path is at least $C^{-1}r$, where C is the constant from (3.106). It follows that $d(p,x) \geq C^{-1}r$, which finishes the proof. □

For any $x \in M$ and $r > 0$, denote by $B(x,r)$ the *geodesic ball* of radius r centered at $x \in M$, that is

$$B(x,r) = \{y \in M : d(x,y) < r\}.$$

In other words, $B(x,r)$ are the metric balls in the metric space (M,d). By definition, the topology of any metric space is generated by metric balls, which form a base of this topology. Note that the metric balls are open sets in this topology.

COROLLARY 3.26. *The topology of the metric space (M,d) coincides with the original topology of the smooth manifold M.*

PROOF. Since the topology of M in any chart U coincides with the Euclidean topology in U, it suffices to show that the geodesic balls form a local base of the Euclidean topology in U. Fix a point $p \in M$ and let U be a chart constructed in Lemma 3.24, where (3.106) holds. Considering U as a part of $\mathbb{R}^n$, recall that the Euclidean balls $B_r(p)$ form a local base of the Euclidean topology at the point p. For some $\varepsilon > 0$, the ball $B_\varepsilon(p)$ is contained in U and, hence, can be regarded as a subset of M. The result will follow if we show that, for any $r \leq \frac{\varepsilon}{2C}$, the geodesic ball $B(p,r)$ is sandwiched between two Euclidean balls as follows:

$$B_{C^{-1}r}(p) \subset B(p,r) \subset B_{Cr}(p), \tag{3.108}$$

where C is the constant from (3.106). Indeed, if $x \in B_{C^{-1}r}(p)$ then $x \in U$ and

$$d(x,p) \leq C|x - p| < r,$$

whence $x \in B(p,r)$. To prove the second inclusion in (3.108), let us first verify that $B(p,r) \subset U$. Indeed, if $x \notin U$ then any path γ connecting x and p contains a point $y \in U$ such that $|y - p| = \varepsilon/2$ (see Fig. 3.7).

By (3.106), we obtain

$$\ell_{\mathbf{g}}(\gamma) \geq d(y,p) \geq C^{-1}|y - p| = \frac{\varepsilon}{2C} \geq r,$$

whence $d(x,p) \geq r$ and $x \notin B(p,r)$. Therefore, $x \in B(p,r)$ implies $x \in U$ and, hence,

$$|x - p| \leq Cd(x,p) < Cr,$$

that is, $x \in B_{Cr}(p)$. □

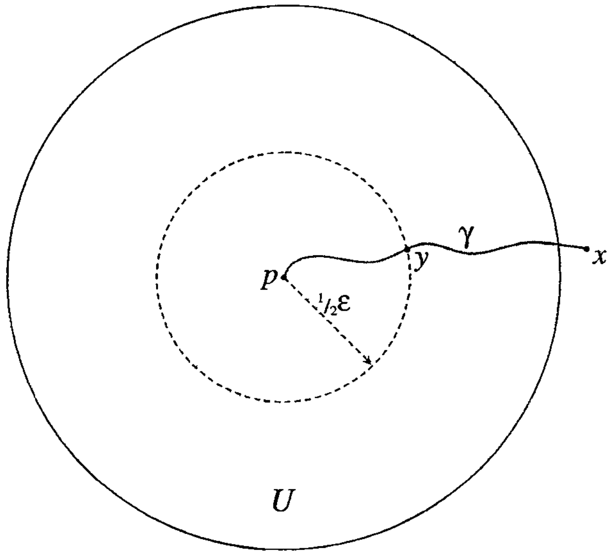

FIGURE 3.7. If $x \notin U$ then any path γ connecting x and p contains a point $y \in U$ such that $|y-p| = \varepsilon/2$

Exercises.

3.34. Prove that the length $\ell(\gamma)$ does not depend on the parametrization of the path γ as long as the change of the parameter is monotone.

3.35. Prove that the geodesic distance $d(x,y)$ is finite if and only if the points x, y belong to the same connected component of M.

3.36. Let (M, g) be a Riemannian model, and let x', x'' be two points on M with the polar coordinates (r', θ') and (r'', θ''), respectively.

(a) Prove that, for any smooth path γ on M connecting the points x' and x'',
$$\ell(\gamma) \geq |r' - r''|.$$
Consequently, $d(x', x'') \geq |r' - r''|$.

(b) Show that if $\theta' = \theta''$ then there exists a path γ of length $|r' - r''|$ connecting the points x' and x''. Consequently, $d(x', x'') = |r' - r''|$.

3.37. Let (M, g) be a Riemannian model. Prove that, for any point $x = (r, \theta)$, we have $d(0, x) = r$.

Hence or otherwise prove that in $\mathbb{R}^n$ the geodesic distance $d(x,y)$ coincides with $|x-y|$.

3.38. Let γ be a shortest geodesics between points x, y and let z be a point on the image of γ. Prove that the part of γ connecting x and z is a shortest geodesics between x and z.

3.39. Fix a point p on a Riemannian manifold M and consider the function $f(x) = d(x,p)$. Prove that if $f(x)$ is finite and smooth in a neighborhood of a point x then $|\nabla f(x)| \leq 1$.

3.40. Let (M, g) be a Riemannian model with infinite radius. Prove that, for any smooth even function a on $\mathbb{R}$, the function $a \circ r$ is smooth on M, where r is the polar radius on (M, g).

3.41. Denote by $\mathcal{S}$ the class of all smooth, positive, even functions a on $\mathbb{R}$, such that

$$\int_0^\infty a(t)\,dt = \infty. \tag{3.109}$$

For any function $a \in \mathcal{S}$, let C_a be the conformal transformation of the metric of a Riemannian model $(M, \mathbf{g})$ with infinite radius given by

$$C_a\mathbf{g} = a^2(r)\,\mathbf{g}.$$

(*a*) Prove that $(M, C_a\mathbf{g})$ is also a Riemannian model with infinite radius and that the polar radius $\widetilde{r}$ on $(M, C_a\mathbf{g})$ is related to the polar radius r on $(M, \mathbf{g})$ by the identity

$$\widetilde{r} = \int_0^r a(s)\,ds.$$

(*b*) For any two functions $a, b \in \mathcal{S}$, consider the operation $a \star b$ defined by

$$(a \star b)(t) = a\left(\int_0^t b(s)\,ds\right) b(t). \tag{3.110}$$

Prove that $(\mathcal{S}, \star)$ is a group.

(*c*) Fix $m \in \mathbb{N}$ and set for any $v \in \mathbb{R}^m$

$$\log^{[v]} r = (\log r)^{v_1} (\log\log r)^{v_2} \ldots (\underbrace{\log \ldots \log}_{m \text{ times}} r)^{v_m}$$

assuming that r is a large enough positive number. Let a and b be functions from $\mathcal{S}$ such that, for large enough r,

$$a(r) \simeq r^{\alpha-1} \log^{[u]} r \quad \text{and} \quad b(r) \simeq r^{\beta-1} \log^{[v]} r,$$

for some $\alpha, \beta \in \mathbb{R}_+$ and $u, v \in \mathbb{R}^m$. Prove that

$$a \star b \simeq r^{\gamma-1} \log^{[w]} r,$$

where

$$\gamma = \alpha\beta \text{ and } w = u + \alpha v. \tag{3.111}$$

REMARK. The identity (3.111) leads to the operation

$$(u, \alpha) \star (v, \beta) = (u + \alpha v, \alpha\beta),$$

that coincides with the group operation in the semi-direct product $\mathbb{R}^m \rtimes \mathbb{R}_+$, where the multiplicative group $\mathbb{R}_+$ acts on the additive group $\mathbb{R}^m$ by the scalar multiplication.

3.12. Smooth mappings and isometries

Let M and N be two smooth manifolds of dimension m and n, respectively. A mapping $J : M \to N$ is called smooth if it is represented in any charts of M and N by smooth functions. More precisely, this means the following. Let $x^1, ..., x^m$ be the local coordinates in a chart $U \subset M$, and $y^1, ..., y^n$ be the local coordinates in a chart $V \subset N$, and let $J(U) \subset V$. Then the mapping $J|_U$ is given by n functions $y^j(x^1, ..., x^m)$, and all they must be smooth.

A smooth mapping $J : M \to N$ allows to transfer various objects and structures either from M to N, or back from N to M. The corresponding operators in the case "from M to N" are called "*push forward*" operators, and in the case "from N to M" they are called "*pullback*" operators and are denoted by J_*. For example, any function f on N induces the pullback function J_*f on M by $J_*f = f \circ J$, that is

$$J_*f\,(x) = f\,(Jx) \text{ for all } x \in M.$$

Clearly, if f is smooth then J_*f is also smooth. This allows to push forward a tangent vector $\xi \in T_xM$ to the tangent vector in $T_{Jx}N$, which is denoted by $dJ\xi$ and is defined as an $\mathbb{R}$-differentiation by

$$dJ\xi\ (f) = \xi\,(J_*f) \quad \text{for any } f \in C^\infty(N)\,. \tag{3.112}$$

The push forward operator

$$dJ : T_xM \to T_{J(x)}N \tag{3.113}$$

is called the *differential* or the *tangent map* of J at the point x. In the local coordinates, we have

$$\xi\,(J_*f) = \xi^i \frac{\partial}{\partial x^i} f\,(J\,(x)) = \xi^i \frac{\partial y^j}{\partial x^i}\frac{\partial f}{\partial y^j},$$

that is,

$$(dJ\xi)^j = \frac{\partial y^j}{\partial x^i}\xi^i.$$

In terms of the differentials dx^i and dy^j, this equation becomes

$$dy^j = \frac{\partial y^j}{\partial x^i}dx^i. \tag{3.114}$$

Given a Riemannian metric tensor $\mathbf{g}$ on N, define its pullback $J_*\mathbf{g}$ by

$$J_*\mathbf{g}\,(x)\,(\xi,\eta) = \mathbf{g}\,(Jx)\,(dJ\xi, dJ\eta)\,, \tag{3.115}$$

for all $x \in M$ and $\xi, \eta \in T_xM$. Obviously, $J_*\mathbf{g}\,(x)$ is a symmetric, non-negative definite, bilinear form on T_xM, and it is positive definite provided the differential (3.113) is injective. In the latter case, $J_*\mathbf{g}$ is a Riemannian metric on M.

In the local coordinates, we have

$$J_*\mathbf{g} = g_{ij}dy^idy^j = g_{ij}\frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^l}dx^kdx^l$$

whence

$$(J_*\mathbf{g})_{kl} = g_{ij}\frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^l}. \tag{3.116}$$

Assume from now on that M and N have the same dimension n. A mapping $J : M \to N$ is called a *diffeomorphism* if it is smooth and the inverse mapping $J^{-1} : N \to M$ exists and is also smooth. In this case, the differentials dJ and dJ^{-1} are mutually inverse, which implies that dJ is injective.

Two Riemannian manifolds $(M, \mathbf{g}_M)$ and $(N, \mathbf{g}_N)$ are called *isometric* if there is a diffeomorphism $J : M \to N$ such that

$$J_*\mathbf{g}_N = \mathbf{g}_M.$$

Such a mapping J is called a *Riemannian isometry.* Two weighted manifolds $(M, \mathbf{g}_M, \mu_M)$ and $(N, \mathbf{g}_N, \mu_N)$ are called isometric if there is a Riemannian isometry $J : M \to N$ such that

$$J_* \Upsilon_N = \Upsilon_M,$$

where Υ_N and Υ_M are the density functions of μ_N and μ_M, respectively.

Similarly, two weighted manifolds $(M, \mathbf{g}_M, \mu_M)$ and $(N, \mathbf{g}_N, \mu_N)$ are called *quasi-isometric* if there is a diffeomorphism $J : M \to N$ such that

$$J_* \mathbf{g}_N \simeq \mathbf{g}_M \quad \text{and} \quad J_* \Upsilon_N \simeq \Upsilon_M, \tag{3.117}$$

where the sign $\simeq$ means that the ratio of the both sides is bounded by positive constants from above and below. Such a mapping J is called a *quasi-isometry.*

LEMMA 3.27. *Let J be an isometry of two weighted manifolds as above. Then the following is true:*

(*a*) *For any integrable function f on N,*

$$\int_M (J_* f) \, d\mu_M = \int_N f \, d\mu_N. \tag{3.118}$$

(*b*) *For any $f \in C^\infty(N)$,*

$$J_* (\Delta_N f) = \Delta_M (J_* f), \tag{3.119}$$

where Δ_M and Δ_N are the weighted Laplace operators on M and N, respectively.

PROOF. By definition of the integral, it suffices to prove (3.118) for functions f with compact supports. Using then a partition of unity of Theorem 3.5, we see that it is enough to consider the case when $\operatorname{supp} f$ is contained in a chart. Let V be a chart on N with coordinates $y^1, ..., y^n$. By shrinking it if necessary, we can assume that $U = J^{-1}(V)$ is a chart on M; let its coordinates be $x^1, ..., x^n$. By pushing forward functions $x^1, ..., x^n$ to N, we can consider $x^1, ..., x^n$ as new coordinates in V.

With this identification of U and V, the operator J_* becomes the identity operator. Hence, (3.118) amounts to proving that measures μ_M and μ_N coincide in V. Let g^y_{ij} be the components of the tensor $\mathbf{g}_N$ in V in the coordinates $y^1, ..., y^n$, and let g^x_{kl} be the components of the tensor $\mathbf{g}_N$ in V in the coordinates $x^1, ..., x^n$. By (3.26), we have

$$g^x_{kl} = g^y_{ij} \frac{\partial y^i}{\partial x^k} \frac{\partial y^j}{\partial x^l}. \tag{3.120}$$

Let $\widetilde{g}_{kl}$ be the components of the tensor $\mathbf{g}_M$ in U in the coordinates $x^1, ..., x^n$. Since $\mathbf{g}_M = J_* \mathbf{g}_N$, we have by (3.116) that

$$\widetilde{g}_{kl} = g^y_{ij} \frac{\partial y^i}{\partial x^k} \frac{\partial y^j}{\partial x^l}$$

whence

$$\widetilde{g}_{kl} = g^x_{kl}. \tag{3.121}$$

Since measures μ_M and μ_N have the same density function, say Υ, it follows that

$$\begin{aligned} d\mu_N &= \Upsilon\sqrt{\det g^x} dx^1...dx^n \\ &= \Upsilon\sqrt{\det \widetilde{g}} dx^1...dx^n \\ &= d\mu_M, \end{aligned}$$

which proves the identity of measures μ_M and μ_N. In the same way, we have

$$\Delta_N = \frac{1}{\Upsilon\sqrt{\det g^x}} \frac{\partial}{\partial x^i}\left(\Upsilon\sqrt{\det g^x}\,(g^x)^{ij}\,\frac{\partial}{\partial x^j}\right) = \Delta_M. \tag{3.122}$$

□

Define the push forward measure $J\mu_M$ on N by

$$(J\mu_M)(A) = \mu_M\left(J^{-1}(A)\right)$$

where A is a subset of N. The identity (3.118) means that

$$J\mu_M = \mu_N,$$

provided J is an isometry.

A typical situation when Lemma 3.27 may be useful is the following. Let J be an isometry of a weighed manifold $(M, \mathbf{g}, \mu)$ onto itself. Then (3.119) means that Δ_μ commutes with J_*. Alternatively, (3.119) can be written in the form

$$(\Delta_\mu f)\circ J = \Delta_\mu (f\circ J).$$

In $\mathbb{R}^n$ with the canonical Euclidean metric $\mathbf{g}_{\mathbb{R}^n}$, a translation is a trivial example of a Riemannian isometry. Another example is an element of the orthogonal group $O(n)$ (in particular, a rotation). The latter is also an isometry of $\mathbb{S}^{n-1}$ with the canonical spherical metric $\mathbf{g}_{\mathbb{S}^{n-1}}$.

Let (M, g, μ) be a weighted model with polar coordinates (r, θ) (see Section 3.10) and let J be an isometry of $\mathbb{S}^{n-1}$. Then J induces an isometry of $(M, \mathbf{g}, \mu)$ by

$$J(r,\theta) = (r, J\theta),$$

which implies that Δ_μ commutes with the rotations of the polar angle θ.

Exercises.

3.42. Let $J: M \to M$ be a Riemannian isometry and let S be a submanifold of M such that $J(S) = S$. Prove that $J|_S$ is a Riemannian isometry of S with respect to the induced metric of S.

3.43. Let $(M, \mathbf{g}_M)$ and $(N, \mathbf{g}_N)$ be Riemannian manifolds and $J: M \to N$ be a Riemannian isometry. Prove the following identities:

(a) For any smooth path γ on M,

$$\ell_{\mathbf{g}_M}(\gamma) = \ell_{\mathbf{g}_N}(J\circ\gamma).$$

(b) For any two points $x, y \in M$,

$$d_M(x,y) = d_N(Jx, Jy),$$

where d_M, d_N are the geodesic distances on M and N, respectively.

3.44. Let $(M, \mathbf{g}_M, \mu_M)$ and $(N, \mathbf{g}_N, \mu_N)$ be two weighted manifolds and $J : M \to N$ be a quasi-isometry. Prove the following relations.

(a) For all smooth paths γ on M,

$$\ell_{\mathbf{g}_M}(\gamma) \simeq \ell_{\mathbf{g}_N}(J \circ \gamma).$$

(b) For all couples of points $x, y \in M$,

$$d_M(x, y) \simeq d_N(Jx, Jy).$$

(c) For all non-negative measurable functions f on N,

$$\int_M (J_* f)\, d\mu_M \simeq \int_N f\, d\mu_N. \tag{3.123}$$

(d) For all smooth functions f on N,

$$\int_M |\nabla (J_* f)|^2_{\mathbf{g}_M}\, d\mu_M \simeq \int_N |\nabla f|^2_{\mathbf{g}_N}\, d\mu_N. \tag{3.124}$$

3.45. For any real α, consider the mapping $y = Jx$ of $\mathbb{R}^{n+1}$ onto itself given by

$$\begin{aligned} y^1 &= x^1 \\ &\dots \\ y^{n-1} &= x^{n-1} \\ y^n &= x^n \cosh\alpha + x^{n+1}\sinh\alpha \\ y^{n+1} &= x^n \sinh\alpha + x^{n+1}\cosh\alpha, \end{aligned} \tag{3.125}$$

which is called a hyperbolic rotation.

(a) Prove that J is an isometry of $\mathbb{R}^{n+1}$ with respect to the Minkowski metric

$$\mathbf{g}_{Mink} = (dx^1)^2 + \dots + (dx^n)^2 - (dx^{n+1})^2.$$

(b) Prove that $J|_{\mathbb{H}^n}$ is a Riemannian isometry of the hyperbolic space $\mathbb{H}^n$ (cf. Section 3.9).

3.46. Prove that, for any four points $p, q, p', q' \in \mathbb{H}^n$ such that

$$d(p', q') = d(p, q), \tag{3.126}$$

there exists a Riemannian isometry J of $\mathbb{H}^n$ such that $Jp' = p$ and $Jq' = q$.

Notes

Most of the material of this Chapter belongs to the basics of Riemannian geometry and can be found in many textbooks, see for example **[45]**, **[51]**, **[52]**, **[200]**, **[213]**, **[228]**, **[227]**, **[244]**, **[299]**, **[326]**, **[329]**.

The presentation of model manifolds follows **[155]**.

CHAPTER 4

Laplace operator and heat equation in $L^2(M)$

We use here quite substantially measure theory, integration, the theory of Hilbert spaces, and the spectral theory of self-adjoint operators. The reader is referred to Appendix A for the necessary reference material. All subsequent Chapters (except for Chapter 6) depend upon and use the results of this Chapter.

In Section 4.1, we introduce the Lebesgue spaces, distributions, and Sobolev spaces on a weighted manifold. This material is similar to the corresponding parts of Chapter 2, although technically we use from Chapter 2 only Corollary 2.5.

The key Sections 4.2 and 4.3 rest on Section 3.6 from Chapter 3, especially on the Green formulas (3.43).

4.1. Distributions and Sobolev spaces

For any smooth manifold M, define the space of test functions $\mathcal{D}(M)$ as $C_0^\infty(M)$ with the following *convergence*: $\varphi_k \xrightarrow{\mathcal{D}} \varphi$ if the following conditions are satisfied:

1. In any chart U and for any multiindex α, $\partial^\alpha \varphi_k \rightrightarrows \partial^\alpha \varphi$ in U.
2. All supports $\operatorname{supp} \varphi_k$ are contained in a compact subset of M.

A *distribution* is a continuous linear functional on $\mathcal{D}(M)$. If u is a distribution then its value at a function $\varphi \in \mathcal{D}$ is denoted by (u, φ). The set $\mathcal{D}'(M)$ of all distributions is obviously a linear space. The convergence in $\mathcal{D}'(M)$ is defined as follows: $u_k \xrightarrow{\mathcal{D}'} u$ if $(u_k, \varphi) \to (u, \varphi)$ for all $\varphi \in \mathcal{D}(M)$.

Since any open set $\Omega \subset M$ is a manifold itself, the spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ are defined as above. In any chart $U \subset M$, the spaces $\mathcal{D}(U)$ and $D'(U)$ are identical to those defined in U as a part of $\mathbb{R}^n$ (cf. Section 2.4).

A distribution $u \in \mathcal{D}'(M)$ *vanishes* in an open set $\Omega \subset M$ if $(u, \varphi) = 0$ for any $\varphi \in \mathcal{D}(\Omega)$. It is proved in the same way as in $\mathbb{R}^n$ (cf. Exercise 2.10) that if u vanishes in a family of open sets then it vanishes also in their union. Hence, there is a maximal open set in M where u vanishes, and its complement in M is called the *support* of u and is denoted by $\operatorname{supp} u$. By construction, $\operatorname{supp} u$ is a closed subset of M.

Next, we would like to identify a function on M as a distribution, and for that we need a measure on M. Assume in the sequel that $(M, \mathbf{g}, \mu)$ is a weighted manifold. The couple (M, μ) can also be considered as a measure space. Hence, the notions of measurable and integrable functions are defined as well as the Lebesgue function spaces $L^p(M) = L^p(M, \mu)$, $1 \le p \le \infty$ (see Section A.4). Note that $L^p(M)$ are Banach spaces, and $L^2(M)$ is a Hilbert space. Sometimes it is useful to know that if $1 \le p < \infty$ then $L^p(M)$ is separable and $\mathcal{D}(M)$ is dense in $L^p(M)$ (cf. Theorem 2.3 and Exercise 4.4)

Denote by $L^p_{loc}(M)$ the space of all measurable functions f on M such that $f \in L^p(\Omega)$ for any relatively compact open set $\Omega \subset M$. The space $L^p_{loc}(M)$ is linear space, and the topology of $L^p_{loc}(M)$ is defined by the family of seminorms $\|f\|_{L^p(\Omega)}$ for all open $\Omega \Subset M$. Clearly, we have the following embeddings:

$$L^p(M) \hookrightarrow L^p_{loc}(M) \hookrightarrow L^1_{loc}(M)$$

(cf. Section 2.1).

Now we can associate any function $u \in L^1_{loc}(M)$ with a distribution by the following rule:

$$(u, \varphi) = \int_M u\varphi d\mu \quad \text{for any } \varphi \in \mathcal{D}(M). \tag{4.1}$$

LEMMA 4.1. *Let $u \in L^1_{loc}(M)$. Then $u = 0$ a.e. if and only if $u = 0$ in $\mathcal{D}'(M)$, that is, if*

$$\int_M u\varphi \, d\mu = 0 \ \text{ for any } \varphi \in \mathcal{D}(M). \tag{4.2}$$

Note that if $u \in C(M)$ then this was proved in Lemma 3.13.

PROOF. Let $U \subset M$ be any chart and λ be the Lebesgue measure in U. Since the density $\frac{d\mu}{d\lambda}$ is a smooth positive function, the condition (4.2) implies that

$$\int_U u\varphi \, d\lambda = 0 \ \text{ for any } \varphi \in \mathcal{D}(U).$$

By Corollary 2.5, we obtain $u = 0$ a.e. in U. Since M can be covered by a countable family of charts, we obtain $u = 0$ a.e. on M. □

Lemma 4.1 implies that the linear mapping $L^1_{loc}(M) \to \mathcal{D}'(M)$ defined by (4.1) is an injection, which enables us to identify $L^1_{loc}(M)$ as a subspace of $\mathcal{D}'(M)$. Since the convergence in $L^1_{loc}(M)$ obviously implies the convergence in $\mathcal{D}'(M)$, we obtain the embedding

$$L^1_{loc}(M) \hookrightarrow \mathcal{D}'(M).$$

In particular, this allows to define the support $\operatorname{supp} u$ of any function $u \in L^1_{loc}(M)$ as that of the associated distribution.

Let us introduce the vector field versions of all the above spaces. Let $\vec{\mathcal{D}}(M)$ be the space of all smooth vector fields on M with compact supports endowed with the convergence similar to that in $\mathcal{D}(M)$.

The elements of the dual space $\vec{\mathcal{D}}'(M)$ are called *distributional vector fields.* The convergence in $\vec{\mathcal{D}}'(M)$ is defined in the same way as in $\mathcal{D}'(M)$.

A vector field v on M is called measurable if all its components in any chart are measurable functions. By definition, the space $\vec{L}^p(M)$ consists of (the equivalence classes of) measurable vector fields v such that $|v| \in L^p(M)$ (where $|v| = \langle v, v \rangle_{\mathbf{g}}^{1/2}$ is the length of v).

Similarly, the space $\vec{L}^p_{loc}(M)$ is determined by the condition $|v| \in L^p_{loc}(M)$. The norm in $\vec{L}^p(M)$ is defined by

$$\|v\|_{\vec{L}^p} := \| \, |v| \, \|_{L^p}.$$

The spaces $\vec{L}^p(M)$ are also complete (see Exercise 4.9). In particular, $\vec{L}^2(M)$ is a Hilbert space with the inner product

$$(v, w)_{\vec{L}^2} = \int_M \langle v, w \rangle d\mu.$$

Any vector field $v \in \vec{L}^1_{loc}(M)$ determines a distributional vector field by

$$(v, \psi) = \int_M \langle v, \psi \rangle d\mu \quad \text{for any } \psi \in \vec{\mathcal{D}}(M),$$

which defines the embedding $\vec{L}^1_{loc}(M) \hookrightarrow \vec{\mathcal{D}}'(M)$.

Let us define some operators in $\mathcal{D}'(M)$ and $\vec{\mathcal{D}}'(M)$. For any distribution $u \in \mathcal{D}'(M)$, its *distributional Laplacian* $\Delta_\mu u \in \mathcal{D}'(M)$ is defined by means of the identity

$$(\Delta_\mu u, \varphi) = (u, \Delta_\mu \varphi) \quad \text{for all } \varphi \in \mathcal{D}(M). \tag{4.3}$$

Note that the right hand side makes sense because $\Delta_\mu \varphi \in \mathcal{D}(M)$, and it determines a continuous linear functional of $\varphi \in \mathcal{D}(M)$. Indeed, it is easy to see that $\varphi_k \xrightarrow{\mathcal{D}} \varphi$ implies $\Delta_\mu \varphi_k \xrightarrow{\mathcal{D}} \Delta_\mu \varphi$ and, hence, $(u, \Delta_\mu \varphi_k) \to (u, \Delta_\mu \varphi)$.

If u is a smooth function then its *classical* Laplacian $\Delta_\mu u$ satisfies (4.3), because by the Green formula (3.43)

$$(\Delta_\mu u, \varphi) = \int_M (\Delta_\mu u) \varphi d\mu = \int_M u \Delta_\mu \varphi d\mu = (u, \Delta_\mu \varphi).$$

Hence, in this case the distributional Laplacian coincides with the classical Laplacian, which justifies the usage of the same notation $\Delta_\mu u$ for the both of them.

If $u \in L^2_{loc}(M)$ and the distribution $\Delta_\mu u$ can be identified as a function from $L^2_{loc}(M)$, also denoted by $\Delta_\mu u$, then the latter is called the *weak Laplacian* of u. Alternatively, the weak Laplacian $\Delta_\mu u$ can be defined as a function from $L^2_{loc}(M)$ that satisfies the identity (4.3). The weak Laplacian does not always exist unlike the distributional Laplacian.

For any distribution $u \in \mathcal{D}'(M)$, define its *distributional gradient* $\nabla u \in \vec{\mathcal{D}}'(M)$ by means of the identity

$$(\nabla u, \psi) = -(u, \operatorname{div}_\mu \psi) \quad \text{for all } \psi \in \vec{\mathcal{D}}(M). \tag{4.4}$$

If u is a smooth function then its *classical* gradient satisfies (4.4) by Corollary 3.15.

If $u \in L^2_{loc}(M)$ and $\nabla u \in \vec{L}^2_{loc}(M)$ then ∇u is called the *weak gradient* of u.

We will always understand the operators Δ_μ, ∇ in the distributional sense unless otherwise stated.

The following lemma is frequently useful.

LEMMA 4.2. *If $u_k \xrightarrow{\mathcal{D}'} u$ then $\nabla u_k \xrightarrow{\vec{\mathcal{D}}'} \nabla u$. Consequently, if $u_k \xrightarrow{\mathcal{D}'} u$ and $\nabla u_k \xrightarrow{\vec{\mathcal{D}}'} v$ then $\nabla u = v$.*

PROOF. By the definition of the distributional gradient, we have, for any $\psi \in \vec{\mathcal{D}}(M)$,

$$(\nabla u_k, \psi) = -(u_k, \operatorname{div}_\mu \psi).$$

Passing to the limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} (\nabla u_k, \psi) = -(u, \operatorname{div}_\mu \psi) = (\nabla u, \psi),$$

which implies

$$\nabla u_k \xrightarrow{\vec{\mathcal{D}}'} \nabla u.$$

The second claim is obvious. □

Define the following *Sobolev space*

$$W^1(M) = W^1(M, \mathrm{g}, \mu) := \{u \in L^2(M) : \nabla u \in \vec{L}^2(M)\}.$$

That is, $W^1(M)$ consists of those functions $u \in L^2(M)$, whose weak gradient ∇u exists and is in $\vec{L}^2(M)$. It is easy to see that $W^1(M)$ is a linear space. Furthermore, $W^1(M)$ has a natural inner product

$$(u, v)_{W^1} := (u, v)_{L^2} + (\nabla u, \nabla v)_{L^2} = \int_M uv \, d\mu + \int_M \langle \nabla u, \nabla v \rangle \, d\mu, \tag{4.5}$$

and the associated norm

$$\|u\|^2_{W^1} = \|u\|^2_{L^2} + \|\nabla u\|^2_{L^2} = \int_M u^2 d\mu + \int_M |\nabla u|^2 \, d\mu. \tag{4.6}$$

LEMMA 4.3. *$W^1(M)$ is a Hilbert space.*

PROOF. It follows from (4.6) that the convergence $u_k \xrightarrow{W^1} u$ in $W^1(M)$ is equivalent to

$$u_k \xrightarrow{L^2} u \quad \text{and} \quad \nabla u_k \xrightarrow{L^2} \nabla u. \tag{4.7}$$

Let $\{u_k\}$ be a Cauchy sequence in $W^1(M)$. Then the sequence $\{u_k\}$ is Cauchy also in $L^2(M)$ and, hence, converges in L^2-norm to a function $u \in L^2(M)$. Similarly, the sequence $\{\nabla u_k\}$ is Cauchy in $\vec{L}^2(M)$ and, hence, converges in $\vec{L}^2$-norm to a vector field $v \in \vec{L}^2(M)$. Since convergence in L^2 is stronger than the convergence in $\mathcal{D}'$, we conclude by Lemma 4.2 that $\nabla u = v$. It follows that the conditions (4.7) are satisfied and the sequence $\{u_k\}$ converges in $W^1(M)$. □

In the case when M is an open subset of $\mathbb{R}^n$, the above definition of $W^1(M)$ matches the one from Section 2.6.1 – see Exercise 4.11.

It is obvious from the definition of the norm (4.6) that $\|u\|_{L^2} \leq \|u\|_{W^1}$, which implies that the identical mapping $W^1(M) \to L^2(M)$ is a bounded injection, that is, an embedding.

Exercises.

4.1. Prove that if $\varphi_k \xrightarrow{\mathcal{D}} \varphi$ then

(a) $\varphi_k \rightrightarrows \varphi$ on M;

(b) $\Delta_\mu \varphi_k \xrightarrow{\mathcal{D}} \Delta_\mu \varphi$;

(c) $f\varphi_k \xrightarrow{\mathcal{D}} f\varphi_k$ for any $f \in C^\infty(M)$.

4.2. For any function $f \in C^\infty(M)$ and a distribution $u \in \mathcal{D}'(M)$, their product fu is defined as a distribution by

$$(fu, \varphi) = (u, f\varphi) \quad \text{for any } \varphi \in \mathcal{D}(M). \tag{4.8}$$

Prove the following assertions.

(a) If $u_k \xrightarrow{\mathcal{D}'} u$ then $fu_k \xrightarrow{\mathcal{D}'} fu$.

(b) $\operatorname{supp}(fu) \subset \operatorname{supp} f \cap \operatorname{supp} u$.

(c) Product rule:

$$\nabla(fu) = f\nabla u + (\nabla f)\, u,$$

where the product $f\nabla u$ of a smooth function by a distributional vector field and the product $(\nabla f)\, u$ of a smooth vector field by a distribution are defined similarly to (4.8).

4.3. Prove that if $f \in C^\infty(M)$ is such that $|f|$ and $|\nabla f|$ are bounded, and $u \in W^1(M)$ then $fu \in W^1(M)$ and

$$\|fu\|_{W^1} \leq C\|u\|_{W^1},$$

where $C = 2\max(\sup|f|, \sup|\nabla f|)$.

4.4. Prove the extension of Theorem 2.3 to manifold: for any $1 \leq p < \infty$ and for any weighted manifold $(M, \mathbf{g}, \mu)$, $\mathcal{D}(M)$ is dense in $L^p(M)$, and the space $L^p(M)$ is separable.

4.5. Prove that $\mathcal{D}(M)$ is dense in $C_0(M)$, where $C_0(M)$ is the space of continuous functions with compact support, endowed with the sup-norm.

4.6. Let $u \in \mathcal{D}'(M)$ and $(u, \varphi) = 0$ for all non-negative functions $\varphi \in \mathcal{D}(M)$. Prove that $u = 0$.

4.7. Let $u \in L^1_{loc}(M)$.

(a) Prove that if $(u, \varphi) \geq 0$ for all non-negative functions $\varphi \in \mathcal{D}(M)$, then $u \geq 0$ a.e.

(b) Prove that if $(u, \varphi) = 0$ for all non-negative functions $\varphi \in \mathcal{D}(M)$, then $u = 0$ a.e.

4.8. Let $\{u_k\}$ be a sequence from $L^2(M)$ such that $u_k \xrightarrow{\mathcal{D}'} u$, where $u \in \mathcal{D}'(M)$.

(a) Prove that if the sequence of norms $\|u_k\|_{L^2}$ is bounded then $u \in L^2(M)$ and

$$\|u\|_{L^2} \leq \liminf_{k\to\infty} \|u_k\|_{L^2}.$$

(b) Assume in addition that $\nabla u_k \in \vec{L}^2$ and that the sequence of norms $\|\nabla u_k\|_{L^2}$ is bounded. Prove that $u \in W^1(M)$ and

$$\|\nabla u\|_{L^2} \leq \liminf_{k\to\infty} \|\nabla u_k\|_{L^2}.$$

4.9. Prove that the space $\vec{L}^p(M, \mu)$ is complete.

4.10. Define the divergence of a distributional vector field $v \in \vec{\mathcal{D}}'$ by

$$(\operatorname{div}_\mu v, \varphi) = -(v, \nabla\varphi) \text{ for all } \varphi \in \mathcal{D}.$$

Prove that, for any distribution $u \in \mathcal{D}'$,

$$\Delta_\mu u = \operatorname{div}_\mu(\nabla u),$$

where all operators Δ_μ, ∇, and div_μ are understood in the distributional sense.

4.11. Let (M, g, μ) be a weighted manifold and U be a chart on M with coordinates $x^1, ..., x^n$. Let $f \in L^2_{loc}(U)$.

(a) Assume that all distributional partial derivatives $\frac{\partial f}{\partial x^j}$ are in $L^2_{loc}(U)$, considering U as a part of $\mathbb{R}^n$. Prove that the distributional gradient $\nabla_{\mathrm{g}} f$ in U is given by

$$(\nabla_{\mathrm{g}} f)^i = g^{ij} \frac{\partial f}{\partial x^j},$$

and

$$|\nabla_{\mathrm{g}} f|^2_{\mathrm{g}} = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}. \tag{4.9}$$

Conclude that $\nabla_{\mathrm{g}} f \in \vec{L}^2_{loc}(U)$.

(b) Assuming that $\nabla_{\mathrm{g}} f \in \vec{L}^2_{loc}(U)$, prove that distributional partial derivatives $\frac{\partial f}{\partial x^j}$ are given by

$$\frac{\partial f}{\partial x^j} = g_{ij} (\nabla_{\mathrm{g}} f)^i$$

and that the identity (4.9) holds. Conclude that $\frac{\partial f}{\partial x^j} \in L^2_{loc}(U)$.

4.12. For an open set $\Omega \subset \mathbb{R}^n$, let $W^1(\Omega)$ be the Sobolev space defined in Section 2.6.1, and $W^1(\Omega, \mathrm{g}, \lambda)$ be the Sobolev space defined in Section 4.1, where g is the canonical Euclidean metric and λ is the Lebesgue measure. Prove that these two Sobolev spaces are identical.

4.13. Denote by ∇_{dist} the distributional gradient in $\mathbb{R}^n$ $(n \geq 2)$ reserving ∇ for the gradient in the classical sense, and the same applies to the Laplace operators Δ_{dist} and Δ.

(a) Let $f \in C^1(\mathbb{R}^n \setminus \{o\})$ and assume that

$$f \in L^2_{loc}(\mathbb{R}^n) \text{ and } \nabla f \in \vec{L}^2_{loc}(\mathbb{R}^n).$$

Prove that $\nabla_{dist} f = \nabla f$.

(b) Let $f \in C^2(\mathbb{R}^n \setminus \{o\})$ and assume that

$$f \in L^2_{loc}(\mathbb{R}^n), \quad \nabla f \in \vec{L}^2_{loc}(\mathbb{R}^n), \quad \text{and } \Delta f \in L^2_{loc}(\mathbb{R}^n).$$

Prove that $\Delta_{dist} f = \Delta f$.

(c) Consider in $\mathbb{R}^3$ the function $f(x) = |x|^{-1}$. Show that $f \in L^2_{loc}(\mathbb{R}^3)$ and $\Delta f = 0$ in $\mathbb{R}^3 \setminus \{o\}$. Prove that $\Delta_{dist} f = -4\pi\delta$ where δ is the Dirac delta-function at the origin o.

4.14. Consider in $\mathbb{R}^n$ $(n \geq 2)$ the function $f(x) = |x|^\alpha$, where α is a real parameter.

(a) Prove that $f \in L^2_{loc}$ provided $\alpha > -n/2$.

(b) Prove that $f \in L^2_{loc}$ and $\nabla f \in \vec{L}^2_{loc}$ provided $\alpha > 1 - n/2$. Show that in this case $\nabla_{dist} f = \nabla f$.

(c) Prove that $f \in L^2_{loc}$, $\nabla f \in \vec{L}^2_{loc}$, and $\Delta u \in L^2_{loc}$ provided $\alpha > 2 - n/2$. Show that in this case $\Delta_{dist} f = \Delta f$..

4.15. Prove that if $\{u_k\}$ is a sequence of functions from W^1 that is bounded in the norm of W^1 then there exists a subsequence $\{u_{k_i}\}$ that converges to a function $u \in W^1$ weakly in W^1 and weakly in L^2.

4.16. Prove that if $\{u_k\}$ is a sequence of functions from W^1 that converges weakly in W^1 to a function $u \in W^1$ then there is a subsequence $\{u_{k_i}\}$ such that

$$u_{k_i} \overset{L^2}{\rightharpoonup} u \text{ and } \nabla u_{k_i} \overset{L^2}{\rightharpoonup} \nabla u,$$

where $\rightharpoonup$ stands for the weak convergence.

4.17. Let $\{u_k\}$ be a sequence of functions from W^1 that converges weakly in L^2 to a function $u \in L^2$.

(a) Prove that if the sequence $\{u_k\}$ is bounded in the norm W^1 then $u \in W^1$ and $u_k \overset{W^1}{\rightharpoonup} u$.

(b) Prove that if in addition $\|u_k\|_{W^1} \to \|u\|_{W^1}$ then $u_k \overset{W^1}{\to} u$.

4.18. Let $\{u_k\}$ be an increasing sequence of non-negative functions from W^1 that converges almost everywhere to a function $u \in L^2$. Prove that if

$$\|\nabla u_k\|_{L^2} \le c$$

for some constant c and all k, then $u \in W^1$, $u_k \overset{W^1}{\rightharpoonup} u$, and $\|\nabla u\|_{L^2} \le c$.

4.2. Dirichlet Laplace operator and resolvent

Let (M, g, μ) be a weighted manifold. The purpose of this section is to extend the Laplace operator Δ_μ to a self-adjoint operator in the Hilbert space $L^2 = L^2(M, \mu)$.

Initially, the Laplace operator Δ_μ is defined on smooth functions, in particular, on the space $\mathcal{D} = \mathcal{D}(M)$. Since $\mathcal{D}$ is a dense subspace of L^2, we can say that Δ_μ is a densely defined operator in L^2 with the domain $\mathcal{D}$. Denote this operator by $A = \Delta_\mu|_{\mathcal{D}}$. This operator is symmetric, because, by Green's formulas (3.43),

$$(\Delta_\mu u, v)_{L^2} = (u, \Delta_\mu v)_{L^2} \quad \text{for all } u, v \in \mathcal{D}.$$

However, A is not self-adjoint, which follows from the following statement.

CLAIM. *For the operator $A = \Delta_\mu|_{\mathcal{D}}$, the adjoint operator A^* has the domain*

$$\operatorname{dom} A^* = \left\{u \in L^2 : \Delta_\mu u \in L^2\right\}, \tag{4.10}$$

and in this domain $A^* u = \Delta_\mu u$.

PROOF. Recall that the adjoint operator is defined by

$$\operatorname{dom} A^* = \left\{u \in L^2 : \exists f \in L^2 \;\; \forall v \in \operatorname{dom} A \quad (Av, u)_{L^2} = (v, f)_{L^2}\right\},$$

and

$$A^* u = f.$$

The equation $(Av, u)_{L^2} = (v, f)_{L^2}$ is equivalent to $(u, \Delta_\mu v)_{\mathcal{D}} = (f, v)_{\mathcal{D}}$ which means that $\Delta_\mu u = f$ in the distributional sense. Hence, $u \in \operatorname{dom} A^*$ if and only if $\Delta_\mu u \in L^2$, and in this domain $A^* u = \Delta_\mu u$, which was to be proved. $\square$

It is clear from (4.10) that $\operatorname{dom} A^*$ contains functions, which are not compactly supported, and, hence, $\operatorname{dom} A^*$ strictly larger than $\operatorname{dom} A$. For example, in $\mathbb{R}$, the function $u(x) = e^{-x^2}$ belongs to $\operatorname{dom} A^*$ but not to $\operatorname{dom} A$.

If A is a densely defined symmetric operator in a Hilbert space $\mathcal{H}$ then we always have

$$A \subset A^*.$$

If B is a self-adjoint extension of A then

$$A \subset B = B^* \subset A^*,$$

which implies

$$\operatorname{dom} A \subset \operatorname{dom} B \subset \operatorname{dom} A^*.$$

Hence, the problem of constructing a self-adjoint extension of A amounts to an appropriate choice of $\operatorname{dom} B$ between $\operatorname{dom} A$ and $\operatorname{dom} A^*$ because then the action of B can defined by restricting A^* to $\operatorname{dom} B$.

Consider the following functional spaces on a weighted manifold $(M, \mathbf{g}, \mu)$:

$$W_0^1(M) = \text{ the closure of } \mathcal{D}(M) \text{ in } W^1(M)$$

and

$$W_0^2(M) = \left\{u \in W_0^1(M) : \Delta_\mu u \in L^2(M)\right\}.$$

That is, W_0^2 consists of those functions $u \in W_0^1$, whose weak Laplacian $\Delta_\mu u$ exists and belongs to L^2. Clearly, $\mathcal{D} \subset W_0^2$.

The space W_0^1 has the same inner product as W^1 and is a Hilbert space as a closed subspace of W^1. It is natural to consider also the following space[1]

$$W^2(M) = \{u \in W^1(M) : \Delta_\mu u \in L^2(M)\}. \tag{4.11}$$

Consider the operator $\Delta_\mu|_{W_0^2}$ as a densely defined operator in L^2, which obviously extends the operator $\Delta_\mu|_{\mathcal{D}}$. As will we prove below, the operator $\Delta_\mu|_{W_0^2}$ is actually self-adjoint. Let us first verify that it is symmetric.

LEMMA 4.4. (The Green formula) *For all functions $u \in W_0^1(M)$ and $v \in W^2(M)$, we have*

$$\int_M u\,\Delta_\mu v\,d\mu = -\int_M \langle\nabla u, \nabla v\rangle\,d\mu. \tag{4.12}$$

PROOF. Indeed, if $u \in \mathcal{D}$ then by the definitions of the distributional Laplacian $\Delta_\mu v$ and the distributional gradient ∇v, we have

$$\begin{aligned}\int_M u\Delta_\mu v\,d\mu &= (\Delta_\mu v, u) = (v, \Delta_\mu u)\\ &= (v, \operatorname{div}_\mu(\nabla u)) = -(\nabla v, \nabla u) = -\int_M \langle\nabla u, \nabla v\rangle d\mu.\end{aligned}$$

[1] *Warning.* If Ω is an open subset of $\mathbb{R}^n$ then, in general, the space $W^2(\Omega)$ defined by (4.11) does not match the space $W^2(\Omega)$ introduced in Section 2.6.1.

For any $u \in W_0^1$ there is a sequence $\{u_k\} \subset \mathcal{D}$ that converges to u in W^1. Applying (4.12) to u_k and passing to the limit we obtain the same identity for u because the both sides of (4.12) are continuous functionals of $u \in W^1$. $\square$

In particular, (2.39) applies when $u \in W_0^1$ and $v \in W_0^2$. If the both functions u, v are in W_0^2 then we can switch them in (2.27), which yields

$$(\Delta_\mu u, v)_{L^2} = (u, \Delta_\mu v)_{L^2}. \tag{4.13}$$

Hence, $\Delta_\mu|_{W_0^2}$ is a symmetric operator.

The identity (4.12) also implies that, for any $u \in W_0^2$,

$$\int_M u\,\Delta_\mu u\,d\mu = -\int_M |\nabla u|^2\,d\mu \le 0, \tag{4.14}$$

that is, the operator $\Delta_\mu|_{W_0^2}$ is non-positive definite. It is frequently more convenient to work with a non-negative definite operator, so set

$$\mathcal{L} = -\,\Delta_\mu|_{W_0^2}.$$

The operator $\mathcal{L}$ (or its negative $\Delta_\mu|_{W_0^2}$) is called the *Dirichlet Laplace operator* of the weighted manifold (M, g, μ).

This terminology is motivated by the following observation. Let Ω be a bounded open subset of $\mathbb{R}^n$. Given a function f in Ω, the problem of finding a function u in Ω satisfying the conditions

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.15}$$

is refereed to as the *Dirichlet problem*. In the classical understanding of this problem, the function u is sought in the class $C^2(\Omega) \cap C\left(\overline{\Omega}\right)$. However, in general the Dirichlet problem has no solution in this class unless the boundary of Ω possesses a certain regularity.

It is more profitable to understand (4.15) in a *weak* sense. Firstly, the Laplace operator in the equation $\Delta u = f$ can be understood in the distributional sense. Secondly, the boundary condition $u = 0$ can be replaced by the requirement that u belongs to a certain functional class. It turns out that a good choice of this class is $W_0^1(\Omega)$. The fact that $W_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^1(\Omega)$ allows to show that functions from $W_0^1(\Omega)$ do tend to 0 in a certain average sense when approaching the boundary $\partial\Omega$.

Hence, the *weak Dirichlet problem* in Ω is stated as follows: assuming that $f \in L^2(\Omega)$, find a function $u \in W_0^1(\Omega)$ such that $\Delta u = f$. Obviously, if u solves this problem then $\Delta u \in L^2$ and hence $u \in W_0^2(\Omega)$. We see that the space W_0^2 appears naturally when solving the Dirichlet problem, which explains the above terminology. Replacing the boundary condition in (4.15) by the requirement $u \in W_0^1$ allows to generalize the weak Dirichlet problem to an arbitrary manifold. Namely, for a weighted manifold (M, g, μ), consider the following problem:

$$\begin{cases} -\Delta_\mu u + \alpha u = f & \text{on } M, \\ u \in W_0^1(M), \end{cases} \tag{4.16}$$

where α is a given constant and $f \in L^2(M)$ is a given function. If $\alpha \le 0$ then this problem may have more that one solution. For example, if M is a compact manifold, $\alpha = 0$ and $f = 0$ then $u = \mathrm{const}$ is a solution. As we will see in the next statement, for $\alpha > 0$ this problem has always exactly one solution (see Exercise 4.28 for the uniqueness conditions when $\alpha = 0$, and Exercise 4.29 for a more general version of the weak Dirichlet problem with non-zero boundary condition).

Consider the *resolvent* $R_\alpha = (\mathcal{L} + \alpha\,\mathrm{id})^{-1}$ of the Dirichlet Laplace operator $\mathcal{L}$, which is defined whenever the operator $\mathcal{L} + \alpha\,\mathrm{id}$ is invertible in L^2.

THEOREM 4.5. *For any $\alpha > 0$, the resolvent $R_\alpha := (\mathcal{L} + \alpha\,\mathrm{id})^{-1}$ exists and is a bounded non-negative definite self-adjoint operator in L^2. Moreover, $\|R_\alpha\| \le \alpha^{-1}$.*

PROOF. Let us show that, for any $f \in L^2$, there exists a unique function $u \in W_0^2$ such that $(\mathcal{L} + \alpha\,\mathrm{id})\,u = f$, that is,

$$-\Delta_\mu u + \alpha u = f. \tag{4.17}$$

This will prove that the resolvent R_α exists and $R_\alpha f = u$. The requirement $u \in W_0^2$ here can be relaxed to $u \in W_0^1$. Indeed, if $u \in W_0^1$ and u satisfies (4.17) then $\Delta_\mu u = \alpha u - f \in L^2$, whence $u \in W_0^2$ (in particular, this will imply that the problem (4.16) has a unique solution).

Considering the both sides of (4.17) as distributions and applying them to a test function $\varphi \in \mathcal{D}$, we obtain that (4.17) is equivalent to the equation

$$-(u, \Delta_\mu \varphi) + \alpha(u, \varphi) = (f, \varphi), \tag{4.18}$$

for any $\varphi \in \mathcal{D}$, where we have used the definition (4.3) of the distributional Δ_μ. Next, using the fact that $u \in W_0^1$ and the definition of the distributional gradient ∇u, rewrite (4.18) in the equivalent form

$$(\nabla u, \nabla \varphi) + \alpha\,(u, \varphi) = (f, \varphi). \tag{4.19}$$

Now, let us interpret the brackets in (4.19) as inner products in L^2. Since $\mathcal{D}$ is dense in W_0^1, we can extend to W_0^1 the class of test functions φ for which (4.19) holds. Hence, the problem amounts to proving the existence and uniqueness of a solution $u \in W_0^1$ to (4.19) assuming that (4.19) holds for all $\varphi \in W_0^1$.

Denote the left hand side of (4.19) by $[u, \varphi]_\alpha$, that is,

$$[u, \varphi]_\alpha := (\nabla u, \nabla \varphi) + \alpha\,(u, \varphi),$$

and observe that $[\cdot, \cdot]_\alpha$ is an inner product in W_0^1. If $\alpha = 1$ then $[\cdot, \cdot]_\alpha$ coincides with the standard inner product in W_0^1. For any $\alpha > 0$ and $u \in W_0^1$, we have

$$\min(\alpha, 1)\, \|u\|_{W^1}^2 \le [u, u]_\alpha \le \max(\alpha, 1)\, \|u\|_{W^1}^2.$$

Therefore, the space W_0^1 with the inner product $[\cdot,\cdot]_\alpha$ is complete. Rewriting the equation (4.19) in the form

$$[u,\varphi]_\alpha = (f,\varphi)\,, \tag{4.20}$$

we obtain by the Riesz representation theorem, that a solution $u \in W_0^1$ exists and is unique provided the right hand side of (4.20) is a bounded functional of $\varphi \in W_0^1$. The latter immediately follows from the estimate

$$|(f,\varphi)| \le \|f\|_{L^2}\|\varphi\|_{L^2} \le \alpha^{-1/2}\|f\|_{L^2}\,[\varphi,\varphi]_\alpha^{1/2}\,,$$

which finishes the proof of the existence of the resolvent.

Substituting $\varphi = u$ in (4.19) we obtain

$$\|\nabla u\|_{L^2}^2 + \alpha\|u\|_{L^2}^2 = (f,u)\,, \tag{4.21}$$

whence it follows $(R_\alpha f,f) = (u,f) \ge 0$, that is, R_α is non-negative definite. Another consequence of (4.21) is

$$\alpha\|u\|_{L^2}^2 \le \|f\|_{L^2}\|u\|_{L^2},$$

which implies $\|R_\alpha f\|_2 \le \alpha^{-1}\|f\|_2$ and, hence, $\|R_\alpha\| \le \alpha^{-1}$.

Since R is a bounded operator, in order to prove that it is self-adjoint it suffices to prove that it is symmetric, that is

$$(Rf,g) = (f,Rg) \quad \text{for all } f,g \in L^2.$$

Setting $R_\alpha f = u$, $R_\alpha g = v$, and choosing $\varphi = v$ in (4.19), we obtain

$$(\nabla u,\nabla v) + \alpha\,(u,v) = (f,R_\alpha g)\,.$$

Since the left hand side is symmetric in u,v, we conclude that the right hand side is symmetric in f,g, which implies that R_α is symmetric. □

Now we can prove the main result of this section.

THEOREM 4.6. *On any weighted manifold, the operator Dirichlet Laplace operator* $\mathcal{L} = -\Delta_\mu|_{W_0^2}$ *is a self-adjoint non-negative definite operator in* L^2, *and* $\operatorname{spec}\mathcal{L} \subset [0,+\infty)$. *Furthermore,* $\Delta_\mu|_{W_0^2}$ *is a unique self-adjoint extension of* $\Delta_\mu|_{\mathcal{D}}$ *whose domain is contained in* W_0^1.

PROOF. The fact that $\mathcal{L}$ is symmetric and non-negative definite was already verified (see (4.13) and (4.14)).

Self-adjointness. By Theorem 4.5, the resolvent $R = R_1 = (\mathcal{L}+\mathrm{id})^{-1}$ exists and is a bounded self-adjoint operator. Let us show that $\mathcal{L} = R^{-1} - \mathrm{id}$ is also a self-adjoint operator. It suffices to prove that R^{-1} with the domain W_0^2 is a self-adjoint operator. The symmetry of R^{-1} easily follows from the symmetry of $\mathcal{L}$. Therefore, $\left(R^{-1}\right)^*$ is an extension of R^{-1}, and all we need to show is that

$$\operatorname{dom}\left(R^{-1}\right)^* \subset \operatorname{dom}\left(R^{-1}\right).$$

By the definition of the adjoint operator,

$$\operatorname{dom}\left(R^{-1}\right)^* = \left\{u \in L^2 : \exists f \in L^2 \;\; \forall v \in \operatorname{dom} R^{-1} \quad \left(R^{-1}v,u\right) = (v,f)\,\right\}.$$

Since Rf is defined and is in $\operatorname{dom} R^{-1}$, we have, by the symmetry of R^{-1},

$$\left(R^{-1}v, Rf\right) = \left(v, R^{-1}Rf\right) = (v, f).$$

Comparing the above two lines, we conclude

$$\left(R^{-1}v, u\right) = \left(R^{-1}v, Rf\right) \text{ for all } v \in W_0^2,$$

whence $u = Rf$ and $u \in \operatorname{dom} R^{-1}$.

Spectrum of H. Since the inverse operator $(\mathcal{L} + \alpha \operatorname{id})^{-1}$ exists and is bounded for any $\alpha > 0$, we see that $-\alpha$ is a regular value of $\mathcal{L}$ and hence $\operatorname{spec} \mathcal{L} \subset [0, +\infty)$. The latter follows also from a general fact that the spectrum of a self-adjoint non-negative definite operator is contained in $[0, +\infty)$ (see Exercise A.26).

Uniqueness of extension. Set $\mathcal{L}_0 := -\Delta_\mu|_{\mathcal{D}}$ and suppose that $\mathcal{L}_1$ is a self-adjoint extension of $\mathcal{L}_0$ such that

$$\operatorname{dom} \mathcal{L}_1 \subset W_0^1. \tag{4.22}$$

We need to prove that $\mathcal{L}_1 = \mathcal{L}$. By (4.10), we have

$$\operatorname{dom} \mathcal{L}_0^* = \left\{u \in L^2 : \Delta_\mu u \in L^2\right\}$$

and in this domain

$$\mathcal{L}_0^* u = -\Delta_\mu u.$$

The inclusion $\mathcal{L}_0 \subset \mathcal{L}_1$ implies $\mathcal{L}_1 \subset \mathcal{L}_0^*$. Combining with (4.22) we obtain

$$\operatorname{dom} \mathcal{L}_1 \subset W_0^1 \cap \operatorname{dom} \mathcal{L}_1^* = \left\{u \in W_0^1 : \Delta_\mu u \in L^2\right\} = \operatorname{dom} \mathcal{L}.$$

Also, if $u \in \operatorname{dom} \mathcal{L}_1$ then

$$\mathcal{L}_1 u = \mathcal{L}_0^* u = -\Delta_\mu u = \mathcal{L} u,$$

whence it follows that $\mathcal{L}$ is an extension of $\mathcal{L}_1$, that is, $\mathcal{L}_1 \subset \mathcal{L}$. This implies $\mathcal{L}_1^* \supset \mathcal{L}^*$ and, hence, $\mathcal{L}_1 = \mathcal{L}$, because both operators $\mathcal{L}_1$ and $\mathcal{L}$ are self-adjoint. □

SECOND PROOF. This proof does not use Theorem 4.5 and is overall shorter but at the expense of using the theory of quadratic forms. Consider the quadratic form

$$\mathcal{E}(u, v) = (\nabla u, \nabla v)_{\vec{L}^2}$$

with the domain W_0^1. This form is obviously symmetric and, as follows from Lemma 4.3, it is closed in L^2. Hence, the form $\mathcal{E}$ has a self-adjoint generator $\mathcal{L}$ such that for all $u \in \operatorname{dom} \mathcal{L}$ and $v \in \operatorname{dom} \mathcal{E}$,

$$\mathcal{E}(u, v) = (\mathcal{L} u, v).$$

The operator $\mathcal{L}$ is non-negative definite because, for all $u \in \operatorname{dom} \mathcal{L}$,

$$(\mathcal{L} u, u) = \mathcal{E}(u, u) \geq 0.$$

The domain of $\mathcal{L}$ is dense in W_0^1 and is defined by

$$\operatorname{dom} \mathcal{L} = \left\{u \in W_0^1 : v \mapsto \mathcal{E}(u, v) \text{ is a bounded linear functional of } v \in W_0^1 \text{ in } L^2\right\}.$$

This condition means, by the Riesz representation theorem, that there exists a unique function $f \in L^2$ such that

$$\mathcal{E}(u,v) = (f,v) \quad \text{for all } v \in W_0^1. \tag{4.23}$$

Since $\mathcal{D}$ is dense in W_0^1, we can rewrite (4.23) as follows:

$$\mathcal{E}(u,v) = (f,v) \quad \text{for all } v \in \mathcal{D}. \tag{4.24}$$

Using the definitions of the distributional Laplacian and gradient, we obtain, for any $u \in \operatorname{dom}\mathcal{L}$ and $v \in \mathcal{D}$,

$$\mathcal{E}(u,v) = (\nabla u, \nabla v) = -(u, \operatorname{div}_\mu \nabla v) = -(u, \Delta_\mu v) = -(\Delta_\mu u, v),$$

and, comparing with (4.24), we see that for the distribution $\Delta_\mu u$ and for any $v \in \mathcal{D}$,

$$-(\Delta_\mu u, v) = (f,v),$$

whence $-\Delta_\mu u = f$. In particular, this means that $\Delta_\mu u \in L^2$ and hence $u \in W_0^2$; furthermore, $\mathcal{L}u = f = -\Delta_\mu u$. Conversely, it is easy to see that $u \in W_0^2$ implies $u \in \operatorname{dom}\mathcal{L}$. Hence, $W_0^2 = \operatorname{dom}\mathcal{L}$ and $\mathcal{L} = -\Delta_\mu|_{W_0^2}$, which finishes the proof of self-adjointness of $\Delta_\mu|_{W_0^2}$. □

THIRD PROOF. Here we provide yet another proof of the self-adjointness of $\mathcal{L}$, based on some properties of *closed* operators. Let us consider gradient ∇ as an operator from L^2 to $\vec{L}^2$ with the domain W_0^1. We claim that ∇ is a closed operator. Indeed, if a sequence $\{f_k\} \subset W_0^1$ is such that $f_k \to f$ in L^2 and $\nabla f_n \to \omega$ in $\vec{L}^2$ then $\{f_k\}$ is Cauchy in W_0^1 and hence converges to f in W_0^1. Therefore, $\nabla f_n \to \nabla f$ in $\vec{L}^2$ and $\omega = \nabla f$, whence we conclude that ∇ is closed.

Consider the adjoint operator ∇^* acting from $\vec{L}^2$ to L^2. By definition, we have

$$\operatorname{dom}\nabla^* = \left\{\omega \in \vec{L}^2\colon \quad f \mapsto \int_M \langle \nabla f, \omega\rangle\, d\mu \ \text{ is a bounded functional of } f \in \operatorname{dom}\nabla\right\},$$

and $\nabla^*\omega$ is a unique function from L^2 such that, for all $f \in W_0^1$,

$$\int_M \langle \nabla f, \omega\rangle\, d\mu = \int_M f\, \nabla^*\omega\, d\mu.$$

Since $\mathcal{D}$ is dense in W_0^1, we can allow f here to vary in $\mathcal{D}$ instead of $\operatorname{dom}\nabla$. Then the above identity is equivalent to the fact that $\nabla^*\omega = -\operatorname{div}_\mu \omega$ where div_μ is understood in the distributional sense. Hence, we obtain that

$$\operatorname{dom}\nabla^* = \left\{\omega \in \vec{L}^2 : \operatorname{div}_\mu \omega \in L^2\right\},$$

and $\nabla^*\omega = -\operatorname{div}_\mu \omega$ in this domain.

Finally, let us show that $\mathcal{L} = \nabla^*\nabla$, which will imply that $\mathcal{L}$ is self-adjoint. Indeed, we have

$$\begin{aligned}
\operatorname{dom}(\nabla^*\nabla) &= \{f \in \operatorname{dom}\nabla : \nabla f \in \operatorname{dom}\nabla^*\} \\
&= \{f \in W_0^1 : \operatorname{div}_\mu \nabla f \in L^2\} \\
&= \{f \in W_0^1 : \Delta_\mu f \in L^2\} \\
&= \operatorname{dom}\mathcal{L},
\end{aligned}$$

and in this domain $\nabla^*\nabla f = -\operatorname{div}_\mu \nabla f = -\Delta_\mu f = \mathcal{L}f$, which finishes the proof. □

EXAMPLE 4.7. Let us prove that in $\mathbb{R}^n$ the domain $W_0^2(\mathbb{R}^n)$ of the Dirichlet Laplace operator coincides with the Sobolev space $W^2(\mathbb{R}^n)$ (cf. Section 2.6). By Exercise 2.30, in $\mathbb{R}^n$ the space $\mathcal{D}$ is dense in W^1, which implies that $W_0^1 = W^1$. Therefore,

$$u \in W^2 \implies u \in W_0^1 \text{ and } \Delta u \in L^2 \implies u \in W_0^2$$

which means that $W^2 \subset W_0^2$.

To prove the opposite inclusion $W_0^2 \subset W^2$, we need to show that if $u \in W^1$ and $\Delta u \in L^2$ then all the second derivatives of u are also in L^2. Let us first show that if $u \in \mathcal{D}$ then

$$\sum_{i,j=1}^n \|\partial_i\partial_j u\|_{L^2}^2 = \|\Delta u\|_{L^2}^2. \tag{4.25}$$

By the definition of the Laplace operator, we have

$$\|\Delta u\|_{L^2}^2 = \int_{\mathbb{R}^n}\left(\sum\nolimits_i \partial_i^2 u\right)^2 dx = \int_{\mathbb{R}^n}\sum\nolimits_{i,j}\partial_i^2 u\,\partial_j^2 u\,dx. \tag{4.26}$$

For any two indices i,j, we obtain, using integration by parts,

$$\left(\partial_i^2 u, \partial_j^2 u\right) = -\left(\partial_i u, \partial_i\partial_j^2 u\right) = \left(\partial_j\partial_i u, \partial_i\partial_j u\right) = \|\partial_i\partial_j u\|_{L^2}^2,$$

whence (4.25) follows.

Let us now prove that if $u \in W_0^2$ and $\operatorname{supp} u$ is compact then $u \in W^2$ and

$$\sum_{i,j=1}^n \|\partial_i\partial_j u\|_{L^2}^2 \le \|\Delta u\|_{L^2}^2. \tag{4.27}$$

Fix a mollifier φ and consider the sequence of functions

$$u_k = u * \varphi_{1/k}.$$

By Lemma 2.9, $u_k \in \mathcal{D}$ and

$$\Delta u_k = (\Delta u) * \varphi_{1/k}.$$

By Theorem 2.11, we obtain

$$\|\Delta u_k\|_{L^2} \le \|\Delta u\|_{L^2},$$

which together with (4.27) implies

$$\sum_{i,j=1}^n \|\partial_i\partial_j u_k\|_{L^2}^2 \le \|\Delta u\|_{L^2}.$$

Since all norms $\|\partial_i\partial_j u_k\|_{L^2}^2$ are bounded uniformly in k, we conclude by Theorem 2.11 that $\partial_i\partial_j u \in L^2$ and (4.27) is satisfied. In particular, we have $u \in W^2$.

Now let us prove that any function $u \in W_0^2$ belongs to W^2. Let ψ be a cutoff function of the unit ball $B_1 = \{|x| < 1\}$ in $\mathbb{R}^n$ so that $\psi \in \mathcal{D}(\mathbb{R}^n)$ and $\psi \equiv 1$ on B_1. Set

$$u_k(x) = \psi\left(\frac{x}{k}\right)u(x) \tag{4.28}$$

where k is a positive integer. Then $u_k \in W^1$ and $\operatorname{supp} u_k$ is compact. Let us show that $\Delta u_k \in L^2$. Using the product rule for second order derivatives (see Exercise (2.12)), we obtain

$$\Delta u_k = k^{-2}(\Delta\psi)\left(\frac{x}{k}\right)u + 2k^{-1}(\partial_i\psi)\left(\frac{x}{k}\right)\partial_i u + \psi\left(\frac{x}{k}\right)\Delta u. \tag{4.29}$$

Obviously, the right hand side here is in L^2 and, hence, $\Delta u_k \in L^2$. Therefore, $u_k \in W_0^2$, and, by the previous part, we conclude $u_k \in W^2$.

It also follows from (4.29) that

$$\|\Delta u_k\|_{L^2} \le C\left(\|u\|_{W^1} + \|\Delta u\|_{L^2}\right),$$

where C does not depend on k. From (4.27), we obtain that, for any any multiindex α of order 2,

$$\|\partial^\alpha u_k\|_{L^2} \le C\left(\|u\|_{W^1} + \|\Delta u\|_{L^2}\right),$$

that is, the sequence $\{\partial^\alpha u_k\}_{k=1}^\infty$ is bounded in L^2. Since $u_k \xrightarrow{\mathcal{D}'} u$, we have also $\partial^\alpha u_k \xrightarrow{\mathcal{D}'} \partial^\alpha u$. By Exercise 4.8, we conclude that $\partial^\alpha u \in L^2$ and, hence, $u \in W^2$.

Exercises.

4.19. Let M be the unit ball B in $\mathbb{R}^n$. Prove that the Laplace operator Δ with domain

$$\{f \in C^2(B) : \Delta f \in L^2(B)\}$$

is not symmetric in $L^2(B)$.

4.20. Let A be an operator in $L^2(M)$ defined by $Af = -\Delta_\mu f$ with $\operatorname{dom} A = C_0^\infty(M)$. Prove that operator A is unbounded.

4.21. Prove that if $f \in C_0^\infty(M)$ and $u \in W_0^1$ then $fu \in W_0^1(M)$.

4.22. Prove that the spaces $W_0^2(M)$ and $W^2(M)$, endowed with the inner product

$$(u,v)_{W^2} = (u,v)_{W^1} + (\Delta_\mu u, \Delta_\mu v)_{L^2}, \tag{4.30}$$

are Hilbert spaces.

4.23. Prove that, for any $u \in W_0^2(M)$,

$$\|u\|_{W^1}^2 \le c\left(\|u\|_{L^2}^2 + \|\Delta_\mu u\|_{L^2}^2\right), \tag{4.31}$$

where $c = \frac{1+\sqrt{2}}{2}$.

4.24. Let $\{E_\lambda\}$ be the spectral resolution of the Dirichlet Laplace operator $\mathcal{L}$ in $L^2(M)$. Prove that, for any $f \in W_0^2(M)$,

$$\|\nabla f\|_{L^2}^2 = \int_0^\infty \lambda \|dE_\lambda f\|_{L^2}^2. \tag{4.32}$$

4.25. Prove that $\operatorname{dom} \mathcal{L}^{1/2} = W_0^1(M)$ and that (4.32) holds for any $f \in W_0^1(M)$.
HINT. Use Exercise A.13.

4.26. Prove that $\operatorname{dom} \mathcal{L}^{1/2} = \operatorname{dom}(\mathcal{L} + \mathrm{id})^{1/2}$ and, for any $f \in W_0^1(M)$,

$$\|f\|_{W^1} = \|(\mathcal{L} + \mathrm{id})^{1/2} f\|_{L^2}. \tag{4.33}$$

4.27. Prove that, for all $f \in W_0^1(M)$,

$$\|\nabla f\|_{L^2}^2 \ge \lambda_{\min} \|f\|_{L^2}^2, \tag{4.34}$$

where

$$\lambda_{\min} := \inf \operatorname{spec} \mathcal{L}. \tag{4.35}$$

4.28. Assuming that $\lambda_{\min} > 0$, prove that the weak Dirichlet problem on M

$$\begin{cases} -\Delta_\mu u = f, \\ u \in W_0^1(M), \end{cases} \tag{4.36}$$

has a unique solution u for any $f \in L^2(M)$, and that for this solution

$$\|u\|_{L^2} \le \lambda_{\min}^{-1} \|f\|_{L^2} \tag{4.37}$$

and

$$\|\nabla u\|_{L^2} \le \lambda_{\min}^{-1/2} \|f\|_{L^2}. \tag{4.38}$$

4.29. Consider the following version of the weak Dirichlet problem: given a real constant α and functions $f \in L^2(M)$, $w \in W^1(M)$, find a function $u \in L^2(M)$ that satisfies the conditions

$$\begin{cases} \Delta_\mu u + \alpha u = f, \\ u = w \mod W_0^1(M), \end{cases} \tag{4.39}$$

where the second condition means $u - w \in W_0^1(M)$. Prove that if $\alpha < \lambda_{\min}$ then the problem (4.39) has exactly one solution.

4.30. Let A be a bounded self-adjoint operator in L^2 such that, for a constant $\alpha > 0$ and for any function $f \in L^2(M)$,

$$\alpha^{-1}\|f\|_2^2 \le (Af, f)_{L^2} \le \alpha\|f\|_2^2.$$

(a) Prove that the bilinear form

$$\{f, g\} := (\nabla f, \nabla g) + (Af, g)$$

defines an inner product in W_0^1, and that W_0^1 with this inner product is a Hilbert space.

(b) Prove that, for any $h \in L^2$, the equation

$$-\Delta_\mu u + Au = h$$

has exactly one solution $u \in W_0^2$.

4.31. Prove that, for any $\alpha > 0$ and $f \in L^2(M)$, the function $u = R_\alpha f$ is the only minimizer of the functional

$$E(v) := \|\nabla v\|_2^2 + \alpha\|v - f\|_2^2,$$

in the domain $v \in W_0^1(M)$.

4.32. Prove that for any $\alpha > 0$ the operators $\nabla \circ R_\alpha : L^2(M) \to \vec{L}^2(M)$ and $\mathcal{L} \circ R_\alpha : L^2(M) \to L^2(M)$ are bounded and

$$\|\nabla \circ R_\alpha\| \le \alpha^{-1/2}, \tag{4.40}$$

$$\|\mathcal{L} \circ R_\alpha\| \le 1. \tag{4.41}$$

4.33. Prove that, for any $f \in L^2(M)$,

$$\alpha R_\alpha f \xrightarrow{L^2} f \text{ as } \alpha \to +\infty.$$

Prove that if $f \in \operatorname{dom}\mathcal{L}$ then

$$\|\alpha R_\alpha f - f\|_{L^2} \le \frac{1}{\alpha}\|\mathcal{L}f\|_{L^2}.$$

4.34. Prove that, for all $\alpha, \beta > 0$,

$$R_\alpha - R_\beta = (\beta - \alpha) R_\alpha R_\beta. \tag{4.42}$$

4.3. Heat semigroup and L^2-Cauchy problem

Let $(M, \mathbf{g}, \mu)$ be a weighted manifold. The classical Cauchy problem is the problem of finding a function $u(t, x) \in C^2(\mathbb{R}_+ \times M)$ such that

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_\mu u, & t > 0, \\ u|_{t=0} = f, \end{cases} \tag{4.43}$$

where f is a given continuous function on M and the the initial data is understood in the sense that $u(t, x) \to f(x)$ as $t \to 0$ locally uniformly in x. Obviously, if a solution $u(t, x)$ exists then it can be extended to $t = 0$ by setting $u(0, x) = f(x)$ so that $u(t, x)$ becomes continuous in $[0, +\infty) \times M$.

The techniques that has been developed so far enables us to solve an L^2-version of this problem, which is stated as follows. Consider the Dirichlet Laplace operator $\mathcal{L} = -\Delta_\mu|_{W_0^2}$ on M. The *L^2-Cauchy problem* is the problem of finding a function $u(t, x)$ on $(0, +\infty) \times M$ such that $u(t, \cdot) \in L^2(M)$ for any $t > 0$ and the following properties are satisfied:

- The mapping $t \mapsto u(t,\cdot)$ is strongly differentiable in $L^2(M)$ for all $t > 0$.
- For any $t > 0$, $u(t,\cdot) \in \operatorname{dom} \mathcal{L}$ and
$$\frac{du}{dt} = -\mathcal{L}u,$$
where $\frac{du}{dt}$ is the strong derivative of u in $L^2(M)$.
- $u(t,\cdot) \xrightarrow{L^2} f$ as $t \to 0$ where f is a given function from $L^2(M)$.

Shortly, the L^2-Cauchy problem can be written in the form
$$\begin{cases} \dfrac{du}{dt} = -\mathcal{L}u, & t > 0, \\ u|_{t=0} = f, \end{cases} \tag{4.44}$$
where all the parts are understood as above.

The problem (4.44) is reminiscent of a system of linear ordinary differential equations of the first order. Indeed, assume for a moment that $u = u(t)$ is a path in $\mathbb{R}^N$, $\mathcal{L}$ is a linear operator in $\mathbb{R}^N$, and $f \in \mathbb{R}^N$. Then the system (4.44) has a unique solution u given by
$$u = e^{-t\mathcal{L}}f,$$
where the exponential of an operator A in $\mathbb{R}^N$ is defined by
$$e^A = \mathrm{id} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots .$$
In the case when $A = -t\mathcal{L}$ is an *unbounded* operator in L^2, the exponential series does not help because the domain of the series, that is, the set of functions f where all the powers $A^k f$ are defined and the series converges, is by far too small. However, one can apply the spectral theory to define $e^{-t\mathcal{L}}$ provided $\mathcal{L}$ is a self-adjoint operator.

Let us briefly summarize the necessary information from the spectral theory (see Section A.5.4 for more details). Let $\mathcal{L}$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$, and let $\operatorname{spec}\mathcal{L}$ be its spectrum. Then any real-valued Borel function φ on $\operatorname{spec}\mathcal{L}$ determines a self-adjoint operator $\varphi(\mathcal{L})$ in $\mathcal{H}$ defined by
$$\varphi(\mathcal{L}) := \int_{\operatorname{spec}\mathcal{L}} \varphi(\lambda) dE_\lambda = \int_{-\infty}^{\infty} \varphi(\lambda) dE_\lambda \,, \tag{4.45}$$
where $\{E_\lambda\}$ is the spectral resolution of $\mathcal{L}$. The domain of $\varphi(\mathcal{L})$ is defined by
$$\operatorname{dom}\varphi(\mathcal{L}) := \left\{ f \in \mathcal{H} \colon \int_{\operatorname{spec}\mathcal{L}} |\varphi(\lambda)|^2 \, d\|E_\lambda f\|^2 < \infty \right\}, \tag{4.46}$$
and, for any $f \in \operatorname{dom}\varphi(\mathcal{L})$, we have
$$\|\varphi(A)f\|^2 = \int_{\operatorname{spec} A} |\varphi(\lambda)|^2 \, d\|E_\lambda f\|^2. \tag{4.47}$$

If φ is bounded on $\operatorname{spec}\mathcal{L}$ then the operator $\varphi(\mathcal{L})$ is bounded and

$$\|\varphi(\mathcal{L})\| \leq \sup_{\operatorname{spec}\mathcal{L}} |\varphi|.$$

In this case, $\operatorname{dom}\varphi(\mathcal{L}) = \mathcal{H}$. If φ is continuous on $\operatorname{spec}\mathcal{L}$ then

$$\|\varphi(\mathcal{L})\| = \sup_{\operatorname{spec}\mathcal{L}} |\varphi|. \tag{4.48}$$

If φ and ψ are Borel functions on $\operatorname{spec}\mathcal{L}$ and ψ is bounded then

$$\varphi(\mathcal{L}) + \psi(\mathcal{L}) = (\varphi + \psi)(\mathcal{L}) \tag{4.49}$$

and

$$\varphi(\mathcal{L})\,\psi(\mathcal{L}) = (\varphi\psi)(\mathcal{L}) \tag{4.50}$$

(cf. Exercise A.23). The relations (4.49) and (4.50) include also the identity of the domains of the both sides.

The following version of the *bounded convergence theorem* is frequently useful.

LEMMA 4.8. *Let $\mathcal{L}$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and $\{\varphi_k\}_{k=1}^{\infty}$ be a sequence of Borel functions on* $\operatorname{spec}\mathcal{L}$. *If $\{\varphi_k\}$ is uniformly bounded and converges pointwise to a Borel function φ on* $\operatorname{spec}\mathcal{L}$, *then, for any $f \in \mathcal{H}$,*

$$\varphi_k(\mathcal{L})\, f \to \varphi(\mathcal{L})\, f \quad \text{as } k \to \infty. \tag{4.51}$$

Note that the operators $\varphi_k(\mathcal{L})$ and $\varphi(\mathcal{L})$ are bounded and their common domain is $\mathcal{H}$. The convergence in (4.51) is understood in the norm of $\mathcal{H}$, which means that the sequence of operators $\varphi_k(\mathcal{L})$ converges to $\varphi(\mathcal{L})$ in the strong operator topology. In terms of the spectral resolution $\{E_\lambda\}$ of the operator $\mathcal{L}$, (4.51) can be stated as follows:

$$\int_{\operatorname{spec}\mathcal{L}} \varphi_k(\lambda)\, dE_\lambda f \longrightarrow \int_{\operatorname{spec}\mathcal{L}} \varphi(\lambda) dE_\lambda f, \tag{4.52}$$

which explains the reference to the bounded convergence theorem.

PROOF. It follows from (4.49) that

$$\varphi_k(\mathcal{L}) - \varphi(\mathcal{L}) = (\varphi_k - \varphi)(\mathcal{L}),$$

and (4.47) yields

$$\|\varphi_k(\mathcal{L})\, f - \varphi(\mathcal{L})\, f\|^2 = \|(\varphi_k - \varphi)(\mathcal{L})\, f\|^2 = \int_{\operatorname{spec}\mathcal{L}} |\varphi_k(\lambda) - \varphi(\lambda)|^2\, d\|E_\lambda f\|^2.$$

The sequence $|\varphi_k(\lambda) - \varphi(\lambda)|$ tends to 0 as $k \to \infty$ for any $\lambda \in \operatorname{spec}\mathcal{L}$. Since this sequence is bounded and the measure $d\|E_\lambda f\|^2$ is finite (its total being $\|f\|^2$), the classical bounded convergence theorem yields that

$$\int_{\operatorname{spec}\mathcal{L}} |\varphi_k(\lambda) - \varphi(\lambda)|^2\, d\|E_\lambda f\|^2 \longrightarrow 0 \text{ as } k \to \infty,$$

whence (4.51) follows. $\square$

Given an operator $\mathcal{L}$ in a Hilbert space $\mathcal{H}$ and a vector $f \in \mathcal{H}$, consider the associated Cauchy problem to find a path $u : (0, +\infty) \to \mathcal{H}$ so that the following conditions are satisfied:

- $u(t)$ is strongly differentiable for all $t > 0$.
- For any $t > 0$, $u(t) \in \operatorname{dom} \mathcal{L}$ and
$$\frac{du}{dt} = -\mathcal{L}u,$$
where $\frac{du}{dt}$ is the strong derivative of u.
- $u(t) \to f$ as $t \to 0$, where the convergence is strong, that is, in the norm of $\mathcal{H}$.

If $\mathcal{L}$ is a self-adjoint, non-negative definite operator, that is, $\operatorname{spec} \mathcal{L} \subset [0, +\infty)$, then this problem is solved by means of the following family $\{P_t\}_{t \geq 0}$ of operators:
$$P_t := e^{-t\mathcal{L}} = \int_{\operatorname{spec} \mathcal{L}} e^{-t\lambda} dE_\lambda = \int_0^\infty e^{-t\lambda} dE_\lambda.$$
The family $\{P_t\}_{t \geq 0}$ is called the *heat semigroup* associated with $\mathcal{L}$. In particular, we have $P_0 = \mathrm{id}$.

THEOREM 4.9. *For any non-negative definite, self-adjoint operator $\mathcal{L}$ in a Hilbert space $\mathcal{H}$, the heat semigroup $P_t = e^{-t\mathcal{L}}$ satisfies the following properties.*

(*i*) *For any $t \geq 0$, P_t is a bounded self-adjoint operator, and*
$$\|P_t\| \leq 1. \tag{4.53}$$

(*ii*) *The family $\{P_t\}$ satisfies the semigroup identity:*
$$P_t P_s = P_{t+s}, \tag{4.54}$$
for all $t, s \geq 0$.

(*iii*) *The mapping $t \mapsto P_t$ is strongly continuous on $[0, +\infty)$. That is, for any $t \geq 0$ and $f \in \mathcal{H}$,*
$$\lim_{s \to t} P_s f = P_t f, \tag{4.55}$$
where the limit is understood in the norm of $\mathcal{H}$. In particular, for any $f \in \mathcal{H}$,
$$\lim_{t \to 0+} P_t f = f. \tag{4.56}$$

(*iv*) *For all $f \in \mathcal{H}$ and $t > 0$, we have $P_t f \in \operatorname{dom} \mathcal{L}$ and*
$$\frac{d}{dt}(P_t f) = -\mathcal{L}(P_t f), \tag{4.57}$$
where $\frac{d}{dt}$ is the strong derivative in $\mathcal{H}$.

Consequently, the path $u = P_t f$ solves the Cauchy problem in $\mathcal{H}$ for any $f \in \mathcal{H}$.

The properties (i)-(iv) mean that $\{P_t\}_{t\geq 0}$ is a strongly continuous contraction semigroup in $\mathcal{H}$ with generator $\mathcal{L}$.

PROOF. The fact that $u = P_t f$ solves the Cauchy problem is obviously contained in (iii) and (iv).

(i) By (4.48), we have

$$\|P_t\| = \|e^{-t\mathcal{L}}\| = \sup_{\lambda\in\operatorname{spec}\mathcal{L}} e^{-t\lambda} \leq \sup_{\lambda\in[0,+\infty)} e^{-t\lambda} = 1.$$

(ii) This follows from the property of the exponential function $e^{-t\lambda}e^{-s\lambda} = e^{-(t+s)\lambda}$ and from (4.50).

(iii) The family of functions $\left\{e^{-s\lambda}\right\}_{s\geq 0}$ is uniformly bounded in $\lambda \in [0,+\infty)$ and tends pointwise to $e^{-t\lambda}$ as $s \to t$. Hence, by Lemma 4.8, for any $f \in \mathcal{H}$,

$$P_s f = \int_0^\infty e^{-s\lambda} dE_\lambda f \longrightarrow \int_0^\infty e^{-t\lambda} dE_\lambda f = P_t f.$$

(iv) Fix $t > 0$ and consider the functions $\varphi(\lambda) = \lambda$, $\psi(\lambda) = e^{-t\lambda}$, and

$$\Phi(\lambda) := \varphi(\lambda)\psi(\lambda) = \lambda e^{-t\lambda}.$$

Since $\psi(\lambda)$ is bounded on $[0,+\infty)$, (4.50) yields

$$\varphi(\mathcal{L})\psi(\mathcal{L}) = (\varphi\psi)(\mathcal{L}) = \Phi(\mathcal{L}),$$

that is

$$\mathcal{L}e^{-t\mathcal{L}} = \Phi(\mathcal{L}).$$

Since $\Phi(\lambda)$ is bounded on $[0,+\infty)$, the operator $\Phi(\mathcal{L})$ is bounded and, hence, $\operatorname{dom}\Phi(\mathcal{L}) = \mathcal{H}$. Therefore, $\operatorname{dom}\left(\mathcal{L}e^{-t\mathcal{L}}\right) = \mathcal{H}$ whence it follows that $\operatorname{ran} e^{-t\mathcal{L}} \subset \operatorname{dom}\mathcal{L}$, that is, $P_t f \in \operatorname{dom}\mathcal{L}$ for any $f \in \mathcal{H}$.

For any $f \in \mathcal{H}$, we have

$$\frac{d}{dt}(P_t f) = \lim_{s\to 0}\frac{P_{t+s}f - P_t f}{s} = \lim_{s\to 0}\int_0^\infty \frac{e^{-s\lambda}-1}{s}e^{-t\lambda} dE_\lambda f, \tag{4.58}$$

where the limit is understood in the norm of $\mathcal{H}$. Obviously, we have

$$\lim_{s\to 0}\frac{e^{-s\lambda}-1}{s}e^{-t\lambda} = -\lambda e^{-t\lambda}.$$

We claim that the function

$$\lambda \mapsto \frac{e^{-s\lambda}-1}{s}e^{-t\lambda}$$

is bounded on $[0,+\infty)$ uniformly in $s \in [-\varepsilon,\varepsilon]$ where ε is fixed in the range $0 < \varepsilon < t$, say $\varepsilon = t/2$. To prove this, let us apply the inequality

$$\left|e^{\theta}-1\right| \leq |\theta| e^{|\theta|}, \tag{4.59}$$

for any $\theta \in \mathbb{R}$, which follows from the mean value theorem. Setting $\theta = -\lambda s$ in (4.59), we obtain

$$\left|e^{-s\lambda}-1\right| \leq \lambda|s| e^{\lambda|s|},$$

whence

$$\left|\frac{e^{-\lambda s}-1}{s}e^{-t\lambda}\right| \leq \lambda e^{-\lambda t}e^{\lambda|s|} \leq \lambda e^{-\lambda(t-\varepsilon)}. \tag{4.60}$$

Since the right hand side is a bounded function of λ, the above claim is proved.

By Lemma 4.8, the right hand side of (4.58) is equal to

$$\int_0^\infty (-\lambda)\, e^{-t\lambda} dE_\lambda f = -\Phi(\mathcal{L})\, f = -\mathcal{L}(P_t f),$$

which was to be proved. □

The existence in the Cauchy problem in $\mathcal{H}$, which follows from Theorem 4.9, is complemented by the following uniqueness result.

THEOREM 4.10. *Let $\mathcal{L}$ be a non-negative definite operator in a Hilbert space $\mathcal{H}$. Then the corresponding Cauchy problem in $\mathcal{H}$ has at most one solution for any initial vector $f \in \mathcal{H}$.*

Note that operator $\mathcal{L}$ here is not necessarily self-adjoint.

PROOF. Assuming that u solves the Cauchy problem, let us prove that the function

$$J(t) := \|u(t,\cdot)\|^2 = (u(t), u(t))$$

is decreasing in $t \in (0,+\infty)$. For that, we use the following product rule for strong derivatives: if $u(t)$ and $v(t)$ are strongly differentiable paths in $\mathcal{H}$ then the numerical function $t \mapsto (u(t), v(t))$ is differentiable and

$$\frac{d}{dt}(u,v) = (\frac{d}{dt}u, v) + (u, \frac{d}{dt}v)$$

(cf. Exercise 4.46). In particular, we obtain that the function $J(t)$ is differentiable on $(0,+\infty)$ and

$$J'(t) = \frac{d}{dt}(u,u) = 2(u, \frac{du}{dt}) = -2(u, \mathcal{L}u) \leq 0,$$

where we have used the fact that the operator $\mathcal{L}$ is non-negative definite. We conclude that $J(t)$ is a decreasing function.

To prove the uniqueness of the solution is suffices to show that $f = 0$ implies $u = 0$. Indeed, if $u(t) \to 0$ as $t \to 0$ then also $J(t) \to 0$. Since $J(t)$ is non-negative and decreasing, we conclude $J(t) \equiv 0$ and $u(t) = 0$, which was to be proved. □

On any weighted manifold M, set

$$P_t = e^{-t\mathcal{L}},$$

where $\mathcal{L} = -\Delta_\mu|_{W_0^2}$ is the Dirichlet Laplace operator. Theorems 4.9 and 4.10 immediately imply the following result.

COROLLARY 4.11. *For any function $f \in L^2(M)$, the L^2-Cauchy problem (4.44) has a solution. Moreover, this solution is unique and is given by $u = P_t f$.*

EXAMPLE 4.12. Let $p_t(x)$ be the Gauss-Weierstrass function defined by (1.8), that is,

$$p_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right), \tag{4.61}$$

where $t > 0$ and $x \in \mathbb{R}^n$. By Lemma 2.18, for any $f \in L^2(\mathbb{R}^n)$ and any $t > 0$, the function $u_t = p_t * f$ is in $L^2(\mathbb{R}^n)$ and

$$u_t \xrightarrow{L^2} f \text{ as } t \to 0.$$

It follows from Theorem 2.22, that if $f \in W^2(\mathbb{R}^n)$ then $u_t \in W^2(\mathbb{R}^n)$, u_t is strongly differentiable in L^2, and

$$\frac{du_t}{dt} = \Delta u_t.$$

As it was shown in Example 4.7, the domain $\operatorname{dom}\mathcal{L} = W_0^2(\mathbb{R}^n)$ of the Dirichlet Laplace operator in $\mathbb{R}^n$ coincides[2] with $W^2(\mathbb{R}^n)$. Hence, $u_t \in \operatorname{dom}\mathcal{L}$ and we obtain that the path $t \mapsto u_t$ solves the L^2-Cauchy problem.

By Corollary 4.11, the unique solution to the L^2-Cauchy problem is given by $e^{-t\mathcal{L}}f$. We conclude that, for all $t > 0$,

$$e^{-t\mathcal{L}}f = p_t * f = \int_{\mathbb{R}^n} p_t(x-y) f(y)\, dy, \tag{4.62}$$

for any $f \in W^2$. Since W^2 is dense in L^2 and all parts of (4.62) are continuous in $f \in L^2$, we obtain that (4.62) holds for all $f \in L^2$.

Recall that, in Section 2.7, the heat semigroup $\{P_t\}$ in $\mathbb{R}^n$ was defined by $P_t f = p_t * f$, whereas in the present context, we have defined it by $P_t = e^{-t\mathcal{L}}$. The identity (4.62) shows that these two definitions are equivalent. Another point of view on (4.62) is that the operator $P_t = e^{-t\mathcal{L}}$ in $\mathbb{R}^n$ has the *integral kernel* $p_t(x-y)$. As we will see in Chapter 7, the heat semigroup has the integral kernel on any manifold, although no explicit formula can be obtained.

Exercises.

4.35. Fix a function $f \in L^2$.

(a) Prove that the function $\varphi(t) := (P_t f, f)$ on $t \in [0, +\infty)$ is non-negative, decreasing, continuous, and log-convex.

(b) Prove that the function $\psi(t) := \|\nabla P_t f\|_2^2$ is decreasing on $(0, +\infty)$ and

$$\int_0^\infty \psi(t)\, dt \le \frac{1}{2}\|f\|_{L^2}^2.$$

4.36. Prove that, for any $f \in W_0^1$, such that $\|f\|_{L^2} = 1$,

$$\|P_t f\|_{L^2} \ge \exp\left(-t\int_M |\nabla f|^2 d\mu\right), \tag{4.63}$$

for any $t > 0$.

HINT. Use Exercise 4.25 and 4.35.

[2] In fact, we need here only the inclusion $W^2(\mathbb{R}^n) \subset \operatorname{dom}\mathcal{L}$, which follows from $W^1(\mathbb{R}^n) = W_0^1(\mathbb{R}^n)$ (see Exercise 2.30).

4.37. Prove that, for all $f \in L^2$ and all $t > 0$,

$$\|\Delta_\mu (P_t f)\|_{L^2} \leq \frac{e}{t}\|f\|_{L^2}. \tag{4.64}$$

and

$$\|\nabla (P_t f)\|_{L^2} \leq \sqrt{\frac{e}{t}}\|f\|_{L^2}. \tag{4.65}$$

4.38. For any $t > 0$, define a quadratic form $\mathcal{E}_t(f)$ by

$$\mathcal{E}_t(f) = \left(\frac{f - P_t f}{t}, f\right)_{L^2}, \tag{4.66}$$

for all $f \in L^2$.

(a) Prove that $\mathcal{E}_t(f)$ is increasing as t is decreasing.

(b) Prove that $\lim_{t\to 0} \mathcal{E}_t(f)$ is finite if and only if $f \in W_0^1$, and

$$\lim_{t\to 0} \mathcal{E}_t(f) = \int_M |\nabla f|^2 \, d\mu.$$

(c) Define a bilinear form $\mathcal{E}_t(f, g)$ in L^2 by

$$\mathcal{E}_t(f, g) = \left(\frac{f - P_t f}{t}, g\right)_{L^2}.$$

Prove that if $f, g \in W_0^1$ then

$$\mathcal{E}_t(f, g) \to \int_M \langle \nabla f, \nabla g\rangle \, d\mu \text{ as } t \to 0. \tag{4.67}$$

4.39. Prove that if $f \in W_0^2$ then, for all $t > 0$,

$$\|P_t f - f\|_{L^2} \leq t\|\Delta_\mu f\|_{L^2}, \tag{4.68}$$

REMARK. *Recall that, by Theorem 4.9, if $f \in L^2$ then $P_t f \xrightarrow{L^2} f$ as $t \to 0$. The estimate (4.68) implies a linear decay of $\|P_t f - f\|_{L^2}$ as $t \to 0$ provided $f \in W_0^2$.*

4.40. Prove that if $f \in W_0^1$ then

$$\|P_t f - f\|_{L^2} \leq t^{1/2}\|\nabla f\|_{L^2}. \tag{4.69}$$

HINT. Use Exercise 4.25 or argue as in Lemma 2.20.

4.41. Prove that if $f \in W_0^2$ then

$$\frac{P_t f - f}{t} \xrightarrow{L^2} \Delta_\mu f \text{ as } t \to 0. \tag{4.70}$$

4.42. Prove that, for any $f \in L^2$,

$$\frac{P_t f - f}{t} \xrightarrow{\mathcal{D}'} \Delta_\mu f \text{ as } t \to 0,$$

where $\Delta_\mu f$ is understood in the distributional sense.

4.43. Prove that if $f \in L^2$ and, for some $g \in L^2$,

$$\frac{P_t f - f}{t} \xrightarrow{L^2} g \text{ as } t \to 0$$

then $f \in W_0^2$ and $g = \Delta_\mu f$.

4.44. Let $f \in W_0^2$ be such that $\Delta_\mu f = 0$ in an open set $\Omega \subset M$. Consider a path

$$u(t) = \begin{cases} P_t f, & t > 0, \\ f, & t \leq 0. \end{cases}$$

Prove that $u(t)$ satisfies in $\mathbb{R} \times \Omega$ the heat equation $\frac{du}{dt} = \Delta_\mu u$ in the following sense: the path $t \mapsto u(t)$ is strongly differentiable in $L^2(\Omega)$ for all $t \in \mathbb{R}$ and the derivative $\frac{du}{dt}$ is equal to $\Delta_\mu u$ where Δ_μ is understood in the distributional sense.

4.45. Prove that if $f \in W_0^1$ then

$$P_t f \xrightarrow{W^1} f \text{ as } t \to 0.$$

and if $f \in W_0^2$ then

$$P_t f \xrightarrow{W^2} f \text{ as } t \to 0.$$

4.46. (*Product rule for strong derivatives*)

(a) Let $\mathcal{H}$ be a Hilbert space, I be an interval in $\mathbb{R}$, and $u(t), v(t) : I \to \mathcal{H}$ be strongly differentiable paths. Prove that

$$\frac{d}{dt}(u, v) = (u, \frac{dv}{dt}) + (\frac{du}{dt}, v).$$

(b) Consider the mappings $u : I \to L^p(M)$ and $v : I \to L^q(M)$ where I is an interval in $\mathbb{R}$ and $p, q \in [1, +\infty]$. Prove that if u and v are continuous then the function $w(t) = u(t)v(t)$ is continuous as a mapping from I to $L^r(M)$ where r is defined by the equation

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

(c) Prove that if u and v as above are strongly differentiable then w is also strongly differentiable and

$$\frac{dw}{dt} = u\frac{dv}{dt} + \frac{du}{dt}v.$$

4.47. For any open set $\Omega \subset M$, denote by $C_b(\Omega)$ the linear space of all bounded continuous functions on Ω with the sup-norm. Let $u(t, x)$ be a continuous function on $I \times M$ where I is an open interval in $\mathbb{R}$, and let the partial derivative $\frac{\partial u}{\partial t}$ be also continuous in $I \times M$. Prove that, for any relatively compact open set $\Omega \subset M$, the path $u(t, \cdot) : I \to C_b(\Omega)$ is strongly differentiable, and its strong derivative $\frac{du}{dt}$ coincides with the partial derivative $\frac{\partial u}{\partial t}$.

4.48. Let $\mathcal{H}$ be a Hilbert space.

(a) Let $u(t) : [a, b] \to \mathcal{H}$ be a continuous path. Prove that, for any $x \in \mathcal{H}$, the functions $t \mapsto (u(t), x)$ and $t \mapsto \|u(t)\|$ are continuous in $t \in [a, b]$, and

$$\left|\int_a^b (u(t), x)\, dt\right| \leq \left(\int_a^b \|u(t)\| dt\right) \|x\|.$$

Conclude that there exists a unique vector $U \in \mathcal{H}$ such that

$$\int_a^b (u(t), x)\, dt = (U, x) \text{ for all } x \in \mathcal{H},$$

which allows to define $\int_a^b u(t)\, dt$ by

$$\int_a^b u(t)\, dt := U.$$

Prove that

$$\left\|\int_a^b u(t)\, dt\right\| \leq \int_a^b \|u(t)\|\, dt.$$

(b) (*Fundamental theorem of calculus*) Let $u(t) : [a, b] \to \mathcal{H}$ be a strongly differentiable path. Prove that if the strong derivative $u'(t)$ is continuous in $[a, b]$ then

$$\int_a^b u'(t)\, dt = u(b) - u(a).$$

4.49. Let $u : [a, b] \to L^1(M, \mu)$ be a continuous paths in L^1. Prove that there exists an function $w \in L^1(N, d\nu)$ where $N = [a, b] \times M$ and $d\nu = dt d\mu$, such that $w(t, \cdot) = u(t)$ for any $t \in [a, b]$.

4.50. (*Chain rule for strong derivatives*) Let $u(t) : (a,b) \to L^2(M)$ be a strongly differentiable path. Consider a function $\psi \in C^1(\mathbb{R})$ such that

$$\psi(0) = 0 \text{ and } \sup|\psi'| < \infty. \tag{4.71}$$

Prove that the path $\psi(u(t))$ is also strongly differentiable in $t \in (a,b)$ and

$$\frac{d\psi(u)}{dt} = \psi'(u)\frac{du}{dt}.$$

4.51. Let $\Phi(\lambda)$ be a continuous function on $[0,+\infty)$ of a subexponential growth; that is, for any $\varepsilon > 0$,

$$\Phi(\lambda) = o\left(e^{\varepsilon\lambda}\right) \quad \text{as } \lambda \to +\infty. \tag{4.72}$$

Let $\mathcal{L}$ be a non-negative definite self-adjoint operator in a Hilbert space $\mathcal{H}$. Fix $f \in \mathcal{H}$ and consider the path $v : \mathbb{R}_+ \to \mathcal{H}$ defined by

$$v(t) := \int_0^\infty \Phi(\lambda) e^{-t\lambda} dE_\lambda f, \tag{4.73}$$

where $\{E_\lambda\}$ is the spectral resolution of $\mathcal{L}$. Prove that, for any $t > 0$, $v(t) \in \operatorname{dom}\mathcal{L}$, the strong derivative $\frac{dv}{dt}$ exists, and

$$\frac{dv}{dt} = -\int_0^\infty \lambda\Phi(\lambda) e^{-t\lambda} dE_\lambda f = -\mathcal{L}v(t). \tag{4.74}$$

Conclude that the strong derivative $\frac{d^k v}{dt^k}$ of any order $k \in \mathbb{N}$ exists and

$$\frac{d^k v}{dt^k} = (-\mathcal{L})^k v(t). \tag{4.75}$$

4.52. Let $\mathcal{L}$ be a non-negative definite self-adjoint operator in a Hilbert space $\mathcal{H}$. For any $t \in \mathbb{R}$, consider *the wave operators*

$$C_t = \cos\left(t\mathcal{L}^{1/2}\right) \quad \text{and} \quad S_t = \sin\left(t\mathcal{L}^{1/2}\right).$$

(a) Prove that C_t and S_t are bounded self-adjoint operators.

(b) Prove that, for all $f, g \in \operatorname{dom}\mathcal{L}^{1/2}$, the function

$$u(t) = C_t f + S_t g$$

is strongly differentiable in t and satisfies the initial data

$$u|_{t=0} = f \quad \text{and} \quad \left.\frac{du}{dt}\right|_{t=0} = \mathcal{L}^{1/2} g.$$

(c) Prove that, for any $f \in \operatorname{dom}\mathcal{L}$, both functions $C_t f$ and $S_t f$ are twice strongly differentiable in t and satisfy the wave equation

$$\frac{d^2 u}{dt^2} = -\mathcal{L}u,$$

where $\frac{d^2}{dt^2}$ is the second strong derivative.

(d) (*A transmutation formula*) Prove the following relation between the heat and wave operators:

$$e^{-t\mathcal{L}} = \int_0^\infty \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{s^2}{4t}\right) C_s ds, \tag{4.76}$$

where the integral is understood in the sense of the weak operator topology (cf. Lemma 5.10).

4.53. Let $\varphi(t)$ be a continuous real-valued function on an interval (a,b), $a < b$, and assume that $\varphi(t)$ is right differentiable at any point $t \in (a,b)$. Prove that if $\varphi'(t) \leq 0$ for all $t \in (a,b)$ (where φ' stands for the right derivative) then function φ is monotone decreasing on (a,b).

4.54. Consider the right Cauchy problem in a Hilbert space $\mathcal{H}$: to find a path $u : (0, +\infty) \to \mathcal{H}$ so that the following conditions are satisfied:

(i) $u(t)$ is continuous and strongly right differentiable for all $t > 0$;

(ii) For any $t > 0$, $u(t) \in \operatorname{dom} \mathcal{L}$ and

$$\frac{du}{dt} = -\mathcal{L}u,$$

where $\frac{du}{dt}$ is the strong right derivative of u.

(iii) $u(t) \to f$ as $t \to 0$, where f is a given element of $\mathcal{H}$.

Prove the uniqueness of the path $u(t)$ for any given f.

Notes

The main result of this Chapter is Theorem 4.6 that guaranties the self-adjointness of the Dirichlet Laplace operator. In the present form it was proved in [**58**]. We give three different proofs using different tools: resolvent, quadratic forms, and the adjoint operator ∇^*, respectively (the latter being from [**58**]).

Construction of the heat semigroup in Theorem 4.9 follows the standard routine of the spectral theory. Different methods for the construction of the heat semigroup (concurrently with the associated diffusion process on M) can be found in [**16**], [**271**].

CHAPTER 5

Weak maximum principle and related topics

Here we study those properties of the heat semigroup that are related to inequalities. Recall that if $u(t,x)$ is a solution the classical bounded Cauchy problem in $\mathbb{R}^n$ with the initial function f, then by Theorem 1.3 $f \geq 0$ implies $u \geq 0$ and $f \leq 1$ implies $u \leq 1$. Our purpose is to obtain similar results for the heat semigroup $P_t = e^{-t\mathcal{L}}$ on any weighted manifold, where $\mathcal{L}$ is the Dirichlet Laplace operator. Such properties of the heat semigroup are called *Markovian*. Obviously, the Markovian properties cannot be extracted just from the fact that $\mathcal{L}$ is a non-negative definite self-adjoint operator; one has to take into account the fact that the solutions are *numerical* functions, but not just elements of an abstract Hilbert space.

5.1. Chain rule in W_0^1

Let $(M, \mathbf{g}, \mu)$ be a weighted manifold.

LEMMA 5.1. *Let ψ be a C^∞-function on $\mathbb{R}$ such that*

$$\psi(0) = 0 \ \text{ and } \ \sup_{t\in\mathbb{R}} |\psi'(t)| < \infty. \tag{5.1}$$

Then $u \in W_0^1(M)$ implies $\psi(u) \in W_0^1(M)$ and

$$\nabla\psi(u) = \psi'(u)\nabla u. \tag{5.2}$$

PROOF. If $u \in C_0^\infty$ then obviously $\psi(u)$ is also in C_0^∞ and hence in W_0^1, and the chain rule (5.2) is trivial (cf. Exercise 3.4).

An arbitrary function $u \in W_0^1$ can be approximated by a sequence $\{u_k\}$ of C_0^∞-functions, which converges to u in W^1-norm, that is,

$$u_k \xrightarrow{L^2} u \ \text{ and } \ \nabla u_k \xrightarrow{L^2} \nabla u.$$

By selecting a subsequence, we can assume that also $u_k(x) \to u(x)$ for almost all $x \in M$.

By (5.1) we have $|\psi(u)| \leq C|u|$ where $C = \sup|\psi'|$, whence it follows that $\psi(u) \in L^2$. The boundedness of ψ' implies also that $\psi'(u)\nabla u \in \vec{L}^2$. Let us show that

$$\psi(u_k) \xrightarrow{L^2} \psi(u) \quad \text{and} \quad \nabla\psi(u_k) \xrightarrow{L^2} \psi'(u)\nabla u, \tag{5.3}$$

which will imply that the distributional gradient of $\psi(u)$ is equal to $\psi'(u)\nabla u$ (see Lemma 4.2). The latter, in turn, yields that $\psi(u)$ is in W^1 and, moreover, in W_0^1.

The convergence $\psi(u_k) \xrightarrow{L^2} \psi(u)$ trivially follows from $u_k \xrightarrow{L^2} u$ and

$$|\psi(u_k) - \psi(u)| \leq C|u_k - u|.$$

To prove the second convergence in (5.3) observe that

$$\begin{aligned} |\nabla\psi(u_k) - \psi'(u)\nabla u| &= |\psi'(u_k)\nabla u_k - \psi'(u)\nabla u| \\ &\leq |\psi'(u_k)(\nabla u_k - \nabla u)| + |(\psi'(u_k) - \psi'(u))\nabla u|, \end{aligned}$$

whence

$$\|\nabla\psi(u_k) - \psi'(u)\nabla u\|_{L^2} \leq C\|\nabla u_k - \nabla u\|_{L^2} + \|(\psi'(u_k) - \psi'(u))\nabla u\|_{L^2}. \tag{5.4}$$

The first term on the right hand side of (5.4) goes to 0 because $\nabla u_k \xrightarrow{L^2} \nabla u$. By construction, we have also $u_k(x) \to u(x)$ a.e. , whence

$$\psi'(u_k) - \psi'(u) \longrightarrow 0 \text{ a.e.}$$

Since

$$|\psi'(u_k) - \psi'(u)|^2 |\nabla u|^2 \leq 4C^2 |\nabla u|^2$$

and the function $|\nabla u|^2$ is integrable on M, we conclude by the dominated convergence theorem that

$$\int_M |\psi'(u_k) - \psi'(u)|^2 |\nabla u|^2 d\mu \longrightarrow 0,$$

which finishes the proof. □

LEMMA 5.2. *Let $\{\psi_k(t)\}$ be a sequence of C^∞-smooth functions on $\mathbb{R}$ such that*

$$\psi_k(0) = 0 \quad \textit{and} \quad \sup_k \sup_{t\in\mathbb{R}} |\psi_k'(t)| < \infty. \tag{5.5}$$

Assume that, for some functions $\psi(t)$ and $\varphi(t)$ on $\mathbb{R}$,

$$\psi_k(t) \to \psi(t) \quad \textit{and} \quad \psi_k'(t) \to \varphi(t) \quad \textit{for all } t \in \mathbb{R}. \tag{5.6}$$

Then, for any $u \in W_0^1(M)$, the function $\psi(u)$ is also in $W_0^1(M)$ and

$$\nabla\psi(u) = \varphi(u)\nabla u.$$

PROOF. The function $\psi(u)$ is the pointwise limit of measurable functions $\psi_k(u)$ and, hence, is measurable; by the same argument, $\varphi(u)$ is also measurable. By (5.5), there is a constant C such that

$$|\psi_k(t)| \leq C|t|, \tag{5.7}$$

for all k and $t \in \mathbb{R}$, and the same holds for function ψ. Therefore, $|\psi(u)| \leq C|u|$, which implies $\psi(u) \in L^2(M)$. By (5.5), we have also $|\varphi(t)| \leq C$, whence $\varphi(u)\nabla u \in \vec{L}^2$.

Since each function ψ_k is smooth and satisfies (5.1), Lemma 5.1 yields that

$$\psi_k(u) \in W_0^1(M) \text{ and } \nabla\psi_k(u) = \psi_k'(u)\nabla u.$$

Let us show that

$$\psi_k(u) \xrightarrow{L^2} \psi(u) \quad \text{and} \quad \nabla\psi_k(u) \xrightarrow{L^2} \varphi(u)\nabla u, \tag{5.8}$$

which will settle the claim. The dominated convergence theorem implies that

$$\int_M |\psi_k(u) - \psi(u)|^2 \, d\mu \longrightarrow 0,$$

because the integrand functions tend pointwise to 0 as $k \to \infty$ and, by (5.7),

$$|\psi_k(u) - \psi(u)|^2 \le 4C^2u^2,$$

whereas u^2 is integrable on M. Similarly, we have

$$\int_M |\nabla\psi_k(u) - \varphi(u)\nabla u|^2 \, d\mu = \int_M |\psi_k'(u) - \varphi(u)|^2 |\nabla u|^2 \, d\mu \longrightarrow 0,$$

because the sequence of functions $|\psi_k'(u) - \varphi(u)|^2 |\nabla u|^2$ tends pointwise to 0 as $k \to \infty$ and is uniformly bounded by the integrable function $4C^2 |\nabla u|^2$. $\square$

EXAMPLE 5.3. Consider the functions

$$\psi(t) = t_+ \quad \text{and} \quad \varphi(t) = \begin{cases} 1, & t > 0, \\ 0, & t \le 0, \end{cases}$$

which can be approximated as in (5.6) as follows. Choose $\psi_1(t)$ to be any smooth function on $\mathbb{R}$ such that

$$\psi_1(t) = \begin{cases} t-1, & t \ge 2, \\ 0, & t \le 0 \end{cases}$$

(see Fig. 5.1). Such function $\psi_1(t)$ can be obtained by twice integrating a suitable function from $C_0^\infty(0,2)$.

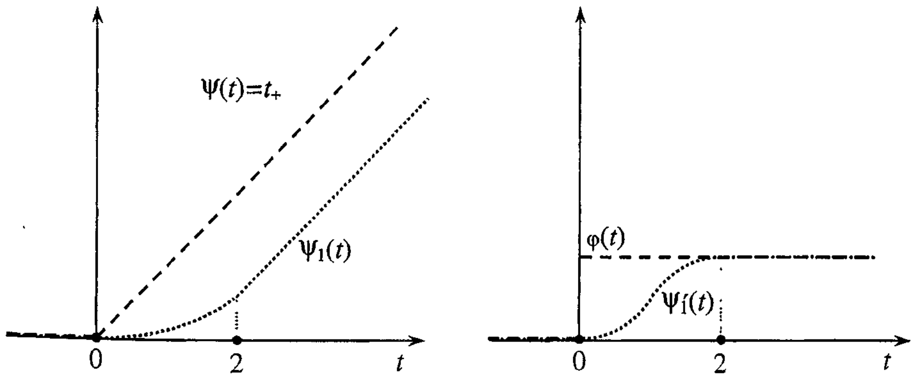

FIGURE 5.1. Functions $\psi(t) = t_+$ and $\psi_1(t)$ and their derivatives

Then define ψ_k by

$$\psi_k(t) = \frac{1}{k}\psi_1(kt).$$

If $t \le 0$ then $\psi_k(t) = 0$. If $t > 0$ then, for large enough k, we have $kt > 2$ whence

$$\psi_k(t) = \frac{1}{k}(kt - 1) = t - \frac{1}{k} \to t \text{ as } k \to \infty.$$

Hence, $\psi_k(t) \to \psi(t)$ for all $t \in \mathbb{R}$. Similarly, if $t \le 0$ then $\psi_k'(t) = 0$, and, for $t > 0$,

$$\psi_k'(t) = \psi_1'(kt) \to 1 \text{ as } k \to \infty.$$

Hence, $\psi_k'(t) \to \varphi(t)$ for all $t \in \mathbb{R}$.

By Lemma 5.2, we obtain that $u_+ \in W_0^1$ and

$$\nabla u_+ = \begin{cases} \nabla u, & u > 0, \\ 0, & u \le 0. \end{cases} \tag{5.9}$$

Applying this to function $(-u)$, we obtain $u_- \in W_0^1$ and

$$\nabla u_- = \begin{cases} 0, & u \ge 0, \\ -\nabla u, & u < 0. \end{cases} \tag{5.10}$$

Consequently, since $\nabla u_+ = \nabla u_- = 0$ on the set $\{u = 0\}$, we obtain

$$\nabla u = 0 \text{ on } \{u = 0\}. \tag{5.11}$$

Of course, if the set $\{u = 0\}$ has measure 0 then (5.11) is void because ∇u is defined up to a set of measure 0, anyway. However, if the set $\{u = 0\}$ has a positive measure then the identity (5.11) is highly non-trivial. In particular, (5.11) implies that if u, v are two functions from W_0^1 such that $u = v$ on some set S then $\nabla u = \nabla v$ on S.

Similarly, $u \in W_0^1$ implies $(u - c)_+ \in W_0^1$ for any $c \ge 0$, and

$$\nabla (u - c)_+ = \begin{cases} \nabla u, & u > c, \\ 0, & u \le c. \end{cases} \tag{5.12}$$

Since $|u| = u_+ + u_-$, it follows from (10.20), (5.10), (5.11) that

$$\nabla |u| = \operatorname{sgn}(u) \nabla u. \tag{5.13}$$

Alternatively, this can be obtained directly from Lemma 5.2 with functions $\psi(t) = |t|$ and $\varphi(t) = \operatorname{sgn}(t)$.

LEMMA 5.4. *Let u be non-negative a function from $W_0^1(M)$. Then there exists a sequence $\{u_k\}$ of non-negative functions from $C_0^\infty(M)$ such that $u_k \stackrel{W^1}{\to} u$.*

PROOF. By definition, there is a sequence $\{v_k\}$ of functions from $C_0^\infty(M)$ such that $v_k \stackrel{W^1}{\to} u$. Let ψ be a smooth non-negative function on $\mathbb{R}$ satisfying (5.1). By (5.3), we have

$$\psi(v_k) \stackrel{W^1}{\to} \psi(u).$$

Observe that $\psi(v_k) \ge 0$ and $\psi(v_k) \in C_0^\infty(M)$. Hence, the function $\psi(u) \in W_0^1$ can be approximated in W^1-norm by a sequence of non-negative functions from $C_0^\infty(M)$. We are left to show that u can be approximated in

W^1-norm by functions like $\psi(u)$, that is, there exists a sequence $\{\psi_k\}$ functions as above such that

$$\psi_k(u) \overset{W^1}{\to} u.$$

Indeed, consider the functions

$$\psi(t) = t_+ \text{ and } \quad \varphi(t) = 1_{(0,+\infty)}, \tag{5.14}$$

and let ψ_k be a sequence of non-negative smooth functions satisfying (5.5) and (5.6). Then, by (5.8),

$$\psi_k(u) \xrightarrow{L^2} \psi(u) = u \text{ and } \nabla\psi_k(u) \xrightarrow{L^2} \varphi(u)\nabla u = \nabla u,$$

which finishes the proof. □

5.2. Chain rule in W^1

The main result of this section is Theorem 5.7 that extends Lemma 5.2 to $W^1(M)$.

Denote by $W_c^1(M)$ the class of functions from $W^1(M)$ with compact support.

LEMMA 5.5. $W_c^1(M) \subset W_0^1(M)$.

PROOF. Set $K = \operatorname{supp} u$ and let $\{U_i\}$ be a finite family of charts covering K. By Theorem 3.5, there exists a family $\{\psi_i\}$ of functions $\psi_i \in \mathcal{D}(U_i)$ such that $\sum_i \psi_i \equiv 1$ in a neighborhood of K. Then we have $\psi_i u \in W_c^1(U_i)$ (cf. Exercise 4.2). Since $u = \sum_i \psi_i u$, it suffices to prove that $\psi_i u \in W_0^1(U_i)$.

Hence, the problem amounts to showing that, for any chart U,

$$W_c^1(U) \subset W_0^1(U),$$

that is, for any function $u \in W_c^1(U)$, there exists a sequence $\{\varphi_k\} \subset \mathcal{D}(U)$ such that $\varphi_k \xrightarrow{W^1} u$. Since U is a chart and, hence, can be considered as a part of $\mathbb{R}^n$, also the Euclidean Sobolev space $W_{eucl}^1(U)$ is defined in U (see Section 2.6.1). In general, the spaces $W_{eucl}^1(U)$ and $W^1(U)$ are different. However, by Exercise 4.11(*b*), the fact that $u \in W^1(U)$ implies that u and all distributional partial derivatives $\partial_j u$ belong to $L_{loc}^2(U)$. Since the support of u is compact, we obtain that u and $\partial_j u$ belong to $L_{eucl}^2(U)$, whence $u \in W_{eucl}^1(U)$.

By Exercise 2.30, there exists a sequence $\{\varphi_k\}_{k=1}^{\infty}$ of functions from $\mathcal{D}(U)$ that converges to u in $W_{eucl}^1(U)$, and the supports of φ_k can be assumed to be in an arbitrarily small open neighborhood V of $\operatorname{supp} u$.

We are left to show that the convergence $\varphi_k \to u$ in $W_{eucl}^1(U)$ implies that in $W^1(U)$. By Exercise 4.11, for any $v \in W_{eucl}^1(U)$ we have

$$|\nabla_{\mathbf{g}} v|_{\mathbf{g}}^2 = g^{ij}\partial_i v \partial_j v,$$

where ∇v is the weak gradient of v in metric $\mathbf{g}$ of the manifold M. It follows that, for any fixed open set $V \Subset U$,

$$\|v\|_{W^1(V)} \le C\|v\|_{W_{eucl}^1(V)}$$

where the constant C depends on the supremums of $\left|g^{ij}\right|$ and $\sqrt{\det g}$ in V. Hence, choosing $V \Subset U$ to contain all the supports of φ_k and u, we obtain

$$\|u-\varphi_k\|_{W^1(V)} \leq C\|u-\varphi_k\|_{W^1_{eucl}(V)} \to 0,$$

that is, $\varphi_k \to u$ in $W^1(U)$. □

Define the space $W^1_{loc}(M)$ by

$$W^1_{loc}(M) = \left\{u \in L^2_{loc}(M) : \nabla u \in \vec{L}^2_{loc}(M)\right\}. \tag{5.15}$$

Clearly, if $u \in W^1_{loc}(M)$ then $u \in W^1(\Omega)$ for any relatively compact open set $\Omega \subset M$. Conversely, if u is a function on M such that $u \in W^1(\Omega_k)$ for an exhaustion sequence $\{\Omega_k\}$ then $u \in W^1_{loc}(M)$.

COROLLARY 5.6. *If $u \in W^1_{loc}(M)$ and $f \in C_0^\infty(M)$ then $fu \in W_0^1(M)$.*

PROOF. Let Ω be any relatively compact open set containing $\operatorname{supp} f$. Then $u \in W^1(\Omega)$ and $f \in C_0^\infty(\Omega)$, whence we obtain by Exercise 4.3 that $fu \in W^1(\Omega)$. Since $\operatorname{supp}(fu)$ is compact and is contained in Ω, we obtain $fu \in W_c^1(\Omega)$ whence by Lemma 5.5 $fu \in W_0^1(\Omega)$. It follows that $fu \in W_0^1(M)$. □

The following result extends Lemma 5.2 to functions from $W^1(M)$.

THEOREM 5.7. *Let $\{\psi_k(t)\}$ be a sequence of C^∞-smooth functions on $\mathbb{R}$ such that*

$$\psi_k(0) = 0 \quad \text{and} \quad \sup_k \sup_{t\in\mathbb{R}} \left|\psi_k'(t)\right| < \infty. \tag{5.16}$$

Assume that, for some functions $\psi(t)$ and $\varphi(t)$ on $\mathbb{R}$,

$$\psi_k(t) \to \psi(t) \quad \text{and} \quad \psi_k'(t) \to \varphi(t) \quad \text{for all } t \in \mathbb{R}, \tag{5.17}$$

as $k \to \infty$.

(i) *If $u \in W^1(M)$ then $\psi(u) \in W^1(M)$ and*

$$\nabla\psi(u) = \varphi(u)\nabla u. \tag{5.18}$$

(ii) *Assume in addition that function $\varphi(t)$ is continuous in $\mathbb{R}\setminus F$ for some finite or countable set F. If $u_k, u \in W^1(M)$ then*

$$u_k \xrightarrow{W^1} u \quad \Longrightarrow \quad \psi(u_k) \xrightarrow{W^1} \psi(u).$$

REMARK. The conditions (5.16) and (5.17) are identical to the conditions (5.5) and (5.6) of Lemma 5.2.

PROOF. (i) As in the proof of Lemma 5.2, we have $\psi(u) \in L^2(M)$ and $\varphi(u)\nabla u \in \vec{L}^2(M)$. The identity (5.18) means that, for any vector field $\omega \in \vec{\mathcal{D}}(M)$,

$$(\psi(u), \operatorname{div}_\mu \omega) = -(\varphi(u)\nabla u, \omega). \tag{5.19}$$

Fix $\omega \in \vec{\mathcal{D}}(M)$ and let $f \in \mathcal{D}(M)$ be a cutoff function of $\operatorname{supp}\omega$. By Corollary 5.6, the function $u_0 := fu$ is in $W_0^1(M)$. Therefore, by Lemma 5.2,

$$\nabla\psi(u_0) = \varphi(u_0)\nabla u_0$$

and, hence,

$$(\psi(u_0), \operatorname{div}_\mu\omega) = -(\varphi(u_0)\nabla u_0, \omega).$$

Since $u = u_0$ in a neighborhood of $\operatorname{supp}\omega$, this identity implies (5.19).

(*ii*) It suffices to prove that a subsequence of $\{\psi(u_k)\}$ converges to $\psi(u)$ (cf. Exercise 2.14). Since $u_k \to u$ in L^2, there is a subsequence $\{u_k\}$ that converges to u almost everywhere. Hence, renaming this subsequence back to $\{u_k\}$, we can assume that $u_k \to u$ a.e..

What follows is similar to the proof of Lemma 5.1. It suffices to show that

$$\psi(u_k) \xrightarrow{L^2} \psi(u) \quad\text{and}\quad \nabla\psi(u_k) \xrightarrow{L^2} \nabla\psi(u). \tag{5.20}$$

By (5.16), there is a constant C such that

$$|\psi_k'(t)| \le C \quad\text{and}\quad |\psi_k(t)| \le C|t|,$$

for all k and $t \in \mathbb{R}$. Therefore,

$$\|\psi(u_k) - \psi(u)\| \le C\|u_k - u\|$$

(where all norms are L^2), which implies the first convergence in (5.20).

Next, using (5.18), we obtain

$$\begin{aligned}\|\nabla\psi(u_k) - \nabla\psi(u)\| &\le \|\varphi(u_k)(\nabla u_k - \nabla u)\| + \|(\varphi(u_k) - \varphi(u))\nabla u\| \\ &\le C\|(\nabla u_k - \nabla u)\| + \|(\varphi(u_k) - \varphi(u))\nabla u\|. \end{aligned} \tag{5.21}$$

The first term in (5.21) tends to 0 by hypothesis. The second term is equal to

$$\left(\int_M |\varphi(u_k(x)) - \varphi(u(x))|^2 |\nabla u|^2 d\mu(x)\right)^{1/2}. \tag{5.22}$$

Consider the following two sets:

$$\begin{aligned} S_1 &= \{x \in M : u_k(x) \not\to u(x) \text{ as } k \to \infty\}, \\ S_2 &= \{x \in M : u(x) \in F\}, \end{aligned}$$

where F is set where φ is discontinuous. By construction, $\mu(S_1) = 0$. Since $\nabla u = 0$ on any set of the form $\{u = \text{const}\}$ (cf. Example 5.3) and S_2 is a countable union of such sets, it follows that $\nabla u = 0$ on S_2. Hence, the domain of integration in (5.22) can be reduced to $M \setminus (S_1 \cup S_2)$. In this domain, we have

$$u_k(x) \to u(x) \notin F,$$

which implies by the continuity of φ in $\mathbb{R} \setminus F$ that

$$\varphi(u_k(x)) \to \varphi(u(x)).$$

Since the functions under the integral sign in (5.22) are uniformly bounded by the integrable function $4C^2|\nabla u|^2$, the dominated convergence theorem implies that the integral (5.22) tends to 0 as $k \to \infty$, which proves the second relation in (5.20). $\square$

EXAMPLE 5.8. Fix $c \geq 0$ and consider functions

$$\psi(t) = (t-c)_+ \quad \text{and} \quad \varphi(t) = 1_{(c,+\infty)} \tag{5.23}$$

(cf. Example 5.3). Since these functions satisfy all the hypotheses of Theorem 5.7, we obtain that if $u \in W^1$ then $(u-c)_+ \in W^1$, and $\nabla(u-c)_+$ is given by (5.12). Furthermore, by Theorem 5.7, $u_k \xrightarrow{W^1} u$ implies $(u_k - c)_+ \xrightarrow{W^1} (u-c)_+$.

Exercises.

5.1. Let $\psi(t)$ and $\varphi(t)$ be functions satisfying the conditions (5.16) and (5.17) of Theorem 5.7. Prove that $\psi'_{dist} = \varphi$.

5.2. Let $\psi \in C^1(\mathbb{R})$ be such that

$$\psi(0) = 0 \quad \text{and} \quad \sup|\psi'| < \infty.$$

Prove that the functions ψ and $\varphi := \psi'$ satisfy the conditions (5.16) and (5.17) of Theorem 5.7.

5.3. Prove that if $u, v \in W_0^1(M)$ then also $\max(u,v)$ and $\min(u,v)$ belong to $W_0^1(M)$.

5.4. Prove that if M is a compact manifold then $W^1(M) = W_0^1(M)$.

5.5. Prove that if $u \in W^1(M)$ then, for any real constant c, $\nabla u = 0$ a.e. on the set $\{x \in M : u(x) = c\}$.

5.6. Prove that, for any $u \in W^1(M)$,

$$(u-c)_+ \xrightarrow{W^1} u_+ \text{ as } c \to 0+.$$

5.7. Let $f \in W^1(M)$ and assume that $f(x) \to 0$ as $x \to \infty$ (the latter means that, for any $\varepsilon > 0$, the set $\{|f| \geq \varepsilon\}$ is relatively compact). Prove that $f \in W_0^1(M)$.

5.8. Prove that if $u \in W_{loc}^1(M)$ and φ, ψ are functions on $\mathbb{R}$ satisfying the conditions of Theorem 5.7 then $\psi(u) \in W_{loc}^1(M)$ and $\nabla\psi(u) = \varphi(u)\nabla u$.

5.9. Define the space $W_{loc}^2(M)$ by

$$W_{loc}^2 = \{f \in W_{loc}^1 : \Delta_\mu f \in L_{loc}^2\}.$$

Prove the Green formula (4.12) for any two functions $u \in W_c^1$ and $v \in W_{loc}^2$.

5.3. Markovian properties of resolvent and the heat semigroup

Set as before $\mathcal{L} = -\Delta_\mu|_{W_0^2}$ and recall that the heat semigroup is defined by $P_t = e^{-t\mathcal{L}}$ for all $t \geq 0$, and the resolvent is defined by

$$R_\alpha = (\mathcal{L} + \alpha \,\mathrm{id})^{-1} \tag{5.24}$$

for all $\alpha > 0$. Both operators P_t and R_α are bounded self-adjoint operator on $L^2(M)$ (cf. Theorems 4.9 and 4.5). Here we consider the properties of the operators P_t and R_α related to inequalities between functions.

THEOREM 5.9. *Let $(M, \mathbf{g}, \mu)$ be a weighted manifold, $f \in L^2(M)$, and $\alpha > 0$.*

(*i*) *If $f \geq 0$ then $R_\alpha f \geq 0$.*
(*ii*) *If $f \leq 1$ then $R_\alpha f \leq \alpha^{-1}$.*

PROOF. We will prove that, for any $c > 0$, $f \leq c$ implies $R_\alpha f \leq c\alpha^{-1}$, which will settle (*ii*) when $c = 1$, and settle (*i*) when $c \to 0+$. Without loss of generality, it suffices to consider the case $c = \alpha$, that is, to prove that $f \leq \alpha$ implies $R_\alpha f \leq 1$. Set $u = R_\alpha f$ and recall that $u \in \operatorname{dom} \mathcal{L} = W_0^2$ and

$$\mathcal{L}u + \alpha u = f. \tag{5.25}$$

To prove that $u \leq 1$ is suffices to show that the function $v := (u-1)_+$ identically vanishes. By Example 5.3 and (5.12), we have $v \in W_0^1$ and

$$\nabla v = \begin{cases} \nabla u, & u > 1, \\ 0, & u \leq 1. \end{cases} \tag{5.26}$$

Multiplying (5.25) by v and integrating, we obtain

$$(\mathcal{L}u, v)_{L^2} + \alpha (u, v)_{L^2} = (f, v)_{L^2}. \tag{5.27}$$

By Lemma 4.4 and (5.26), we have

$$(\mathcal{L}u, v)_{L^2} = -(\Delta_\mu u, v)_{L^2} = (\nabla u, \nabla v)_{\vec{L}^2} = \int_{\{u>1\}} |\nabla u|^2 \, d\mu \geq 0,$$

whereas

$$(u, v)_{L^2} = \int_{\{v>0\}} (v+1)\, v d\mu = \|v\|_{L^2}^2 + \int_M v d\mu.$$

Hence, it follows from (5.27) and $f \leq \alpha$ that

$$\alpha \|v\|_{L^2}^2 + \alpha \int_M v d\mu \leq (f, v)_{L^2} \leq \alpha \int_M v d\mu,$$

whence we conclude $\|v\|_{L^2} = 0$ and $v = 0$. □

Many properties of the heat semigroup can be proved using the corresponding properties of the resolvent and the following identities.

LEMMA 5.10. *For an arbitrary weighted manifold $(M, \mathbf{g}, \mu)$ the following identity hold.*

(*i*) *For any $\alpha > 0$,*

$$R_\alpha = \int_0^\infty e^{-\alpha t} P_t dt. \tag{5.28}$$

where the integral is understood in the following sense: for all $f, g \in L^2(M)$,

$$(R_\alpha f, g)_{L^2} = \int_0^\infty e^{-\alpha t} (P_t f, g)_{L^2} \, dt. \tag{5.29}$$

(*ii*) *For any* $t > 0$,

$$P_t = \lim_{k\to\infty}\left(\frac{k}{t}\right)^k R^k_{k/t}, \tag{5.30}$$

where the limit is understood in the strong operator topology.

PROOF. Let $\{E_\lambda\}$ be the spectral resolution of the Dirichlet Laplace operator $\mathcal{L}$. Then by (4.45) we have

$$P_t f = \int_0^\infty e^{-\lambda t} dE_\lambda f \tag{5.31}$$

and

$$R_\alpha f = \int_0^\infty (\alpha+\lambda)^{-1}\, dE_\lambda f \tag{5.32}$$

(see also Exercise A.24).

(*i*) Substituting (5.31) to the right hand side of (5.29) with $g = f$ and using Fubini's theorem, we obtain

$$\begin{aligned}
\int_0^\infty e^{-\alpha t}\,(P_t f, f)_{L^2}\,dt &= \int_0^\infty e^{-\alpha t}\left(\int_0^\infty e^{-\lambda t} d\,(E_\lambda f, f)_{L^2}\right) dt \\
&= \int_0^\infty \left(\int_0^\infty e^{-(\alpha+\lambda)t} dt\right) d\,(E_\lambda f, f)_{L^2} \\
&= \int_0^\infty (\alpha+\lambda)^{-1}\, d\,(E_\lambda f, f)_{L^2} \\
&= (R_\alpha f, f)_{L^2}\,,
\end{aligned}$$

which proves (5.29) for the case $f = g$. Then (5.29) extends to arbitrary f, g using the identity

$$(P_t f, g) = \frac{1}{2}\,(P_t\,(f+g)\,, f+g) - \frac{1}{2}\,(P_t\,(f-g)\,, f-g)$$

and a similar i[illegible] ntity for R_α

(*ii*) [illegible] he classical identity

$$e^{-t\lambda} = \lim_{k\to\infty}\left(1+\frac{t\lambda}{k}\right)^{-k}$$

[illegible] ergence theorem, we obtain

(5.29)

$$[illegible]_\lambda = \lim_{k\to\infty}\int_0^\infty \left(1+\frac{t\lambda}{k}\right)^{-k} dE_\lambda,$$

for all [illegible]
on $L^2(M)$ [illegible] n the strong sense. This implies (5.30) because
the operators P_t [illegible] to P_t, and the integral in the right hand side

is equal to

$$\int_0^\infty \left(1+\frac{t\lambda}{k}\right)^{-k} dE_\lambda = \left(\frac{k}{t}\right)^k \int_0^\infty \left(\frac{k}{t}+\lambda\right)^{-k} dE_\lambda$$
$$= \left(\frac{k}{t}\right)^k \left(\frac{k}{t}\operatorname{id}+\mathcal{L}\right)^{-k} = \left(\frac{k}{t}\right)^k R_{k/t}^k .$$

□

THEOREM 5.11. *Let $(M, \mathbf{g}, \mu)$ be a weighted manifold, $f \in L^2(M)$, and $t > 0$.*

(i) If $f \geq 0$ then $P_t f \geq 0$.

(iii) If $f \leq 1$ then $P_t f \leq 1$.

PROOF. (*i*) If $f \geq 0$ then, by Theorem 5.9, $R_\alpha f \geq 0$, which implies that $R_\alpha^k f \geq 0$ for all positive integers k. It follows from (5.30) that $P_t f \geq 0$.

(*ii*) If $f \leq 1$ then, by Theorem 5.9, $R_\alpha f \leq \alpha^{-1}$, which implies that $R_\alpha^k f \leq \alpha^{-k}$ for all positive k. Hence, (5.30) implies

$$P_t f \leq \lim_{k\to\infty} \left(\frac{k}{t}\right)^k \left(\frac{k}{t}\right)^{-k} = 1,$$

which was to be proved. □

Exercises.

5.10. Let R_α be the resolvent defined by (5.24).

(*a*) Prove that, for any $f \in L^2$ and $\alpha > 0$,

$$P_t f = \lim_{\alpha\to+\infty} e^{-\alpha t} \sum_{k=0}^{\infty} \frac{\alpha^{2k} t^k}{k!} R_\alpha^k f . \tag{5.33}$$

(*b*) Using (5.33), give an alternative proof of the fact that $f \leq 1$ implies $P_t f \leq 1$.

5.11. For all $\alpha, k > 0$, define R_α^k as $\varphi(R_\alpha)$ where $\varphi(\lambda) = \lambda^k$.

(*a*) Prove that, for all $\alpha, k > 0$,

$$R_\alpha^k = \int_0^\infty \frac{t^{k-1}}{\Gamma(k)} e^{-\alpha t} P_t dt, \tag{5.34}$$

where the integral is understood in the weak sense, as in Lemma 5.10, and Γ is the gamma function (cf. Section A.6).

(*b*) Write for simplicity $R_1 = R$. Prove that

$$R^k R^l = R^{k+l} \text{ for all } k, l > 0.$$

Prove that if $f \in L^2(M)$ then $f \geq 0$ implies $R^k f \geq 0$ and $f \leq 1$ implies $R^k f \leq 1$, for all $k \geq 0$.

(*c*) Prove that $R^k = e^{-kL}$ where $L = \log(\operatorname{id}+\mathcal{L})$ and $\mathcal{L}$ is the Dirichlet Laplace operator.

REMARK. The semigroup $\{R^k\}_{k\geq 0}$ is called the Bessel semigroup, and the operator $\log(\operatorname{id}+\mathcal{L})$ is its generator.

5.12. Prove that, for any non-negative function $f \in L^2(M)$ and all $t, \alpha > 0$,

$$P_t R_\alpha f \leq e^{\alpha t} R_\alpha f.$$

5.13. Let $\mathcal{L}$ be the Dirichlet Laplace operator on $\mathbb{R}^1$.

(a) Prove that the resolvent $R_\lambda = (\mathcal{L} + \lambda \operatorname{id})^{-1}$ is given for any $\lambda > 0$ by the following formula:

$$R_\lambda f = \frac{1}{2\sqrt{\lambda}} \int_{-\infty}^{+\infty} e^{-\sqrt{\lambda}|x-y|} f(y)\, dy, \tag{5.35}$$

for any and $f \in L^2(\mathbb{R}^1)$.

(b) Comparing (5.35) with

$$R_\lambda = \int_0^\infty e^{-\lambda t} P_t dt$$

and using the explicit formula for the heat kernel in $\mathbb{R}^1$, establish the following identity:

$$e^{-t\sqrt{\lambda}} = \int_0^\infty \frac{t}{\sqrt{4\pi s^3}} \exp\left(-\frac{t^2}{4s}\right) e^{-s\lambda} ds, \tag{5.36}$$

for all $t > 0$ and $\lambda \geq 0$.

REMARK. The function $s \mapsto \frac{t}{\sqrt{4\pi s^3}} \exp\left(-\frac{t^2}{4s}\right)$ is the density of a probability distribution on $\mathbb{R}_+$, which is called the *Levy distribution*.

5.14. Let $\mathcal{L}$ be the Dirichlet Laplace operator on an arbitrary weighted manifold, and consider the family of operators $Q_t = \exp\left(-t\mathcal{L}^{1/2}\right)$, where $t \geq 0$.

(a) Prove the identity

$$Q_t = \int_0^\infty \frac{t}{\sqrt{4\pi s^3}} \exp\left(-\frac{t^2}{4s}\right) P_s ds. \tag{5.37}$$

(b) Let $f \in L^2(M)$. Prove that $f \geq 0$ implies $Q_t f \geq 0$ and $f \leq 1$ implies $Q_t f \leq 1$.

(c) Prove that in the case $M = \mathbb{R}^n$, Q_t is given explicitly by

$$Q_t f = \int_{\mathbb{R}^n} q_t(x-y) f(y)\, dy$$

where

$$q_t(x) = \frac{2}{\omega_{n+1}} \frac{t}{\left(t^2 + |x|^2\right)^{\frac{n+1}{2}}}. \tag{5.38}$$

REMARK. The semigroup $\{Q_t\}_{k \geq 0}$ is called the Cauchy semigroup, and the operator $\mathcal{L}^{1/2}$ is its generator.

5.15. Let Ψ be a C^∞-function on $\mathbb{R}$ such that $\Psi(0) = \Psi'(0) = 0$ and $0 \leq \Psi''(s) \leq 1$ for all s.

(a) Prove that, for any $f \in L^2(M)$, the following function

$$F(t) := \int_M \Psi(P_t f)\, d\mu \tag{5.39}$$

is continuous and decreasing in $t \in [0, +\infty)$.

(b) Using part (a), give yet another proof of the fact that $f \leq 1$ implies $P_t f \leq 1$, without using the resolvent.

5.4. Weak maximum principle

5.4.1. Elliptic problems. Given functions $f \in L^2(M)$, $w \in W^1(M)$, and a real constant α, consider the following weak Dirichlet problem:

$$\begin{cases} \Delta_\mu u + \alpha u = f, \\ u = w \mod W_0^1(M), \end{cases} \tag{5.40}$$

where the second line in (5.40) means that

$$u = w + w_0 \text{ for some } w_0 \in W_0^1(M),$$

and can be regarded as the "boundary condition" for u.

If $\alpha < 0$ and $w = 0$ then the problem (5.40) has exactly one solution $u = -R_{-\alpha} f$ by Theorem 4.5. Set

$$\beta = \inf \operatorname{spec} \mathcal{L},$$

where $\mathcal{L}$ is the Dirichlet Laplace operator on M. Then, by Exercise 4.29, the problem (5.40) has exactly one solution for any $\alpha < \beta$ and $w \in W^1(M)$.

Here we are interested in the sign of a solution assuming that it already exists. In fact, we consider a more general situation when the equations in (5.40) are replaced by inequalities. If u, w are two measurable functions on M then we write

$$u \leq w \mod W_0^1(M)$$

if

$$u \leq w + w_0 \text{ for some } w_0 \in W_0^1(M).$$

The opposite inequality $u \geq w \mod W_0^1(M)$ is defined similarly.

LEMMA 5.12. *If $u \in W^1(M)$ then the relation*

$$u \leq 0 \mod W_0^1(M) \tag{5.41}$$

holds if and only if $u_+ \in W_0^1(M)$.

PROOF. If $u_+ \in W_0^1$ then (5.41) is satisfied because $u \leq u_+$. Conversely, we need to prove that if $u \leq v$ for some $v \in W_0^1$ then $u_+ \in W_0^1$.

Assume first that $v \in C_0^\infty$, and let φ be a cutoff function of $\operatorname{supp} v$ (see Fig. 5.2). Then we have the following identity:

$$u_+ = ((1-\varphi) v + \varphi u)_+ . \tag{5.42}$$

Indeed, if $\varphi = 1$ then (5.42) is obviously satisfied. If $\varphi < 1$ then $v = 0$ and, hence, $u \leq 0$, so that the both sides of (5.42) vanish. By Corollary 5.6, we have $\varphi u \in W_0^1$. Since $(1-\varphi) v \in C_0^\infty$, it follows that

$$(1-\varphi) v + \varphi u \in W_0^1.$$

By Lemma 5.2 and (5.42) we conclude that $u_+ \in W_0^1$.

For a general $v \in W_0^1$, let $\{v_k\}$ be a sequence of functions from C_0^∞ such that $v_k \xrightarrow{W^1} v$. Then we have

$$u_k := u + (v_k - v) \leq v_k,$$

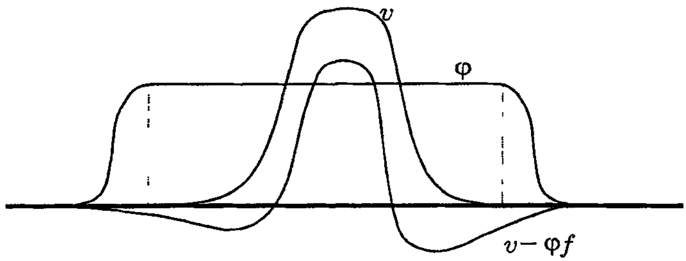

FIGURE 5.2. Functions u, v, φ

which implies by the first part of the proof that $(u_k)_+ \in W_0^1$. Since $u_k \xrightarrow{W^1} u$, it follows by Theorem 5.7 that $(u_k)_+ \xrightarrow{W^1} u_+$, whence we conclude that $u \in W_0^1$. □

We say that a distribution $u \in \mathcal{D}'(M)$ is *non-negative*[1] and write $u \geq 0$ if $(u, \varphi) \geq 0$ for any non-negative function $\varphi \in \mathcal{D}(M)$.

Of course, if $u \in L^1_{loc}(M)$ then $u \geq 0$ in the sense of distributions if and only if $u \geq 0$ a.e.(cf. Exercise 4.7). Similarly, one defines the inequalities $u \geq v$ and $u \leq v$ between two distributions. It is possible to prove that $u \geq v$ and $u \leq v$ imply $u = v$ (cf. Exercise 4.6).

THEOREM 5.13. (Weak maximum principle) *Set $\beta = \inf \operatorname{spec} \mathcal{L}$ and assume that, for some real $\alpha < \beta$, a function $u \in W^1(M)$ satisfies in the distributional sense the inequality*

$$\Delta_\mu u + \alpha u \geq 0 \tag{5.43}$$

and the boundary condition

$$u \leq 0 \mod W_0^1(M). \tag{5.44}$$

Then $u \leq 0$ in M.

REMARK. If $\alpha \geq \beta$ then the statement of Theorem 5.13 may fail. For example, if M is compact then $\beta = 0$ because the constant is an eigenfunction of $\mathcal{L}$ with the eigenvalue 0. Then both (5.43) and (5.44) hold with $\alpha = 0$ for any constant function u so that the sign of u can be both positive and negative.

REMARK. Theorem 5.13 can be equivalently stated as a weak *minimum* principle: if $u \in W^1(M)$ and

$$\begin{cases} \Delta_\mu u + \alpha u \leq 0, \\ u \geq 0 \mod W_0^1(M), \end{cases}$$

then $u \geq 0$ in M.

[1]It is known that any non-negative distribution is given by a measure but we will not use this fact.

REMARK. Consider the weak Dirichlet problem (5.40) with the boundary function $w = 0$. By Theorem 4.5, if $\alpha < 0$ then the problem (5.40) has a unique solution $u = -R_{-\alpha}f$. By Theorem 5.13, we obtain in this case that $f \geq 0$ implies $R_{-\alpha}f \geq 0$, which, hence, recovers Theorem 5.9(i).

PROOF OF THEOREM 5.13. By definition, (5.43) is equivalent to the inequality

$$(\Delta_\mu u, \varphi) + \alpha (u, \varphi) \geq 0$$

for any non-negative $\varphi \in \mathcal{D}(M)$, which, in turn, is equivalent to

$$(\nabla u, \nabla \varphi) - \alpha (u, \varphi) \leq 0. \tag{5.45}$$

Considering the round brackets here also as the inner products in $L^2(M)$ and noticing that all terms in (5.45) are continuous in φ in the norm of $W^1(M)$, we obtain that (5.45) holds for all non-negative $\varphi \in W_0^1(M)$ (cf. Lemma 5.4).

By Lemma 5.12 we have $u_+ \in W_0^1(M)$. Setting (5.45) $\varphi = u_+$, we obtain

$$\int_M \langle \nabla u, \nabla u_+ \rangle d\mu - \alpha \int_M u\, u_+ \, d\mu \leq 0.$$

It follows by (5.9), that

$$\int_M |\nabla u_+|^2 \, d\mu - \alpha \int_M u_+^2 d\mu \leq 0. \tag{5.46}$$

By Exercise 4.27, we have

$$\int_M |\nabla u_+|^2 \, d\mu \geq \beta \int_M u_+^2 d\mu,$$

which together with (5.46) implies

$$(\beta - \alpha) \int_M u_+^2 d\mu \leq 0.$$

Since $\alpha < \beta$, we obtain $\|u_+\|_{L^2} = 0$ and, hence, $u \leq 0$. $\square$

COROLLARY 5.14. (Comparison principle) *Assume that, for some $\alpha < \beta$, functions $u, v \in W^1(M)$ satisfy the conditions*

$$\begin{cases} \Delta_\mu u + \alpha u \geq \Delta_\mu v + \alpha v, \\ u \leq v \mod W_0^1(M). \end{cases}$$

Then $u \leq v$.

PROOF. Indeed, the function $u - v$ satisfies all the conditions of Theorem 5.13, which implies $u - v \leq 0$. $\square$

COROLLARY 5.15. (The minimality of resolvent) *Assume that a function $u \in W^1(M)$ satisfies the inequality*

$$-\Delta_\mu u + \gamma u \geq f, \tag{5.47}$$

where $f \in L^2(M)$ and $\gamma > 0$, and the boundary condition

$$u \geq 0 \mod W_0^1(M).$$

Then $u \geq R_\gamma f$.

PROOF. Using Theorem 4.5, we obtain that

$$\begin{cases} -\Delta_\mu (R_\gamma f) + \gamma R_\gamma f = f \leq -\Delta_\mu u + \gamma u \\ R_\gamma f = 0 \mod W_0^1 (M) \leq u \mod W_0^1 (M), \end{cases}$$

whence we conclude by Corollary 5.14 with $\alpha = -\gamma$ that $R_\gamma f \leq u$. □

If $f \geq 0$ then, by Theorem 5.9, $R_\gamma f \geq 0$. The statement of Corollary 5.15 implies that $u = R_\gamma f$ is the *minimal* function satisfying (5.47) among all non-negative functions $u \in W^1 (M)$.

Exercises.

5.16. Give an example of a manifold M and a non-negative function $u \in W_{loc}^1 (M)$ such that

$$u \leq 0 \mod W_0^1 (M)$$

but $u \notin W^1 (M)$.

5.4.2. Parabolic problems. Now we turn to the weak maximum principle for the heat equation. In Section 4.3, we have considered the L^2-Cauchy problem related to the Dirichlet Laplace operator $\mathcal{L}$. In the next statement, we consider a more general version of this problem, where the requirement to be in $\operatorname{dom} \mathcal{L}$ is dropped.

THEOREM 5.16. (Weak parabolic maximum principle) *Let* $u : (0, T) \to W^1 (M)$ *be a path that satisfies the following conditions:*

(*i*) *For any* $t \in (0, T)$, *the strong derivative* $\frac{du}{dt}$ *exists in* $L^2 (M)$ *and satisfies the inequality*

$$\frac{du}{dt} - \Delta_\mu u \leq 0, \tag{5.48}$$

where Δ_μ *is understood as an operator in* $\mathcal{D}' (M)$.

(*ii*) *For any* $t \in (0, T)$,

$$u (t, \cdot) \leq 0 \mod W_0^1 (M). \tag{5.49}$$

(*iii*) $u_+ (t, \cdot) \xrightarrow{L^2(M)} 0$ *as* $t \to 0$.

Then $u (t, \cdot) \leq 0$ *for all* $t \in (0, T)$.

REMARK. By Theorem 4.9, the function $u = P_t f$ satisfies all the above conditions, provided $f \leq 0$. Hence, we conclude by Theorem 5.16 that $f \leq 0$ implies $P_t f \leq 0$, which recovers Theorem 5.11(*i*).

Let u be a solution to the L^2-Cauchy problem with the initial function f, as stated in Section 4.3. Then, for any $t > 0$, $u (t, \cdot) \in \operatorname{dom} \mathcal{L}$ which implies

$$u (t, \cdot) = 0 \mod W_0^1 (M).$$

Applying Theorem 5.16 to u and $-u$, we see that $f = 0$ implies $u = 0$, which recovers Theorem 4.10.

Similarly to Corollary 5.14, one can state in an obvious way a comparison principle associated with Theorem 5.16.

PROOF. The inequality (5.48) means that, for any fixed $t \in (0,T)$ and any non-negative function $v \in \mathcal{D}(M)$,

$$\left(u', v\right) \leq (\Delta_\mu u, v),$$

where $u' \equiv \frac{du}{dt}$, which implies

$$\left(u', v\right) \leq -(\nabla u, \nabla v). \tag{5.50}$$

Considering the both sides here as the inner products in $L^2(M)$, we extend (5.50) to all non-negative functions $v \in W_0^1(M)$.

Let a function $\varphi \in C^\infty(\mathbb{R})$ be such that, for some positive constant C,

$$\begin{cases} \varphi(s) = 0, & s \leq 0, \\ \varphi(s) \geq 0, & s > 0, \\ 0 \leq \varphi'(s) \leq C, & s \in \mathbb{R}. \end{cases} \tag{5.51}$$

By (5.49) and Lemma 5.12, we have $u_+(t,\cdot) \in W_0^1(M)$, for any $t \in (0,T)$. Therefore, by Lemma 5.1, the function $\varphi(u(t,\cdot)) = \varphi(u_+(t,\cdot))$ is also in $W_0^1(M)$ and

$$\nabla \varphi(u) = \varphi'(u_+)\nabla u_+ = \varphi'(u)\nabla u$$

(cf. (5.9)). Setting $v = \varphi(u(t,\cdot))$ in (5.50), we obtain

$$\left(u', \varphi(u)\right)_{L^2} \leq -\left(\nabla u, \varphi'(u)\nabla u\right)_{L^2} = -\int_M \varphi'(u)|\nabla u|^2 d\mu \leq 0. \tag{5.52}$$

Using the product rule (Exercise 4.46) and the chain rule (Exercise 4.50) for strong derivatives, we obtain

$$\begin{aligned} \frac{d}{dt}(u, \varphi(u))_{L^2} &= \left(u', \varphi(u)\right)_{L^2} + \left(u, \varphi'(u)u'\right)_{L^2} \\ &= \left(u', \varphi(u)\right)_{L^2} + \left(u', \psi(u)\right)_{L^2} \end{aligned} \tag{5.53}$$

where

$$\psi(s) = \varphi'(s)s. \tag{5.54}$$

Next, we specify function φ as follows. Define first its second derivative φ'' as a non-negative smooth function on $\mathbb{R}$, such that

$$\begin{aligned} &\varphi''(s) = 0, \quad s \leq 0 \text{ or } s \geq 1, \\ &\varphi''(s) > 0, \quad 0 < s < 1. \end{aligned}$$

Then φ is obtained by two integrations of φ'' keeping the value 0 at 0. Clearly, φ satisfies (5.51). Also function $\psi(s)$ from (5.54) satisfies (5.51), because its derivative

$$\psi'(s) = \varphi''(s)s + \varphi'(s)$$

is obviously non-negative and bounded. By (5.52), we conclude that the right hand side of (5.53) is non-positive, that is, $(u, \varphi(u))_{L^2}$ as a function of t is decreasing in $(0, T)$. Since $\varphi(s) \leq Cs$ for any $s \geq 0$, we obtain that

$$(u, \varphi(u))_{L^2} = (u_+, \varphi(u_+))_{L^2} \leq C(u_+, u_+)_{L^2}.$$

By hypothesis, $(u_+, u_+)_{L^2} \to 0$ as $t \to 0$. Hence, the function $t \mapsto (u_+, \varphi(u_+))_{L^2}$ is non-negative, decreasing on $(0, T)$ and goes to 0 as $t \to 0$. It follows that $(u_+, \varphi(u_+))_{L^2} = 0$ for all $t \in (0, T)$, which implies that $u_+(t, \cdot) = 0$ for all $t \in (0, T)$. $\square$

COROLLARY 5.17. (The minimality of $P_t f$) *Let $u : (0, T) \to W^1(M)$ be a path in $W^1(M)$ such that*

$$\begin{cases} \dfrac{du}{dt} \geq \Delta_\mu u & \textit{for all } t \in (0, T), \\ u(t, \cdot) \geq 0 \mod W_0^1(M) & \textit{for all } t \in (0, T), \\ u(t, \cdot) \xrightarrow{L^2} f \in L^2(M) & \textit{as } t \to 0. \end{cases} \tag{5.55}$$

Then, for all $t \in (0, T)$,

$$u(t, \cdot) \geq P_t f. \tag{5.56}$$

PROOF. Using Theorem 4.9, we obtain that the function $v(t, \cdot) = u(t, \cdot) - P_t f$ satisfies the conditions

$$\begin{cases} \dfrac{dv}{dt} \geq \Delta_\mu v & \text{for all } t \in (0, T), \\ v(t, \cdot) \geq 0 \mod W_0^1(M), & \text{for all } t \in (0, T), \\ v(t, \cdot) \xrightarrow{L^2} 0 & \text{as } t \to 0, \end{cases}$$

whence (5.56) follows by Theorem 5.16. $\square$

Corollary 5.17 implies the following minimality property of $P_t f$: if $f \geq 0$ then the function $u(t, \cdot) = P_t f$ is the minimal non-negative solution to the Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = \Delta_\mu u, & t > 0, \\ u(t, \cdot) \xrightarrow{L^2} f, & \text{as } t \to 0. \end{cases}$$

Exercises.

5.17. Let the paths $w : (0, T) \to W^1(M)$ and $v : (0, T) \to W_0^1(M)$ satisfy the same heat equation

$$\frac{du}{dt} = \Delta_\mu u \text{ for all } t \in (0, T),$$

where $\frac{du}{dt}$ is the strong derivative in $L^2(M)$ and $\Delta_\mu u$ is understood in the distributional sense. Prove that if

$$w(t, \cdot) - v(t, \cdot) \xrightarrow{L^2(M)} 0 \text{ as } t \to 0,$$

and $w \geq 0$ then $w(t, \cdot) \geq v(t, \cdot)$ for all $t \in (0, T)$.

5.4.3. The pointwise boundary condition at ∞.

DEFINITION 5.18. The one point compactification of a smooth manifold M is the topological space $M\cup\{\infty\}$ where ∞ is the ideal *infinity point* (that does not belong to M) and the family of open sets in $M\cup\{\infty\}$ consists of the open sets in M and the sets of the form $(M\setminus K)\cup\{\infty\}$ where K is an arbitrary compact subset of M.

It is easy to check that this family of open sets determines the Hausdorff topology in $M\cup\{\infty\}$ and the topological space $M\cup\{\infty\}$ is compact. Note that if M is compact then ∞ is disconnected from M.

If M is non-compact and $v(x)$ is a function on M then it follows from the definition of the topology of $M\cup\{\infty\}$ that, for a real c,

$$v(x)\to c \text{ as } x\to\infty \tag{5.57}$$

if, for any $\varepsilon>0$, there is a compact set $K_\varepsilon\subset M$ such that

$$\sup_{x\in M\setminus K_\varepsilon}|v(x)-c|<\varepsilon. \tag{5.58}$$

EXAMPLE 5.19. If $M=\mathbb{R}^n$ then any compact set is contained in a ball $\{x\in\mathbb{R}^n:|x|\le r\}$, which implies that (5.57) is equivalent to

$$v(x)\to c \text{ as } |x|\to\infty,$$

so that $x\to\infty$ means in this case $|x|\to\infty$.

If $M=\Omega$ where Ω is a bounded open set in $\mathbb{R}^n$ then every compact set in M is contained in

$$\Omega_\delta=\{x\in\Omega:d(x,\partial\Omega)\ge\delta\}$$

for some $\delta>0$. Then $x\to\infty$ in M means $d(x,\partial\Omega)\to 0$ that is, $x\to\partial\Omega$.

If $M=\Omega$ where Ω is an arbitrary open set in $\mathbb{R}^n$ then arguing similarly we obtain that $x\to\infty$ in M means that

$$|x|+\frac{1}{d(x,\partial\Omega)}\to\infty.$$

If $v_\alpha(x)$ is a function on a manifold M that depends on a parameter α varying in a set A, then define the uniform convergence in α as follows: $v(x)\rightrightarrows c$ as $x\to\infty$ uniformly in $\alpha\in A$ if the condition (5.58) holds uniformly in α, that is,

$$\sup_{\alpha\in A}\sup_{x\in M\setminus K_\varepsilon}|v_\alpha(x)-c|<\varepsilon. \tag{5.59}$$

For $c=\pm\infty$ the conditions (5.58) and (5.59) should be appropriately modified.

The next statement is a rather straightforward consequence of Theorem 5.16 for the classical (sub)solutions to the heat equation.

COROLLARY 5.20. (Parabolic maximum principle) *Set $I=(0,T)$ where $T\in(0,+\infty]$. Let a function $u(t,x)\in C^2(I\times M)$ satisfy the following conditions:*

- $\dfrac{\partial u}{\partial t} - \Delta_\mu u \le 0$ *in* $I \times M$.
- $u_+(x,t) \rightrightarrows 0$ *as* $x \to \infty$ *in* M, *where the convergence is uniform in* $t \in I$.
- $u_+(t,\cdot) \overset{L^2_{loc}(M)}{\longrightarrow} 0$ *as* $t \to 0$.

Then $u \le 0$ *in* $I \times M$.

PROOF. The hypothesis $u_+(x,t) \rightrightarrows 0$ as $x \to \infty$ means that, for any $\varepsilon > 0$ there is a compact set $K \subset M$ such that

$$\sup_{t \in I} \sup_{x \in M \setminus K} u(t,x) < \varepsilon.$$

Let Ω be any relatively compact open subset of M containing K. Then $u(t,\cdot) \in W^1(\Omega)$ for any $t \in I$ and the partial derivative $\frac{\partial u}{\partial t}$ coincides with the strong derivative $\frac{du}{dt}$ in $L^2(\Omega)$ (cf. Exercise 4.47). Therefore, u satisfies the hypothesis (i) of Theorem 5.16 in Ω. It follows that $u - \varepsilon$ also satisfies that condition.

Since $(u-\varepsilon)_+$ is supported in $K \subset \Omega$, we obtain by Theorem 5.7 and Lemma 5.5 that $(u-\varepsilon)_+ \in W_0^1(\Omega)$. This constitutes the hypothesis (ii) of Theorem 5.16.

Finally, we have $(u-\varepsilon)_+ \overset{L^2(\Omega)}{\to} 0$ as $t \to 0$, which gives the hypothesis (iii) of Theorem 5.16.

Hence, we conclude by Theorem 5.16 that $u - \varepsilon \le 0$ in $I \times \Omega$. Finally, letting $\varepsilon \to 0$ and exhausting M by sets like Ω, we obtain $u \le 0$ in $I \times M$. □

REMARK 5.21. Corollary 5.20 remains true if u_+ satisfies the initial condition in the L^1_{loc} sense rather than in L^2_{loc} sense, that is, if

$$u_+(t,\cdot) \overset{L^1_{loc}(M)}{\longrightarrow} 0 \text{ as } t \to 0.$$

See Exercise 5.21.

Exercises.

5.18. Let $v_\alpha(x)$ be a real valued function on a non-compact smooth manifold M depending on a parameter $\alpha \in A$, and let $c \in \mathbb{R}$. Prove that the following conditions are equivalent (all convergences are inform in $\alpha \in A$):

(i) $v_\alpha(x) \rightrightarrows c$ as $x \to \infty$.

(ii) For any sequence $\{x_k\}_{k=1}^\infty$ that eventually leaves any compact set $K \subset M$, $v_\alpha(x_k) \rightrightarrows c$ as $k \to \infty$.

(iii) For any sequence $\{x_k\}$ on M that eventually leaves any compact set $K \subset M$, there is a subsequence $\{x_{k_i}\}$ such that $v_\alpha(x_{k_i}) \rightrightarrows c$ as $i \to \infty$.

(iv) For any $\varepsilon > 0$, the set

$$V_\varepsilon = \left\{ x \in M : \sup_{\alpha \in A} |v_\alpha(x) - c| \ge \varepsilon \right\} \tag{5.60}$$

is relatively compact.

Show that these conditions are also equivalent for $c = \pm\infty$ provided (5.60) is appropriately adjusted.

5.19. Referring to Exercise 5.18, let $M=\Omega$ where Ω is an unbounded open subset of $\mathbb{R}^n$. Prove that the condition (i) is equivalent to

(v) $v_\alpha(x_k) \rightrightarrows c$ for any sequence $\{x_k\}\subset\Omega$ such that either $x_k\to x\in\partial\Omega$ or $|x_k|\to\infty$.

5.20. Let a function $v\in C^2(M)$ satisfy the conditions:

(i) $-\Delta_\mu v+\alpha v\le 0$ on M, for some $\alpha>0$;

(ii) $v_+(x)\to 0$ as $x\to\infty$ in M.

Prove that $v\le 0$ in M.

5.21. Prove that the statement of Corollary 5.20 remains true if the condition $u_+(t,\cdot)\overset{L^2_{loc}(M)}{\longrightarrow} 0$ as $t\to 0$ is replaced by

$$u_+(t,\cdot)\overset{L^1_{loc}(M)}{\longrightarrow} 0 \text{ as } t\to 0.$$

5.5. Resolvent and the heat semigroup in subsets

Any open subset Ω of a weighted manifold $(M,\mathbf{g},\mu)$ can be regarded as a weighted manifold $(\Omega,\mathbf{g},\mu)$. We will write shortly $L^2(\Omega)$ for $L^2(\Omega,\mu)$, and the same applies to $W_0^1(\Omega)$ and other Sobolev spaces.

Given a function f on Ω, its *trivial extension* is a function $\widetilde{f}$ on M defined by

$$\widetilde{f}(x)=\begin{cases} f(x), & x\in\Omega,\\ 0, & x\in M\setminus\Omega.\end{cases}$$

The same terminology and notation apply to extension of a vector field in Ω by setting it 0 in $M\setminus\Omega$. It is obvious that if $f\in C_0^\infty(\Omega)$ then $\widetilde{f}\in C_0^\infty(M)$ and

$$\nabla\widetilde{f}=\widetilde{\nabla f} \text{ and } \Delta_\mu\widetilde{f}=\widetilde{\Delta_\mu f}.$$

The space $L^2(\Omega)$ can be considered as a subspace of $L^2(M)$ by identifying any function $f\in L^2(\Omega)$ with its trivial extension.

CLAIM. *For any $f\in W_0^1(\Omega)$, its trivial extension $\widetilde{f}$ belongs to $W_0^1(M)$ and*

$$\nabla\widetilde{f}=\widetilde{\nabla f}. \tag{5.61}$$

Note for comparison that if $f\in W^1(\Omega)$ then $\widetilde{f}$ does not have to be in $W^1(M)$ – see Exercise 7.7.

PROOF. If $f\in C_0^\infty(\Omega)$ then obviously $\widetilde{f}\in C_0^\infty(M)$ and hence $\widetilde{f}\in W_0^1(M)$. For any $f\in W_0^1(\Omega)$, there exists a sequence $\{f_k\}$ of functions from $C_0^\infty(\Omega)$ such that

$$f_k\to f \text{ in } W_0^1(\Omega).$$

Clearly,

$$\widetilde{f_k}\to\widetilde{f} \text{ in } L^2(M).$$

On the other hand, the sequence $\{\widetilde{f}_k\}$ is obviously Cauchy in $W_0^1(M)$ and hence converges in $W_0^1(M)$. We conclude that the limit must be $\widetilde{f}$, whence it follows that $\widetilde{f} \in W_0^1(M)$ and

$$\widetilde{f}_k \to \widetilde{f} \text{ in } W_0^1(M).$$

In particular, this implies that

$$\nabla \widetilde{f}_k \to \nabla \widetilde{f} \text{ in } \vec{L}^2(M). \tag{5.62}$$

On the other hand,

$$\nabla f_k \to \nabla f \text{ in } \vec{L}^2(\Omega),$$

whence it follows that

$$\widetilde{\nabla f_k} \to \widetilde{\nabla f} \text{ in } \vec{L}^2(M). \tag{5.63}$$

Since $\nabla \widetilde{f}_k = \widetilde{\nabla f_k}$, we conclude from (5.62) and (5.63) that $\nabla \widetilde{f} = \widetilde{\nabla f}$. $\square$

Hence, the space $W_0^1(\Omega)$ can be considered as a (closed) subspace of $W_0^1(M)$ by identifying any function $f \in W_0^1(\Omega)$ with its trivial extension. This identification is norm preserving, which in particular, implies that we have an embedding $W_0^1(\Omega) \hookrightarrow W_0^1(M)$. In what follows, we will follow the convention to denote the trivial extension of a function by the same letter as the function, unless otherwise mentioned.

Consider the Dirichlet Laplace operator in Ω

$$\mathcal{L}^\Omega = -\Delta_\mu|_{W_0^2(\Omega)},$$

as well as the associated resolvent

$$R_\alpha^\Omega = \left(\mathcal{L}^\Omega + \alpha \,\mathrm{id}\right)^{-1}$$

and the heat semigroup

$$P_t^\Omega = \exp\left(-t\mathcal{L}^\Omega\right).$$

A sequence $\{\Omega_i\}_{i=1}^\infty$ of open subsets of M is called an *exhaustion sequence* if $\Omega_i \subset \Omega_{i+1}$ for all i and the union of all sets Ω_i is M.

THEOREM 5.22. *Let $f \in L^2(M)$ be a non-negative function, and $\alpha > 0$.*

(*i*) *For any open set $\Omega \subset M$,*

$$R_\alpha^\Omega f \le R_\alpha f.$$

(*ii*) *For any exhaustion sequence $\{\Omega_i\}_{i=1}^\infty$ in M,*

$$R_\alpha^{\Omega_i} f \xrightarrow{W^1} R_\alpha f \ \text{ as } i \to \infty. \tag{5.64}$$

Note that $R_\alpha^\Omega f$ is a short form of $R_\alpha^\Omega(f|_\Omega)$.

PROOF. (*i*) By Theorem 5.9, the functions $u = R_\alpha f$ and $v = R_\alpha^\Omega f$ are non-negative. By convention, $v \equiv 0$ outside Ω, so that we only need to prove that $u \ge v$ in Ω. By the definition of resolvent, $u \in W_0^2(M)$ and u satisfies in M the equation

$$-\Delta_\mu u + \alpha u = f.$$

In particular, we have $u \in W^1(\Omega)$. Applying Corollary 5.15 to the manifold Ω, we obtain $u \geq R_\alpha^\Omega f$, which was to be proved.

Alternative proof of (*i*). This proof is longer but it uses fewer tools from the present Chapter confining them to Lemmas 5.1 and 5.4. By the definition of resolvent, we have $u \in W_0^2(M)$, $v \in W_0^2(\Omega)$, and

$$\begin{array}{ll} \mathcal{L}^\Omega v + \alpha v = f & \text{in } \Omega, \\ \mathcal{L}u + \alpha u = f & \text{in } M. \end{array} \tag{5.65}$$

By Lemma 5.4, there are sequences of non-negative functions $\{u_k\} \subset C_0^\infty(M)$ and $\{v_k\} \subset C_0^\infty(\Omega)$ which converge, respectively, to u and v in W^1-norm. Let ψ be a smooth non-negative function on $\mathbb{R}$ such that

$$\psi(t) \equiv 0 \text{ for } t \leq 0, \ \ \psi(t) > 0 \text{ and } 0 \leq \psi'(t) \leq 1 \text{ for all } t > 0. \tag{5.66}$$

One can think of $\psi(t)$ as a smooth approximation to t_+ (see Fig. 5.1). Let us show that $w := \psi(v-u) \in W_0^1(\Omega)$. For that, set $w_k = \psi(v_k - u_k)$ and observe that $w_k \in C_0^\infty(M)$ and $0 \leq w_k \leq v_k$. The latter implies that $\operatorname{supp} w_k$ is contained in Ω and hence $w_k \in C_0^\infty(\Omega)$ (see Fig. 5.3).

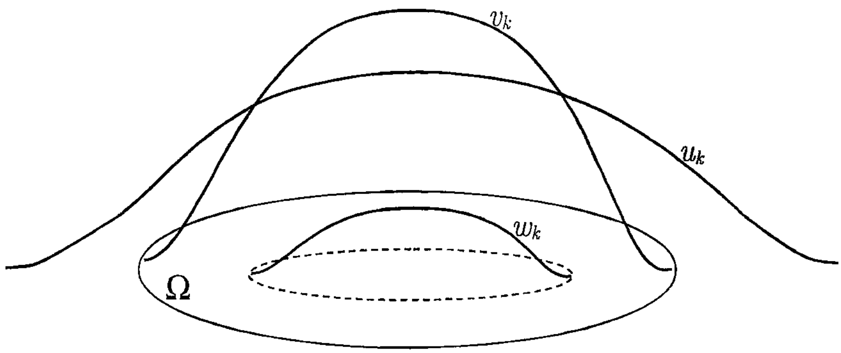

FIGURE 5.3. Function $w_k = \psi(v_k - u_k)$

On the other hand, by (5.3) (see Lemma 5.1) we have

$$\psi(v_k - u_k) \stackrel{W^1(M)}{\longrightarrow} \psi(v-u),$$

which yields $w \in W_0^1(\Omega)$. Subtracting the equations in (5.65) and multiplying them by w, we obtain

$$\left(\mathcal{L}^\Omega v, w\right)_{L^2(\Omega)} - (\mathcal{L}u, w)_{L^2(\Omega)} + \alpha(v-u, w)_{L^2(\Omega)} = 0.$$

By Lemma 4.4, we have

$$\left(\mathcal{L}^\Omega v, w\right)_{L^2(\Omega)} = (\nabla v, \nabla w)_{\vec{L}^2(\Omega)} = \int_\Omega \langle \nabla v, \nabla w \rangle d\mu.$$

and

$$(\mathcal{L}u, w)_{L^2(\Omega)} = (\mathcal{L}u, w)_{L^2(M)} = (\nabla u, \nabla w)_{\vec{L}^2(M)} = \int_\Omega \langle \nabla u, \nabla w \rangle d\mu,$$

where in the last equality we have used (5.61). Hence, we obtain from the above three lines that

$$\int_\Omega \langle \nabla(v-u), \nabla w \rangle d\mu + \alpha \int_\Omega (v-u) w d\mu = 0. \tag{5.67}$$

By (5.2), we have

$$\nabla w = \nabla \psi (v-u) = \psi' (v-u) \nabla (v-u)$$

whence, by $\psi'(t) \geq 0$,

$$\int_\Omega \langle \nabla (v-u), \nabla w \rangle d\mu = \int_\Omega \psi' (v-u) |\nabla (v-u)|^2 d\mu \geq 0.$$

Since $t\psi(t) \geq 0$ for all $t \in \mathbb{R}$, we obtain

$$\int_\Omega (v-u) w d\mu = \int_\Omega (v-u) \psi (v-u) d\mu \geq 0.$$

Therefore, the equation (5.67) is possible only when

$$\int_\Omega (v-u) \psi (v-u) d\mu = 0,$$

that is, when $(v-u) \psi (v-u) = 0$ in Ω, whence $v - u \leq 0$.

(*ii*) Set $u_i = R_\alpha^{\Omega_i} f$ and observe that, by part (*i*) and Theorem 5.9, the sequence $\{u_i\}$ is increasing and

$$0 \leq u_i \leq R_\alpha f.$$

Therefore, u_i converges almost everywhere to a function u such that

$$0 \leq u \leq R_\alpha f,$$

which implies that $u \in L^2(M)$ and, by the dominated convergence theorem, $u_i \to u$ in $L^2(M)$.

Note that the function u_i is in $W_0^1(\Omega_i)$ and, hence, is in $W_0^1(M)$. Let us show that the sequence $\{u_i\}$ is Cauchy in $W_0^1(M)$. Each function u_i satisfies the equation

$$(\nabla u_i, \nabla \varphi) + \alpha (u_i, \varphi) = (f, \varphi), \tag{5.68}$$

for any $\varphi \in W_0^1(\Omega_i)$ (where $(\cdot, \cdot)$ is the inner product in $L^2(M)$). Choosing here $\varphi = u_i$, we obtain

$$(\nabla u_i, \nabla u_i) + \alpha (u_i, u_i) = (f, u_i).$$

Fix $k > i$ and observe that the function $\varphi = u_k - 2u_i$ belongs to $W_0^1(\Omega_k)$. Therefore, by the analogous equation for u_k, we obtain

$$(\nabla u_k, \nabla (u_k - 2u_i)) + \alpha (u_k, u_k - 2u_i) = (f, u_k - 2u_i).$$

Adding up the above two lines yields

$$\|\nabla u_k\|^2 + \|\nabla u_i\|^2 - 2(\nabla u_k, \nabla u_i) + \alpha \left(\|u_k\|^2 + \|u_i\|^2 - 2(u_k, u_i)\right) = (f, u_k - u_i),$$

whence

$$\|\nabla (u_k - u_i)\|^2 + \alpha \|u_k - u_i\|^2 = (f, u_k - u_i) \leq \|f\| \|u_k - u_i\|.$$

Since $\|u_k - u_i\| \to 0$ as $k, i \to \infty$, we conclude that also $\|\nabla (u_k - u_i)\| \to 0$. Therefore, the sequence $\{u_i\}$ is Cauchy in $W_0^1(M)$ and, hence, converges in $W_0^1(M)$. Since its limit in $L^2(M)$ is u, we conclude that the limit of $\{u_i\}$ in $W_0^1(M)$ is also u. In particular, $u \in W_0^1(M)$.

We are left to show that $u = R_\alpha f$. Fix a function $\varphi \in C_0^\infty(M)$ and observe that the support of φ is contained in Ω_i when i is large enough. Therefore, (5.68) holds for this φ for all large enough i. Passing to the limit as $i \to \infty$, we obtain that the same equation holds for u instead of u_i, that is,

$$(\nabla u, \nabla \varphi) + \alpha (u, \varphi) = (f, \varphi). \tag{5.69}$$

Since $C_0^\infty(M)$ is dense in $W_0^1(M)$, this identity holds for all $\varphi \in W_0^1(M)$. By Theorem 4.5, the equation (5.69) has a unique solution $u \in W_0^1(M)$, and this solution is $R_\alpha f$, which finishes the proof. □

THEOREM 5.23. *Let $f \in L^2(M)$ be a non-negative function, and $t > 0$.*

(*i*) *For any open set $\Omega \subset M$,*

$$P_t^\Omega f \le P_t f. \tag{5.70}$$

(*ii*) *For any exhaustion sequence $\{\Omega_i\}_{i=1}^\infty$ in M,*

$$P_t^{\Omega_i} f \xrightarrow{L^2} P_t f \quad \text{as } i \to \infty.$$

REMARK. As we will see from the proof, we have also $P_t^{\Omega_i} f \xrightarrow{\text{a.e.}} P_t f$. It will be shown in Chapter 7 that the functions $P_t^{\Omega_i} f$, $P_t f$ are C^∞-smooth and, in fact, $P_t^{\Omega_i} f \xrightarrow{C^\infty} P_t f$ (see Theorem 7.10 and Exercise 7.18).

PROOF. (*i*) For any $\alpha > 0$, we have by Theorem 5.22 $R_\alpha^\Omega f \le R_\alpha f$. By Theorem 5.9, the operators R_α and R_α^Ω preserve inequalities. Therefore, we obtain by iteration that $(R_\alpha^\Omega)^k f \le R_\alpha^k f$, whence by (5.30)

$$P_t^\Omega f = \lim_{k\to\infty} \left(\frac{k}{t}\right)^k \left(R_{\frac{k}{t}}^\Omega\right)^k f \le \lim_{k\to\infty} \left(\frac{k}{t}\right)^k R_{\frac{k}{t}}^k f = P_t f$$

(cf. Exercise 2.2 for preserving inequalities by convergence in L^2).

Alternative proof of (*i*). By Theorem 4.9, function $u(t,\cdot) := P_t f$ satisfies the conditions

$$\begin{cases} \frac{du}{dt} = \Delta_\mu u, & \text{for all } t > 0, \\ u(t,\cdot) \in W_0^1(M), & \text{for all } t > 0, \\ u(t,\cdot) \xrightarrow{L^2(M)} f & \text{as } t \to 0, \end{cases}$$

where $\frac{du}{dt}$ is the strong derivative in $L^2(M)$. It follows that the restriction of u to Ω (also denoted by u) belongs to $W^1(\Omega)$ for any $t > 0$ and solves the Cauchy problem in Ω with the initial function f. Since by Theorem 5.11 $u \ge 0$, we conclude by Corollary 5.17 that $u \ge P_t^\Omega f$, which was to be proved.

(*ii*) By part (*i*), the sequence of functions $\{P_t^{\Omega_i} f\}_{i=1}^\infty$ is increasing and is bounded by $P_t f$. Hence, for any $t > 0$, the sequence $\{P_t^{\Omega_i} f\}$ converges almost everywhere to a function u_t such that

$$0 \le u_t \le P_t f.$$

Since $P_t f \in L^2$, we conclude that $u_t \in L^2$ and, by the dominated convergence theorem,

$$P_t^{\Omega_i} f \xrightarrow{L^2} u_t.$$

We need to show that $u_t = P_t f$.

Fix a non-negative function $\varphi \in C_0^\infty(M)$ and observe that $\varphi \in C_0^\infty(\Omega_i)$ for large enough i. It follows from (5.29) and the monotone convergence theorem that, for any $\alpha > 0$,

$$(R_\alpha^{\Omega_i} f, \varphi) = \int_0^\infty e^{-\alpha t} (P_t^{\Omega_i} f, \varphi) dt \longrightarrow \int_0^\infty e^{-\alpha t} (u_t, \varphi)\, dt,$$

as $i \to \infty$. On the other hand, by Theorem 5.22 and (5.29),

$$(R_\alpha^{\Omega_i} f, \varphi) \to (R_\alpha f, \varphi) = \int_0^\infty e^{-\alpha t} (P_t f, \varphi)\, dt.$$

We conclude that

$$\int_0^\infty e^{-\alpha t} (u_t, \varphi)\, dt = \int_0^\infty e^{-\alpha t} (P_t f, \varphi)\, dt,$$

which, in the view of inequality $(u_t, \varphi) \le (P_t f, \varphi)$, is only possible when

$$(u_t, \varphi) = (P_t f, \varphi) \tag{5.71}$$

for *almost* all $t > 0$. Let us show that, in fact, both functions $t \mapsto (P_t f, \varphi)$ and $t \mapsto (u_t, \varphi)$ are continuous in $t > 0$, which will imply that (5.71) holds for *all* $t > 0$.

By Theorem 4.9, the function $(P_t f, \varphi)$ is even differentiable in $t > 0$. The same theorem also yields

$$\frac{d}{dt}(P_t^{\Omega_i} f, \varphi) = -(\mathcal{L}^{\Omega_i} P_t^{\Omega_i} f, \varphi) = (P_t^{\Omega_i} f, \Delta_\mu \varphi),$$

where $(\cdot,\cdot)$ is the inner product in L^2. Using $\|P_t^{\Omega_i}\| \le 1$, we obtain

$$\left| \frac{d}{dt}(P_t^{\Omega_i} f, \varphi) \right| \le \|P_t^{\Omega_i} f\|_{L^2(\Omega_i)} \|\Delta_\mu \varphi\|_{L^2(\Omega_i)} \le \|f\|_{L^2(M)} \|\Delta_\mu \varphi\|_{L^2(M)}.$$

Since the right hand side here does not depend on i, we see that all the functions $(P_t^{\Omega_i} f, \varphi)$ have uniformly bounded derivatives in t and, hence, are Lipschitz functions with the same Lipschitz constant. Therefore, the limit function (u_t, φ) is also Lipschitz and, in particular, continuous.

Finally, since (5.71) holds for an arbitrary non-negative function $\varphi \in C_0^\infty(M)$, we conclude that $u_t = P_t f$ (cf. Exercise 4.7). $\square$

Exercises.

5.22. Let u be a function from $C(M) \cap W_0^1(M)$. For any $a > 0$, set

$$U_a = \{x \in M : u(x) > a\}.$$

Prove that $(u-a)_+ \in W_0^1(U_a)$.

5.23. Let Ω be an open subset of a weighted manifold M and K be a compact subset of Ω. Let f be a non-negative function from $L^2(M)$. Prove that, for all $\alpha > 0$,

$$R_\alpha f - R_\alpha^\Omega f \le \operatorname*{esup}_{M\setminus K} R_\alpha f. \tag{5.72}$$

5.24. Under the hypotheses of Exercise 5.23, prove that, for all $t > 0$,

$$P_t f - P_t^\Omega f \le \sup_{s\in[0,t]} \operatorname*{esup}_{M\setminus K} P_s^\Omega f. \tag{5.73}$$

5.25. Let $\{\Omega_i\}_{i=1}^{\infty}$ be an increasing sequence of open subsets of M, $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, and $f \in L^2(\Omega_1)$. Prove that the family of functions $\left\{P_t^{\Omega_i} f\right\}_{i=1}^{\infty}$ considered as the paths in $L^2(\Omega)$, is equicontinuous in $t \in [0, +\infty)$ with resect to the norm in $L^2(\Omega)$.

5.26. Let A be the multiplication operator by a bounded,non-negative measurable function a on M.

(a) Prove that A is a bounded, non-negative definite, self-adjoint operator in L^2 and, for any non-negative $f \in L^2$ and $t \geq 0$,

$$0 \leq e^{-tA} f \leq f. \tag{5.74}$$

(b) Prove that, for any non-negative $f \in L^2$ and $t \geq 0$,

$$0 \leq e^{-t(\mathcal{L}+A)} f \leq e^{-t\mathcal{L}} f. \tag{5.75}$$

(c) Using part (b), give an alternative proof of the fact that $P_t^{\Omega} f \leq P_t f$.

HINT. In part (b) use the Trotter product formula:

$$e^{-t(A+B)} f = \lim_{n\to\infty} \left(e^{-\frac{t}{n}A} e^{-\frac{t}{n}B}\right)^n f, \tag{5.76}$$

that is true for any two non-negative definite self-adjoint operators A, B in L^2.

Notes

There are different approaches to the maximum principle. The classical approach as in Lemma 1.5 applies to smooth solutions of the Laplace and heat equations and uses the fact that the derivatives at the extremal points have certain signs. We will use this approach in Chapter 8 again, after having established the smoothness of the solutions.

In the present Chapter we work with weak solutions, and the boundary values are also understood in a weak sense, so that other methods are employed. It was revealed by Beurling and Deny [**39**], [**106**] that the Markovian properties of the heat semigroup originate from certain properties of the Dirichlet integral $\int_M |\nabla u|^2 \, d\mu$, which in turn follow from the chain rule for the gradient ∇. This is why the chain rule for the weak gradient is discussed in details in Sections 5.1 and 5.2 (see also [**130**]).

Another useful tool is the resolvent R_α. The use of the resolvent for investigation of the heat semigroup goes back to the Hille-Yoshida theorem. Obtaining the Markovian properties of P_t via those of R_α is a powerful method that we have borrowed from [**124**]. Theorems 5.16, 5.22, 5.23 in the present forms as well as their proofs were taken from [**162**].

The reader is referred [**41**], [**115**], [**124**] for the Markovian properties in the general context of Markov semigroups and Markov processes.

CHAPTER 6

Regularity theory in $\mathbb{R}^n$

We present here the regularity theory for second order elliptic and parabolic equations in $\mathbb{R}^n$ with smooth coefficients. In the next Chapter 7, this theory will be transplanted to manifolds and used, in particular, to prove the existence of the heat kernel.

We use here the same notation as in Chapter 2.

6.1. Embedding theorems

6.1.1. Embedding $W^k_{loc} \hookrightarrow C^m$. In this section, we prove the Sobolev embedding theorem (known also as the Sobolev lemma), which provides the link between the classical and weak derivatives. Let us first mention the following trivial embedding.

CLAIM. *For any open set $\Omega \subset \mathbb{R}^n$ and any non-negative integer m, we have an embedding*

$$C^m(\Omega) \hookrightarrow W^m_{loc}(\Omega). \tag{6.1}$$

PROOF. Indeed, if $u \in C^m(\Omega)$ then any classical derivative $\partial^\alpha u$ of order $|\alpha| \le m$ is also a weak derivative from $L^2_{loc}(\Omega)$ and, for any open set $\Omega' \Subset \Omega$,

$$\|\partial^\alpha u\|_{L^2(\Omega')} \le C \sup_{\Omega'} |\partial^\alpha u|,$$

which implies

$$\|u\|_{W^m(\Omega')} \le C\|u\|_{C^m(\Omega')}.$$

Hence, the identical mapping $C^m(\Omega) \to W^m_{loc}(\Omega)$ is not only a linear injection but is also continuous, which means that it is an embedding. □

The next theorem provides a highly non-trivial embedding of $W^k_{loc}(\Omega)$ to $C^m(\Omega)$.

THEOREM 6.1. (Sobolev embedding theorem) *Let Ω be an open subset of $\mathbb{R}^n$. If k and m are non-negative integers such that*

$$k > m + \frac{n}{2}$$

then $u \in W^k_{loc}(\Omega)$ implies $u \in C^m(\Omega)$. Moreover, for all relatively compact open sets Ω', Ω'' such that $\Omega' \Subset \Omega'' \Subset \Omega$

$$\|u\|_{C^m(\Omega')} \le C\|u\|_{W^k(\Omega'')}, \tag{6.2}$$

where the constant C depends on $\Omega', \Omega'', k, m, n$.

More precisely, the first claim means that, for every $u \in W^k_{loc}(\Omega)$, there is a version of u that belongs to $C^m(\Omega)$, which defines a linear injection from $W^k_{loc}(\Omega)$ to $C^m(\Omega)$. The estimate (6.2) means that this injection is continuous, so that we have an embedding

$$W^k_{loc}(\Omega) \hookrightarrow C^m(\Omega). \tag{6.3}$$

Set

$$W^\infty(\Omega) = \bigcap_{k=0}^{\infty} W^k(\Omega)$$

and

$$W^\infty_{loc}(\Omega) = \bigcap_{k=0}^{\infty} W^k_{loc}(\Omega).$$

The topology in the space $W^\infty(\Omega)$ is defined by the family of seminorms

$$\|u\|_{W^k(\Omega)},$$

for all positive integers k, and the topology in $W^\infty_{loc}(\Omega)$ is defined by the family of seminorms

$$\|u\|_{W^k(\Omega')},$$

where k is a positive integer and $\Omega' \Subset \Omega$ is an open set.

It follows from (6.1) and (6.3) that

$$W^\infty_{loc}(\Omega) = C^\infty(\Omega),$$

where the equality means also the identity of the topologies.

Hence, in order to prove that a function from L^2_{loc} belongs to C^∞, it suffices to show that it has weak derivatives of all orders. Although the latter may be difficult as well, the existence of weak derivatives can be frequently proved using the tools of the theory of *Hilbert* spaces, which are not available for the spaces C^k.

EXAMPLE 6.2. Let us show that $u \in L^1_{loc}(\Omega)$ implies $u \in W^{-k}_{loc}(\Omega)$, for any $k > n/2$. Indeed, fix an open set $\Omega' \Subset \Omega$ and observe that, for any $\varphi \in \mathcal{D}(\Omega')$, we have by Theorem 6.1

$$(u, \varphi) = \int_{\Omega'} u\varphi d\mu \le \sup_{\Omega'} |\varphi| \, \|u\|_{L^1(\Omega')} \le C\|\varphi\|_{W^k(\Omega')}\|u\|_{L^1(\Omega')},$$

where C depends on Ω' and n. It follows that

$$\|u\|_{W^{-k}(\Omega')} \le C\|u\|_{L^1(\Omega')}$$

and, hence, $u \in W^{-k}_{loc}$.

PROOF OF THEOREM 6.1. We split the proof into a series of claims. Recall that $B_R = \{x \in \mathbb{R}^n : |x| < R\}$.

CLAIM 1. *For any $u \in \mathcal{D}(B_R)$ and $k > n/2$,*

$$|u(0)| \le C\|u\|_{W^k}, \tag{6.4}$$

where the constant C depends on k, n, R.

We use for the proof the polar coordinates (r, θ) centered at the origin $0 \in \mathbb{R}^n$ (cf. Section 3.9), and write $u = u(r, \theta)$ away from 0. The relations between the Cartesian and polar coordinates are given by the identities

$$x^j = r f^j(\theta),$$

where f^j are the smooth functions of $\theta \in \mathbb{S}^{n-1}$ such that

$$\sum_j \left(f^j\right)^2 \equiv 1 \tag{6.5}$$

(cf. (3.61)). This implies that

$$\partial_r = f^j(\theta)\,\partial_j, \tag{6.6}$$

whence it follows by induction that, for any positive integer k,

$$\partial_r^k = f^{j_1} \dots .f^{j_k} \partial_{j_1} \dots \partial_{j_k}.$$

Applying the Cauchy-Schwarz inequality and (6.5), we obtain

$$\left|\partial_r^k u\right|^2 \le \sum_{|\alpha| \le k} |\partial^\alpha u|^2. \tag{6.7}$$

In particular, we see that the function $\partial_r^k u$ is bounded in $\mathbb{R}^n \setminus \{0\}$ (note that this function is not defined at 0), which allows to integrate $\partial_r^k u$ in r over the interval $[0, R]$.

For any $\theta \in \mathbb{S}^{n-1}$, we have $u(R, \theta) = 0$ whence we obtain by the fundamental theorem of calculus

$$u(0) = -\int_0^R \partial_r u(r, \theta)\, dr.$$

Integration by parts yields

$$u(0) = -\left[\partial_r u(r, \theta)\, r\right]_0^R + \int_0^R r \partial_r^2 u(r, \theta)\, dr = \int_0^R r \partial_r^2 u(r, \theta)\, dr,$$

and continuing by induction, we arrive at

$$u(0) = \frac{(-1)^k}{(k-1)!} \int_0^R r^{k-1} \partial_r^k u(r, \theta)\, dr.$$

Integrating this identity in θ over $\mathbb{S}^{n-1}$ and using

$$r^{n-1} dr d\theta = d\mu,$$

where μ is the Lebesgue measure (cf. (3.82)), we obtain

$$\omega_n u(0) = \frac{(-1)^k}{(k-1)!} \int_{B_R} r^{k-n} \partial_r^k u\, d\mu.$$

The Cauchy-Schwarz inequality yields then

$$|u(0)|^2 \le C \int_{B_R} r^{2k-2n} d\mu \int_{B_R} \left|\partial_r^k u\right|^2 d\mu. \tag{6.8}$$

The first integral in (6.8) is evaluated in the polar coordinates as follows:

$$\int_{B_R} r^{2k-2n} d\mu = \omega_n \int_0^R r^{2k-2n} r^{n-1} dr = \omega_n \int_0^R r^{2k-n-1} r dr = CR^{2k-n} < \infty,$$

where we have used the hypothesis $k > n/2$. Hence, this integral is just a constant depending on R. By (6.7), the second integral in (6.8) is bounded by

$$\int_{B_r} \sum_{|\alpha| \le k} |\partial^\alpha u|^2 \, d\mu = \|u\|_{W^k}^2.$$

Therefore, (6.4) follows from (6.8).

For the next Claims 2-4, $\Omega \subset \mathbb{R}^n$ is a bounded open set.

CLAIM 2. *For any $u \in \mathcal{D}(\Omega)$ and $k > n/2$, we have*

$$\sup |u| \le C \|u\|_{W^k} \tag{6.9}$$

where the constant C depends on k, n, and[1] $\operatorname{diam} \Omega$.

Indeed, let x be a point of maximum of $|u|$ and $R = \operatorname{diam} \Omega$. Applying Claim 1 in the ball $B_R(x)$, we obtain (6.9).

CLAIM 3. *Assume that $u \in W^k(\Omega)$, where $k > n/2$, and let the support of u be a compact set in Ω. Then $u \in C(\Omega)$ and the estimate (6.9) holds.*

Let φ be a mollifier and set $u_j = u * \varphi_{1/j}$ where j is a positive integer. By Lemma 2.9, we have $u_j \in \mathcal{D}(\Omega)$, provided j is large enough, and by Theorem 2.13, $u_j \to u$ in W^k when $j \to \infty$. Applying (6.9) to the difference $u_i - u_j$, we obtain

$$\sup |u_i - u_j| \le C \|u_i - u_j\|_{W^k}.$$

Since the right hand side tends to 0 as $i, j \to \infty$, we obtain that the sequence $\{u_j\}$ is Cauchy with respect to the sup-norm and, hence, converges uniformly to a continuous function. Hence, the function u has a continuos version, which satisfies (6.9).

CLAIM 4. *Assume that $u \in W^k(\Omega)$, where $k > n/2 + m$ and m is a positive integer, and let the support of u be a compact set in Ω. Then $u \in C^m(\Omega)$ and*

$$\|u\|_{C^m} \le C \|u\|_{W^k}, \tag{6.10}$$

where the constant C depends on k, m, n, and $\operatorname{diam} \Omega$.

Indeed, if $|\alpha| \le m$ then $\partial_\alpha u \in W^{k-m}$, which yields by Claim 3 that $\partial_\alpha u \in C(\Omega)$ and

$$\sup |\partial^\alpha u| \le C \|\partial^\alpha u\|_{W^{k-m}} \le C \|u\|_{W^k}, \tag{6.11}$$

whence the claim follows[2].

---

[1] In fact, the constant C in (6.9) can be chosen independently of Ω, as one can see from the second proof of Theorem 6.1.

[2] See Claim in the proof of Lemma 2.9.

Finally, let us prove the statement of Theorem 6.1. Assume $u \in W^k_{loc}(\Omega)$ where Ω is an open subset of $\mathbb{R}^n$ and $k > n/2 + m$. Choose open sets $\Omega' \Subset \Omega'' \Subset \Omega$ and a function $\psi \in C_0^\infty(\Omega'')$ such that $\psi \equiv 1$ on Ω'. Then $\psi u \in W^k(\Omega'')$ (cf. Exercise 2.26) and the support of ψu is a compact subset of Ω''. By Claim 4, we conclude that $\psi u \in C^m(\Omega'')$. In particular, $u \in C^m(\Omega')$ because $u = \psi u$ in Ω'. Since Ω' may be chosen arbitrarily, we conclude that $u \in C^m(\Omega)$. It follows from (6.10) and (2.38) that

$$\|u\|_{C^m(\Omega')} \leq \|\psi u\|_{C^m(\Omega'')} \leq C\|\psi u\|_{W^k(\Omega'')} \leq C'\|u\|_{W^k(\Omega'')},$$

which finishes the proof. $\square$

Second proof. We use here the Fourier transform and the results of Exercise 2.34. Assume first that $u \in W^k(\mathbb{R}^n)$ with $k > n/2$ and prove that $u \in C(\mathbb{R}^n)$. Since $u \in L^2$, the Fourier transform $\widehat{u}(\xi)$ is defined and is also in L^2. By (2.42) we have

$$\int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 \left(1 + |\xi|^2\right)^k d\xi \leq C\|u\|^2_{W^k}. \tag{6.12}$$

By the Cauchy-Schwarz inequality,

$$\left(\int_{\mathbb{R}^n} |\widehat{u}(\xi)|\, d\xi\right)^2 \leq \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 \left(1 + |\xi|^2\right)^k d\xi \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^{-k} d\xi. \tag{6.13}$$

The condition $k > n/2$ implies that the last integral in (6.13) converges, which together with (6.12) yields

$$\int_{\mathbb{R}^n} |\widehat{u}(\xi)|\, d\xi \leq C\|u\|_{W^k}. \tag{6.14}$$

In particular, we see that $\widehat{u} \in L^1$ and, hence, u can be obtain from $\widehat{u}$ by the inversion formula

$$u(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{u}(\xi)\, e^{ix\xi} d\xi, \tag{6.15}$$

for almost all x. Let us show that the right hand side of (6.15) is a continuous function. Indeed, for all $x, y \in \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} \widehat{u}(\xi)\, e^{ix\xi} d\xi - \int_{\mathbb{R}^n} \widehat{u}(\xi)\, e^{iy\xi} d\xi = \int_{\mathbb{R}^n} \widehat{u}(\xi) \left(e^{ix\xi} - e^{iy\xi}\right) d\xi.$$

If $y \to x$ then the function under the integral in the right hand side tends to 0 and is bounded by the integrable function $2|\widehat{u}(\xi)|$. We conclude by the dominated convergence theorem that the integral tends to 0 and, hence, the function u has a continuous version, given by the right hand side of (6.15). It also follows from (6.14) and (6.15) that

$$\sup_{\mathbb{R}^n} |u| \leq C\|u\|_{W^k}.$$

Since this proves Claim 3 from the first proof, the rest follows in the same way. $\square$

THIRD PROOF. We will prove here a somewhat weaker version of Theorem 6.1, when the hypothesis $k > m + n/2$ is replaced by the stronger condition $k \geq m + 2l$, where l is the minimal integer that is greater than $n/4$. This version of Theorem 6.1 is sufficient for all our applications. The advantage of this proof is that it can be carried over to manifolds under a mild assumption that the heat kernel satisfies a certain upper estimate; besides, it can be enhanced to work also for the full range of k (cf. Exercises 7.43, 7.44, 7.46).

We start with the following claim. Recall that P_t is the heat semigroup defined in Section 2.7.

CLAIM. *If $u \in \mathcal{D}(\mathbb{R}^n)$, k is a positive integer, and*

$$f = (-\Delta + \mathrm{id})^k u, \tag{6.16}$$

then, for any $x \in \mathbb{R}^n$,

$$u(x) = \int_0^\infty \frac{t^{k-1}e^{-t}}{(k-1)!} P_t f(x)\, dt. \tag{6.17}$$

By Lemma 2.17, we have in $[0, +\infty) \times \mathbb{R}^n$ the identity

$$P_t f = P_t\left((-\Delta + \mathrm{id})^k u\right) = (-\Delta + \mathrm{id})^k P_t u.$$

Since $P_t u$ satisfies the heat equation and, hence,

$$(-\Delta + \mathrm{id})^k P_t u = (-\partial_t + \mathrm{id})^k P_t u,$$

we obtain

$$P_t f = (-\partial_t + \mathrm{id})^k P_t u.$$

Therefore, the right hand side of (6.17) is equal to

$$\int_0^\infty \frac{t^{k-1}e^{-t}}{(k-1)!}(-\partial_t + \mathrm{id})^k P_t u\, dt. \tag{6.18}$$

Integrating by parts in (6.18) and using the identity

$$(\partial_t + \mathrm{id})\frac{t^{k-1}e^{-t}}{(k-1)!} = \frac{t^{k-2}e^{-t}}{(k-2)!},$$

which holds for any $k \geq 2$, we obtain

$$\begin{aligned}
&\int_0^\infty \frac{t^{k-1}e^{-t}}{(k-1)!}(-\partial_t + \mathrm{id})^k P_t u\, dt \\
= {} & \left[\frac{t^{k-1}e^{-t}}{(k-1)!}(-\partial_t + \mathrm{id})^{k-1} P_t u\right]_0^\infty \\
&+ \int_0^\infty (\partial_t + \mathrm{id})\frac{t^{k-1}e^{-t}}{(k-1)!}(-\partial_t + \mathrm{id})^{k-1} P_t u\, dt \\
= {} & \int_0^\infty \frac{t^{k-2}e^{-t}}{(k-2)!}(-\partial_t + \mathrm{id})^{k-1} P_t u\, dt,
\end{aligned}$$

where the limits at 0 and ∞ vanish due to $k > 1$ and the boundedness of the function $(-\partial_t + \mathrm{id})^{k-1} P_t u(x)$ in $[0, +\infty) \times \mathbb{R}^n$ (cf. Lemma 2.17).

Hence, the integral in (6.18) reduces by induction to a similar integral with $k = 1$. Integrating by parts once again and using $(\partial_t + \mathrm{id})\, e^{-t} = 0$ and $P_t u \to u$ as $t \to 0$ (cf. Theorem 1.3), we obtain

$$\int_0^\infty e^{-t}(-\partial_t + \mathrm{id}) P_t u\, dt = \left[e^{-t} P_t u\right]_0^\infty = u,$$

which proves (6.17).

Let now l be the minimal integer that is greater than $n/4$, $u \in \mathcal{D}(\mathbb{R}^n)$ and

$$f = (-\Delta + \mathrm{id})^{\,l}\, u.$$

It is easy to see that, by (1.22),

$$\int_{\mathbb{R}^n} p_t^2(z)\, dz = p_t * p_t(0) = p_{2t}(0) = (8\pi t)^{-n/2}.$$

Hence, for all $x \in \mathbb{R}^n$ and $t > 0$,

$$\begin{aligned} |P_t f(x)| &\leq \left(\int_{\mathbb{R}^n} p_t^2(x-y)dy\right)^{1/2} \left(\int_{\mathbb{R}^n} f^2(y)dy\right)^{1/2} \\ &= (8\pi t)^{-n/4} \|f\|_{L^2}, \end{aligned} \tag{6.19}$$

which together with (6.17) yields

$$|u(x)| \leq \int_0^\infty \frac{t^{l-1}e^{-t}}{(l-1)!} |P_t f(x)|\, dt \leq \|f\|_{L^2} \int_0^\infty \frac{t^{l-1}e^{-t}}{(l-1)!} (8\pi t)^{-n/4}\, dt.$$

The condition $l > n/4$ implies that the above integral converges, whence we obtain

$$|u(x)| \leq C\|f\|_{L^2},$$

where the constant C depends only on n. Since f can be represented as a combination of the derivatives of u up to the order $2l$, it follows that

$$\sup_{\mathbb{R}^n} |u| \leq C\|u\|_{W^{2l}}. \tag{6.20}$$

The proof is finished in the same way as the first proof after Claim 2. $\square$

Exercises.

6.1. Show that the delta function δ in $\mathbb{R}^n$ belongs to W^{-k} for any $k > n/2$.

6.1.2. Compact embedding $W_0^1 \hookrightarrow L^2$. Define the space $W_0^1(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^1(\Omega)$. Clearly, $W_0^1(\Omega)$ is a Hilbert space with the same inner product as in $W^1(\Omega)$.

THEOREM 6.3. (Rellich compact embedding theorem) *If Ω is a relatively compact open subset of $\mathbb{R}^n$ then the identical embedding*

$$W_0^1(\Omega) \hookrightarrow L^2(\Omega)$$

is a compact operator.

PROOF. Any function $f \in W_0^1(\Omega)$ can be extended to $\mathbb{R}^n$ by setting $f = 0$ outside Ω. Clearly, $f \in L^2 \cap L^1(\mathbb{R}^n)$. Moreover, $f \in W^1(\mathbb{R}^n)$ because f is the limit in $W^1(\Omega)$ of a sequence $\{\varphi_k\} \subset C_0^\infty(\Omega)$, and this sequence converges also in $W^1(\mathbb{R}^n)$.

Let $\{f_k\}$ be a bounded sequence in $W_0^1(\Omega)$. Extending f_k to $\mathbb{R}^n$ as above, we can assume that also $f_k \in W^1(\mathbb{R}^n)$. Since $\{f_k\}$ is bounded in $L^2(\mathbb{R}^n)$, there exists a subsequence, denoted again by $\{f_k\}$, which converges *weakly* in $L^2(\mathbb{R}^n)$ to a function $f \in L^2(\mathbb{R}^n)$. Let us show that, in fact, $\{f_k\}$ converges to f in $L^2(\Omega)$-norm, which will settle the claim.

Let us use the heat semigroup P_t as in the third proof of Theorem 6.1. For any $t > 0$, we have by the triangle inequality

$$\|f_k - f\|_{L^2(\Omega)} \leq \|f_k - P_t f_k\|_{L^2(\Omega)} + \|P_t f_k - P_t f\|_{L^2(\mathbb{R}^n)} + \|P_t f - f\|_{L^2(\mathbb{R}^n)}. \tag{6.21}$$

Let C be a constant that bounds $\|f_k\|_{W^1}$ for all k. Then Lemma 2.20 yields

$$\|f_k - P_t f_k\|_{L^2(\mathbb{R}^n)} \leq \sqrt{t}\,\|f_k\|_{W^1(\mathbb{R}^n)} \leq C\sqrt{t}. \tag{6.22}$$

Since $\{f_k\}$ converges to f weakly in $L^2(\mathbb{R}^n)$ as $k \to \infty$, we obtain that, for all $x \in \mathbb{R}^n$,

$$P_t f_k(x) = (f_k, p_t(x - \cdot))_{L^2} \to (f, p_t(x - \cdot))_{L^2} = P_t f(x).$$

On the other hand, by (6.19)

$$\sup_{\mathbb{R}^n} |P_t f_k| \leq (8\pi t)^{-n/4} \|f_k\|_{L^2(\mathbb{R}^n)} \leq C (8\pi t)^{-n/4}.$$

Hence, for any fixed $t > 0$, the sequence $\{P_t f_k\}$ is bounded in sup-norm and converges to $P_t f$ pointwise in $\mathbb{R}^n$. Since $\mu(\Omega) < \infty$, the dominated convergence theorem yields

$$\|P_t f_k - P_t f\|_{L^2(\Omega)} \to 0 \text{ as } k \to \infty. \tag{6.23}$$

From (6.21), (6.22), and (6.23), we obtain that, for any $t > 0$,

$$\limsup_{k\to\infty} \|f_k - f\|_{L^2(\Omega)} \leq C\sqrt{t} + \|P_t f - f\|_{L^2(\mathbb{R}^n)}.$$

The proof is finished by letting $t \to 0$ because $\|P_t f - f\|_{L^2(\mathbb{R}^n)} \to 0$ by Lemma 2.18. $\square$

Exercises.

6.2. The purpose of this problem is to give an alternative proof of Theorem 6.3 by means of the Fourier transform. Let Ω be a bounded open set in $\mathbb{R}^n$. Recall that $W_0^1(\Omega)$ can be considered as a subspace of $W^1(\mathbb{R}^n)$ by extending functions by 0 outside Ω.

(a) Prove that, for all $f \in W_0^1(\Omega)$ and $g \in C^\infty(\mathbb{R}^n)$,

$$\int_\Omega (\partial_j f)\, g\, dx = -\int_\Omega f\, \partial_j g\, dx. \tag{6.24}$$

(b) Prove that, for any $f \in W_0^1(\Omega)$ and for any $\xi \in \mathbb{R}^n$,

$$\left(f, e^{i\xi x}\right)_{W^1(\Omega)} = \left(1 + |\xi|^2\right) \widehat{f}(\xi), \tag{6.25}$$

where $\widehat{f}(\xi)$ is the Fourier transform of f.

(c) Let $\{f_k\}$ be a sequence from $W_0^1(\Omega)$ such that f_k converges weakly in $W^1(\mathbb{R}^n)$ to a function $f \in W^1(\mathbb{R}^n)$. Prove that $\widehat{f_k}(\xi) \to \widehat{f}(\xi)$, for any $\xi \in \mathbb{R}^n$. Prove that also $\widehat{f_k} \to \widehat{f}$ in $L^2_{loc}(\mathbb{R}^n)$.

(d) Finally, prove that if $\{f_k\}$ is a bounded sequence in $W_0^1(\Omega)$ then $\{f_k\}$ contains a subsequence that converges in $L^2(\Omega)$.

HINT. Use Exercises 2.28 and 2.34.

6.2. Two technical lemmas

LEMMA 6.4. (Friedrichs-Poincaré inequality) *Let Ω be a bounded open set in $\mathbb{R}^n$. Then, for any $\varphi \in \mathcal{D}(\Omega)$ and any index $j = 1, ..., n$,*

$$\int_\Omega \varphi^2 d\mu \le (\operatorname{diam}\Omega)^2 \int_\Omega (\partial_j \varphi)^2\, d\mu. \tag{6.26}$$

PROOF. Set $l = \operatorname{diam}\Omega$. Consider first the case $n = 1$ when we can assume that Ω is the interval $(0, l)$ (note that we can always expand Ω to an interval of the same diameter since a function $\varphi \in \mathcal{D}(\Omega)$ can be extended to a function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ by setting $\varphi = 0$ outside Ω). Since $\varphi(0) = 0$, we have, for any $x \in (0, l)$,

$$\varphi^2(x) = \left(\int_0^x \varphi'(s)\, ds\right)^2 \le l \int_0^l (\varphi')^2(s)\, ds,$$

whence, integrating in x,

$$\int_0^l \varphi^2(x)\, dx \le l^2 \int_0^l (\varphi')^2(s)\, ds,$$

which is exactly (6.26) for the case $n = 1$.

In the case $n > 1$, first apply the one-dimensional Friedrichs' inequality to the function $\varphi(x)$ with respect to the variable x^j considering all other variables frozen, and then integrate in all other variables, which yields (6.26). $\square$

Recall that, for any mollifier φ, we denote by φ_ε the function $\varepsilon^{-n}\varphi(x/\varepsilon)$ (see Chapter 2).

LEMMA 6.5. (Friedrichs lemma) *Let Ω be an open set in $\mathbb{R}^n$ and let $a \in C^\infty(\Omega)$. Consider the operator $\mathcal{A}$ in Ω defined by*

$$\mathcal{A}u = a\partial_j u,$$

for some $j = 1, ..., n$. Then, for any function $u \in L^2(\Omega)$ with compact support in Ω and for any mollifier φ in $\mathbb{R}^n$, we have

$$\|\mathcal{A}(u * \varphi_\varepsilon) - (\mathcal{A}u) * \varphi_\varepsilon\|_{L^2(\Omega)} \longrightarrow 0 \quad as\ \varepsilon \to 0. \tag{6.27}$$

PROOF. Let Ω_0 be the ε_0-neighborhood of $\operatorname{supp} u$, where $\varepsilon_0 > 0$ is so small that $\Omega_0 \Subset \Omega$. Let us extend u to a function in $\mathbb{R}^n$ by setting $u = 0$ outside $\operatorname{supp} u$. Then the convolution $u * \varphi_\varepsilon$ is defined as a smooth function in $\mathbb{R}^n$ and, if $\varepsilon < \varepsilon_0$ then $u * \varphi_\varepsilon$ is supported in Ω. In turn, this implies that the expression $\mathcal{A}(u * \varphi_\varepsilon)$ defines a function from $\mathcal{D}(\Omega)$. Similarly, $\mathcal{A}u$ is a distribution supported by $\operatorname{supp} u$, and $(\mathcal{A}u) * \varphi_\varepsilon$ is a function from $\mathcal{D}(\Omega)$.

Let us show that, for $\varepsilon < \varepsilon_0$,

$$\|\mathcal{A}(u * \varphi_\varepsilon) - (\mathcal{A}u) * \varphi_\varepsilon\|_{L^2} \le K\|u\|_{L^2}, \tag{6.28}$$

where

$$K = \sup_{\Omega_0} |\nabla a| \left(1 + \int_{\mathbb{R}^n} |x|\,|\partial_j \varphi|\,dx\right). \tag{6.29}$$

The point of the inequality (6.28) is that although the constant K depends on functions a, φ and on the set Ω_0, it is still independent of u and ε. We have, for any $x \in \Omega$,

$$\begin{aligned}\mathcal{A}(u * \varphi_\varepsilon)(x) &= a(\partial_j(u * \varphi_\varepsilon))(x) = a(u * \partial_j \varphi_\varepsilon)(x) \\ &= \int_\Omega a(x)u(y)\partial_j\varphi_\varepsilon(x-y)dy \end{aligned} \tag{6.30}$$

and

$$\begin{aligned}(\mathcal{A}u) * \varphi_\varepsilon(x) &= (\mathcal{A}u, \varphi_\varepsilon(x - \cdot)) \\ &= (\partial_j u, a(\cdot)\varphi_\varepsilon(x-\cdot)) \\ &= -(u, \partial_j(a(\cdot)\varphi_\varepsilon(x-\cdot))) \\ &= \int_\Omega u(y)a(y)\partial_j\varphi_\varepsilon(x-y)dy \\ &\quad - \int_\Omega u(y)\partial_j a(y)\varphi_\varepsilon(x-y)dy. \end{aligned} \tag{6.31}$$

Setting

$$\mathcal{A}_\varepsilon u := \mathcal{A}(u * \varphi_\varepsilon) - (\mathcal{A}u) * \varphi_\varepsilon,$$

we obtain from (6.30) and (6.31)

$$\begin{aligned}\mathcal{A}_\varepsilon u(x) &= \int_\Omega (a(x) - a(y))\,\partial_j\varphi_\varepsilon(x-y)u(y)dy \\ &\quad + \int_\Omega \partial_j a(y)\varphi_\varepsilon(x-y)u(y)dy. \end{aligned} \tag{6.32}$$

Note that the domains of integration in (6.32) can be restricted to

$$y \in \operatorname{supp} u \cap B_\varepsilon(x). \tag{6.33}$$

If $x \notin \Omega_0$ then the set (6.33) is empty and, hence, $\mathcal{A}_\varepsilon u(x) = 0$. Therefore, $\mathcal{A}_\varepsilon u(x) \neq 0$ implies $x \in \Omega_0$. In this case, for any y is as in (6.33), we have $x \in B_\varepsilon(y) \subset \Omega_0$ whence it follows that

$$|a(x) - a(y)| \leq \sup_{\Omega_0} |\nabla a| \, |x - y| = C \, |x - y| \, , \tag{6.34}$$

where $C := \sup_{\Omega_0} |\nabla a|$ (see Fig. 6.1).

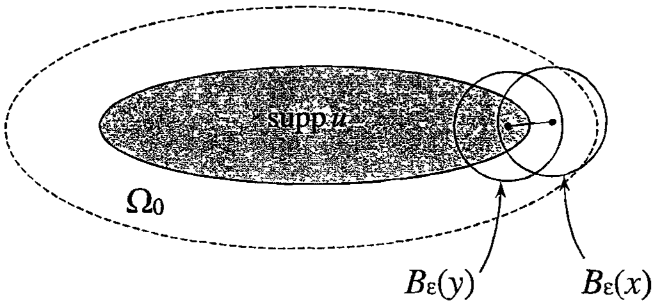

FIGURE 6.1. If $y \in \operatorname{supp} u \cap B_\varepsilon(x)$ then $x \in B_\varepsilon(y) \subset \Omega_0$ and, hence, the straight line segment between x and y is contained in Ω_0, which implies (6.34).

Since also $|\partial_j a(y)| \leq C$, we obtain from (6.32)

$$\begin{aligned} |\mathcal{A}_\varepsilon u(x)| &\leq C \int_\Omega (|x - y| \, |\partial_j \varphi_\varepsilon| \, (x - y) + \varphi_\varepsilon(x - y) \, |u(y)| \, dy \\ &= C \int_\Omega \psi_\varepsilon(x - y) \, |u(y)| \, dy, \end{aligned}$$

where

$$\psi_\varepsilon(x) := |x| \, |\partial_j \varphi_\varepsilon| \, (x) + \varphi_\varepsilon(x).$$

Hence, for all $x \in \mathbb{R}^n$, we have

$$|\mathcal{A}_\varepsilon u\,(x)| \leq C \, |u| * \psi_\varepsilon\,(x) \, ,$$

which implies by rescaling the inequality (2.25) of Theorem 2.11 (see also Remark 2.12) that

$$\|\mathcal{A}_\varepsilon u\|_{L^2} \leq C \left[\int_{\mathbb{R}^n} \psi_\varepsilon(x) dx \right] \|u\|_{L^2}.$$

Evaluating the integral of ψ_ε by changing $z = x/\varepsilon$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \psi_\varepsilon(x) dx &= 1 + \int_{\mathbb{R}^n} |x| \left| \varepsilon^{-n} \partial_{x^j} \varphi(\frac{x}{\varepsilon}) \right| dx \\ &= 1 + \int_{\mathbb{R}^n} |\varepsilon z| \left| \varepsilon^{-1} \partial_{z^j} \varphi(z) \right| dz = 1 + \int_{\mathbb{R}^n} |z| \, |\partial_j \varphi(z)| \, dz, \end{aligned}$$

which gives

$$\|\mathcal{A}_\varepsilon u\|_{L^2} \leq K \|u\|_{L^2}, \tag{6.35}$$

that is (6.28).

Let us now prove (6.27), that is,

$$\|\mathcal{A}_\varepsilon u\|_{L^2} \to 0 \text{ as } \varepsilon \to 0.$$

If $u \in \mathcal{D}(\Omega)$ then, by Lemma 2.10, $u * \varphi_\varepsilon \xrightarrow{\mathcal{D}} u$ and, hence,

$$\mathcal{A}(u * \varphi_\varepsilon) \xrightarrow{\mathcal{D}} \mathcal{A}u.$$

Applying Lemma 2.10 to $\mathcal{A}u$, we obtain

$$(\mathcal{A}u) * \varphi_\varepsilon \xrightarrow{\mathcal{D}} \mathcal{A}u,$$

which together with the previous line implies $\mathcal{A}_\varepsilon u \xrightarrow{\mathcal{D}} 0$. For an arbitrary function $u \in L^2(\Omega)$ with compact support, choose a sequence $u_k \in \mathcal{D}(\Omega)$ such that $u_k \to u$ in L^2 (for example, take $u_k = u * \varphi_{1/k}$ – cf. Theorem 2.3) and observe that

$$\mathcal{A}_\varepsilon u = \mathcal{A}_\varepsilon u_k + \mathcal{A}_\varepsilon (u - u_k).$$

To estimate the second term here, we will apply (6.35) to the difference $u - u_k$. If k is large enough then $\operatorname{supp}(u - u_k)$ is contained in a small neighborhood of $\operatorname{supp} u$ and, hence, the constant K from (6.29) can be chosen the same for all such k. Hence, we obtain

$$\|\mathcal{A}_\varepsilon u\|_{L^2} \leq \|\mathcal{A}_\varepsilon u_k\|_{L^2} + \|\mathcal{A}_\varepsilon (u - u_k)\|_{L^2} \leq \|\mathcal{A}_\varepsilon u_k\|_{L^2} + K\|u - u_k\|_{L^2}.$$

Since $\|u - u_k\|_{L^2} \to 0$ as $k \to \infty$ and, for any fixed k, $\|\mathcal{A}_\varepsilon u_k\|_{L^2} \to 0$ as $\varepsilon \to 0$, we conclude that $\|\mathcal{A}_\varepsilon u\|_{L^2} \to 0$, which was to be proved. □

6.3. Local elliptic regularity

Fix an open set $\Omega \subset \mathbb{R}^n$ and L be the following differential operator in Ω:

$$L = \partial_i \left(a^{ij}(x)\, \partial_j\right),$$

where $a^{ij}(x)$ are smooth functions in Ω such that the matrix $\left(a^{ij}(x)\right)_{i,j=1}^n$ is symmetric and positive definite, for any $x \in \Omega$. Any such operator with a positive definite matrix $\left(a^{ij}\right)$ is referred to as an *elliptic operator*. The fact that the matrix $\left(a^{ij}\right)$ is positive definite means that, for any point $x \in \Omega$ there is a number $c(x) > 0$ such that

$$a^{ij}(x)\, \xi_i \xi_j \geq c(x) |\xi|^2 \text{ for any } \xi \in \mathbb{R}^n. \tag{6.36}$$

The number $c(x)$ is called the *ellipticity constant* of operator L at x. Clearly, $c(x)$ can be chosen to be a continuous function of x. This implies that, for any compact set $K \subset \Omega$, $c(x)$ is bounded below by a positive constant for all $x \in K$, which is called the ellipticity constant of operator L in K.

The symmetry of the matrix a^{ij} implies that the operator L is symmetric with respect to the Lebesgue measure μ in the following sense: for any functions $u, v \in \mathcal{D}(\Omega)$,

$$\int_\Omega Lu\, v\, d\mu = -\int_\Omega a^{ij}\partial_i u\, \partial_j v\, d\mu = \int_\Omega u\, Lv\, d\mu, \tag{6.37}$$

which follows immediately from the integration-by-parts formula or from the divergence theorem.

The operator L is obviously defined on $\mathcal{D}'(\Omega)$ because all parts of the expression $\partial_i\left(a^{ij}\partial_j\right)$ are defined as operators in $\mathcal{D}'(\Omega)$ (see Section 2.4). For all $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned}(Lu, \varphi) &= \left(\partial_i\left(a^{ij}\partial_j u\right), \varphi\right) = -\left(a^{ij}\partial_j u, \partial_i\varphi\right) \\ &= -\left(\partial_j u, a^{ij}\partial_i\varphi\right) = \left(u, \partial_j\left(a^{ij}\partial_i\varphi\right)\right),\end{aligned}$$

that is,

$$(Lu, \varphi) = (u, L\varphi).$$

This identity can be also used as the definition of Lu for a distribution u.

6.3.1. Solutions from L^2_{loc}.

LEMMA 6.6. *If a function $u \in L^2(\Omega)$ is compactly supported in Ω and $Lu \in W^{-1}(\Omega)$, then*

$$L(u * \varphi_\varepsilon) \overset{W^{-1}}{\longrightarrow} Lu \text{ as } \varepsilon \to 0. \tag{6.38}$$

PROOF. Consider the difference

$$L(u * \varphi_\varepsilon) - (Lu) * \varphi_\varepsilon = \partial_i\left(a^{ij}\partial_j(u * \varphi_\varepsilon)\right) - \partial_i\left(\left(a^{ij}\partial_j u\right) * \varphi_\varepsilon\right) = \partial_i f^i_\varepsilon$$

where

$$f^i_\varepsilon := a^{ij}\partial_j(u * \varphi_\varepsilon) - \left(a^{ij}\partial_j u\right) * \varphi_\varepsilon.$$

As follows from Lemma 6.5, $\|f^i_\varepsilon\|_{L^2} \to 0$ as $\varepsilon \to 0$ whence

$$\|L(u * \varphi_\varepsilon) - (Lu) * \varphi_\varepsilon\|_{W^{-1}} = \|\partial_i f^i_\varepsilon\|_{W^{-1}} \le \sum_i \|f^i_\varepsilon\|_{L^2} \longrightarrow 0.$$

Since by Theorem 2.16

$$(Lu) * \varphi_\varepsilon \overset{W^{-1}}{\longrightarrow} Lu,$$

we obtain (6.38). □

LEMMA 6.7. (A priori estimate) *For any open set $\Omega' \Subset \Omega$ and for any $u \in \mathcal{D}(\Omega')$,*

$$\|u\|_{W^1} \le C\|Lu\|_{W^{-1}},$$

where the constant C depends on $\operatorname{diam}\Omega'$ *and on the ellipticity constant of L in Ω'.*

PROOF. Lemma 6.4 implies

$$\|u\|_{W^1} \le C\|\nabla u\|_{L^2},$$

where C depends on $\operatorname{diam}\Omega'$. Setting $f = -Lu$, we obtain by (6.37)

$$(f,u) = -\int_\Omega (Lu)\, u\, d\mu = \int_{\Omega'} a^{ij}(x)\, \partial_i u\, \partial_j u\, d\mu.$$

Let $c > 0$ be the ellipticity constant of L in $\overline{\Omega'}$ so that, for any $x \in \Omega'$,

$$a^{ij}(x)\, \partial_i u\, \partial_j u \ge c\,|\nabla u|^2.$$

Combining with the previous lines, we obtain

$$(f,u) \ge c\int_\Omega |\nabla u|^2\, d\mu \ge c'\|u\|_{W^1}^2, \tag{6.39}$$

for some $c' > 0$. On the other hand, by the definition of the norm W^{-1},

$$(f,u) \le \|f\|_{W^{-1}}\|u\|_{W^1},$$

which implies

$$c'\|u\|_{W^1}^2 \le \|f\|_{W^{-1}}\|u\|_{W^1},$$

whence the claim follows. □

LEMMA 6.8. *For any integer $m \ge -1$, if $u \in W^{m+1}_{loc}(\Omega)$ and $Lu \in W^m_{loc}(\Omega)$ then $u \in W^{m+2}_{loc}(\Omega)$. Moreover, for all open subsets $\Omega' \Subset \Omega'' \Subset \Omega$,*

$$\|u\|_{W^{m+2}(\Omega')} \le C\left(\|u\|_{W^{m+1}(\Omega'')} + \|Lu\|_{W^m(\Omega'')}\right), \tag{6.40}$$

where C is a constant depending on Ω', Ω'', L, m.

PROOF. The main difficulty lies in the proof of the inductive basis for $m = -1$, whereas the inductive step is straightforward.

The inductive basis for $m = -1$. Assuming that $u \in L^2_{loc}(\Omega)$ and $Lu \in W^{-1}_{loc}(\Omega)$, let us show that $u \in W^1_{loc}(\Omega)$ and that the following estimate holds:

$$\|u\|_{W^1(\Omega')} \le C\left(\|u\|_{L^2(\Omega'')} + \|Lu\|_{W^{-1}(\Omega'')}\right). \tag{6.41}$$

Let $\psi \in \mathcal{D}(\Omega'')$ be a cutoff function of $\overline{\Omega'}$ in Ω'' (see Theorem 2.2). Then the function $v := \psi u$ obviously belongs to $L^2(\Omega'')$ and $\operatorname{supp} v$ is a compact subset of Ω''. We claim that $Lv \in W^{-1}(\Omega'')$. Indeed, observe that

$$Lv = \psi Lu + 2a^{ij}\partial_i\psi\partial_j u + (L\psi)\, u, \tag{6.42}$$

and, by Lemma 2.14, the right hand side belongs to $W^{-1}(\Omega'')$ because Lu, $\partial_j u$, u belong to W^{-1}_{loc}, whereas ψ, $a^{ij}\partial_i\psi$, $L\psi$ belong to $\mathcal{D}(\Omega'')$.

Let φ be a mollifier in $\mathbb{R}^n$. By Lemma 6.7, we have

$$\|v * \varphi_\varepsilon\|_{W^1} \le C\|L(v * \varphi_\varepsilon)\|_{W^{-1}},$$

whereas by Lemma 6.38

$$L(v * \varphi_\varepsilon) \xrightarrow{W^{-1}} Lv \text{ as } \varepsilon \to 0,$$

which implies that

$$\limsup_{\varepsilon \to 0} \|v * \varphi_\varepsilon\|_{W^1} \leq C\|Lv\|_{W^{-1}}.$$

By Theorem 2.13, we conclude that $v \in W^1(\Omega'')$ and

$$\|v\|_{W^1(\Omega'')} \leq C\|Lv\|_{W^{-1}(\Omega'')}.$$

Since $u = v$ on Ω', we obtain that $u \in W^1(\Omega')$ and

$$\|u\|_{W^1(\Omega')} \leq \|v\|_{W^1(\Omega'')}.$$

By varying the set Ω' we conclude $u \in W^1_{loc}(\Omega)$. Finally, observing that by (6.42)

$$\|Lv\|_{W^{-1}(\Omega'')} \leq C\left(\|u\|_{L^2(\Omega'')} + \|Lu\|_{W^{-1}(\Omega'')}\right)$$

and combining this with the two previous lines, we obtain (6.41).

The inductive step from $m-1$ to m, where $m \geq 0$. For an arbitrary distribution $u \in \mathcal{D}'(\Omega)$, we have

$$\begin{aligned} \partial_l(Lu) - L(\partial_l u) &= \partial_l\partial_i\left(a^{ij}\partial_j u\right) - \partial_i\left(a^{ij}\partial_j\partial_l u\right) \\ &= \partial_i\left[\partial_l\left(a^{ij}\partial_j u\right) - a^{ij}\partial_l\partial_j u\right] \\ &= \partial_i\left[\left(\partial_l a^{ij}\right)\partial_j u\right]. \end{aligned} \tag{6.43}$$

Assuming that $u \in W^{m+1}_{loc}(\Omega)$ and $Lu \in W^m_{loc}(\Omega)$ and noticing that the right hand side of (6.43) contains only first and second derivatives of u, we obtain

$$L(\partial_l u) = \partial_l(Lu) - \partial_i\left[\left(\partial_l a^{ij}\right)\partial_j u\right] \in W^{m-1}_{loc}. \tag{6.44}$$

Since $\partial_l u \in W^m_{loc}$, we can apply the inductive hypothesis to $\partial_l u$, which yields $\partial_l u \in W^{m+1}_{loc}$ and, hence, $u \in W^{m+2}_{loc}$.

Finally, we see from (6.44) that

$$\|L(\partial_l u)\|_{W^{m-1}(\Omega'')} \leq C\|u\|_{W^{m+1}(\Omega'')} + \|Lu\|_{W^m(\Omega'')},$$

whence by the inductive hypothesis

$$\begin{aligned} \|\partial_l u\|_{W^{m+1}(\Omega')} &\leq C\left(\|\partial_l u\|_{W^m(\Omega'')} + \|L(\partial_l u)\|_{W^{m-1}(\Omega'')}\right) \\ &\leq C\left(\|u\|_{W^{m+1}(\Omega'')} + \|Lu\|_{W^m(\Omega'')}\right), \end{aligned}$$

which obviously implies (6.40). □

THEOREM 6.9. *For any integer $m \geq -1$, if $u \in L^2_{loc}(\Omega)$ and $Lu \in W^m_{loc}(\Omega)$ then $u \in W^{m+2}_{loc}(\Omega)$. Moreover, for all open subsets $\Omega' \Subset \Omega'' \Subset \Omega$,*

$$\|u\|_{W^{m+2}(\Omega')} \leq C\left(\|u\|_{L^2(\Omega'')} + \|Lu\|_{W^m(\Omega'')}\right), \tag{6.45}$$

where C is a constant depending on Ω', Ω'', L, m.

Note the hypotheses of Theorem 6.9 are weaker than those of Lemma 6.8 – instead of the requirement $u \in W^{m+1}_{loc}$, we assume here only $u \in L^2_{loc}$.

PROOF. Let k be the largest integer between 0 and $m+2$ such that $u \in W^k_{loc}$. We need to show that $k = m+2$. Indeed, if $k \leq m+1$ then we have also $Lu \in W^{k-1}_{loc}$, whence it follows by Lemma 6.8 that $u \in W^{k+1}_{loc}$, thus contradicting the definition of k.

The estimate (6.45) is proved by improving inductively the estimate (6.40) of Lemma 6.8. For that, consider a decreasing sequence of open sets $\{\Omega_i\}_{i=0}^{m+2}$ such that $\Omega_0 = \Omega''$, $\Omega_{m+2} = \Omega'$ and $\Omega_{i+1} \Subset \Omega_i$. By Lemma 6.8, we obtain, for any $1 \leq k \leq m+2$,

$$\begin{aligned} \|u\|_{W^k(\Omega_k)} &\leq C\left(\|u\|_{W^{k-1}(\Omega_{k-1})} + \|Lu\|_{W^{k-2}(\Omega_{k-1})}\right) \\ &\leq C\left(\|u\|_{W^{k-1}(\Omega_{k-1})} + \|Lu\|_{W^m(\Omega'')}\right), \end{aligned}$$

which obviously implies (6.45). □

COROLLARY 6.10. *If $u \in L^2_{loc}(\Omega)$ and $Lu \in W^m_{loc}(\Omega)$ where*

$$m > l + \frac{n}{2} - 2.$$

and l is a non-negative integer then $u \in C^l(\Omega)$.

Consequently, if $u \in L^2_{loc}(\Omega)$ and $Lu \in C^\infty(\Omega)$ then also $u \in C^\infty(\Omega)$.

PROOF. Indeed, by Theorem 6.9 $u \in W^{m+2}_{loc}(\Omega)$ and, since $m+2 > l+\frac{n}{2}$, Theorem 6.1 implies $u \in C^l(\Omega)$. The second claim is obvious. □

For applications on manifold, we need the following consequence of Theorem 6.9 for a bit more general operator L.

COROLLARY 6.11. *Consider in Ω the following operator*

$$L = b(x)\,\partial_i\left(a^{ij}(x)\,\partial_j\right),$$

where $a^{ij}(x)$ and $b(x)$ belong to $C^\infty(\Omega)$, $b(x) > 0$, and the matrix $\left(a^{ij}(x)\right)_{i,j=1}^n$ is symmetric and positive definite for all $x \in \Omega$. Assume that, for some positive integer k,

$$u,\ Lu,\ \ldots,\ L^k u \in L^2_{loc}(\Omega). \tag{6.46}$$

Then $u \in W^{2k}_{loc}(\Omega)$ and, for all open sets $\Omega' \Subset \Omega'' \Subset \Omega$,

$$\|u\|_{W^{2k}(\Omega')} \leq C\sum_{l=0}^{k} \|L^l u\|_{L^2(\Omega'')}, \tag{6.47}$$

where C depends on Ω', Ω'', L, k.

If m is a non-negative number and

$$k > \frac{m}{2} + \frac{n}{4}, \tag{6.48}$$

then $u \in C^m(\Omega)$ and

$$\|u\|_{C^m(\Omega')} \leq C\sum_{l=0}^{k} \|L^l u\|_{L^2(\Omega'')}, \tag{6.49}$$

where C depends on $\Omega', \Omega'', L, k, m, n$.

PROOF. Note that Theorem 6.9 applies to operator L as well because $b\partial_i\left(a^{ij}\partial_j u\right) \in W^m_{loc}$ if and only if $\partial_i\left(a^{ij}\partial_j u\right) \in W^m_{loc}$, and the $W^m(\Omega')$-norms of these functions are comparable for any $\Omega' \Subset \Omega$.

The inductive basis for $k = 0$ is trivial. Let us prove the inductive step from $k-1$ to k assuming $k \geq 1$. Applying the inductive hypothesis to function $v = Lu$, we obtain $Lu \in W^{2k-2}_{loc}$, whence $u \in W^{2k}_{loc}$ by Theorem 6.9.

To prove (6.47) observe that by (6.40)

$$\|u\|_{W^{2k}(\Omega')} \leq C\left(\|u\|_{L^2(\Omega^*)} + \|Lu\|_{W^{2k-2}(\Omega^*)}\right),$$

where $\Omega' \Subset \Omega^* \Subset \Omega''$, and, by the inductive hypothesis,

$$\|Lu\|_{W^{2k-2}(\Omega^*)} = \|v\|_{W^{2k-2}(\Omega^*)} \leq C\sum_{l=0}^{k-1}\|L^l v\|_{L^2(\Omega'')} = \sum_{l=1}^{k}\|L^l u\|_{L^2(\Omega'')},$$

whence (6.47) follows.

Finally, $u \in W^{2k}_{loc}(\Omega)$ and $2k > m + n/2$ imply by Theorem 6.1 that $u \in C^m(\Omega)$. The estimate (6.49) follows from (6.2) and (6.47). □

6.3.2. Solutions from $\mathcal{D}'$. Here we extend Lemma 6.8 and Theorem 6.9 to arbitrary negative orders m. We start with the solvability of the equation $Lu = f$ (cf. Section 4.2). For an open set $U \subset \mathbb{R}^n$, consider the space $W^1_0(U)$, which is the closure of $\mathcal{D}(U)$ in $W^1(U)$. Clearly, $W^1_0(U)$ is a Hilbert space with the same inner product as $W^1(U)$.

LEMMA 6.12. *Let $U \Subset \Omega$ be an open set. Then there exists a bounded linear operator $\mathcal{R} : L^2(U) \to W^1_0(U)$ such that, for any $f \in L^2(U)$, the function $u = \mathcal{R}f$ solves the equation $Lu = f$.*

The operator $\mathcal{R}$ is called the resolvent of L (cf. Section 4.2).

PROOF. Denote by $[u, v]$ a bilinear form in $W^1_0(U)$ defined by

$$[u, v] := \int_U a^{ij}\,\partial_i u\,\partial_j v\,d\mu.$$

Let us show that $[u, v]$ is, in fact, an inner product, whose norm is equivalent to the standard norm in $W^1_0(U)$. Indeed, using the ellipticity of L, the compactness of $\overline{U}$, and Lemma 6.4, we obtain, for any $u \in \mathcal{D}(U)$,

$$[u, u] = \int_U a^{ij}\partial_i u\,\partial_j u\,d\mu \geq c\int_U |\nabla u|^2\,d\mu \geq c'\int_U u^2 d\mu, \tag{6.50}$$

and

$$[u, u] \leq C\int_U |\nabla u|^2\,d\mu,$$

and these estimates extend by continuity to any $u \in W^1_0(U)$. It follows that $[u, u]$ is in a finite ratio with

$$\|u\|^2_{W^1} = \int_U u^2 d\mu + \int_U |\nabla u|^2\,d\mu,$$

and, hence, the space $W_0^1(U)$ is a Hilbert space with the inner product $[\cdot,\cdot]$.

The equation $Lu = f$ is equivalent to the identity

$$\int_U a^{ij}\partial_i u\, \partial_j \varphi\, d\mu = -(f,\varphi) \text{ for any } \varphi \in \mathcal{D}(U), \tag{6.51}$$

which can be written in the form

$$[u,\varphi] = -(f,\varphi) \text{ for all } \varphi \in W_0^1(U). \tag{6.52}$$

Note that $\varphi \mapsto (f,\varphi)$ is a bounded linear functional of φ in $W_0^1(U)$, because, using again (6.50), we have

$$|(f,\varphi)| \le \|f\|_{L^2}\|\varphi\|_{L^2} \le C\|f\|_{L^2}[\varphi,\varphi]^{1/2}. \tag{6.53}$$

Hence, by the Riesz representation theorem, (6.52) has a unique solution $u \in W_0^1(U)$, which allows to define the resolvent by $\mathcal{R}f = u$.

The linearity $\mathcal{R}$ is obviously follows from the uniqueness of the solution. Using $\varphi = u$ in (6.52) and (6.53) yields

$$[u,u] \le C\|f\|_{L^2}[u,u]^{1/2},$$

and, hence, $[u,u]^{1/2} \le C\|f\|_{L^2}$, which means that the resolvent $\mathcal{R}$ is a bounded operator from $L^2(U)$ to $W_0^1(U)$. □

REMARK 6.13. If $f \in C^\infty \cap L^2(U)$ then, by Corollary 6.10, the function $u = \mathcal{R}f$ also belongs to the class $C^\infty \cap L^2(U)$.

Let us mention for a future reference that $\mathcal{R}$ is a symmetric operator in the sense that

$$(\mathcal{R}f, g) = (f, \mathcal{R}g) \text{ for all } f,g \in L^2(U). \tag{6.54}$$

Indeed, setting $u = \mathcal{R}f$ and $v = \mathcal{R}g$, we obtain from (6.52)

$$[u,v] = -(f,v) = -(f,\mathcal{R}g)$$

and

$$[v,u] = -(g,u) = -(g,\mathcal{R}f),$$

whence (6.54) follows. Since $W_0^1(U)$ is a subspace of $L^2(U)$, we can consider the resolvent as an operator from $L^2(U)$ to $L^2(U)$. Since U is relatively compact and, by Theorem 6.3, the embedding of $W_0^1(U)$ into $L^2(U)$ is a compact operator, we obtain that $\mathcal{R}$, as an operator from $L^2(U)$ to $L^2(U)$, is a compact operator.

LEMMA 6.14. *For any $m \in \mathbb{Z}$, if $u \in W_{loc}^{m+1}(\Omega)$ and $Lu \in W_{loc}^m(\Omega)$ then $u \in W_{loc}^{m+2}(\Omega)$.*

PROOF. If $m \ge -1$ then this was proved in Lemma 6.8. Assume $m \le -2$ and set $k = -m$ so that the statement becomes: if $u \in W_{loc}^{-k+1}$ and $Lu \in W_{loc}^{-k}$ then $u \in W_{loc}^{-k+2}$. It suffices to prove that $\psi u \in W^{-k+2}(\Omega)$ for any $\psi \in \mathcal{D}(\Omega)$. Fix such ψ and set $v = \psi u$. Clearly, $v \in W^{-k+1}(\Omega)$ and, as it follows from (6.42), $Lv \in W^{-k}(\Omega)$.

Let U be a small neighborhood of $\operatorname{supp} v$ such that $U \Subset \Omega$. We need to show that $\|v\|_{W^{-k+2}(U)} < \infty$, and this will follow if we prove that, for any $f \in \mathcal{D}(U)$,

$$(v, f) \leq C\|Lv\|_{W^{-k}}\|f\|_{W^{k-2}}. \tag{6.55}$$

By Lemma 6.12, for any $f \in \mathcal{D}(U)$ there exists a function $w \in C^\infty \cap L^2(U)$ solving the equation $Lw = f$ in U and satisfying the estimate

$$\|w\|_{L^2(U)} \leq C\|f\|_{L^2(U)}. \tag{6.56}$$

Fix a function $\varphi \in \mathcal{D}(U)$ such that $\varphi \equiv 1$ in a neighborhood of $\operatorname{supp} \psi$. Then $\varphi w \in \mathcal{D}(U)$ and

$$\begin{aligned}(v, f) &= (v, Lw) = (v, L(\varphi w)) = (Lv, \varphi w) \\ &\leq \|Lv\|_{W^{-k}(U)}\|\varphi w\|_{W^k(U)} \leq C\|Lv\|_{W^{-k}(U)}\|w\|_{W^k(U')},\end{aligned}$$

where $U' \Subset U$ is a neighborhood of $\operatorname{supp} \varphi$ and the constant C depends only on φ. By Theorem 6.9 and (6.56), we obtain

$$\|w\|_{W^k(U')} \leq C\left(\|w\|_{L^2(U)} + \|Lw\|_{W^{k-2}(U)}\right) \leq C'\|f\|_{W^{k-2}(U)},$$

whence (6.55) follows. $\square$

Finally, we have the following extension of Theorem 6.9.

THEOREM 6.15. *For any $m \in \mathbb{Z}$, if $u \in \mathcal{D}'(\Omega)$ and $Lu \in W^m_{loc}(\Omega)$ then $u \in W^{m+2}_{loc}(\Omega)$.*

PROOF. Let us first show that, for any open set $U \Subset \Omega$ there exists a positive integer l such that $u \in W^{-l}(U)$. By Lemma 2.7, there exist constants N and C such that, for all $\varphi \in \mathcal{D}(U)$,

$$(u, \varphi) \leq C \max_{|\alpha| \leq N} \sup_U |\partial^\alpha \varphi|.$$

It follows from Theorem 6.1 (more precisely, we use the estimate (6.10) from the proof of that theorem) that the right hand side here is bounded above by const $\|\varphi\|_{W^l(U)}$ provided $l > N + n/2$. Hence, we obtain

$$(u, \varphi) \leq C'\|\varphi\|_{W^l(U)},$$

which implies, by the definition of W^{-l},

$$\|u\|_{W^{-l}(U)} \leq C' < \infty$$

and $u \in W^{-l}(U)$.

In particular, we have $u \in W^{-l}_{loc}(U)$. Let k be the maximal integer between $-l$ and $m+2$ such that $u \in W^k_{loc}(U)$. If $k \leq m+1$ then $Lu \in W^{k-1}_{loc}(U)$, which implies by Lemma 6.14 $u \in W^{k+1}_{loc}(U)$, thus contradicting the definition of k. We conclude that $k = m+2$, that is, $u \in W^{m+2}_{loc}(U)$. It follows that $u \in W^{m+2}_{loc}(\Omega)$, which was to be proved. $\square$

Exercises.

6.3. Prove that, for any open set $\Omega' \Subset \Omega$, for any $m \geq -1$, and for any $u \in \mathcal{D}(\Omega')$,

$$\|u\|_{W^{m+2}} \leq C\|Lu\|_{W^m}, \tag{6.57}$$

where a constant C depends on Ω', L, m.

HINT. Use Lemma 6.7 for the inductive basis and prove the inductive step as in Lemma 6.8.

6.4. Consider a more general operator

$$L = \partial_i \left(a^{ij}(x)\,\partial_j\right) + b^j(x)\,\partial_j + c(x), \tag{6.58}$$

where a^{ij} is as before, and b^j and c are smooth functions in Ω. Prove that if $u \in \mathcal{D}'(\Omega)$ and $Lu \in W^m_{loc}(\Omega)$ for some $m \in \mathbb{Z}$ then $u \in W^{m+2}_{loc}(\Omega)$. Conclude that $Lu \in C^\infty$ implies $u \in C^\infty$.

6.4. Local parabolic regularity

6.4.1. Anisotropic Sobolev spaces. Denote the Cartesian coordinates in $\mathbb{R}^{n+1}$ by $t, x^1, ..., x^n$. Respectively, the first order partial derivatives are denoted by $\partial_t \equiv \frac{\partial}{\partial t}$ and $\partial_j \equiv \frac{\partial}{\partial x^j}$ for $j \geq 1$. For any $(n+1)$-dimensional multiindex $\alpha = (\alpha_0, ..., \alpha_n)$, the partial derivative ∂^α is defined by

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{(\partial t)^{\alpha_0}(\partial x^1)^{\alpha_1}...(\partial x^n)^{\alpha_n}} = \partial_t^{\alpha_0}\partial_1^{\alpha_1}...\partial_n^{\alpha_n}.$$

Alongside the order $|\alpha|$ of the multiindex, consider its *weighted* order $[\alpha]$, defined by

$$[\alpha] := 2\alpha_0 + \alpha_1 + ... + \alpha_n.$$

This definition reflect the fact that, in the theory of parabolic equations, the time derivative ∂_t has the same weight as any spatial derivative ∂_j^2 of the *second* order.

Fix an open set $\Omega \subset \mathbb{R}^{n+1}$. The spaces of test functions $\mathcal{D}(\Omega)$ and distributions $\mathcal{D}'(\Omega)$ are defined in the same way as before. Our purpose is to introduce *anisotropic* (parabolic) Sobolev spaces $V^k(\Omega)$ which reflect different weighting of time and space directions. For any non-negative integer k, set

$$V^k(\Omega) = \left\{u \in L^2(\Omega) : \partial^\alpha u \in L^2(\Omega) \text{ for all } \alpha \text{ with } [\alpha] \leq k\right\},$$

and the norm in V^k is defined by

$$\|u\|^2_{V^k(\Omega)} := \sum_{[\alpha]\leq k} \|\partial^\alpha u\|^2_{L^2}.$$

Obviously, $V^0 \equiv L^2$, whereas

$$V^1(\Omega) = \left\{u \in L^2(\Omega) : \partial_j u \in L^2(\Omega)\ \forall j = 1, ..., n\right\}$$

and

$$V^2(\Omega) = \left\{u \in L^2(\Omega) : \partial_t u, \partial_j u, \partial_i\partial_j u \in L^2(\Omega)\ \forall i, j = 1, ..., n\right\}. \tag{6.59}$$

Most facts about the spaces V^k are similar to those of W^k. Let us point out some distinctions between these spaces. For simplicity, we write $V^k \equiv V^k(\Omega)$ unless otherwise stated.

CLAIM.(*a*) *If $u \in V^k$ and $[\alpha] \leq k$ then $\partial^\alpha u \in V^{k-[\alpha]}$.*

(*b*) *If $\partial^\alpha u \in V^k$ for all α with $[\alpha] \leq l$ and one of the numbers k, l is even then $u \in V^{k+l}$.*

PROOF. (*a*) Let β be a multiindex with $[\beta] \leq k - [\alpha]$. Then $[\alpha + \beta] \leq k$ and hence $\partial^{\alpha+\beta} u \in L^2$. Therefore, $\partial^\beta(\partial^\alpha u) \in L^2$, which means that $\partial^\alpha u \in V^{k-[\alpha]}$

(*b*) It is not difficult to verify that if one of the numbers k, l is even, then any multiindex β with $[\beta] \leq k + l$ can be presented in the form $\beta = \alpha + \alpha'$ where $[\alpha] \leq l$ and $[\alpha'] \leq k$. Hence, $\partial^\beta u = \partial^{\alpha'}(\partial^\alpha u) \in V^{k-[\alpha']} \subset L^2$, whence the claim follows. □

It follows from part (*a*) of the above Claim that if $u \in V^k$ then $\partial_j u \in V^{k-1}$ and $\partial_t u \in V^{k-2}$ (provided $k \geq 2$).

We will use below only the case $l = 2$ of part (*b*). Note that if both k, l are odd then the claim of part (*b*) is not true. For example, if $k = l = 1$ then the condition that $\partial^\alpha u \in V^1$ for all α with $[\alpha] \leq 1$ means that the spatial derivatives $\partial_j u$ are in V^1. This implies that $\partial_i u$, $\partial_i \partial_j u$ are in L^2. However, to prove that $u \in V^2$ we need to know that also $\partial_t u \in L^2$, which cannot be derived from the hypotheses.

For any positive integer k and a distribution $u \in \mathcal{D}'(\Omega)$, set

$$\|u\|_{V^{-k}} := \sup_{\varphi \in \mathcal{D}(\Omega) \setminus \{0\}} \frac{(u, \varphi)}{\|\varphi\|_{V^k}}. \tag{6.60}$$

The space $V^{-k}(\Omega)$ is defined by

$$V^{-k}(\Omega) := \left\{u \in \mathcal{D}'(\Omega) : \|u\|_{V^{-k}} < \infty\right\}.$$

Obviously, for all $u \in V^{-k}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, we have

$$|(u, \varphi)| \leq \|u\|_{V^{-k}(\Omega)} \|\varphi\|_{V^k(\Omega)}.$$

The local Sobolev spaces $V^k_{loc}(\Omega)$ are defined similarly to $W^k_{loc}(\Omega)$.

The statements of Lemma 2.14 and Theorems 2.13, 2.16 remain true for the spaces V^k, and the proofs are the same, so we do not repeat them. Observing that $V^{2k}_{loc}(\Omega) \hookrightarrow W^k_{loc}(\Omega)$ and applying Theorem 6.1, we obtain that

$$V^{2k}_{loc}(\Omega) \hookrightarrow C^m(\Omega),$$

provided k and m are non-negative integers such that $k > m + n/2$. Consequently, we have

$$V^\infty_{loc}(\Omega) = C^\infty(\Omega).$$

6.4.2. Solutions from L^2_{loc}. Fix an open set $\Omega \subset \mathbb{R}^{n+1}$ and consider in Ω the differential operator

$$\mathcal{P} = \rho(x)\,\partial_t - \partial_i\left(a^{ij}(x)\,\partial_j\right) \tag{6.61}$$

where ρ and a^{ij} are smooth functions depending *only* on $x = \left(x^1, ..., x^n\right)$ (but not on t), $\rho(x) > 0$, and the matrix $\left(a^{ij}\right)_{i,j=1}^n$ is symmetric and positive definite. The operator $\mathcal{P}$ with such properties belongs to the class of *parabolic operators*. The results of this section remain true if the coefficients a^{ij} and ρ depend also on t but the proofs are simpler if they do not, and this is sufficient for our applications.

Setting as in Section 6.3

$$L = \partial_i\left(a^{ij}(x)\,\partial_j\right),$$

we can write

$$\mathcal{P} = \rho\partial_t - L.$$

The operator $\mathcal{P}$ is defined not only on smooth functions in Ω but also on distributions from $\mathcal{D}'(\Omega)$ because all terms on the right hand side of (6.61) are defined as operators in $\mathcal{D}'(\Omega)$. For all $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, we have

$$(\mathcal{P}u, \varphi) = (\rho\partial_t u, \varphi) - (Lu, \varphi) = -(u, \rho\partial_t\varphi) - (u, L\varphi)$$

whence it follows that

$$(\mathcal{P}u, \varphi) = (u, \mathcal{P}^*\varphi), \tag{6.62}$$

where

$$\mathcal{P}^* = -\rho\partial_t - L$$

is the *dual operator* to $\mathcal{P}$. The identity (6.62) can be also used as the definition of $\mathcal{P}$ on $\mathcal{D}'(\Omega)$.

We start with an analog of Lemma 6.7

LEMMA 6.16. (1st a priori estimate) *For any open set $\Omega' \Subset \Omega$ and for any $u \in \mathcal{D}(\Omega')$,*

$$\|u\|_{V^1} \le C\|\mathcal{P}u\|_{V^{-1}},$$

where the constant C depends on $\operatorname{diam}\Omega'$ *and on the ellipticity constant of L in Ω'.*

PROOF. Setting $f = \mathcal{P}u$ and multiplying this equation by u, we obtain

$$\int_\Omega uf\,d\mu = \int_\Omega \rho u\,\partial_t u\,d\mu - \int_\Omega uLu\,d\mu,$$

where $d\mu = dtdx$ is the Lebesgue measure in $\mathbb{R}^{n+1}$. Since

$$\rho u\partial_t u = \frac{1}{2}\partial_t(\rho u^2),$$

after integrating the function $\rho u\partial_t u$ in dt we obtain 0. Hence,

$$\int_\Omega \rho u\,\partial_t u\,d\mu = 0.$$

Applying the same argument as in the proof Lemma 6.7 (cf. (6.39)), we obtain,

$$-\int_\Omega uLu\,d\mu \geq c\|u\|^2_{V^1(\Omega)}, \tag{6.63}$$

where the constant $c > 0$ depends on the ellipticity constant of L in Ω' and on $\operatorname{diam}\Omega'$. Since

$$-\int_\Omega uLu\,d\mu = \int_\Omega uf\,d\mu = (f,u) \leq \|f\|_{V^{-1}}\|u\|_{V^1}$$

we obtain

$$c\|u\|^2_{V^1} \leq \|f\|_{V^{-1}}\|u\|_{V^1},$$

whence the claim follows. □

LEMMA 6.17. (2nd a priori estimate) *For any open set $\Omega' \Subset \Omega$ and for any $u \in \mathcal{D}(\Omega')$,*

$$\|u\|_{V^2} \leq C\|\mathcal{P}u\|_{L^2}, \tag{6.64}$$

where the constant C depends on $\Omega', \mathcal{P}$.

PROOF. If follows from (6.59) that we have to estimate the L^2-norm of $\partial_t u$ as well as that of $\partial_j u$ and $\partial_i\partial_j u$. Setting $f = \mathcal{P}u$ and multiplying this equation by $\partial_t u$, we obtain

$$\int_\Omega \partial_t u\, f\,d\mu = \int_\Omega \rho\,(\partial_t u)^2\,d\mu - \int_\Omega \partial_t u\,Lu\,d\mu. \tag{6.65}$$

Since ∂_t and L commute, we obtain, using integration by parts and (6.37),

$$\int_\Omega \partial_t u\,Lu\,d\mu = -\int_\Omega u\,\partial_t(Lu)\,d\mu = -\int_\Omega u\,L(\partial_t u)\,d\mu = -\int_\Omega Lu\,\partial_t u\,d\mu,$$

whence it follows that

$$\int_\Omega \partial_t u\,Lu\,d\mu = 0.$$

Since $\rho(x)$ is bounded on Ω' by a positive constant, say c, we obtain

$$\int_\Omega \rho\,(\partial_t u)^2\,d\mu \geq c\|\partial_t u\|^2_{L^2}.$$

Finally, applying the Cauchy-Schwarz inequality to the left hand side of (6.65), we obtain

$$\|\partial_t u\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)} \geq c\|\partial_t u\|^2_{L^2(\Omega)},$$

whence

$$\|\partial_t u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{6.66}$$

To estimate the spatial derivatives, observe that the identity $f = \rho\partial_t u - Lu$ implies

$$\|Lu\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} + C\|\partial_t u\|_{L^2(\Omega)} \leq C'\|f\|_{L^2(\Omega)}. \tag{6.67}$$

Let Q and Q' be the projections of Ω and Ω', respectively, onto the subspace $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ spanned by the coordinates $x^1, ..., x^n$. Obviously, the operator L can be considered as an elliptic operator in Q. Since $Q' \Subset Q$ and $u(t,\cdot) \in$

$\mathcal{D}(Q')$ for any fixed t, the estimate (6.57) of Exercise 6.3 yields, for any fixed t,

$$\|u\|_{W^2(Q)} \le C\|Lu\|_{L^2(Q)}$$

(a somewhat weaker estimate follows also from Theorem 6.9). Integrating this in time and using (6.67), we obtain that the $L^2(\Omega)$-norms of the derivatives $\partial_j u$ and $\partial_i\partial_j u$ are bounded by $C\|f\|_{L^2(\Omega)}$. Combining with (6.66), we obtain (6.64). □

LEMMA 6.18. *If $u \in L^2_{loc}(\Omega)$ and $\mathcal{P}u \in V^{-1}_{loc}(\Omega)$ then $u \in V^1_{loc}(\Omega)$. Moreover, for all open subsets $\Omega' \Subset \Omega'' \Subset \Omega$,*

$$\|u\|_{V^1(\Omega')} \le C\left(\|u\|_{L^2(\Omega'')} + \|\mathcal{P}u\|_{V^{-1}(\Omega'')}\right), \tag{6.68}$$

where C is a constant depending on $\Omega', \Omega'', \mathcal{P}$.

PROOF. Let $\psi \in \mathcal{D}(\Omega'')$ be a cutoff function of $\overline{\Omega}'$ in Ω'', and let us prove that the function $v = \psi u$ belongs to $V^1(\Omega'')$, which will imply $u \in V^1_{loc}(\Omega)$. Clearly, $v \in L^2(\Omega'')$ and $\operatorname{supp} v$ is a compact subset of Ω''. Next, we have

$$\mathcal{P}(\psi u) = \psi\mathcal{P}u - 2a^{ij}\partial_i\psi\partial_i u + (\mathcal{P}\psi)u \tag{6.69}$$

(cf. (6.42)), whence it follows that $\mathcal{P}v \in V^{-1}(\Omega'')$ and

$$\|\mathcal{P}v\|_{V^{-1}(\Omega'')} \le C\left(\|u\|_{L^2(\Omega'')} + \|\mathcal{P}u\|_{V^{-1}(\Omega'')}\right), \tag{6.70}$$

where C depends on $\Omega', \Omega'', \mathcal{P}$.

Fix a mollifier φ in $\mathbb{R}^{n+1}$ and observe that, for small enough $\varepsilon > 0$, $v * \varphi_\varepsilon$ belongs to $\mathcal{D}(\Omega'')$. By Lemma 6.16, we have

$$\|v * \varphi_\varepsilon\|_{V^1} \le C\|\mathcal{P}(v * \varphi_\varepsilon)\|_{V^{-1}}, \tag{6.71}$$

where the constant C depends on Ω'' and $\mathcal{P}$.

Let us show that

$$\|\mathcal{P}(v * \varphi_\varepsilon) - (\mathcal{P}v) * \varphi_\varepsilon\|_{V^{-1}} \to 0 \text{ as } \varepsilon \to 0. \tag{6.72}$$

By Lemma 6.5, we have

$$\|\rho\partial_t(v * \varphi_\varepsilon) - (\rho\partial_t v) * \varphi_\varepsilon\|_{L^2} \to 0. \tag{6.73}$$

As in the proof of Lemma 6.6, we have

$$L(v * \varphi_\varepsilon) - (Lv) * \varphi_\varepsilon = \partial_i f^i_\varepsilon$$

where

$$f^i_\varepsilon := a^{ij}\partial_j(v * \varphi_\varepsilon) - \left(a^{ij}\partial_j v\right) * \varphi_\varepsilon.$$

By Lemma 6.5, $\|f^i_\varepsilon\|_{L^2} \to 0$ whence

$$\|L(v * \varphi_\varepsilon) - (Lv) * \varphi_\varepsilon\|_{V^{-1}} = \|\partial_i f^i_\varepsilon\|_{V^{-1}} \le \sum_i \|f^i_\varepsilon\|_{L^2} \to 0. \tag{6.74}$$

Combining (6.73) and (6.74), we obtain (6.72).

By extension of Theorem 2.16 to the spaces V^{-k}, the condition $\mathcal{P}v \in V^{-1}(\Omega'')$ implies

$$(\mathcal{P}v) * \varphi_\varepsilon \xrightarrow{V^{-1}} \mathcal{P}v,$$

which together with (6.72) yields

$$\mathcal{P}(v * \varphi_\varepsilon) \xrightarrow{V^{-1}} \mathcal{P}v \text{ as } \varepsilon \to 0. \tag{6.75}$$

It follows from (6.71) and (6.75) that

$$\limsup_{\varepsilon \to 0} \|v * \varphi_\varepsilon\|_{V^1} \le C\|\mathcal{P}v\|_{V^{-1}}.$$

By extension of Theorem 2.13 to V^k, we conclude that $v \in V^1(\Omega'')$ and

$$\|v\|_{V^1(\Omega'')} \le C\|\mathcal{P}v\|_{V^{-1}(\Omega'')}.$$

Combining this estimate with (6.70) and $\|u\|_{V^1(\Omega')} = \|v\|_{V^1(\Omega')}$, we obtain (6.68). □

LEMMA 6.19. *If $u \in V^1_{loc}(\Omega)$ and $\mathcal{P}u \in L^2_{loc}(\Omega)$ then $u \in V^2_{loc}(\Omega)$. Moreover, for all open subsets $\Omega' \Subset \Omega'' \Subset \Omega$,*

$$\|u\|_{V^2(\Omega')} \le C\left(\|u\|_{V^1(\Omega'')} + \|\mathcal{P}u\|_{L^2(\Omega'')}\right), \tag{6.76}$$

where C is a constant depending on $\Omega', \Omega'', \mathcal{P}$.

PROOF. Let $\psi \in \mathcal{D}(\Omega'')$ be a cutoff function of $\overline{\Omega}'$ in Ω'', and let us prove that the function $v = \psi u$ belongs to $V^2(\Omega'')$, which will imply $u \in V^2_{loc}(\Omega)$. It follows from (6.69) that $\mathcal{P}v \in L^2(\Omega'')$ and

$$\|\mathcal{P}v\|_{L^2(\Omega'')} \le C\left(\|u\|_{V^1(\Omega'')} + \|\mathcal{P}u\|_{L^2(\Omega'')}\right), \tag{6.77}$$

where C depends on $\Omega', \Omega'', \mathcal{P}$. Function v belongs to $V^1(\Omega'')$ and has a compact support in Ω''.

For any mollifier φ in $\mathbb{R}^{n+1}$ and a small enough $\varepsilon > 0$, we have $v * \varphi_\varepsilon \in \mathcal{D}(\Omega'')$. By Lemma 6.17, we obtain

$$\|v * \varphi_\varepsilon\|_{V^2} \le C\|\mathcal{P}(v * \varphi_\varepsilon)\|_{L^2} \tag{6.78}$$

where C depends on $\Omega'', \mathcal{P}$.

Let us show that

$$\|\mathcal{P}(v * \varphi_\varepsilon) - (\mathcal{P}v) * \varphi_\varepsilon\|_{L^2} \longrightarrow 0 \text{ as } \varepsilon \to 0. \tag{6.79}$$

For that, represent the operator $\mathcal{P}$ in the form

$$\mathcal{P} = -a^{ij}\partial_i\partial_j - b^j\partial_j + \rho\partial_t,$$

where $b_j = \partial_i a^{ij}$. The part of the estimate (6.79) corresponding to the first order terms $b^j\partial_j$ and $\rho\partial_t$, follows from Lemma 6.5 because $v \in L^2(\Omega)$. Next, applying Lemma 6.5 to function $\partial_j v$, which is also in $L^2(\Omega)$, we obtain

$$\|a^{ij}\partial_i(\partial_j v * \varphi_\varepsilon) - \left(a^{ij}\partial_i\partial_j v\right) * \varphi_\varepsilon\|_{L^2} \to 0,$$

whence (6.79) follows.

Since $\mathcal{P}v \in L^2$, we have by Theorem 2.11

$$(\mathcal{P}v) * \varphi_\varepsilon \xrightarrow{L^2} \mathcal{P}v$$

which together with (6.79) yields

$$\mathcal{P}(v * \varphi_\varepsilon) \xrightarrow{L^2} \mathcal{P}v \text{ as } \varepsilon \to 0.$$

Combining with (6.78), we obtain

$$\limsup_{\varepsilon\to 0} \|v * \varphi_\varepsilon\|_{V^2} \le C\|\mathcal{P}v\|_{L^2}.$$

Therefore, by extension of Theorem 2.13 to V^k, we conclude that $v \in V^2(\Omega'')$ and

$$\|v\|_{V^2(\Omega'')} \le C\|\mathcal{P}v\|_{L^2(\Omega'')}.$$

Combining this with (6.77) and $\|u\|_{V^2(\Omega')} = \|v\|_{V^2(\Omega')}$, we obtain (6.76). $\square$

LEMMA 6.20. *For any integer $m \ge -1$, if $u \in V_{loc}^{m+1}(\Omega)$ and $\mathcal{P}u \in V_{loc}^{m}(\Omega)$ then $u \in V_{loc}^{m+2}(\Omega)$. Moreover, for all open subsets $\Omega' \Subset \Omega'' \Subset \Omega$,*

$$\|u\|_{V^{m+2}(\Omega')} \le C\left(\|u\|_{V^{m+1}(\Omega'')} + \|\mathcal{P}u\|_{V^m(\Omega'')}\right), \tag{6.80}$$

where the constant C depends on $\Omega', \Omega'', \mathcal{P}, m$.

PROOF. The case $m = -1$ coincides with Lemma 6.18, and the case $m = 0$ coincides with Lemma 6.19. Let us prove the inductive step from $m-2$ and $m-1$ to m, assuming $m \ge 1$. To show that $u \in V_{loc}^{m+2}$, it suffices to verify that

$$\partial_t u,\ \partial_j u,\ \partial_i\partial_j u \in V_{loc}^m.$$

Since $\partial_t u \in V_{loc}^{m-1}$ and

$$\mathcal{P}(\partial_t u) = \partial_t \mathcal{P}u \in V_{loc}^{m-2}. \tag{6.81}$$

the inductive hypothesis yields $\partial_t u \in V_{loc}^m$.

It follows from (6.43) that

$$\partial_l(\mathcal{P}u) - \mathcal{P}(\partial_l u) = (\partial_l\rho)\,\partial_t u - \partial_i\left[\left(\partial_l a^{ij}\right)\partial_j u\right], \tag{6.82}$$

which implies

$$\mathcal{P}(\partial_l u) \in V_{loc}^{m-1}.$$

Since $\partial_l u \in V_{loc}^m$, the inductive hypothesis yields $\partial_l v \in V_{loc}^{m+1}$. Consequently, all the second order derivatives $\partial_i\partial_j v$ are in V_{loc}^m, which was to be proved.

Let us now prove (6.80). It follows from (6.81) and the inductive hypothesis for $m-2$ that

$$\begin{aligned}\|\partial_t u\|_{V^m(\Omega')} &\le C\left(\|\partial_t u\|_{V^{m-1}(\Omega'')} + \|\mathcal{P}(\partial_t u)\|_{V^{m-2}(\Omega'')}\right)\\ &\le C\left(\|u\|_{V^{m+1}(\Omega'')} + \|\mathcal{P}u\|_{V^m(\Omega'')}\right).\end{aligned} \tag{6.83}$$

it follows from (6.82) that

$$\|\mathcal{P}(\partial_l u)\|_{V^{m-1}(\Omega'')} \le C\left(\|u\|_{V^{m+1}(\Omega'')} + \|\mathcal{P}u\|_{V^m(\Omega'')}\right),$$

whence, by the inductive hypothesis for $m-1$,

$$\begin{aligned}\|\partial_l u\|_{V^{m+1}(\Omega')} &\le C\left(\|\partial_l u\|_{V^m(\Omega'')} + \|\mathcal{P}(\partial_l u)\|_{V^{m-1}(\Omega'')}\right)\\ &\le C\left(\|u\|_{V^{m+1}(\Omega'')} + \|\mathcal{P}u\|_{V^m(\Omega'')}\right).\end{aligned} \tag{6.84}$$

Combining (6.83) and (6.84) yields (6.80). $\square$

THEOREM 6.21. *For any integer $m \geq -1$, if $u \in L^2_{loc}(\Omega)$ and $\mathcal{P}u \in V^m_{loc}(\Omega)$ then $u \in V^{m+2}_{loc}(\Omega)$. Moreover, for all open subsets $\Omega' \Subset \Omega'' \Subset \Omega$,*

$$\|u\|_{V^{m+2}(\Omega')} \leq C\left(\|u\|_{L^2(\Omega'')} + \|\mathcal{P}u\|_{V^m(\Omega'')}\right), \tag{6.85}$$

where C depends on $\Omega', \Omega'', \mathcal{P}, m$.

PROOF. Let k be the largest number between 0 and $m+2$ such that $u \in V^k_{loc}(\Omega)$. If $k \leq m+1$ then we have $\mathcal{P}u \in V^{k-1}_{loc}$, which implies by Lemma 6.20 that $u \in V^{k+1}_{loc}$, thus contradicting the definition of k. Hence, $k = m+2$ and $u \in V^{m+2}_{loc}$, which was to be proved.

To prove the estimate (6.85), consider a decreasing sequence of open sets $\{\Omega_i\}_{i=0}^{m+2}$ such that $\Omega_0 = \Omega''$, $\Omega_{m+2} = \Omega'$ and $\Omega_{i+1} \Subset \Omega_i$. By Lemma 6.20, we obtain, for any $1 \leq k \leq m+2$,

$$\begin{aligned} \|u\|_{V^k(\Omega_k)} &\leq C\left(\|u\|_{V^{k-1}(\Omega_{k-1})} + \|\mathcal{P}u\|_{V^{k-2}(\Omega_{k-1})}\right) \\ &\leq C\left(\|u\|_{V^{k-1}(\Omega_{k-1})} + \|\mathcal{P}u\|_{V^m(\Omega'')}\right), \end{aligned}$$

which obviously implies (6.85). $\square$

COROLLARY 6.22. *(i) If $u \in L^2_{loc}(\Omega)$ and $\mathcal{P}u = f$ where $f \in C^\infty(\Omega)$ then also $u \in C^\infty(\Omega)$.*

(ii) Let $\{u_k\}$ be a sequence of smooth functions in Ω, each satisfying the equation $\mathcal{P}u_k = f$ where $f \in C^\infty(\Omega)$. If $u_k \xrightarrow{L^2_{loc}} u$ where $u \in L^2_{loc}(\Omega)$ then $\mathcal{P}u = 0$, $u \in C^\infty(\Omega)$, and $u_k \xrightarrow{C^\infty} u$.

PROOF. (i) Since $\mathcal{P}u \in V^m_{loc}(\Omega)$ for any positive integer m, Theorem 6.21 yields that also $u \in V^m_{loc}(\Omega)$ for any m. Therefore, $u \in W^m_{loc}(\Omega)$ for any m and, by Theorem 6.1, we conclude $u \in C^\infty(\Omega)$.

(ii) Let us first show that u satisfies the equation $\mathcal{P}u = f$ in the distributional sense, that is,

$$(u, \mathcal{P}^*\varphi) = (f, \varphi) \ \text{ for all } \varphi \in \mathcal{D}(\Omega), \tag{6.86}$$

where $\mathcal{P}^* = -\rho\partial_t - L$ is the dual operator (cf. (6.62)). Indeed, $\mathcal{P}u_k = f$ implies that

$$(u_k, \mathcal{P}^*\varphi) = (f, \varphi),$$

whence (6.86) follows by letting $k \to \infty$. By part (i), we conclude that $u \in C^\infty(\Omega)$.

Setting $v_k = u - u_k$, noticing that $\mathcal{P}v_k = 0$, and applying to v_k the estimate (6.85), we obtain, for all open subsets $\Omega' \Subset \Omega'' \Subset \Omega$ and for any positive integer m,

$$\|v_k\|_{V^m(\Omega')} \leq C\|v_k\|_{L^2(\Omega'')},$$

Since $v_k \to 0$ in $L^2(\Omega'')$, we obtain that $v_k \to 0$ in $V^m(\Omega')$.

Hence, $v_k \to 0$ in $V^m_{loc}(\Omega)$ for any m, which implies that also $v_k \to 0$ in $W^m_{loc}(\Omega)$ for any m and, by the estimate (6.2) of Theorem 6.1, $v_k \to 0$ in $C^\infty(\Omega)$, which was to be proved. □

Exercises.

6.5. Let $\Omega' \Subset \Omega$ be open sets and $m \geq -1$ be an integer.

(a) Prove that, for any $u \in \mathcal{D}(\Omega')$,
$$\|u\|_{V^{m+2}(\Omega)} \leq C\|\mathcal{P}u\|_{V^m(\Omega)}, \tag{6.87}$$
where a constant C depends on $\Omega', \mathcal{P}, m$.

(b) Using part (a), prove that, for any $u \in C^\infty(\Omega)$,
$$\|u\|_{V^{m+2}(\Omega')} \leq C\left(\|u\|_{L^2(\Omega)} + \|\mathcal{P}u\|_{V^m(\Omega)}\right). \tag{6.88}$$

REMARK. The estimate (6.88) was proved in Theorem 6.21. In the case $u \in C^\infty$, it is easier to deduce it from (6.87).

6.4.3. Solutions from $\mathcal{D}'$.

We start with a parabolic analogue of Lemma 6.12.

LEMMA 6.23. *Let $U \Subset \Omega$ be an open set of the form $U = (0,T) \times Q$ where $T > 0$ and Q is an open set in $\mathbb{R}^n$. Then, for any $f \in L^2(U)$, there exists a function $u \in L^2(U)$ solving the equation $\mathcal{P}u = f$ and satisfying the estimate*
$$\|u\|_{L^2(U)} \leq T\|f/\rho\|_{L^2(U)}. \tag{6.89}$$

REMARK 6.24. As it follows from Corollary 6.22, if $f \in C^\infty \cap L^2(U)$ then the solution u also belongs to $C^\infty \cap L^2(U)$.

PROOF. By Lemma 6.12 and Remark 6.13, the resolvent $\mathcal{R}$ of the equation $Lu = f$ is a compact self-adjoint operator in $L^2(Q)$. The multiplication operator by the coefficient $\rho(x)$ is a bounded self-adjoint operator in $L^2(Q)$. Therefore, $\mathcal{R} \circ \rho$ is a compact self-adjoint operator in $L^2(Q)$. By the Hilbert-Schmidt theorem, there exists an orthonormal basis $\{v_k\}$ in $L^2(Q)$, which consists of the eigenfunctions of the operator $\mathcal{R} \circ \rho$. Since $\mathcal{R}(\rho v) = 0$ implies $\rho v = L0 = 0$ and, hence, $v = 0$, zero is not an eigenvalue of $\mathcal{R} \circ \rho$. Therefore, each v_k is also an eigenfunction of the inverse operator $\rho^{-1}L$, and let the corresponding eigenvalue be λ_k, that is,
$$Lv_k = \lambda_k \rho v_k. \tag{6.90}$$
Since $\operatorname{ran}\mathcal{R}$ is contained in $W^1_0(Q)$, we have $v_k \in W^1_0(Q)$. Using the identity (6.51) with $u = \varphi = v_k$ and $f = \lambda_k \rho v_k$, we obtain
$$\int_U a^{ij}\partial_i v_k\, \partial_j v_k\, dx = -\lambda_k \int_U \rho v_k^2\, dx,$$
whence it follows that $\lambda_k < 0$.

Given $f \in \mathcal{D}(U)$, expand function f/ρ in the basis $\{v_k\}$:
$$\frac{f(t,x)}{\rho(x)} = \sum_k f_k(t)\, v_k(x),$$

where $f_k(t) = (f(t,\cdot)/\rho, v_k)$, and set

$$u_k(t) = \int_0^t e^{-\lambda_k s} f_k(s)\, ds. \tag{6.91}$$

We claim that the function

$$u(t,x) = \sum_k e^{\lambda_k t} u_k(t)\, v_k(x) \tag{6.92}$$

belongs to $L^2(U)$ and solves the equation $\mathcal{P}u = f$. Indeed, since $\lambda_k < 0$ and

$$e^{\lambda_k t} u_k(t) = \int_0^t e^{\lambda_k (t-s)} f_k(s)\, ds,$$

we see that, for any $t \in (0,T)$,

$$\left| e^{\lambda_k t} u_k(t) \right| \le \int_0^T |f_k(s)|\, ds,$$

whence by the Parseval identity

$$\begin{aligned} \sum_k \left| e^{\lambda_k t} u_k(t) \right|^2 &\le \sum_k T \int_0^T |f_k(s)|^2\, ds \\ &= T \int_0^T \|f(s,\cdot)\|^2_{L^2(Q)} ds = T \|f/\rho\|^2_{L^2(U)}. \end{aligned}$$

Therefore, the series (6.92) converges in $L^2(Q)$ and, for any $t \in (0,T)$,

$$\|u(t,\cdot)\|^2_{L^2(Q)} \le T\|f/\rho\|^2_{L^2(U)}.$$

Integrating in t, we obtain $u \in L^2(U)$ and the estimate (6.89). The same argument shows that the series (6.92) converges in $L^2(U)$.

Using (6.90) and (6.91) we obtain

$$\mathcal{P}\left(e^{\lambda_k t} u_k v_k\right) = \rho e^{\lambda_k t}(\partial_t u_k)\, v_k + \rho\lambda_k e^{\lambda_k t} u_k v_k - e^{\lambda_k t} u_k L v_k = \rho f_k v_k.$$

Using the convergence of the series (6.92) in $\mathcal{D}'(U)$, we obtain

$$\mathcal{P}u = \sum_k \rho f_k v_k = f,$$

which finishes the proof. $\square$

In the proof of the next statement, we will use the operator

$$\mathcal{P}^* := -\rho\partial_t - L,$$

which is dual to $\mathcal{P}$ in the following sense: for any distribution $u \in \mathcal{D}'(\Omega)$ and a test function $\varphi \in \mathcal{D}(\Omega)$,

$$(\mathcal{P}u, \varphi) = (u, \mathcal{P}^*\varphi)$$

(cf. Section 6.4.2). Let τ be the operator of changing the time direction, that is, for a test function $\varphi \in \mathcal{D}(\Omega)$,

$$\tau\varphi(t,x) = \varphi(-t,x)$$

and, for a distribution $u \in \mathcal{D}'(\Omega)$,

$$(\tau u, \varphi) = (u, \tau\varphi) \quad \text{for all } \varphi \in \mathcal{D}(\tau\Omega).$$

Clearly, we have $\mathcal{P}^* = \mathcal{P} \circ \tau$. Using this relation, many properties of the operator $\mathcal{P}^*$ can be derived from those for $\mathcal{P}$. In particular, one easily verifies that Theorem 6.21 and Lemma 6.23 are valid[3] also for the operator $\mathcal{P}^*$.

THEOREM 6.25. *For any $m \in \mathbb{Z}$, if $u \in \mathcal{D}'(\Omega)$ and $\mathcal{P}u \in V^m_{loc}(\Omega)$ then $u \in V^{m+2}_{loc}(\Omega)$.*

PROOF. As in the proof of Theorem 6.21, let us first show that $u \in V^{m+1}_{loc}(\Omega)$ and $\mathcal{P}u \in V^m_{loc}(\Omega)$ imply $u \in V^{m+2}_{loc}(\Omega)$. For $m \geq -1$, this was proved in Theorem 6.21, so assume $k := -m \geq 2$.

It suffices to prove that $\psi u \in V^{-k+2}(\Omega)$ for any $\psi \in \mathcal{D}(\Omega)$ with sufficiently small support. Fix a cylindrical open set $U \Subset \Omega$, a function $\psi \in \mathcal{D}(U)$, and set $v = \psi u$. It follows from the hypotheses that $v \in V^{-k+1}(\Omega)$ and $\mathcal{P}v \in V^{-k}(\Omega)$ (cf. (6.69)).

We need to show that $\|v\|_{V^{-k+2}(U)} < \infty$, and this will be done if we prove that, for any $f \in \mathcal{D}(U)$,

$$(v, f) \leq C\|\mathcal{P}v\|_{V^{-k}}\|f\|_{V^{k-2}}. \tag{6.93}$$

By Lemma 6.23, for any $f \in \mathcal{D}(U)$, there exists a function $w \in C^\infty \cap L^2(U)$ solving the equation $\mathcal{P}^*w = f$ in U and satisfying the estimate

$$\|w\|_{L^2(U)} \leq C\|f\|_{L^2(U)}. \tag{6.94}$$

Fix a function $\varphi \in \mathcal{D}(U)$ such that $\varphi \equiv 1$ in a neighborhood of $\operatorname{supp}\psi$. Then $\varphi w \in \mathcal{D}(U)$ and

$$\begin{aligned}
(v, f) &= (v, \mathcal{P}^*w) = (v, \mathcal{P}^*(\varphi w)) = (\mathcal{P}v, \varphi w) \\
&\leq \|\mathcal{P}v\|_{V^{-k}(U)}\|\varphi w\|_{V^k(U)} \leq C\|\mathcal{P}v\|_{V^{-k}(U)}\|w\|_{V^k(U')},
\end{aligned}$$

where $U' \Subset U$ is a neighborhood of $\operatorname{supp}\varphi$ and the constant C depends only on φ. Using the estimate (6.85) of Theorem 6.21 (or the estimate (6.88) of Exercise 6.5) and (6.94), we obtain

$$\|w\|_{V^k(U')} \leq C\left(\|w\|_{L^2(U)} + \|\mathcal{P}^*w\|_{V^{k-2}(U)}\right) \leq C'\|f\|_{V^{k-2}(U)},$$

whence (6.93) follows.

Assume now that $u \in \mathcal{D}'(\Omega)$ and $\mathcal{P}u \in V^m_{loc}(\Omega)$, and prove that $u \in V^{m+2}_{loc}(\Omega)$. As was shown in the proof of Theorem 6.15, for any open set $U \Subset \Omega$ there exists $l > 0$ such that $u \in W^{-l}(U)$. Since $\|\cdot\|_{W^l} \leq \|\cdot\|_{V^{2l}}$ and, hence, $\|\cdot\|_{V^{-2l}} \leq \|\cdot\|_{W^{-l}}$, this implies $u \in V^{-2l}(U)$. Let $k \leq m+2$ be the maximal integer such that $u \in V^k_{loc}(U)$. If $k \leq m+1$ then $\mathcal{P}u \in V^{k+1}_{loc}(U)$ whence by the first part of the proof $u \in V^{k+1}_{loc}(U)$. Hence, $k = m+2$, which was to be proved. $\square$

[3]Let us emphasize that the solvability result of Lemma 6.23 is not sensitive to the time direction because we do not impose the initial data.

Combining Theorems 6.25 and 6.1, we obtain the following statement that extends the result of Corollary 6.22(i) from L^2_{loc} to $\mathcal{D}'(\Omega)$.

COROLLARY 6.26. *If* $u \in \mathcal{D}'(\Omega)$ *and* $\mathcal{P}u \in C^\infty(\Omega)$ *then* $u \in C^\infty(\Omega)$.

Exercises.

6.6. Consider a more general parabolic operator

$$\mathcal{P} = \rho\partial_t - \partial_i\left(a^{ij}(x)\,\partial_j\right) - b^j(x)\,\partial_j - c(x),$$

where a^{ij} and ρ are as before, and b^j and c are smooth functions in Ω. Prove that if $u \in \mathcal{D}'(\Omega)$ and $\mathcal{P}u \in V^m_{loc}(\Omega)$ for some $m \in \mathbb{Z}$ then $u \in V^{m+2}_{loc}(\Omega)$. Conclude that $\mathcal{P}u \in C^\infty(\Omega)$ implies $u \in C^\infty(\Omega)$.

Notes

All the material of this chapter is classical although the presentation has some novelties. The theory of distributions was created by L. Schwartz [**327**]. The Sobolev spaces were introduced by S. L. Sobolev in [**328**], where he also proved the Sobolev embedding theorem. We have presented in Section 6.1.1 only a part of this theorem. The full statement includes the claim that if $k < n/2$ then $W^k_{loc} \hookrightarrow L^{\frac{2n}{n-2k}}_{loc}$, and similar results hold for the spaces $W^{k,p}$ based on L^p. The modern proofs of the Sobolev embedding theorem can be found in [**118**], [**130**]; see also [**1**] and [**269**] for further results.

One of the first historical result in the regularity theory (in the present sense) is due to H. Weyl [**357**], who proved that any distribution $u \in \mathcal{D}'$ solving the equation $\Delta u = f$ is a smooth function provided $f \in C^\infty$ (*Weyl's lemma*). This and similar results for the elliptic operators with *constant* coefficient can be verified by means of the Fourier transform (see [**309**]). The regularity theorem for elliptic operators with smooth *variable* coefficients was proved by K. O. Friedrichs [**123**], who introduced for that the techniques of mollifiers in [**122**]. Alternative approaches were developed concurrently by P. D. Lax [**245**] and L. Nirenberg [**293**], [**294**].

Nowadays various approaches are available for the regularity theory. The one we present here makes a strong use of the symmetry of the operator (via the Green formula) and of the mollifiers. The proofs of the Friedrichs lemma (Lemma 6.4) and the key Lemmas 6.7, 6.8 were taken from [**208**]. The reader may notice that Lemma 6.4 is the only technical part of the proof. Other frequently used devices include elementary estimates of the commutators of the differential operators with the operators of convolution and multiplication by a function. The parabolic regularity theory as presented here follows closely its elliptic counterpart, with the Sobolev spaces W^k being replaced by their anisotropic version.

Different accounts of the regularity theory can be found in [**118**], [**121**], [**130**], [**241**], [**241**], [**273**], [**308**], although in the most sources the theory is restricted to solutions from L^2_{loc} or even from W^1_{loc} as opposed to those from $\mathcal{D}'$. A far reaching extension and unification of the elliptic and parabolic regularity theories was achieved in L. Hörmander's theory of hypoelliptic operators [**208**] (see also [**297**], [**348**]).

Another branch of the regularity theory goes in the direction of reducing the smoothness of the coefficients – this theory is covered in [**130**], [**118**], [**242**], [**241**]. If the coefficients are just measurable functions then all that one can hope for is the Hölder continuity of the solutions. The fundamental results in this direction were obtained by E. De Giorgi [**103**] for the elliptic case and by J. Nash [**292**] for the parabolic case. For the operators in non-divergence form, the Hölder regularity of solutions was proved by N. Krylov and

M. Safonov [**236**], partly based on the work of E. M. Landis [**243**]. See [**130**] and [**230**] for a detailed account of these results.

CHAPTER 7

The heat kernel on a manifold

This is a central Chapter of the book, where we prove the existence of the heat kernel and its general properties. From Chapter 6, we use Corollaries 6.11, 6.22, and 6.26.

7.1. Local regularity issues

Let (M, g, μ) be a weighted manifold. The only Sobolev space on M we have considered so far was $W^1(M)$. In general, the higher order Sobolev spaces W^k cannot be defined in the same way as in $\mathbb{R}^n$ because the partial derivatives of higher order are not well-defined on M. Using the Laplace operator, we still can define the spaces of *even* orders as follows. For any non-negative integer k, set

$$\mathcal{W}^{2k}(M) = \mathcal{W}^{2k}(M, \mathrm{g}, \mu) = \left\{u : u,\ \Delta_\mu u,\ \dots,\ \Delta_\mu^k u \in L^2(M)\right\},$$

and

$$\|u\|_{\mathcal{W}^{2k}}^2 = \sum_{l=0}^{k} \|\Delta_\mu^l u\|_{L^2}^2. \tag{7.1}$$

It is easy to check that $\mathcal{W}^{2k}(M)$ with the norm (7.1) is a Hilbert space[1].

Define the local Sobolev space $\mathcal{W}^{2k}_{loc}(M)$ by

$$\mathcal{W}^{2k}_{loc}(M) = \left\{u : u,\ \Delta_\mu u,\ \dots,\ \Delta_\mu^k u \in L^2_{loc}(M)\right\}.$$

Equivalently, $u \in \mathcal{W}^{2k}_{loc}(M)$ if $u \in \mathcal{W}^{2k}(\Omega)$ for any open set $\Omega \Subset M$. The topology in $\mathcal{W}^{2k}_{loc}(M)$ is determined by the family of seminorms $\|u\|_{\mathcal{W}^{2k}(\Omega)}$.

The following theorem is a consequence of the elliptic regularity theory of Section 6.3.1.

THEOREM 7.1. *Let (M, g, μ) be a weighted manifold of dimension n, and let u be a function from $\mathcal{W}^{2k}_{loc}(M)$ for some positive integer k.*

---

[1]By considering in addition the gradient of $\Delta_\mu^k u$, one could define $\mathcal{W}^{2k+1}(M)$ similarly to $W^1(M)$, but we have no need in such space (cf. Exercise 7.1).

The reader should be warned that if M is an open subset of $\mathbb{R}^n$ then $\mathcal{W}^{2k}(\Omega)$ need not match the Euclidean Sobolev space $W^{2k}(\Omega)$, although these two spaces do coincide if $M = \mathbb{R}^n$ (cf. Exercise 2.33(d)).

(*i*) *If $k > n/4$ then $u \in C(M)$. Moreover, for any relatively compact open set $\Omega \subset M$ and any set $K \Subset \Omega$, there is a constant $C = C(K, \Omega, \mathbf{g}, \mu, k, n)$ such that*

$$\sup_K |u| \leq C \|u\|_{\mathcal{W}^{2k}(\Omega)}. \tag{7.2}$$

(*ii*) *If $k > m/2 + n/4$ where m is a positive integer then $u \in C^m(M)$. Moreover, for any relatively compact chart $U \subset M$ and any set $K \Subset U$, there is a constant $C = C(K, U, \mathbf{g}, \mu, k, n, m)$ such that*

$$\|u\|_{C^m(K)} \leq C \|u\|_{\mathcal{W}^{2k}(U)}. \tag{7.3}$$

PROOF. Let U be a chart with coordinates $x^1, ..., x^n$, and let λ be the Lebesgue measure in U. Recall that by (3.21) $d\mu = \rho(x)\, d\lambda$, where $\rho = \Upsilon\sqrt{\det g}$ and Υ is the density function of measure μ. Considering U as a part of $\mathbb{R}^n$, define in U the following operator

$$L = \rho^{-1}\partial_i\left(\rho g^{ij}\partial_j\right). \tag{7.4}$$

By (3.45), we have

$$L\varphi = \Delta_\mu\varphi \text{ for all } \varphi \in \mathcal{D}(U). \tag{7.5}$$

Now let us consider the operators L and Δ_μ in $\mathcal{D}'(U)$. Since we will apply the results of Chapter 6 to the operator L, we need to treat it as an operator in a domain of $\mathbb{R}^n$. Hence, we define L on $\mathcal{D}'(U)$ using the definitions of ∂_i and the multiplication by a function in $\mathcal{D}'(U)$ given in Section 2.4 (cf. Section 6.3).

However, we treat Δ_μ as an operator on M, and Δ_μ extends to $\mathcal{D}'(U)$ by means of the identity (4.3). Then L and Δ_μ are not necessarily equal as operators on $\mathcal{D}'(U)$ because their definitions as operators in $\mathcal{D}'(U)$ depend on the reference measures, which in the case of Δ_μ is μ and in the case of L is λ. Indeed, for any $u \in \mathcal{D}'(U)$ and $\varphi \in \mathcal{D}(U)$, we have

$$\begin{aligned}(Lu, \varphi) &= \left(\rho^{-1}\partial_i(\rho g^{ij}\partial_j), \varphi\right) = \left(\partial_i(\rho g^{ij}\partial_j), \rho^{-1}\varphi\right) \\ &= -\left(\rho g^{ij}\partial_j, \partial_i(\rho^{-1}\varphi)\right) = \left(u, \partial_j(\rho g^{ij}\partial_i(\rho^{-1}\varphi))\right) \\ &= \left(u, \rho L\left(\rho^{-1}\varphi\right)\right),\end{aligned} \tag{7.6}$$

whereas

$$(\Delta_\mu u, \varphi) = (u, \Delta_\mu\varphi). \tag{7.7}$$

Obviously, we have in general $Lu \neq \Delta_\mu u$.

Nevertheless, when the distributions $\Delta_\mu u$ and Lu are identified with L^2_{loc} (or L^1_{loc}) functions, the reference measures are used again and cancel, which leads back to the equality $Lu = \Delta_\mu u$. More precisely, the following it true[2].

CLAIM. *If $u \in L^2_{loc}(U)$ and $\Delta_\mu u \in L^2_{loc}(U)$ then $Lu = \Delta_\mu u$ in U.*

---

[2]Compare this to Exercise 4.11, where a similar identity is proved for the weak gradient.

As follows from (7.6), if $u \in L^2_{loc}(U)$ then, for any $\varphi \in \mathcal{D}(U)$,

$$(Lu, \varphi) = \int_U u\, \rho\, L\left(\rho^{-1}\varphi\right) d\lambda.$$

To prove the claim it suffices to show that

$$\int_U \Delta_\mu u\, \varphi\, d\lambda = \int_U u\, \rho\, L\left(\rho^{-1}\varphi\right) d\lambda.$$

Since both u and $\Delta_\mu u$ are in $L^2_{loc}(M)$, the identity (7.7) becomes

$$\int_U \Delta_\mu u\, \varphi\, d\mu = \int_U u\, \Delta_\mu \varphi\, d\mu.$$

Using this identity and (7.5), we obtain

$$\int_U \Delta_\mu u\, \varphi\, d\lambda = \int_U \Delta_\mu u\, \left(\rho^{-1}\varphi\right) d\mu = \int_U u\, \Delta_\mu(\rho^{-1}\varphi) d\mu = \int_U u\, L(\rho^{-1}\varphi)\rho\, d\lambda,$$

which was to be proved.

The hypothesis $u \in \mathcal{W}^{2k}_{loc}(M)$ and the above claim imply that, in any chart U,

$$u,\ Lu,\ ...,\ L^k u \in L^2_{loc}(U).$$

If $k > m/2 + n/4$ then we conclude by Corollary 6.11 that $u \in C^m(U)$ and, hence, $u \in C^m(M)$.

The estimate (7.3) follows immediately from the estimate (6.49) of Corollary 6.11 and the definition (7.1) of the norm $\|u\|_{\mathcal{W}^{2k}}$. To prove (7.2) observe that there exist two finite families $\{V_i\}$ and $\{U_i\}$ of relatively compact charts such that K is covered by the charts V_i and $V_i \Subset U_i \Subset \Omega$ (cf. Lemma 3.4). Applying the estimate (6.49) of Corollary 6.11 in each chart U_i for the operator $L = \Delta_\mu$ and replacing $L^2(U_i, \lambda)$-norm by $L^2(U_i, \mu)$-norm (which are comparable), we obtain

$$\sup_{V_i} |u| \leq C \sum_{l=0}^{k} \|\Delta^l_\mu u\|_{L^2(U_i,\lambda)} \leq C' \sum_{l=0}^{k} \|\Delta^l_\mu u\|_{L^2(\Omega,\mu)}$$

Finally, taking maximum over all i, we obtain (7.2). □

If we define the topology in $C^m(M)$ by means of the family of seminorms $\|u\|_{C^m(K)}$ where K is a compact subset of a chart then Theorem 7.1 can be shorty stated that we have an embedding

$$\mathcal{W}^{2k}_{loc}(M) \hookrightarrow C^m(M),$$

provided $k > m/2 + n/4$.

Let us introduce the topology in $C^\infty(M)$ by means of the family of seminorms

$$\sup_K |\partial^\alpha u|, \tag{7.8}$$

where K is any compact set that is contained in a chart, and ∂^α is an arbitrary partial derivative in this chart. The convergence in this topology, denoted by

$$v_k \xrightarrow{C^\infty} v,$$

means that v_k converges to v locally uniformly as $k \to \infty$ and, in any chart and for any multiindex α, $\partial^\alpha v_k$ converges to $\partial^\alpha v$ locally uniformly, too.

Denote by $\mathcal{W}^\infty_{loc}(M)$ the intersection of all spaces $\mathcal{W}^{2l}_{loc}(M)$, and define the topology in $\mathcal{W}^\infty_{loc}(M)$ by means of the family of seminorms

$$\|u\|_{\mathcal{W}^{2l}(\Omega)}, \tag{7.9}$$

where l is any positive integer and Ω is a relatively compact open subset of M. The convergence in $\mathcal{W}^\infty_{loc}(M)$, denoted by

$$v_k \xrightarrow{\mathcal{W}^\infty_{loc}} v,$$

means that v_k converges to v in $L^2_{loc}(M)$ and, for any positive integer l, $\Delta^l_\mu v_k$ converges to $\Delta^l_\mu v$ in $L^2_{loc}(M)$.

COROLLARY 7.2. *The natural identity mapping*

$$I : C^\infty(M) \to \mathcal{W}^\infty_{loc}(M) \tag{7.10}$$

is a homeomorphism of the topological spaces $C^\infty(M)$ and $\mathcal{W}^\infty_{loc}(M)$.

PROOF. If $f \in C^\infty$ then $I(f)$ is the same function f considered as an element of L^2_{loc}. Clearly, $I(f) \in \mathcal{W}^\infty_{loc}$ so that the mapping (7.10) is well-defined. The injectivity of I is obvious, the surjectivity follows from Theorem 7.1(ii). The inequality (7.3) means that any seminorm in C^∞ is bounded by a seminorm in $\mathcal{W}^\infty_{loc}$. Hence, the inverse mapping I^{-1} is continuous. Any seminorm (7.9) in $\mathcal{W}^\infty_{loc}$ can be bounded by a finite sum of seminorms (7.8) in C^∞, which can be seen by covering Ω by a finite family of relatively compact charts. Hence, I is continuous, and hence, is a homeomorphism. □

It is tempting to say that the spaces C^∞ and $\mathcal{W}^\infty_{loc}$ are identical. However, this is not quite so because the elements of C^∞ are pointwise functions whereas the element of $\mathcal{W}^\infty_{loc}$ are equivalence classes of measurable functions.

COROLLARY 7.3. *If a function $u \in L^2_{loc}(M)$ satisfies in M the equation $-\Delta_\mu u + \alpha u = f$ where $\alpha \in \mathbb{R}$ and $f \in C^\infty(M)$, then $u \in C^\infty(M)$.*

More precisely, the statement of Corollary 7.3 means that there is a C^∞ smooth version of a measurable function u.

PROOF. By Corollary 7.2, it suffices to prove that $\Delta^k_\mu u \in L^2_{loc}$ for all $k = 1, 2, \ldots$. It is obvious that $\alpha u - f \in L^2_{loc}$ and, hence,

$$\Delta_\mu u = \alpha u - f \in L^2_{loc}.$$

Then we have

$$\Delta^2_\mu u = \alpha \Delta_\mu u - \Delta_\mu f \in L^2_{loc}.$$

Continuing by induction, we obtain

$$\Delta_\mu^k u = \alpha \Delta_\mu^{k-1} u - \Delta_\mu^{k-1} f \in L^2_{loc},$$

which finishes the proof. □

Now we consider the consequences of the parabolic regularity theory of Section 6.4.3. Fix an open interval $I \subset \mathbb{R}$ and consider the product manifold $N = I \times M$ with the measure $d\nu = dt\, d\mu$. The time derivative ∂_t is defined on $\mathcal{D}'(N)$ as follows: for all $u \in \mathcal{D}'(N)$ and $\varphi \in \mathcal{D}(N)$,

$$(\partial_t u, \varphi) = -(u, \partial_t \varphi),$$

and the Laplace operator Δ_μ of M extends to $\mathcal{D}'(N)$ by

$$(\Delta_\mu u, \varphi) = (u, \Delta_\mu \varphi).$$

Hence, the heat operator $\partial_t - \Delta_\mu$ is naturally defined on $\mathcal{D}'(N)$ as follows:

$$((\partial_t - \Delta_\mu) u, \varphi) = -(u, \partial_t \varphi + \Delta_\mu \varphi). \tag{7.11}$$

THEOREM 7.4. *Let $N = I \times M$.*

(i) *If $u \in \mathcal{D}'(N)$ and $\partial_t u - \Delta_\mu u \in C^\infty(N)$ then $u \in C^\infty(N)$.*

(ii) *Let $\{u_k\}$ be a sequence of smooth functions on N, each satisfying the same equation*

$$\partial_t u_k - \Delta_\mu u_k = f,$$

where $f \in C^\infty(N)$. If

$$u_k \overset{L^2_{loc}(N)}{\longrightarrow} u \in L^2_{loc}(N)$$

then (a version of) function u is C^∞-smooth in N, satisfies the equation

$$\partial_t u - \Delta_\mu u = f,$$

and

$$u_k \overset{C^\infty(N)}{\longrightarrow} u.$$

PROOF. (i) As in the proof of Theorem 7.1, let U be a chart on M with coordinates $x^1, ..., x^n$, and λ be the Lebesgue measure in U. Then we have $d\mu = \rho(x)\, d\lambda$, where $\rho(x)$ is a smooth positive function in U, and the Laplace operator Δ_μ on $\mathcal{D}(U)$ has the form

$$\Delta_\mu = \rho^{-1} \partial_i \left(\rho g^{ij} \partial_j\right) = \rho^{-1} L,$$

where

$$L = \partial_i \left(\rho g^{ij} \partial_j\right).$$

Note that $\widetilde{U} := I \times U$ is a chart on N. Using the definition of the operators Δ_μ and ∂_j in $\mathcal{D}'(\widetilde{U})$, we obtain, for all $u \in \mathcal{D}'(\widetilde{U})$ and $\varphi \in \mathcal{D}(\widetilde{U})$,

$$\begin{aligned} (\Delta_\mu u, \varphi) &= (u, \Delta_\mu \varphi) = \left(u, \rho^{-1} \partial_i \left(\rho g^{ij} \partial_j \varphi\right)\right) = \left(\rho^{-1} u, \partial_i \left(\rho g^{ij} \partial_j \varphi\right)\right) \\ &= -\left(\partial_i \left(\rho^{-1} u\right), \rho g^{ij} \partial_j \varphi\right) = \left(\partial_j \left(\rho g^{ij} \partial_i \left(\rho^{-1} u\right)\right), \varphi\right) = (Lv, \varphi), \end{aligned}$$

where $v = \rho^{-1}u \in \mathcal{D}'(\widetilde{U})$. Hence,

$$(\partial_t u - \Delta_\mu u, \varphi) = (\rho\partial_t v - Lv, \varphi)$$

so that we have the identity

$$\partial_t u - \Delta_\mu u = \rho\partial_t v - Lv.$$

The hypothesis $\partial_t u - \Delta_\mu u \in C^\infty(N)$ implies $\rho\partial_t v - Lv \in C^\infty(\widetilde{U})$, and Corollary 6.26 yields $v \in C^\infty(\widetilde{U})$. Hence, we conclude $u \in C^\infty(\widetilde{U})$, which finishes the proof.

(*ii*) It follows from (7.11) that if $\{u_k\}$ is a sequence of distributions on N each satisfying the same equation $\partial_t u_k - \Delta_\mu u_k = f$ and $u_k \xrightarrow{\mathcal{D}'} u$ then u also satisfies $\partial_t u - \Delta_\mu u = f$. In particular, this is the case when $u_k \xrightarrow{L^2_{loc}} u$ as we have now. By part (*i*), we conclude that $u \in C^\infty(N)$, and the convergence $u_k \xrightarrow{C^\infty} u$ follows from Corollary 6.22. $\square$

COROLLARY 7.5. *The statement of Corollary 7.3 remains true if the hypothesis $u \in L^2_{loc}(M)$ is relaxed to $u \in L^1_{loc}(M)$.*

PROOF. Indeed, consider the function $v(t,x) = e^{\alpha t}u(x)$, which obviously belongs to $L^1_{loc}(N)$ where $N = \mathbb{R} \times M$. In particular, $v \in \mathcal{D}'(N)$. We have then

$$\partial_t v - \Delta_\mu v = \alpha e^{\alpha t}u - e^{\alpha t}\Delta_\mu u = e^{\alpha t}f.$$

Since $e^{\alpha t}f \in C^\infty(N)$, we conclude by Theorem 7.4 that $v \in C^\infty(N)$, whence $u \in C^\infty(M)$. $\square$

Exercises.

For any real $s > 0$, define the space $\mathcal{W}_0^s(M)$ as a subspace of $L^2(M)$ by

$$\mathcal{W}_0^s(M) = \operatorname{dom}(\mathcal{L} + \mathrm{id})^{s/2},$$

where $\mathcal{L}$ is the Dirichlet Laplace operator. The norm in this space is defined by

$$\|f\|_{\mathcal{W}_0^s} := \|(\mathcal{L} + \mathrm{id})^{s/2} f\|_{L^2}.$$

7.1. Prove that $\mathcal{W}_0^s$ is a Hilbert space.

7.2. Prove that $\mathcal{W}_0^1 = W_0^1$ and $\mathcal{W}_0^2 = W_0^2$ including the equivalence (but not necessarily the identity) of the norms.

7.3. Prove that if k is a positive integer then $f \in \mathcal{W}_0^{2k}$ if and only if

$$f,\ \mathcal{L}f,\ \ldots,\ \mathcal{L}^{k-1}f \in W_0^1(M) \ \text{ and } \ \mathcal{L}^k f \in L^2(M). \tag{7.12}$$

7.4. Prove that $\mathcal{W}_0^{2k} \subset \mathcal{W}^{2k}$ and that the norms in $\mathcal{W}_0^{2k}$ and $\mathcal{W}^{2k}$ are equivalent.

7.5. Prove that if $f \in \mathcal{W}_0^{2k}$ then, for all integer $0 \le l \le k$,

$$\|\mathcal{L}^l f\|_{L^2} \le \|f\|_{L^2}^{(k-l)/k} \|\mathcal{L}^k f\|_{L^2}^{l/k}. \tag{7.13}$$

7.6. Let M be a connected weighted manifold. Prove that if $f \in L^2_{loc}(M)$ and $\nabla f = 0$ on M then $f = \mathrm{const}$ on M.

7.7. Let M be a connected manifold and Ω be an open subset of M such that and $M\setminus\overline{\Omega}$ is non-empty. Prove that $1_\Omega \notin W^1(M)$ and $1_\Omega \notin W_0^1(\Omega)$.

REMARK. If in addition $\mu(\Omega) < \infty$ then clearly $1_\Omega \in L^2(\Omega)$ and $\nabla 1_\Omega = 0$ in Ω whence $1_\Omega \in W^1(\Omega)$. In this case we obtain an example of a function that is in $W^1(\Omega)$ but not in $W_0^1(\Omega)$.

7.8. *(The exterior maximum principle)*Let M be a connected weighted manifold and Ω be a non-empty open subset of M such that $M\setminus\overline{\Omega}$ is non-empty. Let u be a function from $C(M)\cap W_0^1(M)$ such that $\Delta_\mu u = 0$ in Ω. Prove that

$$\sup_\Omega u = \sup_{\partial\Omega} u.$$

Prove that if in addition Ω is the exterior of a compact set, then the hypothesis $u \in C(M)\cap W_0^1(M)$ can be relaxed to $u \in C(\overline{\Omega})\cap W_0^1(M)$.

7.9. Assume that $u \in L^2_{loc}(M)$ and $\Delta_\mu u \in L^2_{loc}(M)$. Prove that $u \in W^1_{loc}(M)$ and, moreover, for any couple of open sets $\Omega' \Subset \Omega'' \Subset M$,

$$\|u\|_{W^1(\Omega')} \le C\left(\|u\|_{L^2(\Omega'')} + \|\Delta_\mu u\|_{L^2(\Omega'')}\right), \tag{7.14}$$

where the constant C depends on $\Omega', \Omega'', \mathbf{g}, \mu, n$. The space $W^1_{loc}(M)$ is defined in Exercise 5.8 by (5.15).

7.10. Prove that if $u \in \mathcal{D}'(M)$ and $\Delta_\mu u \in C^\infty(M)$ then $u \in C^\infty(M)$.

7.11. A function u on a weighted manifold M is called harmonic if $u \in C^\infty(M)$ and $\Delta_\mu u = 0$. Prove that if $\{u_k\}_{k=1}^\infty$ is a sequence of harmonic functions such that

$$u_k \xrightarrow{L^2_{loc}} u \in L^2_{loc}(M)$$

then (a version of) u is also harmonic. Moreover, prove that, in fact, $u_k \xrightarrow{C^\infty} u$.

7.12. Let $\{u_k\}$ be a sequence of functions from $L^2_{loc}(M)$ such that

$$-\Delta_\mu u_k + \alpha_k u_k = f_k, \tag{7.15}$$

for some $\alpha_k \in \mathbb{R}$ and $f_k \in \mathcal{W}^{2m}_{loc}(M)$, with a fixed non-negative integer m. Assume further that, as $k \to \infty$,

$$\alpha_k \to \alpha, \quad f_k \xrightarrow{\mathcal{W}^{2m}_{loc}} f \text{ and } u_k \xrightarrow{L^2_{loc}} u.$$

Prove that function u satisfies the equation

$$-\Delta_\mu u + \alpha u = f, \tag{7.16}$$

and that

$$u_k \xrightarrow{\mathcal{W}^{2m+2}_{loc}} u. \tag{7.17}$$

Prove that if in addition $f_k \in C^\infty(M)$ and $f_k \xrightarrow{C^\infty} f$ then (versions of) u_k and u belong to $C^\infty(M)$ and $u_k \xrightarrow{C^\infty} u$.

7.13. Let $\{u_k\}$ be a sequence of non-negative functions from $C^\infty(M)$, which satisfy (7.15) with $\alpha_k \in \mathbb{R}$ and $f_k \in C^\infty(M)$. Assume further that, as $k \to \infty$,

$$\alpha_k \to \alpha, \quad f_k \xrightarrow{C^\infty} f \text{ and } u_k(x) \uparrow u(x) \text{ for any } x \in M,$$

where $u(x)$ is a function from L^2_{loc} that is defined pointwise. Prove that $u \in C^\infty(M)$ and $u_k \xrightarrow{C^\infty} u$.

7.14. Prove that, for any relatively compact open set $\Omega \subset M$, for any set $K \Subset \Omega$, and for any $\alpha \in \mathbb{R}$, there exists a constant $C = C(K, \Omega, \alpha)$ such that, for any smooth solution to the equation $-\Delta_\mu u + \alpha u = 0$ on M,

$$\sup_K |u| \le C\|u\|_{L^2(\Omega)}.$$

7.15. Let R_α be the resolvent operator defined in Section 4.2, that is, $R_\alpha = (\mathcal{L} + \alpha\,\mathrm{id})^{-1}$, where $\alpha > 0$. Prove that if $f \in L^2 \cap C^\infty(M)$ then also $R_\alpha f \in L^2 \cap C^\infty(M)$.

7.16. Let $\{\Omega_i\}$ be an exhaustion sequence in M. Prove that, for any non-negative function $f \in L^2 \cap C^\infty(M)$ and any $\alpha > 0$,

$$R_\alpha^{\Omega_i} f \xrightarrow{C^\infty} R_\alpha f \text{ as } i \to \infty.$$

HINT. Use that $R_\alpha^{\Omega_i} f \xrightarrow{L^2} R_\alpha f$ (cf. Theorem 5.22).

7.2. Smoothness of the semigroup solutions

The next theorem is a key technical result, which will have may important consequences.

THEOREM 7.6. *For any $f \in L^2(M)$ and $t > 0$, the function $P_t f$ belongs to $C^\infty(M)$.*

Moreover, for any set $K \Subset M$, the following inequality holds

$$\sup_K |P_t f| \le F_K(t) \|f\|_{L^2(M)}, \tag{7.18}$$

where

$$F_K(t) = C\left(1 + t^{-\sigma}\right), \tag{7.19}$$

σ is the smallest integer larger than $n/4$, and C is a constant depending on $K, \mathbf{g}, \mu, n$.

Furthermore, for any chart $U \Subset M$, a set $K \Subset U$, and a positive integer m, we have

$$\|P_t f\|_{C^m(K)} \le F_K(t) \|f\|_{L^2(M)}, \tag{7.20}$$

where $F_K(t)$ is still given by (7.19) but now σ is the smallest integer larger than $m/2 + n/4$, and $C = C(K, U, \mathbf{g}, \mu, n, m)$.

The estimate (7.18) is true also with $\sigma = n/4$, which is the best possible exponent in (7.19) (cf. Corollary 15.7). However, for our immediate applications, the value of σ is unimportant. Moreover, we will only use the fact that the function $F_K(t)$ in (7.18) and (7.20) is finite and locally bounded in $t \in (0, +\infty)$.

PROOF. Let $\{E_\lambda\}$ be the spectral resolution of the operator $\mathcal{L} = -\Delta_\mu|_{W_0^2}$ in $L^2(M)$. Consider the function $\Phi(\lambda) = \lambda^k e^{-t\lambda}$, where $t > 0$ and k is a positive integer. Observe that by (4.50)

$$\mathcal{L}^k e^{-t\mathcal{L}} = \Phi(\mathcal{L}) = \int_0^\infty \lambda^k e^{-t\lambda} dE_\lambda. \tag{7.21}$$

Since the function $\Phi(\lambda)$ is bounded on $[0, +\infty)$, the operator $\Phi(\mathcal{L})$ is bounded and so is $\mathcal{L}^k e^{-t\mathcal{L}}$. Hence, for any $f \in L^2(M)$, we have

$$\mathcal{L}^k \left(e^{-t\mathcal{L}} f\right) \in L^2(M),$$

that is,

$$\Delta_\mu^k (P_t f) \in L^2(M).$$

Since this is true for any k, we obtain $P_t f \in \mathcal{W}^\infty(M)$. By Theorem 7.1 (or Corollary 7.2) we conclude that $P_t f \in C^\infty(M)$.

Let us prove the estimates (7.18) and (7.20). Observe that the function $\lambda \mapsto \lambda^k e^{-t\lambda}$ takes its maximal value at $\lambda = k/t$, which implies, for any $f \in L^2$,

$$\begin{aligned}\|\Delta_\mu^k P_t f\|_{L^2} &= \|\mathcal{L}^k e^{-t\mathcal{L}} f\|_{L^2} \\ &= \left(\int_0^\infty \left(\lambda^k e^{-t\lambda}\right)^2 d\|E_\lambda f\|_{L^2}^2\right)^{1/2} \\ &\le \sup_{\lambda\ge 0}\left(\lambda^k e^{-t\lambda}\right)\left(\int_0^\infty d\|E_\lambda f\|_{L^2}^2\right)^{1/2} \\ &= \left(\frac{k}{t}\right)^k e^{-k}\|f\|_{L^2}. \end{aligned} \tag{7.22}$$

Using the definition (7.1) of the norm in $\mathcal{W}^{2\sigma}$ and (7.22), we obtain, for any positive integer σ,

$$\begin{aligned}\|P_t f\|_{\mathcal{W}^{2\sigma}} &= \sum_{k=0}^{\sigma} \|\Delta_\mu^k P_t f\|_{L^2} \\ &\le C\left(1+\sum_{k=1}^{\sigma}\left(\frac{k}{t}\right)^k e^{-k}\right)\|f\|_{L^2} \\ &\le C'\left(1+t^{-\sigma}\right)\|f\|_{L^2}. \end{aligned} \tag{7.23}$$

By the estimate (7.2) of Theorem 7.1, we have

$$\sup_K |P_t f| \le C\|P_t f\|_{\mathcal{W}^{2\sigma}(M)},$$

provided $\sigma > n/4$, which together with (7.23) yields (7.18). In the same way, (7.20) follows from (7.3). $\square$

Initially $P_t f$ was defined for as $e^{-t\mathcal{L}} f$, which is an element of $L^2(M)$. By Theorem 7.6, this function has a C^∞-version. From now on, let us redefine $P_t f$ to be the smooth version of $e^{-t\mathcal{L}} f$. Now we are in position to prove that, on any weighted manifold M, the operator P_t possesses an integral kernel.

THEOREM 7.7. *For any $x \in M$ and for any $t > 0$, there exists a unique function $p_{t,x} \in L^2(M)$ such that, for all $f \in L^2(M)$,*

$$P_t f(x) = \int_M p_{t,x}(y) f(y)\, d\mu(y). \tag{7.24}$$

Moreover, for any relatively compact set $K \subset M$ and for any $t > 0$, we have

$$\sup_{x\in K} \|p_{t,x}\|_{L^2(M)} \le F_K(t), \tag{7.25}$$

where $F_K(t)$ is the same function as in the estimate (7.18) *of* Theorem 7.6.

REMARK 7.8. The function $p_{t,x}(y)$ is defined for all $t > 0$, $x \in M$ but for *almost* all $y \in M$. Later on, it will be regularized to obtain a smooth function of all three variables t, x, y.

PROOF. Fix a relatively compact set $K \subset M$. By Theorem 7.6, for all $t > 0$ and $f \in L^2(M)$, the function $P_t f(x)$ is smooth in $x \in M$ and admits the estimate

$$|P_t f(x)| \leq F_K(t) \|f\|_{L^2} \text{ for all } x \in K. \tag{7.26}$$

Therefore, for fixed $t > 0$ and $x \in K$, the mapping $f \mapsto P_t f(x)$ is a bounded linear functional on $L^2(M)$. By the Riesz representation theorem, there exists a function $p_{t,x} \in L^2(M)$ such that

$$P_t f(x) = (p_{t,x}, f)_{L^2} \text{ for all } f \in L^2(M),$$

whence (7.24) follows. The uniqueness of $p_{t,x}$ is clear from (7.24). Since for any point $x \in M$ there is a compact set K containing x (for example, $K = \{x\}$), the function $p_{t,x}$ is defined for all $t > 0$ and $x \in M$.

Taking in (7.26) $f = p_{t,x}$ and using

$$P_t f(x) = (f, f)_{L^2} = \|f\|_{L^2}^2$$

we obtain

$$\|f\|_{L^2}^2 \leq F_K(t) \|f\|_{L^2},$$

whence (7.25) follows. □

EXAMPLE 7.9. Recall that the heat semigroup in $\mathbb{R}^n$ is determined by (4.62), which implies that in this case

$$p_{t,x}(y) = p_t(x-y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

Using the identity $p_t * p_t = p_{2t}$ (see Example 1.9), we obtain

$$\|p_{t,x}\|_{L^2}^2 = \int_{\mathbb{R}^n} p_t^2(x-y)\, dy = (p_t * p_t)(0) = p_{2t}(0) = \frac{1}{(8\pi t)^{n/2}},$$

whence

$$\|p_{t,x}\|_{L^2} = (8\pi t)^{-n/4}.$$

In particular, we see that the estimate (7.25) with $F_K(t) = C(1 + t^{-\sigma})$ and $\sigma > n/4$ is almost sharp for small t.

Now we prove that the function $P_t f(x)$ is, in fact, smooth jointly in t, x. Consider the product manifold $N = \mathbb{R}_+ \times M$ with the metric tensor $\mathbf{g}_N = dt^2 + \mathbf{g}_M$ and with measure $d\nu = dt d\mu$. The Laplace operator Δ_μ of (M, μ), which is obviously defined on $C^\infty(N)$, extends to $\mathcal{D}'(N)$ as follows:

$$(\Delta_\mu v, \varphi) = (v, \Delta_\mu \varphi),$$

for all $v \in \mathcal{D}'(N)$ and $\varphi \in \mathcal{D}(N)$. The time derivative $\frac{\partial}{\partial t}$ is defined in on $\mathcal{D}'(N)$ by

$$(\frac{\partial v}{\partial t}, \varphi) = -(v, \frac{\partial \varphi}{\partial t}).$$

Hence, for a function $u \in L^1_{loc}(N)$, it makes sense to consider the heat equation $\frac{\partial u}{\partial t} = \Delta_\mu u$ as a distributional equation on N.

THEOREM 7.10. *For any $f \in L^2(M)$, the function $u(t,x) = P_t f(x)$ belongs to $C^\infty(N)$ and satisfies in N the heat equation $\frac{\partial u}{\partial t} = \Delta_\mu u$.*

More precisely, the statement means that, for any $t > 0$, there is a pointwise version of the $L^2(M)$-function $u(t,\cdot)$ such that the function $u(t,x)$ belongs to $C^\infty(N)$.

PROOF. We already know by Theorem 7.6 that the function $u(t,x)$ is C^∞ smooth in x for any fixed $t > 0$. To prove that $u(t,x)$ is continuous jointly in t,x, it suffices to show that $u(t,x)$ is continuous in t locally uniformly in x. In fact, we will prove that, for any $t > 0$,

$$u(t+\varepsilon,\cdot) \xrightarrow{C^\infty} u(t,\cdot) \quad \text{as } \varepsilon \to 0, \tag{7.27}$$

which will settle the joint continuity of u. By Corollary 7.2), to prove (7.27) it suffices to show that, for any non-negative integer k,

$$u(t+\varepsilon,\cdot) \xrightarrow{\mathcal{W}^{2k}} u(t,\cdot). \tag{7.28}$$

We know already from the proof of Theorem 7.6 that, for any non-negative integer m, $u(t,\cdot) \in \operatorname{dom} \mathcal{L}^m$ and, hence, $\Delta_\mu^m u = (-\mathcal{L})^m u$. Therefore, it suffices to prove that,

$$\mathcal{L}^m(P_{t+\varepsilon} f) \xrightarrow{L^2} \mathcal{L}^m(P_t f). \tag{7.29}$$

Since by (7.21)

$$\mathcal{L}^m(P_{t+\varepsilon} f) = \int_0^\infty \lambda^m e^{-(t+\varepsilon)\lambda} dE_\lambda f$$

and the function $\lambda^m e^{-(t+\varepsilon)\lambda}$ remains uniformly bounded in λ as $\varepsilon \to 0$, Lemma 4.8 allows to pass to the limit under the integral sign, which yields (7.29).

Since $u(t,x)$ is continuous jointly in (t,x), it makes sense to consider u as a distribution on N. Let us show that the function $u(t,x)$ satisfies on N the heat equation in the distributional sense, which amounts to the equation

$$(u, \frac{\partial \varphi}{\partial t} + \Delta_\mu \varphi) = 0, \tag{7.30}$$

for any $\varphi \in \mathcal{D}'(N)$. Using Fubini's theorem, we obtain

$$\begin{aligned} (u, \frac{\partial \varphi}{\partial t} + \Delta_\mu \varphi) &= \int_N u\,(\frac{\partial \varphi}{\partial t} + \Delta_\mu \varphi)\,d\nu \\ &= \int_{\mathbb{R}_+} (u, \frac{\partial \varphi}{\partial t})_{L^2(M)}\,dt + \int_{\mathbb{R}_+} (u, \Delta_\mu \varphi)_{L^2(M)}\,dt. \end{aligned} \tag{7.31}$$

Considering $\varphi(t,\cdot)$ as a path in $L^2(M)$, observe that the classical partial derivative $\frac{\partial \varphi}{\partial t}$ coincides with the strong derivative $\frac{d\varphi}{dt}$ (cf. Exercise 4.47).

The product rule for the strong derivative (cf. Exercise 4.46) yields

$$(u, \frac{\partial\varphi}{\partial t}) = (u, \frac{d\varphi}{dt}) = \frac{d}{dt}(u,\varphi) - (\frac{du}{dt},\varphi), \tag{7.32}$$

where all the brackets mean the inner product in $L^2(M)$. Since $\varphi(t,\cdot)$ vanishes outside some time interval $[a,b]$ where $0<a<b$, we obtain

$$\int_{\mathbb{R}_+} \frac{d}{dt}(u,\varphi)\,dt = 0. \tag{7.33}$$

To handle the last term in (6.66), recall that, by Theorem 4.9,

$$\frac{du}{dt} = \Delta_\mu u,$$

which yields

$$\int_{\mathbb{R}_+} (\frac{du}{dt},\varphi)dt = \int_{\mathbb{R}_+} (\Delta_\mu u,\varphi)dt = \int_{\mathbb{R}_+} (u,\Delta_\mu\varphi)dt. \tag{7.34}$$

Combining (7.32), (7.33), and (7.34), we obtain that the right hand side of (7.31) vanishes, which proves (7.30).

Applying Theorem 7.4 to function $u(t,x)$, which satisfies the heat equation in the distributional sense, we conclude that $u\in C^\infty(N)$ and u satisfies the heat equation in the classical sense. □

SECOND PROOF. In this proof, we do not use the parabolic regularity theory (Theorem 7.4). However, we still use the first part of the first proof, namely, the convergence (7.27). Let us fix a chart $U\subset M$ so that we can consider the partial derivatives ∂^α with respect to x in this chart. By Theorem 7.6, $\partial^\alpha u$ is C^∞-smooth in x. By (7.27), we have

$$\partial^\alpha u(t+\varepsilon,\cdot) \xrightarrow{C^\infty} \partial^\alpha u(t,\cdot) \quad \text{as } \varepsilon\to 0,$$

which implies that $\partial^\alpha u$ is jointly continuous in t,x.

To handle the time derivative $\partial_t u$, let us first prove that, for any $t>0$,

$$\frac{u(t+\varepsilon,\cdot)-u(t,\cdot)}{\varepsilon} \xrightarrow{C^\infty} \Delta_\mu u(t,\cdot) \quad \text{as } \varepsilon\to 0. \tag{7.35}$$

By Corollary 7.2, it suffices to prove that, for any non-negative integer k,

$$\frac{u(t+\varepsilon,\cdot)-u(t,\cdot)}{\varepsilon} \xrightarrow{W^{2k}} -\mathcal{L}u,$$

and this, in turn, will follows from

$$\mathcal{L}^m\frac{u(t+\varepsilon,\cdot)-u(t,\cdot)}{\varepsilon} \xrightarrow{L^2} -\mathcal{L}^{m+1}u, \tag{7.36}$$

provided (7.36) holds for all non-negative integers m. It follows from (7.21) that

$$\mathcal{L}^m\frac{u(t+\varepsilon,\cdot)-u(t,\cdot)}{\varepsilon} = \int_0^\infty \lambda^m\frac{e^{-\varepsilon\lambda}-1}{\varepsilon}e^{-t\lambda}dE_\lambda f.$$

Since the function under the integration remains uniformly bounded in λ as $\varepsilon\to0$ (cf. the estimate (4.60) from the proof of Theorem 4.9), by Lemma 4.8 we can pass to the limit under the integral sign, which yields (7.36).

It follows from (7.35) that $\partial_t u$ exists in the classical sense for all $t>0$ and $x\in M$, and

$$\partial_t u = \Delta_\mu u. \tag{7.37}$$

Moreover, (7.35) also yields that, for any partial derivative ∂^α in x,

$$\frac{\partial^\alpha u(t+\varepsilon,\cdot)-\partial^\alpha u(t,\cdot)}{\varepsilon}\overset{C^\infty}{\longrightarrow}\partial^\alpha\Delta_\mu u(t,\cdot),$$

which implies that

$$\partial_t(\partial^\alpha u)=\partial^\alpha\Delta_\mu u. \tag{7.38}$$

In particular, we obtain that $\partial_t(\partial^\alpha u)$ is continuous in t,x.

Observe that the function $v=\Delta_\mu u$ can be represented in the form $v=P_{t-s}g$ where $g\in L^2(M)$, which follows from the identity

$$v=-\mathcal{L}u=-\mathcal{L}e^{-(t-s)\mathcal{L}}e^{-s\mathcal{L}}f=-e^{-(t-s)\mathcal{L}}\left(\mathcal{L}e^{-s\mathcal{L}}f\right).$$

Therefore, all the above argument works also for function v instead of u and, hence, for $\Delta_\mu^k u$ instead of u, for any positive integer k. Replacing in (7.38) u by $\Delta_\mu^k u$ we obtain

$$\partial_t\left(\partial^\alpha\Delta_\mu^{k-1}u\right)=\partial^\alpha\left(\Delta_\mu^k u\right). \tag{7.39}$$

Iterating (7.39) for a decreasing sequence of values of k, we obtain

$$\partial^\alpha\Delta_\mu^k u=\partial_t\left(\partial^\alpha\Delta_\mu^{k-1}u\right)=\partial_t^2\left(\partial^\alpha\Delta_\mu^{k-2}u\right)=...=\partial_t^{k-1}(\partial^\alpha\Delta_\mu u)=\partial_t^k\partial^\alpha u.$$

Hence, we have the identity

$$\partial_t^k\partial^\alpha u=\partial^\alpha\Delta_\mu^k u.$$

In particular, this gives $\partial_t^k u=\Delta_\mu^k u$, whence applying ∂^α,

$$\partial^\alpha\partial_t^k u=\partial^\alpha\Delta_\mu^k u.$$

Finally, using the above two identities, any partial derivative

$$\partial_t^{k_1}\partial^{\alpha_1}\partial_t^{k_2}\partial^{\alpha_2}....u$$

can be brought to the form $\partial^\alpha\Delta_\mu^k u$ and, hence, it exists and is continuous in t,x, which finishes the proof. $\square$

THIRD PROOF. Let $\Phi(\lambda)$ be a continuous function on $[0,+\infty)$ of a subexponential growth; that is for any $\varepsilon>0$

$$|\Phi(\lambda)|=o\left(e^{\varepsilon\lambda}\right)\quad\text{as }\lambda\to+\infty. \tag{7.40}$$

Fix $f\in L^2(M)$ and, for any $t>0$, consider the function

$$v(t,\cdot):=\int_0^\infty\Phi(\lambda)e^{-t\lambda}dE_\lambda f, \tag{7.41}$$

where $\{E_\lambda\}$ is the spectral resolution of the Dirichlet Laplace operator $\mathcal{L}$ on M. We will prove that $v(t,x)$ belongs to $C^\infty(N)$ and satisfies the heat equation on N (obviously, this contains Theorem 7.10 as a particular case for $\Phi\equiv 1$).

Fix the numbers $0<a<b$ and consider the open set $N_{a,b}\subset N$ defined by

$$N_{a,b}=(a,b)\times M.$$

LEMMA 7.11. *For any $t\in(a,b)$, function $v(t,\cdot)$ can be modified*[3] *on a subset of μ-measure 0 of M so that $v(t,x)\in L^2(N_{a,b})$. Furthermore, the weak derivatives $\frac{\partial v}{\partial t}$ and $\Delta_\mu v$ exist in $L^2(N_{a,b})$ and satisfy the identity*

$$\frac{\partial v}{\partial t}=\Delta_\mu v=-\int_0^\infty\lambda\Phi(\lambda)e^{-t\lambda}dE_\lambda f. \tag{7.42}$$

[3] Such a modification is necessary because there are non-measurable subsets of N that have μ-measure 0 for any fixed t.

PROOF. By (7.40), the function $\Phi(\lambda) e^{-t\lambda}$ is bounded for any $t > 0$, which implies that the right hand side of (7.41) is defined for all $f \in L^2(M)$ and determines a function from $L^2(M)$. As in the proof of Theorem 4.9, one shows that the mapping $t \mapsto v(t, \cdot)$ as a path in $L^2(M)$ is strongly differentiable and satisfies the equation

$$\frac{dv}{dt}(t, \cdot) = -\mathcal{L}v(t, \cdot) = -\int_0^\infty \lambda\Phi(\lambda) e^{-t\lambda} dE_\lambda f \tag{7.43}$$

(cf. Exercise 4.51).

Consequently, the path $t \mapsto v^2(t, \cdot)$ is continuous in $L^1(M)$. By Exercise 4.49, the function $v(t, \cdot)$ can be modified for any $t \in (a, b)$ on a set of μ-measure 0 in M so that $v^2(t, x) \in L^1(N_{a,b})$. Hence, $v(t, x) \in L^2(N_{a\,b})$.

Since the function $\lambda\Phi(\lambda)$ also satisfies the condition (7.40), we conclude by the above argument that $\frac{dv}{dt}(t, x) \in L^2(N_{a,b})$. Let us show that the distributional derivative $\frac{\partial v}{\partial t}$ coincides with the strong derivative, that is,

$$\frac{\partial v}{\partial t} = \frac{dv}{dt}. \tag{7.44}$$

Indeed, applying the product rule of the strong derivative (see Exercise 4.46), we obtain, for any $\varphi \in C_0^\infty(N_{a,b})$

$$\frac{d}{dt}(v, \varphi)_{L^2(M)} = \left(v, \frac{d\varphi}{dt}\right)_{L^2(M)} + \left(\frac{dv}{dt}, \varphi\right)_{L^2(M)}.$$

Since

$$\int_a^b \frac{d}{dt}(v, \varphi)_{L^2(M)} = 0,$$

it follows that

$$\int_a^b (\frac{dv}{dt}, \varphi)_{L^2(M)} dt = -\int_a^b (v, \frac{d\varphi}{dt})_{L^2(M)} dt.$$

Since $\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial t}$ (cf. Exercise 4.47), we conclude that

$$\left(\frac{dv}{dt}, \varphi\right)_{L^2(N)} = -(v, \frac{\partial \varphi}{\partial t})_{L^2(N)},$$

which proves (7.44).

Let us prove that

$$\Delta_\mu v = -\mathcal{L}v. \tag{7.45}$$

By definition of $\mathcal{L}$, for any fixed $t > 0$, $\Delta_\mu v(t, \cdot)$ as a distribution on M coincides with $-\mathcal{L}v(t, \cdot)$, which implies

$$\begin{aligned} -(\mathcal{L}v, \varphi)_{L^2(N)} &= -\int_a^b (\mathcal{L}v(t, \cdot), \varphi(t, \cdot))_{L^2(M)} dt = \int_a^b (\Delta_\mu v(t, \cdot), \varphi(t, \cdot)) dt \\ &= \int_a^b (v(t, \cdot), \Delta_\mu \varphi(t, \cdot)) dt = (v, \Delta_\mu \varphi)_{L^2(N)}, \end{aligned}$$

whence (7.45) follows. Combining (7.43), (7.44), and (7.45), we obtain (7.42). □

Let $\widetilde{\Delta}$ be the (distributional) Laplace operator on the manifold N, that is,

$$\widetilde{\Delta} = \frac{\partial^2}{\partial t^2} + \Delta_\mu.$$

By Theorem 7.1, in order to prove that $v \in C^\infty(N_{a,b})$, it suffices to show that $\widetilde{\Delta}^k v \in L^2(N_{a,b})$ for all $k \geq 1$.

Since the function $\lambda\Phi(\lambda)$ also satisfies condition (7.40), Lemma 7.11 applies to function

$$\frac{\partial v}{\partial t} = -\int_0^\infty \lambda\Phi(\lambda) e^{-t\lambda} dE_\lambda f$$

and yields that the weak derivative $\frac{\partial^2 v}{\partial t^2}$ exists in $L^2(N_{a,b})$ and

$$\frac{\partial^2 v}{\partial t^2} = \int_0^\infty \lambda^2 \Phi(\lambda) e^{-t\lambda} dE_\lambda f.$$

Since $\frac{\partial^2 v}{\partial t^2}$ and $\Delta_\mu v$ belong to $L^2(N_{a,b})$, we obtain that also $\widetilde{\Delta} v \in L^2(N_{a,b})$ and

$$\widetilde{\Delta} v = \int_0^\infty (\lambda^2 - \lambda) \Phi(\lambda) e^{-t\lambda} dE_\lambda f.$$

Applying the same argument to the function $(\lambda^2 - \lambda)\Phi(\lambda)$ instead of $\Phi(\lambda)$ and then continuing by induction, we obtain, that for all integers $k \geq 1$,

$$\widetilde{\Delta}^k v = \int_0^\infty (\lambda^2 - \lambda)^k \Phi(\lambda) e^{-t\lambda} dE_\lambda f,$$

and, hence, $\widetilde{\Delta}^k v \in L^2(N_{a,b})$. We conclude that $v \in C^\infty(N_{a,b})$ and the equation $\frac{\partial v}{\partial t} = \Delta_\mu v$ (which follows from (7.42)) is satisfied in the classical sense. $\square$

Exercises.

7.17. Prove that, for any compact set $K \subset M$, for any $f \in L^2(M,\mu)$, and for any positive integer m,

$$\sup_K |\Delta_\mu^m (P_t f)| \leq C t^{-m} (1 + t^{-\sigma}) \|f\|_2, \tag{7.46}$$

where σ is the smallest integer larger than $n/4$.

7.18. Let f be a non-negative function from $L^2(M)$ and $\{\Omega_i\}$ be an exhaustion sequence in M. Prove that

$$P_t^{\Omega_i} f \overset{C^\infty(\mathbb{R}_+ \times M)}{\longrightarrow} P_t f \ \text{ as } i \to \infty.$$

Hint. Use the fact that, for any $t > 0$,

$$P_t^{\Omega_i} f \xrightarrow{\text{a.e.}} P_t f \ \text{ as } i \to \infty$$

(cf. Theorem 5.23).

7.19. Prove that if $f \in C_0^\infty(M)$ then

$$P_t f \overset{C^\infty(M)}{\longrightarrow} f \text{ as } t \to 0.$$

7.20. Consider the cos-wave operator

$$C_t = \cos\left(t\mathcal{L}^{1/2}\right)$$

(cf. Exercise 4.52). Prove that, for any $f \in C_0^\infty(M)$, the function

$$u(t,x) = C_t f(x)$$

belongs to $C^\infty(\mathbb{R} \times M)$ and solves in $\mathbb{R} \times M$ the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta_\mu u$$

with the initial conditions

$$u(0,x) = f(x) \text{ and } \frac{\partial u}{\partial t}(0,x) = 0.$$

7.3. The heat kernel

By Theorem 7.7, for any $x \in M$ and $t > 0$, there exists a function $p_{t,x} \in L^2(M, \mu)$ such that, for all $f \in L^2(M, \mu)$,

$$P_t f(x) = (p_{t,x}, f)_{L^2}. \tag{7.47}$$

Note that the function $p_{t,x}(y)$ is defined for all x but for *almost* all y. Here we construct a regularized version of $p_{t,x}(y)$, which will be defined for *all* y. Namely, for any $t > 0$ and all $x, y \in M$, set

$$p_t(x, y) := \left(p_{t/2,x}, p_{t/2,y}\right)_{L^2}. \tag{7.48}$$

DEFINITION 7.12. The function $p_t(x, y)$ is called the *heat kernel* of the weighted manifold $(M, \mathbf{g}, \mu)$.

The main properties of $p_t(x, y)$ are stated in the following theorem.

THEOREM 7.13. *On any weighted manifold $(M, \mathbf{g}, \mu)$ the heat kernel satisfies the following properties.*

- *Symmetry: $p_t(x, y) \equiv p_t(y, x)$ for all $x, y \in M$ and $t > 0$.*
- *For any $f \in L^2$, and for all $x \in M$ and $t > 0$,*
$$P_t f(x) = \int_M p_t(x, y) f(y)\, d\mu(y). \tag{7.49}$$
- *$p_t(x, y) \geq 0$ for all $x, y \in M$ and $t > 0$, and*
$$\int_M p_t(x, y)\, d\mu(y) \leq 1, \tag{7.50}$$
for all $x \in M$ and $t > 0$.
- *The semigroup identity: for all $x, y \in M$ and $t, s > 0$,*
$$p_{t+s}(x, y) = \int_M p_t(x, z)\, p_s(z, y)\, d\mu(z). \tag{7.51}$$
- *For any $y \in M$, the function $u(t, x) := p_t(x, y)$ is C^∞ smooth in $(0, +\infty) \times M$ and satisfies the heat equation*
$$\frac{\partial u}{\partial t} = \Delta_\mu u. \tag{7.52}$$
- *For any function $f \in C_0^\infty(M)$,*
$$\int_M p_t(x, y) f(y)\, d\mu(y) \to f(x) \text{ as } t \to 0, \tag{7.53}$$
where the convergence is in $C^\infty(M)$.

REMARK 7.14. Obviously, a function $p_t(x, y)$, which satisfies (7.49) and is continuous in y for any fixed t, x, is unique. As we will see below in Theorem 7.20, the function $p_t(x, y)$ is, in fact, C^∞ smooth jointly in t, x, y. Note also that $p_t(x, y) > 0$ provided manifold M is connected (see Corollary 8.12).

PROOF. Everywhere in the proof, $(\cdot,\cdot)$ stands for the inner product in $L^2(M)$. The symmetry of $p_t(x,y)$ is obvious from definition (7.48). The latter also implies $p_t(x,y) \geq 0$ provided we show that $p_{t,x} \geq 0$ a.e.. Indeed, by Theorems 5.11 and 7.6, $P_t f(x) \geq 0$ for all non-negative $f \in L^2$ and for all $t > 0$, $x > 0$. Setting $f = (p_{t,x})_-$, we obtain

$$0 \leq P_t f(x) = (p_{t,x}, f) = \left((p_{t,x})_+, f\right) - \left((p_{t,x})_-, f\right) = -(f, f),$$

whence $f = 0$ a.e. and $p_{t,x} \geq 0$ a.e.

The proof of the rest of Theorem 7.13 will be preceded by two claims.

CLAIM 1. *For all $x \in M$, $t, s > 0$, and $f \in L^2(M)$,*

$$P_{t+s} f(x) = \int_M (p_{t,z}, p_{s,x}) f(z)\, d\mu(z). \tag{7.54}$$

Indeed, using $P_{t+s} = P_s P_t$, (7.47), and the symmetry of P_t, we obtain

$$\begin{aligned} P_{t+s} f(x) &= P_s(P_t f)(x) \\ &= (p_{s,x}, P_t f) = (P_t p_{s,x}, f) \\ &= \int_M P_t p_{s,x}(z) f(z)\, d\mu(z) \\ &= \int_M (p_{t,z}, p_{s,x}) f(z)\, d\mu(z), \end{aligned}$$

whence (7.54) follows.

CLAIM 2. *For all $x, y \in M$ and $t > 0$, the inner product $(p_{s,x}, p_{t-s,y})$ does not depend on $s \in (0,t)$.*

Indeed, for all $0 < r < s < t$, we have, using (7.47) and applying (7.54) with $f = p_{r,x}$,

$$\begin{aligned} (p_{s,x}, p_{t-s,y}) &= P_s p_{t-s,y}(x) = P_r(P_{s-r} p_{t-s,y})(x) \\ &= \int_M p_{r,x}(z) (p_{s-r,z}, p_{t-s,y})\, d\mu(z) \\ &= P_{t-r} p_{r,x}(y) \\ &= (p_{t-r,y}, p_{r,x}), \end{aligned}$$

which was to be proved.

Proof of (7.49). Combining (7.54) and (7.48), we obtain

$$P_t f(x) = \int_M \left(p_{t/2,x}, p_{t/2,y}\right) f(y)\, d\mu(y) = \int_M p_t(x,y) f(y)\, d\mu(y). \tag{7.55}$$

Proof of (7.50). By Theorem 5.11, $f \leq 1$ implies $P_t f(x) \leq 1$ for all $x \in M$ and $t > 0$. Taking $f = 1_K$ where $K \subset M$ is a compact set, we obtain

$$\int_K p_t(x,y)\, d\mu(y) \leq 1,$$

whence (7.50) follows.

Proof of (7.51). It follows from Claim 2 that, for all $x, y \in M$ and $0 < s < t$,

$$p_t(x,y) = (p_{s,x}, p_{t-s,y}). \tag{7.56}$$

Indeed, (7.56) holds for $s = t/2$ by definition (7.48), which implies that it holds for all $s \in (0,t)$ because the right hand side of (7.56) does not depend on s. Comparison of (7.47) and (7.49) shows that

$$p_t(x,\cdot) = p_{t,x} \text{ a.e.} \tag{7.57}$$

Using (7.56) and (7.57), we obtain, for all $x, y \in M$ and $t, s > 0$,

$$\int_M p_t(x,z)\, p_s(z,y)\, d\mu(z) = (p_t(x,\cdot), p_s(y,\cdot)) = (p_{t,x}, p_{s,y}) = p_{t+s}(x,y). \tag{7.58}$$

Proof of (7.52). Fix $s > 0$ and $y \in M$ and consider the function $v(t,x) := p_{t+s}(x,y)$. We have by (7.58)

$$v(t,x) = (p_{t,x}, p_{s,y}) = P_t p_{s,y}(x). \tag{7.59}$$

Since $p_{s,y} \in L^2(M)$, Theorem 7.10 yields that the function $v(t,x)$ is smooth in (t,x) and solves the heat equation. Changing t to $t-s$, we obtain that the same is true for the function $p_t(x,y)$.

Proof of (7.53). If $f \in C_0^\infty(M)$ then also $\Delta_\mu f \in C_0^\infty(M)$ whence it follows by induction that $f \in \operatorname{dom}\mathcal{L}^m$ for any positive integer m, where $\mathcal{L}$ is the Dirichlet Laplace operator. By (A.48), this implies that

$$\int_0^\infty \lambda^{2m} d\|E_\lambda f\|^2 < \infty.$$

The identities

$$\mathcal{L}^m f = \int_0^\infty \lambda^m dE_\lambda f$$

and

$$\mathcal{L}^m P_t f = \int_0^\infty \lambda^m e^{-t\lambda} dE_\lambda f$$

imply

$$\|\mathcal{L}^m(P_t f - f)\|_{L^2}^2 = \int_0^\infty \lambda^{2m}\left(1 - e^{-t\lambda}\right)^2 d\|E_\lambda f\|^2.$$

Since the function $\lambda^{2m}\left(1-e^{-t\lambda}\right)^2$ is bounded for all $t > 0$ by the integrable function λ^{2m}, and

$$\lambda^{2m}\left(1 - e^{-t\lambda}\right)^2 \to 0 \text{ as } t \to 0,$$

the dominated convergence theorem implies that

$$\|\mathcal{L}^m(P_t f - f)\|_{L^2} \to 0 \text{ as } t \to 0.$$

We see that $P_t f - f \to 0$ in $\mathcal{W}^\infty_{loc}(M)$, which implies by Corollary 7.2 that

$$P_t f \xrightarrow{C^\infty} f, \tag{7.60}$$

which was to be proved. □

Exercises.

7.21. Prove that, for all $x, y \in M$ and $t > 0$,

$$p_t(x,y) \le \sqrt{p_t(x,x)\, p_t(y,y)}. \tag{7.61}$$

7.22. Prove that, for all $x \in M$, the functions $p_t(x,x)$ and $\|p_{t,x}\|_2$ are non-increasing in t.

7.23. Let $K \subset M$ be a compact set.

(a) Prove that the function

$$S(t) := \sup_{x,y \in K} p_t(x,y)$$

is non-increasing in $t > 0$.

(b) Prove that, for all $t > 0$,

$$S(t) \le C\left(1 + t^{-\alpha}\right),$$

for some constants $\alpha, C > 0$, where C depends on K.

7.24. Let J be an isometry of a weighted manifold M (see Section 3.12). Prove that

$$p_t(Jx, Jy) \equiv p_t(x,y).$$

7.4. Extension of the heat semigroup

So far the operator P_t has been defined on functions $f \in L^2$ so that $P_t f \in L^2 \cap C^\infty$. Using the identity (7.49), we now extend the definition of P_t as follows: set

$$P_t f(x) := \int_M p_t(x,y) f(y)\, d\mu(y), \tag{7.62}$$

for any function f such that the right hand side of (7.62) makes sense. In particular, $P_t f(x)$ will be considered as a function defined pointwise (as opposed to functions defined up to null sets).

7.4.1. Heat semigroup in L^1_{loc}.

THEOREM 7.15. *If $f \in L^1_{loc}(M)$ is a non-negative function on M then the function $P_t f(x)$ is measurable in $x \in M$ (for any $t > 0$) and in $(t,x) \in \mathbb{R}_+ \times M$.*

If, in addition, $P_t f(x) \in L^1_{loc}(I \times M)$ where I is an open interval in $\mathbb{R}_+$, then the function $P_t f(x)$ is C^∞ smooth on $I \times M$ and satisfies the heat equation

$$\frac{\partial}{\partial t}(P_t f) = \Delta_\mu (P_t f).$$

PROOF. Let $\{\Omega_k\}$ be a *compact exhaustion sequence* in M, that is, an increasing sequence of relatively compact open set $\Omega_k \subset M$ such that $\overline{\Omega}_k \subset \Omega_{k+1}$ and the union of all sets Ω_k is M. Set

$$f_k = \min(f,k)\, 1_{\Omega_k}$$

and observe that functions f_k are bounded, compactly supported, and the sequence $\{f_k\}$ is monotone increasing and converges to f a.e.. Since $f_k \in L^2(M)$, by Theorems 7.10 and 7.13, the function $u_k(t,x) = P_t f_k(x)$ is smooth in $N := I \times M$ and satisfies the heat equation in N. Set also $u(t,x) = P_t f(x)$ and observe that, by (7.62) and the monotone convergence theorem,

$$u_k(t,x) \to u(t,x) \text{ for all } (t,x) \in N. \tag{7.63}$$

Hence, $u(t,x)$ is a measurable function both on M and N (note that so far u may take value ∞).

If, in addition, $u \in L^1_{loc}(N)$ then u can be considered as a distribution on N. The heat equation for u_k implies the identity

$$\int_N \left(\frac{\partial \varphi}{\partial t} + \Delta_\mu \varphi\right) u_k \, d\mu dt = 0,$$

for all $\varphi \in \mathcal{D}(N)$. Since the sequence of functions $\left(\frac{\partial \varphi}{\partial t} + \Delta_\mu \varphi\right) u_k$ is uniformly bounded on N by the integrable function $C 1_{\operatorname{supp}\varphi} u$, where $C = \sup|\partial_t \varphi + \Delta_\mu \varphi|$, we can pass to the limit under the integral sign as $k \to \infty$ and obtain that u satisfies the same identity. Hence, u solves the heat equation in the distributional sense and, by Theorem 7.4, u admits a $C^\infty(N)$-modification, which we denote by $\widetilde{u}(t,x)$.

The sequence $\{u_k\}$ is increasing and, by (7.63), converges to $\widetilde{u}$ a.e.. Since $\widetilde{u}$ is smooth and, hence, $\widetilde{u} \in L^2_{loc}(N)$, we obtain by the dominated convergence theorem that $u_k \xrightarrow{L^2_{loc}(N)} \widetilde{u}$. By the second part of Theorem 7.4, we conclude that $u_k \xrightarrow{C^\infty} \widetilde{u}$. Finally, since $u_k \to u$ pointwise, we see that $u(t,x) = \widetilde{u}(t,x)$ for all $(t,x) \in N$, which finishes the proof. $\square$

For applications, Theorem 7.15 should be complemented by the conditions ensuring the finiteness of $P_t f$. It is also important to understand whether $P_t f$ converges to f as $t \to 0$ and in what sense. We present in the next subsections some basic results in this direction.

7.4.2. Heat semigroup in C_b.

Denote by $C_b(M)$ the class of bounded continuous functions on M. The following result extends Theorem 1.3 to arbitrary weighted manifold.

THEOREM 7.16. *For any $f \in C_b(M)$, the function $P_t f(x)$ is finite for all $t > 0$ and $x \in M$, and satisfies the estimate*

$$\inf f \le P_t f(x) \le \sup f. \tag{7.64}$$

Moreover, $P_t f(x)$ is C^∞ smooth in $\mathbb{R}_+ \times M$, satisfies in $\mathbb{R}_+ \times M$ the heat equation, and

$$\lim_{t \to 0} P_t f(x) = f(x), \tag{7.65}$$

where the limit is locally uniform in $x \in M$.

In particular, we see that $P_tf \in C_b(M)$ for any $t > 0$ so that P_t can considered as an operator in $C_b(M)$.

The statement of Theorem 7.16 can be rephrased also as follows: for any $f \in C_b(M)$, the function $u(t,x) = P_tf(x)$ is a bounded solution to the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_\mu u & \text{in } \mathbb{R}_+ \times M, \\ u|_{t=0} = f, \end{cases}$$

understood in the classical sense. The question of uniqueness is quite subtle and will be first addressed in Section 8.4.1.

PROOF. By treating separately f_+ and f_-, we can assume that $f \geq 0$. By (7.50), we obtain, for all $t > 0$ and $x \in M$,

$$P_tf(x) \leq \sup f \int_M p_t(x,y)\,d\mu(y) \leq \sup f, \tag{7.66}$$

which proves the finiteness of P_tf and (7.64). By the first part of Theorem 7.15, $P_tf(x) \in L^\infty(\mathbb{R}_+ \times M)$, and by the second part of Theorem 7.15, $P_tf(x) \in C^\infty(\mathbb{R}_+ \times M)$ and P_tf satisfies the heat equation.

The initial condition (7.65) was proved in Theorem 7.13 for $f \in C_0^\infty(M)$. Assume next that $f \in C_0(M)$, where $C_0(M)$ is the class of continuous functions with compact supports. Since $C_0^\infty(M)$ is dense in $C_0(M)$ (cf. Exercise 4.5), there exists a sequence $\{f_k\}$ of functions from $C_0^\infty(M)$ that converges to f uniformly on M. Obviously, we have

$$P_tf - f = (P_tf - P_tf_k) + (P_tf_k - f_k) + (f_k - f).$$

For a given $\varepsilon > 0$, choose k large enough so that

$$\sup_M |f_k - f| < \varepsilon. \tag{7.67}$$

By (7.64) we have, for all $t > 0$,

$$\sup_M |P_t(f_k - f)| < \varepsilon.$$

By the previous step, $P_tf_k \to f_k$ as $t \to 0$ locally uniformly; hence, for any compact set $K \subset M$ and for small enough $t > 0$,

$$\sup_K |P_tf_k - f_k| < \varepsilon.$$

Combining all the previous lines yields

$$\sup_K |P_tf - f| < 3\varepsilon,$$

whence the claim follows.

Let now $f \in C_b(M)$. Renormalizing f, we can assume $0 \leq f \leq 1$. Fix a compact set $K \subset M$ and a let $\psi \in C_0^\infty(M)$ be a cutoff function of K in M, that is, $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on K (cf. Theorem 3.5). Since $f\psi = f$ on K, we have the identity

$$P_tf - f = (P_tf - P_t(f\psi)) + (P_t(f\psi) - f\psi) \text{ on } K.$$

Since $f\psi \in C_0(M)$, we have by the previous step

$$\sup_K |P_t(f\psi) - f\psi| \to 0 \text{ as } t \to 0. \tag{7.68}$$

To estimate the difference $P_t f - P_t(f\psi)$, observe that, by $0 \le f \le 1$ and (7.64),

$$0 \le P_t(f - f\psi) \le P_t(1-\psi) = P_t 1 - P_t\psi \le 1 - P_t\psi.$$

By the previous part, we have $P_t\psi \to \psi$ as $t \to 0$ locally uniformly. Since $\psi \equiv 1$ on K, we obtain that $P_t\psi \to 1$ uniformly on K, which implies that

$$\sup_K |P_t(f - f\psi)| \to 0 \text{ as } t \to 0,$$

which together with (7.68) implies

$$\sup_k |P_t f \to f| \to \text{ as } t \to 0,$$

which was to be proved. $\square$

REMARK 7.17. Consider the function

$$u(t,x) = \begin{cases} P_t f(x), & t > 0, \\ f(x), & t = 0. \end{cases}$$

It follows from Theorem 7.16 that if $f \in C_b(M)$ then u is continuous in $[0,+\infty) \times M$.

The main difficulty in the proof of Theorem 7.16 was to ensure that the convergence (7.65) is locally uniform. Just pointwise convergence is much simpler – see Lemma 9.2.

7.4.3. Heat semigroup in L^1. Our next goal is to consider $P_t f$ for $f \in L^1(M)$. We will need for that the following lemma.

LEMMA 7.18. *Let $\{v_{ik}\}$ be a double sequence of non-negative functions from $L^1(M)$ such that, for any k,*

$$v_{ik} \xrightarrow{L^1} u_k \in L^1(M) \quad \text{as } i \to \infty$$

and

$$u_k \xrightarrow{L^1} u \in L^1(M) \quad \text{as } k \to \infty.$$

Let $\{w_i\}$ be a sequence of functions from $L^1(M)$ such that, for all i, k,

$$v_{ik} \le w_i \quad \text{and} \quad \|w_i\|_{L^1} \le \|u\|_{L^1}.$$

Then $w_i \xrightarrow{L^1} u$ as $i \to \infty$.

PROOF. All the hypotheses can be displayed in schematic form in the following diagram:

$$\begin{array}{ccc} v_{ik} & \le & w_i \\ \downarrow^{L^1} & & \le_{L^1} \\ u_k & \xrightarrow{L^1} & u \end{array}$$

where all notation are self-explanatory.

Given $\varepsilon > 0$, we have, for large enough k,

$$\|u - u_k\|_{L^1} \leq \varepsilon.$$

Fix one of such indices k. Then, for large enough i, we have

$$\|u_k - v_{ik}\|_{L^1} \leq \varepsilon$$

so that

$$\|u - v_{ik}\|_{L^1} \leq 2\varepsilon.$$

Let us show that, for such i,

$$\|u - w_i\|_{L^1} \leq 4\varepsilon, \tag{7.69}$$

which will settle the claim.

By condition $v_{ik} \leq w_i$, we have

$$u - w_i \leq u - v_{ik}$$

whence

$$(u - w_i)_+ \leq (u - v_{ik})_+$$

and, hence,

$$\| (u - w_i)_+ \|_{L^1} \leq 2\varepsilon.$$

Next, write

$$\begin{aligned}\int_M (u - w_i)\, d\mu &= \int_{\{u \geq w_i\}} (u - w_i)\, d\mu + \int_{\{u < w_i\}} (u - w_i)\, d\mu \\ &= \| (u - w_i)_+ \|_{L^1} - \| (u - w_i)_- \|_{L^1}.\end{aligned}$$

By hypothesis,

$$\int_M (u - w_i)\, d\mu = \|u\|_{L^1} - \|w_i\|_{L^1} \geq 0,$$

whence it follows that

$$\| (u - w_i)_- \|_{L^1} \leq \| (u - w_i)_+ \|_{L^1} \leq 2\varepsilon,$$

and which proves (7.69). $\square$

THEOREM 7.19. *For any $f \in L^1(M)$ and $t > 0$, we have $P_t f \in L^1(M)$ and*

$$\|P_t f\|_{L^1} \leq \|f\|_{L^1}. \tag{7.70}$$

Moreover, $P_t f(x)$ is C^∞ smooth in $\mathbb{R}_+ \times M$, satisfies in $\mathbb{R}_+ \times M$ the heat equation, and

$$P_t f \overset{L^1(M)}{\longrightarrow} f \ \text{ as } t \to 0. \tag{7.71}$$

PROOF. Without loss of generality, we can assume $f \geq 0$ (otherwise, use $f = f_+ - f_-$). Note that by Theorem 7.15, $P_t f(x)$ is a measurable function of x and of (t, x). Using (7.50), we obtain

$$\begin{aligned} \int_M P_t f d\mu &= \int_M \left(\int_M p_t(x,y) f(y) d\mu(y) \right) d\mu(x) \\ &= \int_M \left(\int_M p_t(x,y) d\mu(x) \right) f(y) d\mu(y) \\ &\leq \int_M f(y) d\mu(y) = \|f\|_{L^1}, \end{aligned}$$

which implies $P_t f \in L^1(M)$ and the estimate (7.70). Integrating the latter in dt, we obtain that $P_t f(x) \in L^1_{loc}(\mathbb{R}_+ \times M)$. By Theorem 7.15, we conclude that $P_t f \in C^\infty(\mathbb{R}_+ \times M)$ and $P_t f$ satisfies the heat equation.

Let us now prove the initial condition (7.71). Let $\{\Omega_k\}$ be a compact exhaustion sequence in M. Set

$$f_k = \min(f, k) 1_{\Omega_k}$$

and observe that $f_k \in L^2(\Omega_k)$, which implies by Theorem 4.9 that

$$P_t^{\Omega_k} f_k \overset{L^2(\Omega_k)}{\longrightarrow} f_k \text{ as } t \to 0.$$

Since $\mu(\Omega_k) < \infty$ and, hence, $L^2(\Omega_k) \hookrightarrow L^1(\Omega_k)$, we obtain also

$$P_t^{\Omega_k} f_k \overset{L^1(\Omega_k)}{\longrightarrow} f_k \text{ as } t \to 0.$$

Extending function $P_t^{\Omega_k} f_k(x)$ to M by setting it to 0 outside Ω_k, we obtain

$$P_t^{\Omega_k} f_k \overset{L^1(M)}{\longrightarrow} f_k \text{ as } t \to 0.$$

Obviously, $f_k \overset{L^1(M)}{\longrightarrow} f$ as $k \to \infty$, so that we have the diagram

$$\begin{array}{ccc} P_t^{\Omega_k} f_k & \leq & P_t f \\ \downarrow_{L^1} & & \leq_{L^1} \\ f_k & \overset{L^1}{\longrightarrow} & f \end{array}$$

and conclude by Lemma 7.18 that $P_t f \overset{L^1}{\longrightarrow} f$. □

Exercises.

7.25. Prove that, for any two non-negative measurable functions f and g on M,

$$(P_t(fg))^2 \leq P_t(f^2) P_t(g^2).$$

Prove that

$$(P_t f)^2 \leq P_t(f^2).$$

7.26. Prove that the following dichotomy takes place: either $\sup P_t 1 = 1$ for all $t > 0$ or there is $c > 0$ such that

$$\sup P_t 1 \leq \exp(-ct)$$

for all large enough t.

7.27. Prove that, for any fixed $t > 0$ and $x \in M$, the heat kernel $p_t(x,y)$ is a bounded function of $y \in M$.

7.28. Let $\mathcal{F}$ be a set of functions on M such that $f \in \mathcal{F}$ implies $|f| \in \mathcal{F}$ and $P_t f \in \mathcal{F}$.

(a) Prove that the semigroup identity

$$P_t P_s = P_{t+s}$$

holds in $\mathcal{F}$.

(b) Assume in addition that $\mathcal{F}$ is a normed linear space such that, for any $f \in \mathcal{F}$,

$$\|P_t f\|_{\mathcal{F}} \leq \|f\|_{\mathcal{F}}$$

and

$$\|P_t f - f\|_{\mathcal{F}} \to 0 \text{ as } t \to 0.$$

Prove that, for any $s > 0$,

$$\|P_t f - P_s f\|_{\mathcal{F}} \to 0 \text{ as } t \to s.$$

7.29. Let $f \in W^1_{loc}(M)$ be a non-negative function such that $\Delta_\mu f \leq 0$ in the distributional sense. Prove that $P_t f \leq f$ for all $t > 0$.

7.30. Let $f \in L^1_{loc}(M)$ be a non-negative function such that $P_t f \leq f$ for all $t > 0$.

(a) Prove that $P_t f(x)$ is decreasing in t for any $x \in M$.

(b) Prove that $P_t f$ is a smooth solution to the heat equation in $\mathbb{R}_+ \times M$.

(c) Prove that $P_t f \xrightarrow{L^1_{loc}} f$ as $t \to 0$.

(d) Prove that $\Delta_\mu f \leq 0$ in the distributional sense.

7.31. Under the conditions of Exercise 7.30, assume in addition that $\Delta_\mu f = 0$ in an open set $U \subset M$. Prove that the function

$$u(t,x) = \begin{cases} P_t f(x), & t > 0, \\ f(x), & t \leq 0, \end{cases}$$

is C^∞ smooth in $\mathbb{R} \times U$ and solves the heat equation in $\mathbb{R} \times U$.

REMARK. The assumption $P_t f \leq f$ simplifies the proof but is not essential – cf. Exercise 9.8(c).

7.32. Let $f \in L^1_{loc}(M)$ be a non-negative function such that $P_t f \in L^1_{loc}(M)$ for all $t \in (0,T)$ (where $T > 0$) and $P_t f \geq f$ for all $t \in (0,T)$.

(a) Prove that $P_t f(x)$ is increasing in t for any $x \in M$.

(b) Prove that $P_t f$ is a smooth solution to the heat equation in $(0,T) \times M$.

(c) Prove that $P_t f \xrightarrow{L^1_{loc}} f$ as $t \to 0$.

(d) Prove that $\Delta_\mu f \geq 0$ in the distributional sense.

(e) Show that the function $f(x) = \exp\left(\frac{|x|^2}{4T}\right)$ in $\mathbb{R}^n$ satisfies the above conditions.

7.33. Let $f \in L^\infty(M)$. Prove that $P_t f \in L^\infty(M)$ for any $t > 0$,

$$\|P_t f\|_{L^\infty} \leq \|f\|_{L^\infty},$$

and the function $u(t,x) = P_t f(x)$ is C^∞ smooth in $\mathbb{R}_+ \times M$ and satisfies the heat equation.

7.34. Let $\Omega \subset M$ be an open set, and consider the function

$$f(x) = 1_\Omega(x) := \begin{cases} 1, & x \in \Omega \\ 0, & x \in M \setminus \Omega. \end{cases}$$

Prove that

$$\lim_{t \to 0} P_t f(x) = f(x) \text{ for all } x \in M \setminus \partial\Omega, \tag{7.72}$$

and the convergence is locally uniform in x.

7.35. Prove that if a function $f \in L^{\infty}(M)$ is continuous at a point $x \in M$ then
$$P_t f(x) \to f(x) \text{ as } t \to 0. \tag{7.73}$$

7.36. Let $1 \le r \le \infty$ and $f \in L^r(M)$.

(a) Prove that $P_t f \in L^r(M)$ for any $t > 0$, and
$$\|P_t f\|_{L^r} \le \|f\|_{L^r}. \tag{7.74}$$

(b) Prove that $P_t f(x)$ is a smooth function of $(t, x) \in \mathbb{R}_+ \times M$ and satisfies the heat equation.

7.37. Prove that if $1 < r < \infty$ and $f \in L^r(M)$ then $P_t f \xrightarrow{L^r} f$ as $t \to 0$.

7.38. Assume that
$$F(t) := \sup_{x \in M} p_t(x, x) < \infty.$$
Prove that, for all $1 \le r < s \le +\infty$, $f \in L^r(M)$ implies $P_t f \in L^s(M)$ and
$$\|P_t f\|_{L^s} \le F(t)^{1/r - 1/s} \|f\|_{L^r}. \tag{7.75}$$

7.5. Smoothness of the heat kernel in t, x, y

In this section, we prove the smoothness of the heat kernel $p_t(x, y)$ jointly in t, x, y.

THEOREM 7.20. *The heat kernel $p_t(x, y)$ is C^{∞}-smooth jointly in $t > 0$ and $x, y \in M$. Furthermore, for any chart $U \subset M$ and for any partial differential operator D^{α} in $t \in \mathbb{R}_+$ and $x \in U$,*
$$D^{\alpha} p_t(x, \cdot) \in L^2(M) \tag{7.76}$$
and, for any $f \in L^2(M)$,
$$D^{\alpha} P_t f(x) = \int_M D^{\alpha} p_t(x, y) f(y)\, d\mu(y). \tag{7.77}$$

PROOF. Fix a relatively compact chart $U \subset M$, and let X be a closed ball in U. We will assume that x varies in X and denote by ∂^{α} partial derivatives in x in chart U. Recall that, by Theorem 7.13, $\partial^{\alpha} p_t(x, y)$ is a smooth function in t, x for any fixed y.

Let us first prove the following claim, which constitutes the main part of the proof.

CLAIM. *Function $\partial^{\alpha} p_t(x, y)$ is continuous in x locally uniformly in t, y.*

By Theorem 7.6, we have, for any $f \in L^2(M)$ and any multiindex α,
$$\sup_X |\partial^{\alpha} P_t f| \le F_{X, |\alpha|}(t) \|f\|_{L^2} \tag{7.78}$$
where $F_{X,k}(t)$ is a locally bounded function of $t \in \mathbb{R}_+$. Since (7.78) can be also applied to the derivatives $\partial_j \partial^{\alpha}$, it follows that, for all $x, x' \in X$,
$$\left|\partial^{\alpha} P_t f(x) - \partial^{\alpha} P_t f(x')\right| \le F_{X, |\alpha|+1}(t) \|f\|_{L^2} \left|x - x'\right|, \tag{7.79}$$
where $|x - x'|$ is the Euclidean distance computed in the chart U.

For all $t, s > 0$, $y \in M$, and $x, x' \in X$, we have by (7.56)

$$p_{t+s}(x, y) - p_{t+s}(x', y) = P_t p_{s,y}(x) - P_t p_{s,y}(x'),$$

which implies by (7.79).

$$\left|\partial^\alpha p_{t+s}(x, y) - \partial^\alpha p_{t+s}(x', y)\right| \le F_{X,|\alpha|+1}(t) \|p_{s,y}\|_{L^2} |x - x'|.$$

Restricting y to a compact set $Y \subset M$ and applying the inequality (7.25) of Theorem 7.7 to estimate $\|p_{s,y}\|_{L^2}$, we obtain

$$\left|\partial^\alpha p_{t+s}(x, y) - \partial^\alpha p_{t+s}(x', y)\right| \le F_{X,|\alpha|+1}(t) F_Y(s) |x - x'|, \tag{7.80}$$

for all $x, x' \in X$ and $y \in Y$. Hence, $\partial^\alpha p_{t+s}(x, y)$ is continuous in x locally uniformly in t, y. Since $s > 0$ is arbitrary, the same holds for $\partial^\alpha p_t(x, y)$, which was claimed.

By Theorem 7.13, $p_t(x, y)$ is a continuous function in t, y for a fixed x. By the above Claim, $p_t(x, y)$ is continuous in x locally uniformly in t, y, which implies that $p_t(x, y)$ is continuous jointly in t, x, y.

Denote by Δ_x the operator Δ_μ with respect to the variable x. It follows from the above Claim that $\Delta_x p_t(x, y)$ is continuous in x locally uniformly in t, y. Since by Theorem 7.13

$$\Delta_x p_t(x, y) = \frac{\partial}{\partial t} p_t(x, y) = \Delta_y p_t(x, y) \tag{7.81}$$

and $\Delta_y p_t(x, y)$ is continuous in t, y, we conclude that all three functions in (7.81) are continuous jointly in t, x, y.

Now consider the manifold $N = M \times M$ with the product metric tensor and the product measure $d\nu = d\mu\, d\mu$. Since $p_t(x, y)$ and its derivatives (7.81) are continuous functions on $\mathbb{R}_+ \times N$, all these derivatives are also the distributional derivatives of $p_t(x, y)$ on $\mathbb{R}_+ \times N$. Hence, we have the following equation

$$\frac{\partial}{\partial t} p_t = \frac{1}{2}(\Delta_x + \Delta_y) p_t,$$

which is satisfied in the distributional sense in $\mathbb{R}_+ \times N$. Since

$$\Delta_x + \Delta_y = \Delta_\nu,$$

where Δ_ν is the Laplace operator on (N, ν), the function $p_t(x, y)$ satisfies the heat equation on $\mathbb{R}_+ \times N$ (up to the time change $t \mapsto 2t$). By Theorem 7.4, we conclude that $p_t(x, y)$ is C^∞ smooth on $\mathbb{R}_+ \times N$, which was to be proved.

Let D^α be any partial derivative in t and x. By the previous part of the proof, $D^\alpha p_t(x, y)$ is a smooth function in t, x, y, which implies that, for any $f \in C_0^\infty(M)$

$$D^\alpha \int_M p_t(x, y) f(y)\, d\mu(y) = \int_M D^\alpha p_t(x, y) f(y)\, d\mu(y), \tag{7.82}$$

because the the function $p_t(x, y) f(y)$ is C^∞-smooth in t, x, y and the range of t, x, y can be restricted to a compact set.

Observe that the estimate (7.78) holds also for the derivative D^α in place of ∂^α, because by (7.81) the time derivative operator ∂_t on $p_t(x,y)$ can be replaced by Δ_x and, hence, by a combination of operators ∂^α. Then (7.78) implies that, for fixed t,x, the left hand side of (7.82) is a bounded linear functional on $f\in L^2(M)$. By the Riesz representation theorem, there exists a function $h_{t,x}\in L^2(M)$ such that this functional has the form $(h_{t,x},f)$. It follows from (7.82) that, for all $f\in C_0^\infty(M)$,

$$\int_M D^\alpha p_t(x,y)\,f(y)\,d\mu(y)=(h_{t,x},f)\,.$$

By Lemma 3.13, we conclude that

$$D^\alpha p_t(x,\cdot)=h_{t,x}\ \text{ a.e.},$$

whence

$$D^\alpha p_t(x,\cdot)\in L^2(M)\,. \tag{7.83}$$

Finally, to prove the identity (7.82) for all $f\in L^2(M)$, observe that, by (7.83), the right hand side of (7.82) is also a bounded linear functional on $f\in L^2(M)$. Hence, the identity (7.82) extends by continuity from $C_0^\infty(M)$ to $L^2(M)$ (cf. Exercise 4.4), which finishes the proof. $\square$

In what follows we will give an alternative proof of Theorem 7.20, without the parabolic regularity theory. As we will see, the joint smoothness of the heat kernel in t,x,y follows directly from the smoothness of $P_tf(x)$ for any $f\in L^2(M)$, by means of some abstract result concerning the differentiability of functions taking values in a Hilbert space.

Consider an open set $\Omega\subset\mathbb{R}^n$, a Hilbert space $\mathcal{H}$, and a function $h:\Omega\to\mathcal{H}$. Denote by $(\cdot,\cdot)$ the inner product in $\mathcal{H}$. We say that the function h is *weakly* C^k if, for any $\varphi\in\mathcal{H}$, the numerical function

$$x\mapsto(h(x),\varphi)$$

belongs to $C^k(\Omega)$. The function h is *strongly continuous* if it is continuous with respect to the norm of $\mathcal{H}$, that is, for any $x\in\Omega$,

$$\|h(y)-h(x)\|\to0\text{ as }y\to x.$$

The Gâteaux partial derivative ∂_ih is defined by

$$\partial_ih(x)=\lim_{s\to0}\frac{h(x+se_i)-h(x)}{s},$$

where e_i is the unit vector in the direction of the coordinate x^i and the limit is understood in the norm of $\mathcal{H}$. One inductively defines the Gâteaux partial derivative $\partial^\alpha h$ for any multiindex α. We say that the function h is *strongly* C^k if all Gâteaux partial derivatives $\partial^\alpha h$ up to the order k exist and are strongly continuous.

Since the norm limit commutes with the inner product, one easily obtains that if h is strongly C^k then h is weakly C^k and

$$\partial^\alpha(h(x),\varphi)=(\partial^\alpha h(x),\varphi)\ \text{ for any }\varphi\in\mathcal{H}, \tag{7.84}$$

provided $|\alpha|\le k$. It turns out that a partial converse to this statement is true as well.

LEMMA 7.21. *For any non-negative integer k, if h is weakly C^{k+1} then h is strongly C^k. Consequently, h is weakly C^∞ if and only if h is strongly C^∞.*

PROOF. We use induction in k.

Inductive basis for $k = 0$. Fix a point $x \in \Omega$ and prove that h is strongly continuous at x. Fix also $\varphi \in \mathcal{H}$ and consider a numerical function

$$f(x) = (h(x), \varphi),$$

which, by hypothesis, belongs to $C^1(\Omega)$. Choose $\varepsilon > 0$ so small that the closed Euclidean ball $\overline{B}_\varepsilon(x)$ lies in Ω. Then, for any vector $v \in \mathbb{R}^n$ such that $|v| < \varepsilon$, the straight line segment connecting the points x and $x+v$ lies in Ω. Restricting function f to this segment and applying the mean-value theorem, we obtain

$$|f(x+v) - f(x)| \leq \sup_{\overline{B}_\varepsilon(x)} |\nabla f| \, |v|.$$

Rewrite this inequality in the form

$$\left| \left(\frac{h(x+v) - h(x)}{|v|}, \varphi \right) \right| \leq C(x, \varphi), \tag{7.85}$$

where $C(x, \varphi) := \sup_{\overline{B}_\varepsilon(x)} |\nabla f|$, and consider $\frac{h(x+v)-h(x)}{|v|}$ as a family of vectors in $\mathcal{H}$ parametrized by v (while x is fixed). Then (7.85) means that this family is *weakly* bounded. By the principle of uniform boundedness, any weakly bounded family in a Hilbert space is norm bounded, that is, there is a constant $C = C(x)$ such that

$$\left\| \frac{h(x+v) - h(x)}{|v|} \right\| \leq C(x), \tag{7.86}$$

for all values of the parameter v (that is, $|v| < \varepsilon$ and $v \neq 0$). Obviously, (7.86) implies that h is strongly continuous at x.

Inductive step from $k-1$ *to* k. We assume here $k \geq 1$. Then, for any $\varphi \in \mathcal{H}$, the function $(h(x), \varphi)$ belongs to $C^1(\Omega)$, and consider its partial derivative $\partial_i (h(x), \varphi)$ at a fixed point $x \in \Omega$ as a linear functional of $\varphi \in \mathcal{H}$. This functional is bounded because by (7.86)

$$\left| \left(\frac{h(x + se_i) - h(x)}{s}, \varphi \right) \right| \leq C(x) \|\varphi\|$$

and, hence,

$$|\partial_i (h(x), \varphi)| \leq C(x) \|\varphi\|.$$

By the Riesz representation theorem, there exists a unique vector $h_i = h_i(x) \in \mathcal{H}$ such that

$$\partial_i (h(x), \varphi) = (h_i(x), \varphi) \text{ for all } \varphi \in \mathcal{H}. \tag{7.87}$$

The function $h_i(x)$ is, hence, a *weak* derivative of $h(x)$. The condition that $(h(x), \varphi)$ belongs to $C^{k+1}(\Omega)$ implies that $(h_i(x), \varphi)$ belongs to $C^k(\Omega)$, that is, h_i is weakly C^k. By the inductive hypothesis, we conclude that h_i is strongly C^{k-1}.

To finish the proof, it suffices to show that the Gâteaux derivative $\partial_i h$ exists and is equal to h_i, which will imply that h is strongly C^k. We will verify this for the index $i = n$; for $i < n$, it is done similarly. Consider a piecewise-smooth path $\gamma : [0, T] \to \Omega$ such that $\gamma(0) = x_0$ and $\gamma(T) = x$, and show that

$$\int_0^T h_i(\gamma(t)) \dot{\gamma}^i(t) \, dt = h(x) - h(x_0). \tag{7.88}$$

Denoting the integral in (7.88) by I and using (7.87) and the fundamental theorem of calculus, we obtain, for any $\varphi \in \mathcal{H}$,

$$\begin{aligned}(I,\varphi) &= \int_0^T (h_i(\gamma(t)),\varphi)\dot{\gamma}^i(t)\,dt \\ &= \int_0^T \partial_i (h(\cdot),\varphi)|_{\gamma(t)}\dot{\gamma}^i(t)\,dt \\ &= \int_0^T \frac{d}{dt}(h(\gamma(t)),\varphi)\,dt \\ &= (h(x),\varphi)-(h(x_0),\varphi),\end{aligned}$$

whence (7.88) follows.

Now fix a point $x \in \Omega$ and choose $\varepsilon > 0$ so that the cube $(x-\varepsilon, x-\varepsilon)^n$ lies in Ω. For simplicity of notation, assume that the origin 0 of $\mathbb{R}^n$ is contained in this cube, and consider the polygonal path γ connecting 0 and x inside the cube, whose consecutive vertices are as follows:

$$(0,0,...,0,0)\,,\ \left(x^1,0,...,0,0\right),\ ...\ ,\ \left(x^1,x^2,...,x^{n-1},0\right),\ \left(x^1,x^2,...,x^{n-1},x^n\right).$$

By (7.88), we have

$$h(x) = h(0) + \int_0^T h_i(\gamma(t))\,\dot{\gamma}^i(t)\,dt. \tag{7.89}$$

The integral in (7.89) splits into the sum of n integral over the legs of γ, and only the last one depends on x^n. Hence, to differentiate (7.89) in x^n, it suffices to differentiate the integral over the last leg of γ. Parametrizing this leg by

$$\gamma(t) = \left(x^1,x^2,...,x^{n-1},t\right),\ \ 0 \le t \le x^n,$$

we obtain

$$\partial_n h(x) = \frac{\partial}{\partial x^n}\int_0^{x^n} h_i(\gamma(t))\,\dot{\gamma}^i(t)\,dt = \frac{\partial}{\partial x^n}\int_0^{x^n} h_n\left(x^1,...,x^{n-1},t\right)dt = h_n(x),$$

which was to be proved. □

SECOND PROOF OF THEOREM 7.20. Let Ω be a chart on the manifold $\mathbb{R}_+ \times M$, and consider $p_{t,x}$ as a mapping $\Omega \to L^2(M)$. By Theorem 7.10, for any $f \in L^2(M)$, the function $P_t f(x) = (p_{t,x}, f)_{L^2}$ is C^∞-smooth in t,x. Hence, the mapping $p_{t,x}$ is weakly C^∞. By Lemma 7.21, the mapping $p_{t,x}$ is strongly C^∞. Let Ω' be another chart on $\mathbb{R}_+ \times M$ which will be the range of the variables s,y. Since $p_{s,y}$ is also strongly C^∞ as a mapping from $\Omega' \to L^2(M)$, we obtain by (7.56)

$$p_{t+s}(x,y) = (p_{t,x}, p_{s,y})_{L^2} = C^\infty\left(\Omega \times \Omega'\right),$$

which implies that $p_t(x,y)$ is C^∞-smooth in t,x,y.

Let D^α be a partial differential operator in variables $(t,x) \in \Omega$. By (7.84), we have, for any $f \in L^2(M)$,

$$D^\alpha(p_{t,x}, f) = (D^\alpha p_{t,x}, f), \tag{7.90}$$

where $D^\alpha p_{t,x}$ is understood as the Gâteaux derivative. Since the left hand sides of (7.82) and (7.90) coincide, so do the right hand sides, whence we obtain by Lemma 3.13

$$D^\alpha p_t(x,\cdot) = D^\alpha p_{t,x} \text{ a.e.}$$

Consequently, $D^\alpha p_t(x,\cdot) \in L^2(M)$ and, for any $f \in L^2(M)$,

$$D^\alpha \int_M p_t(x,y) f(y)\,d\mu = D^\alpha(p_{t,x}, f) = (D^\alpha p_{t,x}, f) = \int_M D^\alpha p_t(x,y) f(y)\,d\mu,$$

which finishes the proof. □

Exercises.

7.39. Let $f : M \to [-\infty, +\infty]$ be a measurable function on M.

(a) Prove that, if $f \geq 0$ then the function

$$P_t f(x) := \int_M p_t(x, y) f(y) \, d\mu(y) \tag{7.91}$$

is measurable on M for any $t > 0$.

(b) Prove that if f is signed and the integral (7.91) converges for almost all x then $P_t f(x)$ is measurable on M.

(c) Prove the identity

$$P_{t+s} f = P_t(P_s f)$$

for any non-negative measurable function f.

7.40. For any open set $\Omega \subset M$, denote by $p_t^\Omega(x, y)$ the heat kernel of the manifold $(\Omega, \mathrm{g}, \mu)$.

(a) Prove that $p_t^\Omega(x, y) \leq p_t(x, y)$ for all $x, y \in \Omega$ and $t > 0$.

(b) Let $\{\Omega_i\}$ be an exhaustion sequence in M. Prove that

$$p_t^{\Omega_i}(x, y) \overset{C^\infty(\mathbb{R}_+ \times M \times M)}{\longrightarrow} p_t(x, y) \quad \text{as } i \to \infty.$$

(c) Prove that, for any non-negative measurable function $f(x)$,

$$P_t^{\Omega_i} f(x) \to P_t f(x) \quad \text{as } i \to \infty,$$

for any fixed $t > 0$ and $x \in M$.

(d) Prove that if $f \in C_b(M)$ then

$$P_t^{\Omega_i} f(x) \overset{C^\infty(\mathbb{R}_+ \times M)}{\longrightarrow} P_t f(x) \quad \text{as } i \to \infty.$$

7.41. Let (X, g_X, μ_X) and (Y, g_Y, μ_Y) be two weighted manifold and (M, g, μ) be their direct product (see Section 3.8). Denote by p_t^X and p_t^Y the heat kernels on X and Y, respectively. Prove that the heat kernel p_t on M satisfies the identity

$$p_t((x, y), (x', y')) = p_t^X(x, x') \, p_t^Y(y, y'), \tag{7.92}$$

for all $t > 0$, $x, x' \in X$, $y, y' \in Y$ (note that (x, y) and (x', y') are points on M).

7.42. For any $t > 0$, consider the quadratic form in $L^2(M)$, defined by

$$\mathcal{E}_t(f) = \left(\frac{f - P_t f}{t}, f \right)_{L^2}$$

(cf. Exercise 4.38). Prove that if the heat kernel is stochastically complete, that is, for all $x \in M$ and $t > 0$,

$$\int_M p_t(x, y) \, d\mu(y) = 1, \tag{7.93}$$

then the following identity holds:

$$\mathcal{E}_t(f) = \frac{1}{2t} \int_M \int_M (f(x) - f(y))^2 p_t(x, y) d\mu(y) d\mu(x), \tag{7.94}$$

for all $t > 0$ and $f \in L^2(M)$.

7.43. Prove that, for any real $k > 0$ and for any $f \in L^2(M)$,

$$(\mathcal{L} + \mathrm{id})^{-k} f(x) = \int_0^\infty \frac{t^{k-1}}{\Gamma(k)} e^{-t} P_t f(x) \, dt, \tag{7.95}$$

for almost all $x \in M$, where Γ is the gamma function.

HINT. Use Exercise 5.11.

7.44. Assume that the heat kernel satisfies the following condition

$$p_t(x,x) \le ct^{-\gamma} \text{ for all } x \in M \text{ and } 0 < t < 1. \tag{7.96}$$

where $\gamma, c > 0$. Fix a real number $k > \gamma/2$.

(a) Prove that, for any $f \in L^2(M)$, the function $(\mathcal{L} + \mathrm{id})^{-k} f$ is continuous and

$$\sup_M \left|(\mathcal{L} + \mathrm{id})^{-k} f\right| \le C\|f\|_{L^2}, \tag{7.97}$$

where $C = C(c, \gamma, k)$.

(b) Prove that, for any $u \in \operatorname{dom} \mathcal{L}^k$, we have $u \in C(M)$ and

$$\sup_M |u| \le C\left(\|u\|_{L^2} + \|\mathcal{L}^k u\|_{L^2}\right). \tag{7.98}$$

7.45. Prove that if (7.98) holds for all $u \in \operatorname{dom} \mathcal{L}^k$ with some $k > 0$ then the heat kernel satisfies the estimate (7.96) with $\gamma = 2k$.

7.46. The purpose of this question is to give an alternative proof of Theorem 6.1 (Sobolev embedding theorem).

(a) Prove that if $u \in W^k(\mathbb{R}^n)$ where k is a positive integer then $u \in \operatorname{dom} \mathcal{L}^{k/2}$, where $\mathcal{L}$ is the Dirichlet Laplace operator in $\mathbb{R}^n$. Prove also that, for any $u \in W^k(\mathbb{R}^n)$,

$$\|(\mathcal{L} + \mathrm{id})^{k/2} u\|_{L^2} \le C\|u\|_{W^k},$$

where C is a constant depending only on n and k.

(b) Prove that if $u \in W^k(\mathbb{R}^n)$ where k is an integer such that $k > n/2$ then $u \in C(\mathbb{R}^n)$ and

$$\sup_{\mathbb{R}^n} |u| \le C\|u\|_{W^k}.$$

(c) Prove that if $k > m + n/2$ where m is a positive integer then $u \in W^k(\mathbb{R}^n)$ implies $u \in C^m(\mathbb{R}^n)$ and

$$\|u\|_{C^m(\mathbb{R}^n)} \le C\|u\|_{W^k(\mathbb{R}^n)}.$$

(d) Prove that if Ω is an open subset of $\mathbb{R}^n$ and k and m are non-negative integers such that $k > m + n/2$ then $u \in W^k_{loc}(\Omega)$ implies $u \in C^m(\Omega)$. Moreover, for any open sets $\Omega' \Subset \Omega'' \Subset \Omega$,

$$\|u\|_{C^m(\Omega')} \le C\|u\|_{W^k(\Omega'')},$$

with a constant C depending on $\Omega', \Omega'', k, m, n$.

HINT. Use Exercise 4.25 for part (a) and Exercise 7.44 for part (b)

7.47. (*Compact embedding theorems*)

(a) Assume that $\mu(M) < \infty$ and

$$\sup_{x \in M} p_t(x,x) < \infty \text{ for all } t > 0. \tag{7.99}$$

Prove that the identical embedding $W_0^1(M) \hookrightarrow L^2(M)$ is a compact operator.

(b) Prove that, on any weighted manifold M and for any non-empty relatively open compact set $\Omega \subset M$, the identical embedding $W_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is a compact operator (cf. Theorem 6.3 and Corollary 10.21).

HINT. Use for part (a) the weak compactness of bounded sets in L^2 and Exercises 7.36, 4.40.

7.48. Let I be an open interval in $\mathbb{R}$ and $\mathcal{H}$ be a Hilbert space. Prove that if a mapping $h : I \to \mathcal{H}$ is weakly differentiable then h is strongly continuous.

7.6. Notes

One of the main results of Chapter 7 is the existence of the heat kernel satisfying various nice properties (Theorems 7.7, 7.13, and 7.20). The classical approach to construction of the heat kernel on a Riemannian manifold, which originated from [**275**], [**276**], uses a *parametrix* of the heat equation, that is, a smooth function which satisfies the necessary conditions in some asymptotic sense. The parametrix itself is constructed using as a model the heat kernel in $\mathbb{R}^n$. A detailed account of this approach can be found in many sources, see for example [**36**], [**37**], [**51**], [**58**], [**317**], [**326**]. This approach has certain advantages as it gives immediately the short time asymptotics of the heat kernel and requires less of abstract functional analysis.

On the other hand, the theory of elliptic and parabolic equations with singular (measurable) coefficients, developed by de Giorgi [**103**], Nash [**292**], Moser [**279**], [**280**], and Aronson [**9**], has demonstrated that the fundamental solutions for such equations can be constructed using certain a priori estimates, whereas the parametrix method is not available. The method of construction of the heat kernel via a priori estimates of solutions has been successfully applied in analysis on more general spaces – metric measure spaces with energy forms.

In our approach, the key a priori estimate (7.18) of the heat semigroup is given by Theorem 7.6. The proof of (7.18) uses the elliptic regularity theory and the Sobolev embedding theorem. As soon as one has (7.18), the existence of the heat kernel follows as in Theorem 7.7. This approach gives at the same token the smoothness of $P_t f(x)$ in variable x. There are other proofs of (7.18) based only on the local *isoperimetric* properties of manifolds, which can be used in more general settings (cf. Corollary 15.7 in Chapter 15).

The heat kernel obtained as above is not yet symmetric. Its symmetrization (and regularization) is done in Theorem 7.13 using a general method of J.-A. Yan [**360**]. The smoothness of the heat kernel $p_t(x,y)$ is proved in three installments: first, smoothness of $P_t f(x)$ in x (Theorem 7.6), then smoothness of $p_t(x,y)$ in (t,x) (Theorems 7.10 and 7.13) and, finally, smoothness of $p_t(x,y)$ in (t,x,y) (Theorem 7.20; the second proof of this theorem uses the approach from [**96**, Theorem 5.2.1] and [**92**, Corollary 1.42]).

Other methods for construction of the heat kernel are outlined in Section 16.4.

An somewhat similar approach for construction of the heat kernel via the smoothness of $P_t f$ was used by Strichartz [**330**], although without quantitative estimates of $P_t f$. That method was also briefly outlined in [**96**].

After the heat kernel has been constructed, the heat semigroup $P_t f$ can be extended from L^2 to other function classes as an integral operator. We consider here only extensions to L^1 and C_b. A good account of the properties of the heat semigroup in spaces L^q can be found in [**330**].

CHAPTER 8

Positive solutions

This Chapter can be regarded as a continuation of Chapter 5. However, the treatment of the Markovian properties is now different because of the use of the smoothness of solutions.

8.1. The minimality of the heat semigroup

We say that a smooth function $u(t,x)$ is a *supersolution* of the heat equation if it satisfies the inequality

$$\frac{\partial u}{\partial t} \geq \Delta_\mu u$$

in a specified domain. The following statement can be considered as an extension of Corollary 5.17.

THEOREM 8.1. *Set $I = (0,T)$ where $T \in (0,+\infty]$. Let $u(t,x)$ be a non-negative smooth supersolution to the heat equation in $I \times M$ such that*

$$u(t,\cdot) \overset{L^2_{loc}}{\longrightarrow} f \quad \text{as } t \to 0, \tag{8.1}$$

for some $f \in L^2_{loc}(M)$. Then $P_t f(x)$ is also a smooth solution to the heat equation in $I \times M$, satisfying the initial condition (8.1), and

$$u(t,x) \geq P_t f(x), \tag{8.2}$$

for all $t \in I$ and $x \in M$.

PROOF. Note that $f \geq 0$. Let (8.2) be already proved. Then $P_t f(x)$ is locally bounded and, by Theorem 7.15, it is a smooth solution to the heat equation. Let us verify the initial condition

$$P_t f \overset{L^2_{loc}}{\to} f \quad \text{as } t \to 0. \tag{8.3}$$

Indeed, for any relatively compact open set $\Omega \subset M$, we have

$$P_t(f 1_\Omega) \leq P_t f \leq u(t,\cdot).$$

Since both functions $P_t(f1_\Omega)$ and $u(t,\cdot)$ converge to f in $L^2(\Omega)$ as $t \to 0$, we conclude that $P_t f \overset{L^2(\Omega)}{\to} f$, which implies (8.3).

In order to prove (8.2), we reduce the present setting to the L^2-Cauchy problem (5.55) of Corollary 5.17. Choose an open set $\Omega \Subset M$. The smoothness of u implies that $u(t,\cdot) \in W^1(\Omega)$ and the strong derivative $\frac{du}{dt}$ in $L^2(\Omega)$ obviously coincides with the classical derivative $\frac{\partial u}{\partial t}$. Hence, $u(t,\cdot)$ as a path in $W^1(\Omega)$ satisfies the conditions (5.55), and we conclude by Corollary 5.17, that[1]

$$u(t,\cdot) \geq P_t^{\Omega} f. \tag{8.4}$$

Let $\{\Omega_k\}$ be a compact exhaustion sequence in M, and set $f_k = f 1_{\Omega_k}$. It follows from (8.4) that

$$u(t,\cdot) \geq P_t^{\Omega_k} f_i,$$

for all i and k. Since $f_i \in L^2(M)$, by Theorem 5.23 we obtain

$$P_t^{\Omega_k} f_i \xrightarrow{L^2} P_t f_i \text{ as } k \to \infty,$$

whence it follows that

$$u(t,\cdot) \geq P_t f_i \text{ a.e.}$$

Letting $i \to \infty$, we obtain

$$u(t,\cdot) \geq P_t f \text{ a.e.} \tag{8.5}$$

This implies $P_t f \in L^1_{loc}(I \times M)$, and we conclude by Theorem 7.15 that the function $P_t f(x)$ is smooth in t, x. Hence, (8.5) implies the pointwise estimate (8.2). □

COROLLARY 8.2. *Let $u(t,x)$ be a non-negative smooth solution to the heat equation in $I \times M$ such that*

$$u(t,\cdot) \xrightarrow{L^2_{loc}} f \text{ as } t \to 0$$

for some $f \in L^2_{loc}(M)$, and

$$u(t,x) \rightrightarrows 0 \text{ as } x \to \infty \text{ in } M, \tag{8.6}$$

where the convergence is uniform in $t \in I$. Then $u(t,x) \equiv P_t f(x)$.

The hypotheses of Corollary 8.2 are exactly those of Theorem 8.1 except for the additional condition (8.6), which leads to the identity of $u(t,x)$ and $P_t f(x)$.

PROOF. It follows from Theorem 8.1 that the function $v(t,x) := u(t,x) - P_t f(x)$ satisfies the heat equation in $I \times M$ and the initial condition $v(t,\cdot) \xrightarrow{L^2_{loc}} 0$ as $t \to 0$. Besides, (8.6) implies $v(t,x) \rightrightarrows 0$ as $x \to \infty$. Hence, by Corollary 5.20, $v \equiv 0$, which was to be proved. □

COROLLARY 8.3. *For any non-negative $f \in C_b(M)$, the function $u(t,x) = P_t f(x)$ is the minimal non-negative solution to the following Cauchy problem*

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_\mu u, & \text{in } \mathbb{R}_+ \times M, \\ u|_{t=0} = f, \end{cases} \tag{8.7}$$

where $u|_{t=0} = f$ means that $u(t,\cdot) \to f$ as $t \to 0$ locally uniformly in x.

[1] Corollary 5.17 says that (8.4) holds *almost* everywhere on M (for any $t \in I$). However, since by Theorem 7.10 $P_t^{\Omega} f$ is a smooth function, (8.4) holds, in fact, everywhere on M.

PROOF. Indeed, by Theorem 7.16, the function $P_t f(x)$ does solve (8.7), and by Theorem 8.1, $u(t,x) \geq P_t f(x)$ for any other non-negative solution u. □

Exercises.

8.1. Prove that if h is a non-negative function satisfying on M the equation

$$-\Delta_\mu h + \alpha h = 0,$$

where α is a real constant, then $P_t h \leq e^{\alpha t} h$ for all $t > 0$.

8.2. Extension of resolvent

Our goal here is to extend the resolvent $R_\alpha f$ to a larger class of functions f and to prove the properties of R_α similar to the properties of the heat semigroup P_t given by Theorems 7.15 and 8.1.

Recall that, for any $\alpha > 0$, the resolvent R_α is a bounded operator in $L^2(M)$ defined by

$$R_\alpha = (\mathcal{L} + \alpha \,\mathrm{id})^{-1},$$

where $\mathcal{L}$ is the Dirichlet Laplace operator (cf. Section 4.2). For any $f \in L^2(M)$, the function $u = R_\alpha f$ satisfies the equation

$$-\Delta_\mu u + \alpha u = f \tag{8.8}$$

in the distributional sense.

As an operator in L^2, the resolvent is related to the heat semigroup by the identity

$$(R_\alpha f, g)_{L^2} = \int_0^\infty e^{-\alpha t} (P_t f, g)_{L^2} \, dt, \tag{8.9}$$

for all $f, g \in L^2(M)$ (cf. Theorem 4.5 and Lemma 5.10).

Now we extend $R_\alpha f$ to a more general class of functions f by setting

$$R_\alpha f(x) := \int_0^\infty e^{-\alpha t} P_t f(x) \, dt = \int_0^\infty \int_M e^{-\alpha t} p_t(x,y) f(y) \, d\mu(y) \, dt, \tag{8.10}$$

whenever the right hand side of (8.10) makes sense. Note that the function $R_\alpha f(x)$ is defined by (8.10) pointwise rather than almost everywhere.

If f is a non-negative measurable function then the right hand side of (8.10) is always a measurable function by Fubini's theorem, although it may take value ∞. If, in addition, $f \in L^2(M)$ then substituting $R_\alpha f$ from (8.10) into (8.9), we obtain, again by Fubini's theorem, that (8.9) holds for all non-negative $g \in L^2(M)$, which implies that the new definition of $R_\alpha f$ matches the old one. For a signed $f \in L^2(M)$, the same conclusion follows using $f = f_+ - f_-$.

THEOREM 8.4. *Fix a non-negative function $f \in L^2_{loc}(M)$ and a constant $\alpha > 0$.*

(a) *If* $u \in L^2_{loc}(M)$ *is a non-negative solution to the equation*

$$-\Delta_\mu u + \alpha u = f \tag{8.11}$$

then $u \geq R_\alpha f$.

(b) *If* $R_\alpha f \in L^2_{loc}(M)$ *then the function* $u = R_\alpha f$ *satisfies the equation* (8.11).

PROOF. Let $\{\Omega_k\}$ be a compact exhaustion sequence in M and set

$$f_k = \min(k, f)\, 1_{\Omega_k}.$$

(a) It follows from (8.11) that $\Delta_\mu u \in L^2_{loc}(M)$, and we conclude by Exercise 7.9 that $u \in W^1_{loc}(M)$. Then $u \in W^1(\Omega_k)$ and, applying Corollary 5.15 in Ω_k, we obtain $u \geq R^{\Omega_k}_\alpha f$. Consequently, we have, for all indices i, k,

$$u \geq R^{\Omega_k}_\alpha f_i.$$

Since $f_i \in L^2(M)$, Theorem 5.22 yields $R^{\Omega_k}_\alpha f_i \xrightarrow{L^2} R_\alpha f_i$ as $k \to \infty$, whence it follows $u \geq R_\alpha f_i$. Passing to the limit as $i \to \infty$, we finish the proof.

(b) Since $f_k \in L^2(M)$, the function and $u_k = R_\alpha f_k$ belongs to $L^2(M)$ and satisfies the equation

$$-\Delta_\mu u_k + \alpha u_k = f_k$$

(cf. Theorem 4.5). Hence, for any $\varphi \in \mathcal{D}(M)$, we have

$$\int_M u_k(-\Delta_\mu \varphi + \alpha\varphi)\, d\mu = \int_M f_k \varphi\, d\mu. \tag{8.12}$$

Since the sequence f_k is monotone increasing and converges to f a.e., we obtain by (8.10) and the monotone convergence theorem that $u_k(x) \uparrow u(x)$ pointwise. Since u and f belong to $L^1(\operatorname{supp}\varphi)$, we can pass to the limit in (8.12) by the dominated convergence theorem and obtain that u also satisfies this identity, which is equivalent to the equation (8.11). □

REMARK 8.5. Part (a) of Theorem 8.4 can be modified as follows: if $u \in W^1_{loc}(M)$, $u \geq 0$, and u satisfies the inequality

$$-\Delta_\mu u + \alpha u \geq f,$$

then $u \geq R_\alpha f$. This is proved in the same way because the only place where the equality in (8.11) was used, is to conclude that $\Delta_\mu u \in L^2_{loc}$ and, hence, $u \in W^1_{loc}$.

COROLLARY 8.6. *If* $f \in L^\infty(M)$ *and* $\alpha > 0$ *then* $R_\alpha f$ *is bounded,*

$$\sup|R_\alpha f| \leq \alpha^{-1}\|f\|_{L^\infty}, \tag{8.13}$$

and $u = R_\alpha f$ *is a distributional solution to the equation* (8.8).

PROOF. The estimate (8.13) follows from (8.10) and $\sup|P_t f| \leq \|f\|_{L^\infty}$ (cf. Exercise 7.33). If $f \geq 0$ then the fact that $R_\alpha f$ solves (8.8) follows from Theorem 8.4(b). For a signed f, the same follows from $R_\alpha f = R_\alpha f_+ - R_\alpha f_-$. □

Some cases when one can claim the smoothness of $R_\alpha f$ are stated in the following theorem.

THEOREM 8.7. *Let $f \in C^\infty(M)$ and $\alpha > 0$.*

(i) *If $f \geq 0$ and and $R_\alpha f \in L^2_{loc}(M)$ then $R_\alpha f \in C^\infty(M)$.*
(ii) *If f is bounded then $R_\alpha f \in C^\infty(M)$.*

PROOF. Consider first a special case when $f \geq 0$ and $f \in C_0^\infty(M)$, and prove that $R_\alpha f \in C^\infty(M)$. Since the function $u = R_\alpha f$ satisfies the equation (8.8), it is tempting to conclude that $u \in C^\infty$ applying Corollary 7.3. Indeed, the latter says that every distributional solution to (8.8) is a smooth function, which means that u as a distribution is represented by a C^∞ function, that is, there is a function $\widetilde{u} \in C^\infty$ such that $u = \widetilde{u}$ a.e.. However, our aim now is to show that $u(x)$ itself is C^∞.

Recall that, by Theorem 7.13, function $P_t f$ is C^∞ smooth in $[0, +\infty) \times M$ (cf. (7.53)). Therefore, the function

$$u_l(x) = \int_0^l e^{-\alpha t} P_t f(x)\, dt$$

is C^∞ smooth on M for any finite $l > 0$ and, moreover, any partial derivative of u_l can be computed by differentiating under the integral sign. Using the properties of the heat semigroup, we obtain

$$\begin{aligned} \Delta_\mu u_l &= \int_0^l e^{-\alpha t} \Delta_\mu (P_t f)\, dt = \int_0^l e^{-\alpha t} \frac{\partial}{\partial t} (P_t f)\, dt \\ &= \left[e^{-\alpha t} P_t f\right]_0^l + \alpha \int_0^l e^{-\alpha t} P_t f\, dt, \end{aligned}$$

which implies that

$$-\Delta u_l + \alpha u_l = f_l := f - e^{-\alpha l} P_l f.$$

By the estimate (7.20) of Theorem 7.6, we have, for any compact set K that is contained in a chart, and for any positive integer m,

$$\|P_l f\|_{C^m(K)} \leq F_{K,m}(l)\, \|f\|_{L^2(M)},$$

where $F_{K,m}(l)$ is a function of l that remains bounded as $l \to \infty$. This implies

$$e^{-\alpha l} P_l f \xrightarrow{C^\infty} 0 \text{ as } l \to \infty$$

and, hence, $f_l \xrightarrow{C^\infty} f$. The sequence $\{u_l(x)\}$ increases and converges to $u(x)$ pointwise as $l \to \infty$. Since $u \in L^2(M)$, this implies by Exercise 7.13 that $u(x)$ belongs to C^∞, which finishes the proof in the special case.

(i) Let $\{\Omega_k\}$ be a compact exhaustion sequence in M and let ψ_k be a cutoff function of Ω_k in Ω_{k+1}. Set $f_k = \psi_k f$ so that $f_k \in C_0^\infty(M)$. By the special case above, the function $u_k = R_\alpha f_k$ belongs to C^∞. Since the sequence $\{f_k\}$ increases and converges pointwise to f, by (8.10) the sequence

$u_k(x)$ also increases and converges pointwise to $u = R_\alpha f$. Since $u \in L^2_{loc}$ and $f_k \xrightarrow{C^\infty} f$, we conclude by Exercise 7.13 that $u \in C^\infty$.

(*ii*) By Corollary 8.6, $R_\alpha f$ is bounded and, hence, belongs to L^2_{loc}. If $f \geq 0$ then $R_\alpha f$ is smooth by part (*i*). For a signed f, the smoothness of $R_\alpha f$ follows from the representation

$$f = e^f - \left(e^f - f\right),$$

because both function e^f and $e^f - f$ are bounded, smooth, and non-negative. $\square$

REMARK 8.8. If f is a non-negative function from $C^\infty \cap L^2$ then $R_\alpha f \in L^2$ and we obtain by Theorem 8.7 that $R_\alpha f \in C^\infty$. This was stated in Exercise 7.15, but using the definition of R_α as an operator in L^2. In other words, the statement of Exercise 7.15 means that the L^2-function $R_\alpha f$ has a smooth modification, whereas the statement of Theorem 8.7 means that the function $R_\alpha f$, which is defined pointwise by (8.10), is C^∞ itself.

Exercises.

8.2. If $u \in L^2_{loc}(M)$ is a non-negative solution to the equation

$$-\Delta_\mu u + \alpha u = f$$

where $\alpha > 0$ and $f \in L^2_{loc}(M)$, $f \geq 0$. Prove that if

$$u(x) \to 0 \text{ as } x \to \infty,$$

then $u = R_\alpha f$.

8.3. Let $u \in L^2(M)$ satisfy in M the equation

$$\Delta_\mu u + \lambda u = 0,$$

where $\lambda \in \mathbb{R}$, and

$$u(x) \to 0 \text{ as } x \to \infty.$$

Prove that $u \in W_0^1(M)$.

REMARK. Since by the equation $\Delta_\mu u \in L^2(M)$, it follows that $u \in \operatorname{dom}(\mathcal{L})$ and, hence, u satisfies the equation $\mathcal{L}u = -\lambda u$. Assuming that $u \neq 0$ we obtain that u is an eigenfunction of the Dirichlet Laplace operator.

8.3. Strong maximum/minimum principle

8.3.1. The heat equation. As before, let $(M, \mathbf{g}, \mu)$ be a weighted manifold. For an open set $\Omega \subset \mathbb{R} \times M$, define its *top boundary* $\partial_{top}\Omega$ as the set of points $(t, x) \in \partial\Omega$ for which exists an open neighborhood $U \subset M$ of x and $\varepsilon > 0$ such that the cylinder $(t - \varepsilon, t) \times U$ is contained in Ω (see Fig. 8.1).

For example, if $\Omega = (a, b) \times Q$ where $a < b$ and Q is an open subset of M, then $\partial_{top}\Omega = \{b\} \times Q$. If $M = \mathbb{R}^n$ and Ω is a Euclidean ball in $\mathbb{R}^{n+1}$ then $\partial_{top}\Omega = \emptyset$.

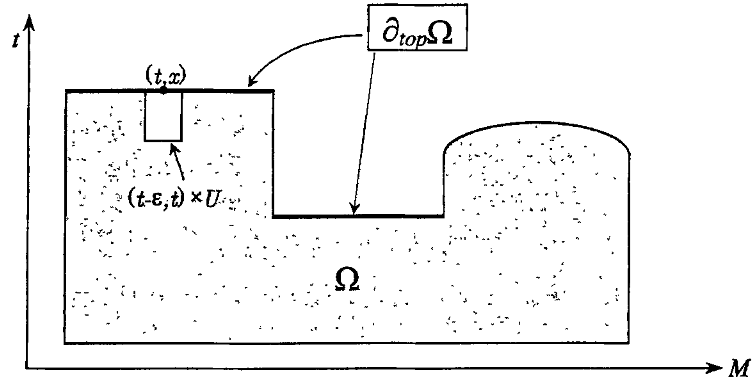

FIGURE 8.1. The top boundary of a set Ω.

DEFINITION 8.9. The parabolic boundary $\partial_p\Omega$ of an open set $\Omega \subset \mathbb{R}\times M$ is defined by

$$\partial_p\Omega := \partial\Omega \setminus \partial_{top}\Omega.$$

If Ω is non-empty and the value of t in Ω has a lower bound then $\partial_p\Omega$ is non-empty – indeed, a point $(t,x) \in \overline{\Omega}$ with the minimal value of t cannot belong to $\partial_{top}\Omega$. The importance of this notion for the heat equation is determined by the following theorem, which generalizes Lemma 1.5.

THEOREM 8.10. (Parabolic minimum principle) *Let Ω be a non-empty relatively compact open set in $\mathbb{R}\times M$, and let a function $u \in C^2\left(\overline{\Omega}\right)$ satisfy in Ω the inequality*

$$\frac{\partial u}{\partial t} \geq \Delta_\mu u. \tag{8.14}$$

Then

$$\inf_{\overline{\Omega}} u = \inf_{\partial_p\Omega} u. \tag{8.15}$$

REMARK. Any function $u \in C^2$ satisfying (8.14) is called a *supersolution* to the heat equation, while a function satisfying the opposite inequality

$$\frac{\partial u}{\partial t} \leq \Delta_\mu u,$$

is called a *subsolution*. Obviously, Theorem 8.10 can be equivalently stated as the *maximum* principle for subsolutions:

$$\sup_{\overline{\Omega}} u = \sup_{\partial_p\Omega} u.$$

In particular, if $u \leq 0$ on the parabolic boundary of Ω then $u \leq 0$ in Ω.

Let $\Omega = (0,T)\times Q$ where Q is a relatively compact open subset of M. Then the condition $u \leq 0$ on $\partial_p\Omega$ can be split into two parts:

- $u\,(t,x) \leq 0$ for all $x \in \partial Q$ and $t \in (0,T)$
- $u\,(0,x) \leq 0$ for all $x \in Q$,

which imply the following (assuming $u \in C(\overline{\Omega})$):

- $u_+(t,x) \rightrightarrows 0$ as $x \to \infty$ in Q (considering Q is a manifold itself)
- $u_+(t,x) \rightrightarrows 0$ as $t \to 0$.

Then the conclusion that $u \le 0$ in Ω follows from Corollary 5.20. Hence, for cylindrical domains Ω, Theorem 8.10 is contained in Corollary 5.20. However, we will need Theorem 8.10 also for non-cylindrical domains, and for this reason we provide below an independent proof of this theorem.

PROOF OF THEOREM 8.10. The proof is similar to that of Lemma 1.5. Assume first that u satisfies a strict inequality in Ω:

$$\frac{\partial u}{\partial t} > \Delta_\mu u. \tag{8.16}$$

Let (t_0, x_0) be a point of minimum of function u in $\overline{\Omega}$. If $(t_0,x_0) \in \partial_p \Omega$ then (8.15) is trivially satisfied. Let us show that, in fact, this point cannot be located elsewhere. Assume from the contrary that (t_0, x_0) is contained in Ω or in $\partial_{top}\Omega$. In the both cases, there exists an open neighborhood $U \subset M$ of x_0 and $\varepsilon > 0$ such that the cylinder $\Gamma := (t_0 - \varepsilon, t_0) \times U$ is contained in Ω. Since function $t \mapsto u(t, x_0)$ in $[t_0 - \varepsilon, t_0]$ takes the minimal value at (t_0, x_0), we necessarily have

$$\frac{\partial u}{\partial t}(t_0, x_0) \le 0. \tag{8.17}$$

By the choice of U, we can assume that U is a chart, with the coordinates $x^1, ..., x^n$. Let g be the matrix of the metric tensor $\mathbf{g}$ in the coordinates $x^1, ..., x^n$ and $\widetilde{g}$ be the matrix of $\mathbf{g}$ in another coordinate system $y^1, ..., y^n$ in U (yet to be defined). By (3.25), we have

$$\widetilde{g} = J^T g J,$$

where J is the Jacobi matrix defined by

$$J_i^k = \frac{\partial x^k}{\partial y^i}.$$

It is well known from linear algebra that any quadratic form can be brought to a diagonal form by a linear change of the variables. The quadratic form $\xi \mapsto g_{ij}(x_0)\xi^i\xi^j$ is positive definite and, hence, can be transform to the form $(\widetilde{\xi}^1)^2 + ... + (\widetilde{\xi}^n)^2$ by a linear change $\xi^i = A^i_j \widetilde{\xi}^j$, where A is a numerical non-singular matrix. This implies that

$$A^T g(x_0) A = \mathrm{id}.$$

Defining the new coordinates y^i by the linear equations $x^i = A^i_j y^j$, we obtain that $J(x_0) = A$ and, hence, $\widetilde{g}(x_0) = \mathrm{id}$.

So, renaming y^i back to x^i, we obtain from (3.46) that the Laplace operator Δ_μ at point x_0 has the form

$$\Delta_\mu|_{x_0} = \sum_i \frac{\partial^2}{(\partial x^i)^2} + b^i \frac{\partial}{\partial x^i},$$

for some constants b^i. Since x_0 is a point of minimum of function $x \mapsto u(t_0, x)$ in $\overline{U}$, we obtain that

$$\frac{\partial u}{\partial x^i}(t_0, x_0) = 0 \text{ and } \frac{\partial^2 u}{(\partial x^i)^2}(t_0, x_0) \geq 0.$$

This implies

$$\Delta_\mu u(t_0, x_0) \geq 0,$$

which together with (8.17) contradicts (8.16).

In the general case of a non-strict inequality in (8.14), consider for any $\varepsilon > 0$ the function $u_\varepsilon = u + \varepsilon t$, which obviously satisfies (8.16). Hence, by the previous case, we obtain

$$\inf_{\overline{\Omega}}(u + \varepsilon t) = \inf_{\partial_p \Omega}(u + \varepsilon t),$$

whence (8.15) follows by letting $\varepsilon \to 0$. □

THEOREM 8.11. (Strong parabolic minimum principle) *Let $(M, \mathbf{g}, \mu)$ be a connected weighted manifold, and $I \subset \mathbb{R}$ be an open interval. Let a non-negative function $u(t, x) \in C^2(I \times M)$ satisfy in $I \times M$ the inequality*

$$\frac{\partial u}{\partial t} \geq \Delta_\mu u.$$

If u vanishes at a point $(t', x') \in I \times M$ then u vanishes at all points $(t, x) \in I \times M$ with $t \leq t'$.

Under the conditions of Theorem 8.11, one cannot claim that $u(t, x) = 0$ for $t > t'$ – see Remark 9.22 in Section 9.3.

Note that the function u in Theorem 8.11 is a supersolution to the heat equation. Since u+const is also supersolution, one can state Theorem 8.11 as follows: if u is a bounded below supersolution then $u(t', x') = \inf u$ at some point implies $u(t, x) = \inf u$ for all $t \leq t'$ and x. Equivalently, Theorem 8.11 can be stated as the strong parabolic *maximum* principle for *subsolutions*: if u is a bounded subsolution in $I \times M$ then $u(t', x') = \sup u$ at some point implies $u(t, x) = \sup u$ for all $t \leq t'$ and x.

PROOF. The main part of the proof is contained in the following claim.

CLAIM. *Let V be a chart in M and x_0, x_1 be two points in V such that the straight line segment between x_0, x_1 is also in V. If u is a function as in the hypotheses of* Theorem 8.11 *then*

$$u(t_0, x_0) > 0 \implies u(t_1, x_1) > 0 \text{ for all } t_1 > t_0, \tag{8.18}$$

assuming that $t_0, t_1 \in I$.

For simplicity of notation, set $t_0 = 0$. By shrinking V, we can assume that V is relatively compact and its closure $\overline{V}$ is contained in a chart. Let $r > 0$ be so small that the Euclidean $2r$-neighborhood of the straight line segment $[x_0, x_1]$ is also in V. Let U be the Euclidean ball in V of radius r centered at x_0. By further reducing r, we can assume that also

$$\inf_{x \in U} u(0, x) > 0. \tag{8.19}$$

Setting $\xi = \frac{1}{t_1}(x_1 - x_0)$ we obtain that the translates $U + t\xi$ are all in V for any $t \in [0, t_1]$. Consider the following tilted cylinder (see Fig. 8.2)

$$\Gamma = \{(t, x) : 0 < t < t_1, \ x \in U + t\xi\}.$$

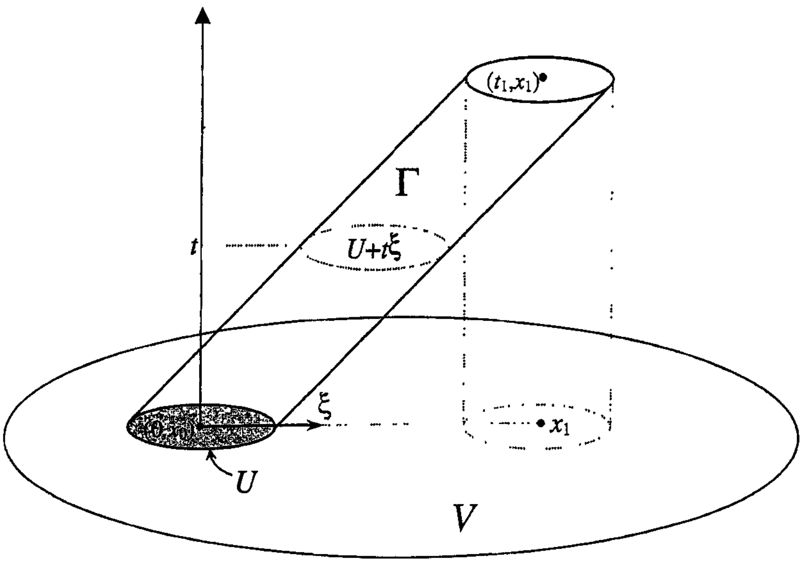

FIGURE 8.2. Tilted cylinder Γ.

This cylinder is chosen so that the center of the bottom is $(0, x_0)$ while the center of the top is (t_1, x_1). We will prove that, under condition (8.19), u is strictly positive in $\overline{\Gamma}$, except for possibly the lateral surface of Γ; in particular, this will imply that $u(t_1, x_1) > 0$.

To that end, construct a non-negative function $v \in C^2(\overline{\Gamma})$ such that

$$\frac{\partial v}{\partial t} \leq \Delta_\mu v \ \text{ in } \Gamma, \tag{8.20}$$

and

$$v = 0 \text{ on the lateral surface of } \Gamma, \text{ and } v > 0 \text{ otherwise.} \tag{8.21}$$

Assuming that such a function v exists, let us compare v and εu, where $\varepsilon > 0$ is chosen so small that

$$\inf_{x \in U} u(0, x) \geq \varepsilon \sup_{x \in U} v(0, x).$$

Due to this choice of ε, we have $u \geq \varepsilon v$ at the bottom of Γ. Since $v = 0$ on the lateral surface of Γ and u is non-negative, we conclude that inequality

$u \geq \varepsilon v$ holds on the whole parabolic boundary of Γ. The function $u - \varepsilon v$ satisfies the hypotheses of Theorem 8.10, and we conclude by this theorem that $u \geq \varepsilon v$ in $\overline{\Gamma}$. By (8.21), this implies that u is positive in $\overline{\Gamma}$ except for the lateral surface, which was to be proved.

Assume for simplicity that $x_0 = 0$ is the origin in the chart V. Let us look for v in the form

$$v(t,x) = e^{-\alpha t} f\left(|x - \xi t|^2\right),$$

where $\alpha > 0$ and function f are to be specified, and $|\cdot|$ stands for the Euclidean length of a vector. Note that $(t,x) \in \Gamma$ implies $x - \xi t \in U$, which means $|x - \xi t|^2 < r^2$. Hence, to satisfy conditions (8.21), function $f(\cdot)$ should be positive in $[0, r^2)$ and vanish at r^2. Let us impose also the conditions

$$f' \leq 0 \text{ and } f'' \geq 0 \text{ in } [0, r^2], \tag{8.22}$$

which will be satisfied in the final choice of f. Denoting by $x^1, ..., x^n$ the coordinates in V and setting

$$w(t,x) := |x - \xi t|^2 = \sum_{i=1}^{n} \left(x^i - \xi^i t\right)^2,$$

we obtain, for $(t,x) \in \Gamma$,

$$\frac{\partial v}{\partial t} = -\alpha e^{-\alpha t} f(w) + e^{-\alpha t} f'(w)\, 2\xi^i \left(\xi^i t - x^i\right) \leq -e^{-\alpha t}\left(\alpha f(w) + C f'(w)\right),$$

where C is a large enough constant (we have used that the ranges of t and x are bounded, ξ is fixed, and $f' \leq 0$).

Observe that, by (3.46),

$$\Delta_\mu w = g^{ij} \frac{\partial^2 w}{\partial x^i \partial x^j} + b^i \frac{\partial w}{\partial x^i},$$

where b^i are smooth functions, which yields

$$\Delta_\mu w = 2g^{ii} + 2b^i\left(x^i - \xi^i t\right) \leq C,$$

where C is a large enough constant. Computing the gradient of w and using the positive definiteness of the matrix $\left(g^{ij}\right)$, we obtain

$$|\nabla w|_g^2 = g^{ij} \frac{\partial w}{\partial x^i} \frac{\partial w}{\partial x^j} \geq c' \sum_{i=1}^{n} \left(\frac{\partial w}{\partial x^i}\right)^2 = cw,$$

where c' and $c = c'/4$ are (small) positive constants. Using the chain rule for Δ_μ (see Exercise 3.9) and (8.22), we obtain from the above estimates

$$\Delta_\mu v = e^{-\alpha t}\left(f''(w)\,|\nabla w|_g^2 + f'(w)\,\Delta_\mu w\right) \geq e^{-\alpha t}\left(cwf''(w) + Cf'(w)\right),$$

which yields

$$\frac{\partial v}{\partial t} - \Delta_\mu v \leq -e^{-\alpha t}\left(\alpha f(w) + Cf'(w) + cwf''(w)\right),$$

where we have merged two similar terms $Cf'(w)$. Now specify f as follows:

$$f(w) = \left(r^2 - w\right)^2.$$

Obviously, this function is smooth, positive in $[0, r^2)$, vanishes at r^2, and satisfies (8.22). We are left to verify that, for a choice of α,

$$\alpha f(w) + Cf'(w) + cwf''(w) \geq 0 \text{ for all } w \in [0, r^2],$$

which will imply (8.20). Indeed, we have

$$\begin{aligned} \alpha f(w) + Cf'(w) + cwf''(w) &= \alpha\left(r^2 - w\right)^2 - 2C\left(r^2 - w\right) + 2cw \\ &= \alpha z^2 - 2(C + c)z + 2cr^2, \end{aligned}$$

where $z = r^2 - w$. Clearly, for large enough α, this quadratic polynomial is positive for all real z, which finishes the proof of the Claim.

The proof of Theorem 8.11 can now be completed as follows. Assuming $u(t', x') = 0$, let us show that $u(t, x) = 0$ for all $(t, x) \in I \times M$ with $t \leq t'$. By the continuity of u, it suffices to prove that for $t < t'$. Since M is connected, it is possible to find a finite sequence $\{x_i\}_{i=0}^k$ so that $x_0 = x$, $x_k = x'$, and any two consecutive points x_i and x_{i+1} are contained in the same chart together with the straight line segment between them.

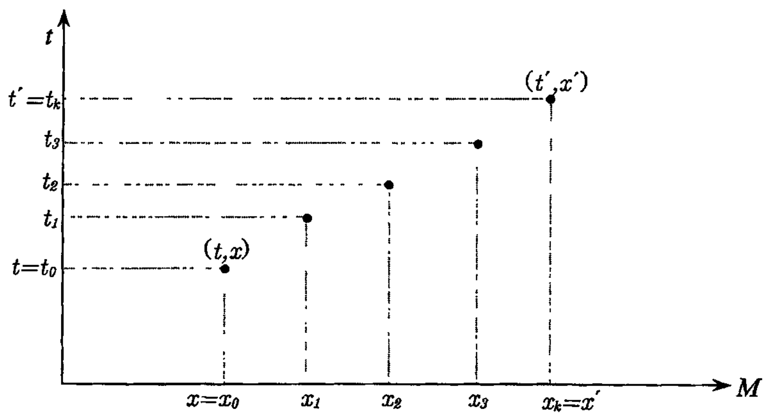

FIGURE 8.3. If non-negative supersolution u vanishes at (t', x') then it vanishes also at any point (t, x) with $t < t'$.

Choosing arbitrarily a sequence of times (see Fig. 8.3)

$$t = t_0 < t_1 < \dots < t_k = t',$$

we can apply the above Claim: if $u(t_0, x_0) = u(t, x) > 0$ then also $u(t_1, x_1) > 0$ and, continuing by induction, $u(t_k, x_k) > 0$, which contradicts the assumption $u(t', x') = 0$. $\square$

The strong maximum/minimum principle has numerous applications. Let us state some immediate consequences.

COROLLARY 8.12. *On a connected manifold* (M, g, μ), *the heat kernel* $p_t(x,y)$ *is strictly positive for all* $t > 0$ *and* $x, y \in M$.

PROOF. Assume that $p_{t'}(x', y') = 0$ for some $t' > 0$, $x', y' \in M$. Since the function $u(t,x) = p_t(x, y')$ satisfies the heat equation (see Theorem 7.20), we obtain by Theorem 8.11 that $p_t(x, y') = 0$ for all $0 < t \le t'$ and all $x \in M$. Consider any function $f \in C_0^\infty(M)$ such that $f(y') \neq 0$. By Theorem 7.13, we have

$$\int_M p_t(x, y') f(x)\, d\mu(x) \to f(y') \quad \text{as } t \to 0,$$

which, however, is not possible if $p_t(x, y') \equiv 0$ for small $t > 0$. □

Note that, by Example 9.10 below, if M is disconnected then $p_t(x,y) = 0$ whenever x and y belong to different connected components of M.

8.3.2. Super- and subharmonic functions.

DEFINITION 8.13. Let $\alpha \in \mathbb{R}$. A function $u \in C^2(M)$ is called α-*superharmonic* on M if it satisfies the inequality $-\Delta_\mu u + \alpha u \ge 0$. It is called α-*subharmonic* if $-\Delta_\mu u + \alpha u \le 0$, and α-*harmonic* if $-\Delta_\mu u + \alpha u = 0$.

Of course, in the latter case $u \in C^\infty(M)$ by Corollary 7.3. If $\alpha = 0$ then the prefix "α-" is suppressed, that is, u is superharmonic if $\Delta_\mu u \le 0$, subharmonic if $\Delta_\mu u \le 0$, and harmonic if $\Delta_\mu u = 0$.

COROLLARY 8.14. (Strong elliptic minimum principle) *Let M be a connected weighted manifold and u be a non-negative α-superharmonic function on M, where $\alpha \in \mathbb{R}$. If $u(x_0) = 0$ at some point $x_0 \in M$ then $u(x) \equiv 0$.*

PROOF. Consider function $v(t,x) = e^{\alpha t} u(x)$. The condition

$$-\Delta_\mu u + \alpha u \ge 0$$

implies that v is a supersolution to the heat equation in $\mathbb{R} \times M$, because

$$\frac{\partial v}{\partial t} - \Delta_\mu v = \alpha e^{\alpha t} u - e^{\alpha t} \Delta_\mu u \ge 0.$$

If $u(x_0) = 0$ then also $v(t, x_0) = 0$ for any t, which implies by Theorem 8.11 that $v(t,x) \equiv 0$ and, hence, $u \equiv 0$. □

There is a direct "elliptic" proof of the strong elliptic minimum principle, which does not use the heat equation and which is simpler than the proof of Theorem 8.11 (see Exercise 8.4).

COROLLARY 8.15. *Let M be a connected weighted manifold. If u is a superharmonic function in M and $u(x_0) = \inf u$ at some point x_0 then $u \equiv \inf u$. If u is a subharmonic function in M and $u(x_0) = \sup u$ as some point x_0 then $u \equiv \sup u$.*

PROOF. The first claim follows from Corollary 8.14 because the function $u - \inf u$ is non-negative and superharmonic. The second claim trivially follows from the first one. □

COROLLARY 8.16. (Elliptic minimum principle) *Let M be a connected weighted manifold and Ω be a relatively compact open subset of M with non-empty boundary. If $u \in C\left(\overline{\Omega}\right) \cap C^2(\Omega)$ is a superharmonic function in Ω then*

$$\inf_{\overline{\Omega}} u = \inf_{\partial\Omega} u.$$

PROOF. Set $m = \inf_{\overline{\Omega}} u$ and consider the set

$$S = \left\{x \in \overline{\Omega} : u(x) = m\right\}.$$

We just need to show that S intersects the boundary $\partial\Omega$. Assuming the contrary, consider any point $x \in S$. Then $x \in \Omega$ and, in any connected open neighborhood $U \subset \Omega$ of x, function u takes its minimal value at the point x. By Corollary 8.15, we conclude that $u \equiv m$ in U, which means that $U \subset S$ and, hence, S is an open set. Since set S is also closed and non-empty, the connectedness of M implies $S = M$, which contradicts to $S \subset \Omega \subset M \setminus \partial\Omega$. □

A companion statement to Corollary 8.16 is the *maximum* principle for subharmonic functions: under the same conditions, if u is subharmonic then

$$\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u.$$

Exercises.

8.4. Let M be a connected weighted manifold and E, F be two compact subsets of M. Prove that, for any real α there is a constant $C = C(\alpha, E, F)$ such that, for any non-negative α-superharmonic function u on M,

$$\inf_E u \leq C \inf_F u.$$

8.5. *(A version of the elliptic minimum principle)* Let M be a non-compact connected weighted manifold and let $u(t,x) \in C^2(M)$ be a superharmonic function. Prove that if

$$\limsup_{k\to\infty} u(x_k) \geq 0 \tag{8.23}$$

for any sequence $\{x_k\}$ such that $x_k \to \infty$ in M, then $u(x) \geq 0$ for all $x \in M$.

8.6. *(A version of the parabolic minimum principle)* Fix $T \in (0, +\infty]$ and consider the manifold $N = (0,T) \times M$. We say that a sequence $\{(t_k, x_k)\}_{k=1}^{\infty}$ of points in N escapes from N if one of the following two alternatives takes place as $k \to \infty$:

1. $x_k \to \infty$ in M and $t_k \to t \in [0,T]$;
2. $x_k \to x \in M$ and $t_k \to 0$.

Let $u(t,x) \in C^2(N)$ be a supersolution to the heat equation in N. Prove that if

$$\limsup_{k\to\infty} u(t_k, x_k) \geq 0 \tag{8.24}$$

for any sequence $\{(t_k, x_k)\}$ that escapes from N, then $u(t,x) \geq 0$ for all $(t,x) \in N$.

8.4. Stochastic completeness

DEFINITION 8.17. A weighted manifold (M, g, μ) is called *stochastically complete* if the heat kernel $p_t(x, y)$ satisfies the identity

$$\int_M p_t(x,y)\,d\mu(y) = 1, \tag{8.25}$$

for all $t > 0$ and $x \in M$.

The condition (8.25) can also be stated as $P_t 1 \equiv 1$. Note that in general we have $0 \le P_t 1 \le 1$ (cf. Theorems 5.11 and 7.16). If the condition (8.25) fails, that is, $P_t 1 \not\equiv 1$ then the manifold M is called *stochastically incomplete.*

Our purpose here is to provide conditions for the stochastic completeness (or incompleteness) in various terms.

8.4.1. Uniqueness for the bounded Cauchy problem. Fix $0 < T \le \infty$, set $I = (0, T)$ and consider the Cauchy problem in $I \times M$

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_\mu u, & \text{in } I \times M, \\ u|_{t=0} = f, \end{cases} \tag{8.26}$$

where f is a given function from $C_b(M)$. The problem (8.26) is understood in the classical sense, that is, $u \in C^\infty(I \times M)$ and $u(t,x) \to f(x)$ locally uniformly in $x \in M$ as $t \to 0$. Here we consider the question of the *uniqueness* of a bounded solution of (8.26).

THEOREM 8.18. *Fix $\alpha > 0$ and $T \in (0, \infty]$. For any weighted manifold (M, g, μ), the following conditions are equivalent.*

(a) *M is stochastically complete.*
(b) *The equation $\Delta_\mu v = \alpha v$ in M has the only bounded non-negative solution $v = 0$.*
(c) *The Cauchy problem in $(0, T) \times M$ has at most one bounded solution.*

REMARK 8.19. As we will see from the proof, in condition (b) the assumption that v is non-negative can be dropped without violating the statement.

PROOF. We first assume $T < \infty$ and prove the following sequence of implications

$$\neg(a) \Longrightarrow \neg(b) \Longrightarrow \neg(c) \Longrightarrow \neg(a),$$

where $\neg$ means the negation of the statement.

Proof of $\neg(a) \Rightarrow \neg(b)$. So, we assume that M is stochastically incomplete and prove that there exists a non-zero bounded solution to the equation $-\Delta_\mu v + \alpha v = 0$. Consider the function

$$u(t,x) = P_t 1(x) = \int_M p_t(x,y)\,d\mu(y),$$

which is C^∞ smooth, $0 \le u \le 1$ and, by the hypothesis of stochastic incompleteness, $u \not\equiv 1$. Consider also the function

$$w(x) = R_\alpha 1(x) = \int_0^\infty e^{-\alpha t} u(t,x)\,dt, \tag{8.27}$$

which, by Theorem 8.7 and Corollary 8.6, is C^∞-smooth, satisfies the estimate

$$0 \le w \le \alpha^{-1} \tag{8.28}$$

and the equation

$$-\Delta_\mu w + \alpha w = 1. \tag{8.29}$$

It follows from $u \not\equiv 1$ that there exist $x \in M$ and $t > 0$ such that $u(t,x) < 1$. Then (8.27) implies that, for this value of x, we have a *strict* inequality $w(x) < \alpha^{-1}$. Hence, $w \not\equiv \alpha^{-1}$.

Finally, consider the function $v = 1 - \alpha w$, which by (8.29) satisfies the equation $\Delta_\mu v = \alpha v$. It follows from (8.28) that $0 \le v \le 1$, and $w \not\equiv \alpha^{-1}$ implies $v \not\equiv 0$. Hence, we have constructed a non-zero non-negative bounded solution to $\Delta_\mu v = \alpha v$, which finishes the proof.

Proof of $\neg(b) \Rightarrow \neg(c)$. Let v be a bounded non-zero solution to equation $\Delta_\mu v = \alpha v$. By Corollary 7.3, $v \in C^\infty(M)$. Then the function

$$u(t,x) = e^{\alpha t} v(x) \tag{8.30}$$

satisfies the heat equation because

$$\Delta_\mu u = e^{\alpha t} \Delta_\mu v = \alpha e^{\alpha t} v = \frac{\partial u}{\partial t}.$$

Hence, u solves the Cauchy problem in $\mathbb{R}_+ \times M$ with the initial condition $u(0,x) = v(x)$, and this solution u is bounded on $(0,T) \times M$ (note that T is *finite*). Let us compare $u(t,x)$ with another bounded solution to the same Cauchy problem, namely with $P_t v(x)$. By Theorem 7.16, we have

$$\sup |P_t v| \le \sup |v|,$$

whereas by (8.30)

$$\sup |u(t,\cdot)| = e^{\alpha t} \sup |v| > \sup |v|.$$

Therefore, $u \not\equiv P_t v$, and the bounded Cauchy problem in $(0,T) \times M$ has two different solutions with the same initial function v.

Proof of $\neg(c) \Rightarrow \neg(a)$. Assume that the problem (4.43) has two different bounded solutions with the same initial function. Subtracting these solutions, we obtain a non-zero bounded solution $u(t,x)$ to (4.43) with the initial function $f = 0$. Without loss of generality, we can assume that $0 < \sup u \le 1$. Consider the function $w = 1 - u$, for which we have $0 \le \inf w < 1$. The function w is a non-negative solution to the Cauchy problem (4.43) with the initial function $f = 1$. By Theorem 8.1 (or Corollary 8.3), we conclude that $w(t,\cdot) \ge P_t 1$. Hence, $\inf P_t 1 < 1$ and M is stochastically incomplete.

Finally, let us prove the equivalence of (a), (b), (c) in the case $T = \infty$. Since the condition (c) with $T = \infty$ is weaker than that for $T < \infty$, it suffices to show that (c) with $T = \infty$ implies (a). Assume from the contrary that M is stochastically incomplete, that is, $P_t 1 \not\equiv 1$. Then the functions $u_1 \equiv 1$ and $u_2 = P_t 1$ are two different bounded solutions to the Cauchy problem (4.43) in $\mathbb{R}_+ \times M$ with the same initial function $f \equiv 1$, so that (a) fails, which was to be proved. $\square$

Exercises.

8.7. Prove that any compact weighted manifold is stochastically complete.

8.8. Prove that $\mathbb{R}^n$ is stochastically complete (cf. Exercise 8.11).

8.9. Prove that if $P_t 1(x) = 1$ for some $t > 0$, $x \in M$ then $P_t 1(x) = 1$ for all $t > 0$, $x \in M$.

8.10. Fix $\alpha > 0$. Prove that M is stochastically complete if and only if $R_\alpha 1 \equiv \alpha^{-1}$.

8.4.2. α-Super- and subharmonic functions. We prove here convenient sufficient conditions for stochastic completeness and incompleteness in terms of the existence of certain α-super- and α-subharmonic functions (see Sections 8.3.2 for the definitions).

THEOREM 8.20. *Let M be a connected weighted manifold and $K \subset M$ be a compact set. Assume that, for some $\alpha \geq 0$, there exists an α-superharmonic function v in $M \setminus K$ such that $v(x) \to +\infty$ as $x \to \infty$*[2]*. Then M is stochastically complete.*

PROOF. By enlarging K, we can assume v is defined also on ∂K and that $v > 0$ in $M \setminus K$. Then v is also β-superharmonic in $M \setminus K$ for any $\beta > \alpha$ so we can assume $\alpha > 0$.

By Theorem 8.18, in order to prove that M is stochastically complete, it suffices to verify that any non-negative bounded solution M to the equation $\Delta_\mu u = \alpha u$ is identical zero. Assume that $0 \leq u \leq 1$ and set

$$m = \max_K u.$$

Then, for any $\varepsilon > 0$, we have

$$\varepsilon v \geq 0 \geq u - m \text{ on } \partial K. \tag{8.31}$$

By hypothesis $v(x) \to +\infty$ as $x \to \infty$ and Exercise 5.18, the set $\{v < \varepsilon^{-1}\}$ is relatively compact; therefore, there exists a relatively compact open set $\Omega \subset M$ that contains $\{v < \varepsilon^{-1}\}$ and K. Compare the functions εv and $u - m$ in $\Omega \setminus K$. By the choice of Ω, we have $v \geq \varepsilon^{-1}$ on $\partial\Omega$ and, consequently,

$$\varepsilon v \geq 1 \geq u - m \text{ on } \partial\Omega. \tag{8.32}$$

In $\Omega \setminus K$, the function εv satisfies the equation

$$-\Delta_\mu (\varepsilon v) + \alpha (\varepsilon v) = 0,$$

[2]See Definition 5.18.

whereas

$$-\Delta_\mu (u-m) + \alpha (u-m) = -\alpha m \leq 0,$$

whence it following that the function $\varepsilon v - (u-m)$ is superharmonic in $\Omega \setminus K$. By (8.31) and (8.32), we have $\varepsilon v \geq u - m$ on the boundary $\partial (\Omega \setminus K)$, which implies by Corollary 8.16 that $\varepsilon v \geq u - m$ in $\Omega \setminus K$. Exhausting M by a sequence of sets Ω, we obtain that $\varepsilon v \geq u - m$ holds in $M \setminus K$; finally, letting $\varepsilon \to 0$, we obtain $u \leq m$ in $M \setminus K$.

Hence, m is the supremum of u on the entire manifold M, and this supremum is attained at a point in K. Since $\Delta_\mu u = \alpha u \geq 0$, that is, the function u is subharmonic, Corollary 8.14 implies that $u \equiv m$ on M. The equation $\Delta_\mu u = \alpha u$ then yields $m = 0$ and $u \equiv 0$, which was to be proved. □

THEOREM 8.21. *Let M be a connected weighted manifold. Assume that there exists a non-negative superharmonic function u on M such that $u \not\equiv$ const and $u \in L^1(M)$. Then M is stochastically incomplete.*

PROOF. Let us first construct another non-negative superharmonic function v on M such that $v \in L^1(M)$ and $\Delta_\mu v \not\equiv 0$. Fix a point $x_0 \in M$ such that $\nabla u(x_0) \neq 0$ and set $c = u(x_0)$. Then the function $\widetilde{u} := \min(u, c)$ is not differentiable at x_0.

Consider the function $P_t \widetilde{u}$. Since u is superharmonic, we have by Exercise 7.29

$$P_t \widetilde{u} \leq P_t u \leq u,$$

which together with

$$P_t \widetilde{u} \leq c P_t 1 \leq c$$

yields

$$P_t \widetilde{u} \leq \widetilde{u}. \tag{8.33}$$

Therefore,

$$P_{t+s} \widetilde{u} = P_s (P_t \widetilde{u}) \leq P_t \widetilde{u},$$

that is,

$$\frac{\partial}{\partial t} P_t \widetilde{u} \leq 0. \tag{8.34}$$

Since $P_t \widetilde{u}$ is a smooth function for any $t > 0$, and $\widetilde{u}$ is not, we see that there is $t > 0$ and $x \in M$ such that

$$\frac{\partial}{\partial t} P_t \widetilde{u}(x) < 0. \tag{8.35}$$

Set $v = P_t \widetilde{u}$ and observe that, by (8.33), (8.34), and (8.35),

$$v \in L^1(M), \quad -\Delta_\mu v \geq 0 \ \text{ and } \ \Delta_\mu v \not\equiv 0.$$

Therefore, there exists a non-negative function $f \in C_0^\infty(M)$ such that $f \not\equiv 0$ and $-\Delta_\mu v \geq f$ on M. Since $v \in W^1_{loc}(M)$ and v satisfies for any $\alpha > 0$ the inequality

$$-\Delta_\mu v + \alpha v \geq f,$$

Theorem 8.4 yields $v \geq R_\alpha f$ a.e.(cf. Remark 8.5). Since $R_\alpha f$ is smooth by Theorem 8.7, we obtain $v \geq R_\alpha f$ pointwise. Letting $\alpha \to 0$ and using (8.10), we obtain, for all $x \in M$,

$$v(x) \geq \int_0^\infty P_t f(x)\, dt = \int_0^\infty \int_M p_t(x,y) f(y)\, d\mu(y)\, dt.$$

Integrating in x and using the condition $v \in L^1(M)$, we obtain

$$\int_M \int_0^\infty \int_M p_t(x,y) f(y)\, d\mu(y)\, dt d\mu(x) < \infty,$$

whence it follows by interchanging the order of integration that

$$\int_0^\infty \int_M P_t 1(y) f(y)\, d\mu(y)\, dt < \infty.$$

However, if M is stochastically complete and, hence, $P_t 1 \equiv 1$, this integral should be equal to

$$\int_0^\infty \left(\int_M f(y)\, d\mu(y) \right) dt = \infty.$$

This contradiction finishes the proof. □

REMARK 8.22. The hypothesis $u \not\equiv \text{const}$ cannot be dropped because it can happen that $1 \in L^1(M)$ and M is stochastically complete, for example, if M is a compact manifold (see Exercise 8.7).

THEOREM 8.23. *Assume that, for some $\alpha > 0$, there exists a non-zero non-negative bounded α-subharmonic function v on M. Then M is stochastically incomplete.*

PROOF. By hypothesis, we have $\Delta_\mu v \geq \alpha v$ and, without loss of generality, we can assume that $0 \leq v \leq 1$. Let $\{\Omega_k\}$ be a compact exhaustion sequence in M. Consider in each Ω_k the following weak Dirichlet problem

$$\begin{cases} -\Delta_\mu u_k + \alpha u_k = 0 \\ u_k = 1 \mod W_0^1(\Omega_k). \end{cases} \tag{8.36}$$

Since $1 \in W^1(\Omega_k)$, this problem has a unique solution u_k, by Exercise 4.29 (or by Theorem 4.5). We will show that

$$v \leq u_{k+1} \leq u_k \leq 1 \tag{8.37}$$

(see Fig. 8.4), which will imply that the sequence $\{u_k\}$ has a limit u that, hence, satisfies the equation $\Delta_\mu u = \alpha u$ on M and $v \leq u \leq 1$. The latter means that u is bounded but non-zero, which implies by Theorem 8.18 that M is stochastically incomplete.

To prove (8.37), observe that function v belongs to $W^1(\Omega_k)$ and obviously satisfies the conditions

$$\begin{cases} -\Delta_\mu v + \alpha v \leq 0, \\ v \leq 1. \end{cases} \tag{8.38}$$

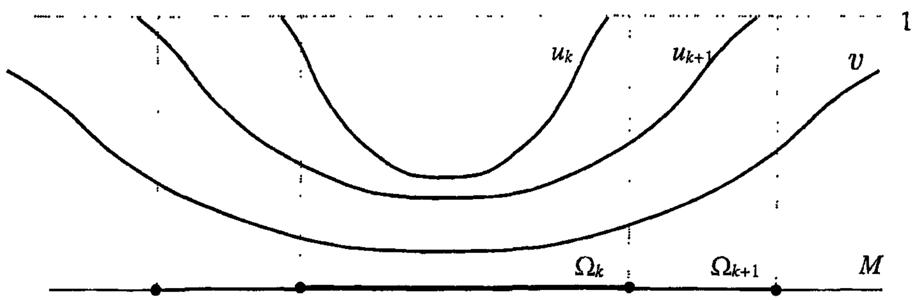

FIGURE 8.4. Functions v, u_k, u_{k+1}

Comparing (8.36) and (8.38) and using Corollary 5.14, we conclude $v \le u_k$. Since the constant function 1 satisfies the condition

$$-\Delta_\mu 1 + \alpha 1 \ge 0,$$

the comparison with (8.36) shows that $u_k \le 1$. Of course, the same applies also to u_{k+1}. Noticing that u_{k+1} satisfies in Ω_k the conditions

$$\begin{cases} -\Delta_\mu u_{k+1} + \alpha u_{k+1} = 0, \\ u_{k+1} \le 1, \end{cases}$$

and comparing them to (8.36), we obtain $u_{k+1} \le u_k$, which finishes the proof. $\square$

Exercises.

8.11. Prove the following claims.

(a) $\mathbb{R}^n$ is stochastically complete for all $n \ge 1$. (cf. Exercise 8.8).

(b) $\mathbb{R}^n \setminus \{0\}$ is stochastically complete if $n \ge 2$, whereas $\mathbb{R}^1 \setminus \{0\}$ is stochastically incomplete.

(c) Any open set $\Omega \subset \mathbb{R}^n$ such that $\overline{\Omega} \ne \mathbb{R}^n$, is stochastically incomplete.

8.12. Let Ω be an open subset of $\mathbb{R}^n$ and h be a positive smooth function in Ω such that

$$\begin{cases} \Delta h = 0 \text{ in } \Omega, \\ h(x) \to 0 \text{ as } x \to \partial\Omega, \\ h(x) = e^{O(|x|)} \text{ as } |x| \to \infty \end{cases}$$

Prove that $P_t^\Omega h = h$ for all $t > 0$.

8.13. Let f be a non-negative superharmonic function on M.

(a) Prove that the function

$$v(x) := \lim_{t\to\infty} P_t f(x) \tag{8.39}$$

satisfies the identity $P_t v = v$ for all $t > 0$ and, hence, is harmonic on M (the limit in (8.39) exists and is finite because by Exercise 7.29 the function $P_t f(x)$ is finite and decreases in t).

(b) Assume in addition that manifold M is stochastically complete and f is bounded. Prove that, for any non-negative harmonic function h on M, the condition $h \le f$ implies $h \le v$.

REMARK. The maximal non-negative harmonic function that is bounded by f is called the largest harmonic minorant of f. Hence, the function v is the largest harmonic minorant of f.

8.14. Set $v(x) = \lim_{t\to\infty} P_t 1(x)$. Prove that either $v \equiv 0$ or $\sup v = 1$. Prove also that either $v \equiv 1$ or $\inf v = 0$.

8.15. Let Ω be the exterior of the unit ball in $\mathbb{R}^n$, $n \geq 2$. Evaluate $\lim_{t\to\infty} P_t^\Omega 1(x)$.

8.4.3. Model manifolds. Let (M, g, μ) be a weighted model based on $M = \mathbb{R}^n$ as it was defined in Section 3.10. This means that, in the polar coordinates (r, θ) in $\mathbb{R}^n$, the metric g and measure μ are expressed as follows:

$$\mathrm{g} = dr^2 + \psi(r)^2 \, \mathrm{g}_{\mathbb{S}^{n-1}}, \tag{8.40}$$

where $\psi(r)$ is a smooth positive function on $(0, +\infty)$, and

$$d\mu = \Upsilon(r)\, \psi(r)^{n-1}\, dr d\theta, \tag{8.41}$$

where $d\theta$ is the Riemannian measure on $\mathbb{S}^{n-1}$ and $\Upsilon(r)$ is a smooth positive function on $(0, +\infty)$. Recall that

$$S(r) := \omega_n \Upsilon(r)\, \psi^{n-1}(r)$$

is the area function of M, and

$$V(r) := \mu(B_r) = \int_0^r S(t)\, dt$$

is the volume function of M. By (3.93), the weighted Laplace operator of (M, g, μ) has in the polar coordinates the following form:

$$\Delta_\mu = \frac{\partial^2}{\partial r^2} + \frac{S'(r)}{S(r)} \frac{\partial}{\partial r} + \frac{1}{\psi^2(r)} \Delta_{\mathbb{S}^{n-1}}. \tag{8.42}$$

It is important to observe that, away from a neighborhood of 0, the functions $\psi(r)$ and $\Upsilon(r)$ can be chosen arbitrarily as long as they are smooth and positive. Near 0 some care should be taken to ensure that the metric g, defined by (8.40), extends smoothly to the origin. If $\psi(r)$ and $\Upsilon(r)$ are prescribed for large r then it is always possible to extend them to all $r > 0$ so that $\psi(r) = r$ and $\Upsilon(r) = 1$ for small enough r. This ensures that the metric and measure in a neighborhood of the origin are exactly Euclidean and, hence, can be extended to the origin.

It follows from this observation that any function $S(r)$ can serve as the area function for large r, as long as $S(r)$ is smooth and positive. Furthermore, setting $\Upsilon \equiv 1$, we can realize $S(r)$ as the area function of a Riemannian model.

Our main result here is the following criterion of the stochastic completeness of the model manifold.

THEOREM 8.24. *The weighted model $(M, \mathbf{g}, \mu)$ as above is stochastically complete if and only if*

$$\int^{\infty} \frac{V(r)}{S(r)} dr = \infty. \tag{8.43}$$

For example, if $V(r) = \exp r^{\alpha}$ for large r then M is stochastically complete if $\alpha \leq 2$ and incomplete if $\alpha > 2$. The manifolds $\mathbb{R}^n$ and $\mathbb{H}^n$ with their canonical metrics satisfy (8.43) because $S(r) = \omega_n r^{n-1}$ for $\mathbb{R}^n$ and $S(r) = \omega_n \sinh^{n-1} r$ for $\mathbb{H}^n$ (see Section 3.10). Hence, both $\mathbb{R}^n$ and $\mathbb{H}^n$ are stochastically complete. Note that $\mathbb{S}^n$ is stochastically complete by Exercise 8.7.

PROOF. Let us show that (8.43) implies the stochastic completeness of M. By Theorem 8.20, it suffices to construct a 1-superharmonic function $v = v(r)$ in the domain $\{r > 1\}$ such that $v(r) \to +\infty$ as $r \to \infty$.

In fact, we construct v as a solution to the equation $\Delta_{\mu} v = v$, which in the polar coordinates has the form

$$v'' + \frac{S'}{S} v' - v = 0. \tag{8.44}$$

So, let v be the solution of the ordinary differential equation (8.44) on $[1, +\infty)$ with the initial values $v(1) = 1$ and $v'(1) = 0$. The function $v(r)$ is monotone increasing because the equation (8.44) after multiplying by Sv and integrating from 1 to R, amounts to

$$Svv'(R) = \int_1^R S\left(v'^2 + v^2\right) dr \geq 0.$$

Hence, we have $v \geq 1$.

Multiplying (8.44) by S, we obtain

$$(Sv')' = Sv,$$

which implies by two integrations

$$v(R) = 1 + \int_1^R \frac{dr}{S(r)} \int_1^r S(t) v(t) dt.$$

Using $v \geq 1$ in the right hand side, we obtain, for $R > 2$,

$$v(R) \geq \int_1^R \frac{dr}{S(r)} \int_1^R S(t) dt = \int_1^R \frac{(V(r) - V(1)) dr}{S(r)} \geq c \int_2^R \frac{V(r) dr}{S(r)},$$

where $c = 1 - \frac{V(1)}{V(2)} > 0$. Finally, (8.43) implies $v(R) \to \infty$ as $R \to \infty$.

Now we assume that

$$\int^{\infty} \frac{V(r)}{S(r)} dr < \infty, \tag{8.45}$$

and prove that M is stochastically incomplete. By Theorem 8.21, it suffices to construct on M a non-negative function $u \in L^1(M)$ such that

$$-\Delta_{\mu} u = f, \tag{8.46}$$

where where $f \in C_0^\infty(M)$, $f \geq 0$ and $f \not\equiv 0$. Both functions u and f will depend only on r so that (8.46) in the domain of the polar coordinates becomes

$$u'' + \frac{S'}{S}u' = f. \tag{8.47}$$

Choose $f(r)$ to be any non-negative non-zero function from $C_0^\infty(1,2)$, and set, for any $R > 0$,

$$u(R) = \int_R^\infty \frac{dr}{S(r)} \int_0^r S(t)\, f(t)\, dt. \tag{8.48}$$

Since f is bounded, the condition (8.43) implies that u is finite. It is easy to see that u satisfies the equation

$$(Su')' = -Sf,$$

which is equivalent to (8.47). The function $u(R)$ is constant on the interval $0 < R < 1$ because $f(t) \equiv 0$ for $0 < t < 1$. Hence, u extends by continuity to the origin and satisfies (8.46) on the whole manifold.

We are left to verify that $u \in L^1(M)$. Since $f(t) \equiv 0$ for $t > 2$, we have for $R > 2$

$$u(R) = C \int_R^\infty \frac{dr}{S(r)}$$

where $C = \int_0^2 S(t)\, f(t)\, dt$. Therefore,

$$\begin{aligned}
\int_{\{R>2\}} u\, d\mu &= \int_2^\infty u(R)\, S(R)\, dR \\
&= C \int_2^\infty \left(\int_R^\infty \frac{dr}{S(r)} \right) S(R)\, dR \\
&= C \int_2^\infty \left(\int_2^r S(R)\, dR \right) \frac{dr}{S(r)} \\
&\leq C \int_2^\infty \frac{V(r)}{S(r)} dr < \infty,
\end{aligned}$$

which gives $u \in L^1(M)$. $\square$

EXAMPLE 8.25. Let us show that, for any continuous positive increasing function $F(r)$ on $(0, +\infty)$ such that $F(r) \to \infty$ as $r \to \infty$, there exists a stochastically complete model M for which

$$F(r) - 1 \leq V(r) \leq F(r), \tag{8.49}$$

for large enough r. Indeed, for large r, the volume function $V(r)$ of a weighted model M may be any smooth positive increasing function. Choose first any such function $V(r)$ satisfying

$$F(r) - 1/2 \leq V(r) \leq F(r), \tag{8.50}$$

and then modify $V(r)$ as follows. Select a sequence of disjoint intervals (a_k, b_k) such that $a_k \to \infty$ and

$$F(b_k) = F(a_k) + 1/2. \tag{8.51}$$

Now reduce $V(r)$ on each interval (a_k, b_k) to create inside (a_k, b_k) the points with very small derivative $V'(r)$ while keeping the values of $V(r)$ at the ends and the monotonicity, which together with (8.50) and (8.51) will ensure (8.49) for the modified V (see Fig. 8.5).

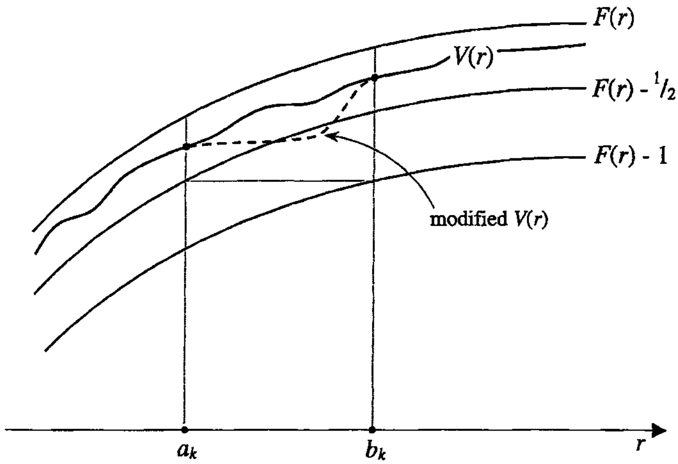

FIGURE 8.5. Modification of function $V(r)$ on the interval (a_k, b_k).

By doing so, we can make the value of integral $\int_{a_k}^{b_k} \frac{V(r)}{V'(r)} dr$ arbitrarily large, say, larger than 1, which implies

$$\int^{\infty} \frac{V(r)}{V'(r)} dr = \infty.$$

Therefore, M is stochastically complete by Theorem 8.24.

Exercises.

8.16. *(A model with two ends)* Set $M = \mathbb{R} \times \mathbb{S}^{n-1}$ (where $n \geq 1$) so that every point $x \in M$ can be represented as a couple (r, θ) where $r \in \mathbb{R}$ and $\theta \in \mathbb{S}^{n-1}$. Fix smooth positive functions $\psi(r)$ and $\Upsilon(r)$ on $\mathbb{R}$, and consider the Riemannian metric on M

$$\mathbf{g} = dr^2 + \psi^2(r)\, \mathbf{g}_{\mathbb{S}^{n-1}},$$

and measure μ on $(M, \mathbf{g})$ with the density function Υ. Define the area function $S(r)$ by

$$S(r) = \omega_n \Upsilon(r) \psi^{n-1}(r)$$

and volume function $V(R)$ by

$$V(R) = \int_{[0,R]} S(r)\, dr,$$

so that $V(R) \geq 0$.

(a) Show that the expression (8.42) for Δ_μ remains true in this setting.

(b) Prove that if function $V(r)$ is even then the following are equivalent:
  (i) $(M, \mathbf{g}, \mu)$ is stochastically complete.
  (ii) There is a non-constant non-negative harmonic function $u \in L^1(M, \mu)$.
  (iii) $\int^\infty \frac{V(r)}{S(r)} dr = \infty$.

(c) Let $S(r)$ satisfy the following relations for some $\alpha > 2$:

$$S(r) = \begin{cases} \exp(r^\alpha), & r > 1, \\ \exp(-|r|^\alpha), & r < -1. \end{cases}$$

Prove that $(M, \mathbf{g}, \mu)$ is stochastically incomplete. Prove that any non-negative harmonic function $u \in L^1(M, \mu)$ is identical zero.

Notes

The material of Sections 8.1 and 8.2 extends the Markovian properties of Chapter 5.

The proofs of the parabolic maximum/minimum principles (Theorems 8.10 and 8.11) are taken from [**243**].

Theorems 8.18, 8.20, 8.23 are due to Khas'minskii [**223**] (see also [**93**], [**155**]), Theorems 8.21, 8.24 were proved in [**142**] (see also [**155**]).

CHAPTER 9

Heat kernel as a fundamental solution

Recall that the heat kernel was introduced in Chapter 7 as the integral kernel of the heat semigroup. Here we prove that the heat kernel can be characterized as the minimal positive fundamental solution of the heat equation. This equivalent definition of the heat kernel is frequently useful in applications.

9.1. Fundamental solutions

DEFINITION 9.1. Any smooth function u on $\mathbb{R}_+ \times M$ satisfying the following conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_\mu u & \text{in } \mathbb{R}_+ \times M, \\ u(t,\cdot) \xrightarrow{\mathcal{D}'} \delta_y & \text{as } t \to 0, \end{cases} \tag{9.1}$$

is called a *fundamental solution* to the heat equation at the point y. If in addition $u \geq 0$ and, for all $t > 0$,

$$\int_M u(t,\cdot)\, d\mu \leq 1, \tag{9.2}$$

then u is called a *regular* fundamental solution.

As it follows from Theorem 7.13, the heat kernel $p_t(x,y)$ as a function of t, x is a regular fundamental solution at y.

The following elementary lemma is frequently useful for checking that a given solution to the heat equation is a regular fundamental solution.

LEMMA 9.2. *Let $u(t,x)$ be a smooth non-negative function on $\mathbb{R}_+ \times M$ satisfying* (9.2). *Then the following conditions are equivalent:*

(a) $u(t,\cdot) \xrightarrow{\mathcal{D}'} \delta_y$ *as* $t \to 0$.
(b) *For any open set U containing y,*

$$\int_U u(t,\cdot)\, d\mu \to 1 \quad \text{as } t \to 0. \tag{9.3}$$

(c) *For any $f \in C_b(M)$,*

$$\int_M u(t,\cdot) f\, d\mu \to f(y) \quad \text{as } t \to 0. \tag{9.4}$$

PROOF. The implication $(c) \Rightarrow (a)$ is trivial because $u(t,\cdot) \xrightarrow{\mathcal{D}'} \delta_y$ is equivalent to (9.4) for all $f \in \mathcal{D}(M)$.

$(a) \Rightarrow (b)$. Let $f \in \mathcal{D}(U)$ be a cutoff function of the set $\{y\}$ in U. Then (9.4) holds for this f. Since $f(y) = 1$ and

$$\int_M u(t,\cdot) f \, d\mu \le \int_U u(t,\cdot) \, d\mu \le 1,$$

(9.3) follows from (9.4).

$(b) \Rightarrow (c)$. For any open set U containing y, we have

$$\begin{aligned}\int_M u(t,x) f(x) \, d\mu(x) &= \int_{M\setminus U} u(t,x) f(x) \, d\mu(x) \\ &\quad + \int_U u(t,x) (f(x) - f(y)) \, d\mu(x) \\ &\quad + f(y) \int_U u(t,x) \, d\mu(x).\end{aligned}$$

The last term here tends to $f(y)$ by (9.3). The other terms are estimated as follows:

$$\left| \int_{M\setminus U} u(t,x) f(x) \, d\mu \right| \le \sup |f| \int_{M\setminus U} u(t,x) \, d\mu(x) \tag{9.5}$$

and

$$\begin{aligned}\left| \int_U u(t,x) (f(x) - f(y)) \, d\mu \right| &\le \sup_{x\in U} |f(x) - f(y)| \int_U u(t,x) \, d\mu(x) \\ &\le \sup_{x\in U} |f(x) - f(y)|. \end{aligned} \tag{9.6}$$

Obviously, the right hand side of (9.5) tends to 0 as $t \to 0$ due to (9.2) and (9.3). By the continuity of f at y, the right hand side of (9.6) can be made arbitrarily small uniformly in t by choosing U to be a small enough neighborhood of y. Combining the above three lines, we obtain (9.4). □

REMARK 9.3. As we see from the last part of the proof, (9.4), in fact, holds for arbitrary $f \in L^\infty(M)$ provided f is continuous at the point y.

The next lemma is needed for the proof of main result of this section – Theorem 9.5.

LEMMA 9.4. *Let $u(t,x)$ be a non-negative smooth function on $\mathbb{R}_+ \times M$ such that $u(t,\cdot) \xrightarrow{\mathcal{D}'} \delta_y$ as $t \to 0$. Then, for any open set $\Omega \Subset M$ and any $f \in C_b(\Omega)$,*

$$\int_\Omega u(t,x) f(x) \, d\mu(x) \to \begin{cases} f(y), & \text{if } y \in \Omega, \\ 0, & \text{if } y \in M \setminus \overline{\Omega}, \end{cases} \tag{9.7}$$

as $t \to 0$.

REMARK. Extend f to M by setting $f = 0$ outside Ω. Then in the both cases in (9.7), f is continuous at y. Hence, (9.7) follows from Lemma 9.2 and Remark 9.3, provided u satisfies (9.2). However, we need (9.7) without the hypothesis (9.2), which requires a more elaborate argument.

PROOF. If $f \in \mathcal{D}(\Omega)$ then, by hypothesis $u(t,\cdot) \xrightarrow{\mathcal{D}'} \delta_y$, we have (9.4), which implies (9.7).

Representing $f \in C_b(\Omega)$ as $f = f_+ - f_-$, it suffices to prove (9.7) for non-negative f. By scaling f, we can assume without loss of generality that $0 \le f \le 1$.

Let us first prove (9.7) for $y \in M \setminus \overline{\Omega}$. Let $\psi \in \mathcal{D}(M)$ be a cutoff function of $\overline{\Omega}$ in $M \setminus \{y\}$ (see Fig. 9.1).

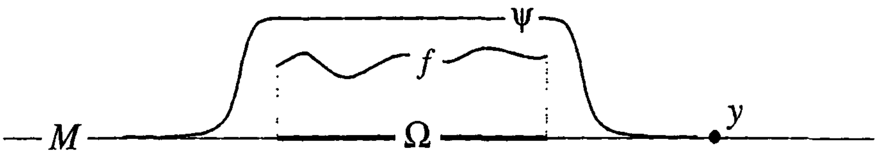

FIGURE 9.1. Functions f and ψ.

Then $f \le \psi$,whence it follows that

$$\int_\Omega u(t,\cdot) f \, d\mu \le \int_\Omega u(t,\cdot) \psi \, d\mu \to \psi(y) = 0 \text{ as } t \to 0,$$

which implies

$$\int_\Omega u(t,\cdot) f \, d\mu \to 0.$$

Assume now $y \in \Omega$ and set $f(y) = a$. By the continuity of f at y, for any $\varepsilon > 0$ there exists an open neighborhood $U \Subset \Omega$ of y such that

$$a - \varepsilon < f < a + \varepsilon \text{ in } U.$$

Let φ be a cutoff function of $\{y\}$ in U and ψ be a cutoff function of $\overline{U}$ in Ω so that

$$(a - \varepsilon)\varphi \le f \le (a + \varepsilon)\psi \text{ in } U.$$

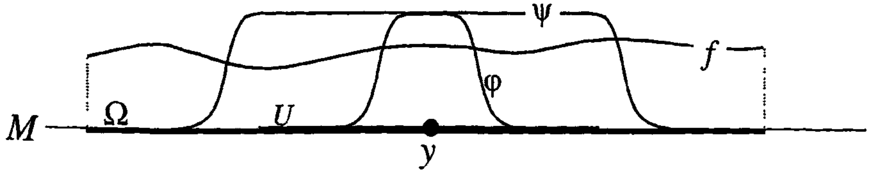

FIGURE 9.2. Functions f, φ, ψ.

Therefore, we have

$$(a-\varepsilon)\int_U u(t,\cdot)\,\varphi d\mu \le \int_U u(t,\cdot)\ f\,d\mu \le (a+\varepsilon)\int_\Omega u(t,\cdot)\,\psi\,d\mu.$$

Passing to the limit as $t\to 0$, we obtain

$$\limsup_{t\to 0}\int_U u(t,\cdot)\ f\,d\mu \le (a+\varepsilon)\,\psi(y) = a+\varepsilon \tag{9.8}$$

and

$$\liminf_{t\to 0}\int_U u(t,\cdot)\ f\,d\mu \ge (a-\varepsilon)\,\varphi(y) = a-\varepsilon. \tag{9.9}$$

Let us show that

$$\int_{\Omega\setminus U} u(t,\cdot)\ f\,d\mu \to 0. \tag{9.10}$$

Indeed, let $V \Subset U$ be an open neighborhood of y. Since $\Omega\setminus\overline{V}$ is a relatively compact open set and $y\notin\Omega\setminus\overline{V}$, we obtain by the first part of the proof that

$$\int_{\Omega\setminus\overline{V}} u(t,\cdot)\,f\,d\mu \to 0,$$

whence (9.10) follows.

Finally, combining together (9.8), (9.9), (9.10) and letting $\varepsilon\to 0$, we obtain

$$\int_\Omega u(t,\cdot)\ f\,d\mu \to a = f(y) \quad \text{as } t\to 0.$$

□

The next theorem provides a characterization of the heat kernel, which can serve as an alternative definition.

THEOREM 9.5. *For any $y\in M$, the heat kernel $p_t(x,y)$ is the minimal non-negative fundamental solution of the heat equation at y.*

PROOF. Let $u(t,x)$ be another non-negative fundamental solution at y, and fix $s>0$. The function $t,x\mapsto u(t+s,x)$ satisfies the heat equation in $\mathbb{R}_+\times M$ and, hence, $u(t+s,x)$ can be considered as a non-negative solution to the Cauchy problem in $\mathbb{R}_+\times M$ with the initial function $f(x)=u(s,x)$. Since u is a smooth function, we have $f\in L^2_{loc}(M)$ and

$$u(t+s,\cdot)\xrightarrow{L^2_{loc}} f \quad \text{as } t\to 0.$$

By Theorem 8.1, we conclude that, for all $t>0$ and $x\in M$,

$$u(t+s,x)\ge P_t f(x) = \int_M p_t(x,z)\,u(s,z)\,d\mu(z). \tag{9.11}$$

Fix now $t>0$, $x\in M$ and choose an open set $\Omega\Subset M$ containing y. Then $p_t(x,\cdot)\in C_b(\Omega)$ and, by Lemma 9.4,

$$\int_\Omega p_t(x,z)\,u(s,z)\,d\mu(z)\to p_t(x,y) \quad \text{as } s\to 0.$$

Hence, letting $s \to 0$ in (9.11), we obtain $u(t,x) \ge p_t(x,y)$, which was to be proved. $\square$

COROLLARY 9.6. *Let $(M, \mathbf{g}, \mu)$ be a stochastically complete weighted manifold. If $u(t,x)$ is a regular fundamental solution at a point $y \in M$, then $u(t,x) \equiv p_t(x,y)$.*

PROOF. By Theorem 9.5, we have $u(t,x) \ge p_t(x,y)$, which implies

$$1 \ge \int_M u(t,x)\, d\mu(x) \ge \int_M p_t(x,y)\, d\mu(x) = 1,$$

where in the last part we have used the stochastic completeness of M. We conclude that all the inequalities above are actually equalities, which is only possible when $u(t,x) = p_t(x,y)$. $\square$

The next theorem helps establishing the identity of a fundamental solution and the heat kernel using the "boundary condition", which may be useful in the case of stochastic incompleteness.

THEOREM 9.7. *Let $u(t,x)$ be a non-negative fundamental solution to the heat equation at $y \in M$. If $u(t,x) \rightrightarrows 0$ as $x \to \infty$ where the convergence is uniform in $t \in (0,T)$ for any $T > 0$, then $u(t,x) \equiv p_t(x,y)$.*

REMARK 9.8. The hypothesis that $u(t,x)$ is non-negative can be relaxed to the assumption that

$$\limsup_{k\to\infty} u(t_k, x_k) \ge 0, \tag{9.12}$$

for any sequence (t_k, x_k) such that $t_k \to 0$ and $x_k \to x \in M$. Indeed, by the maximum principle of Exercise 8.6, (9.12) together with the other hypotheses implies $u \ge 0$.

PROOF. By Theorem 9.5, we have $u(t,x) \ge p_t(x,y)$ so that we only need to prove the opposite inequality. Fix some $s > 0$ and notice that the function $v(t,x) = u(t+s,x)$ solves the heat equation with the initial function $f(x) = v(0,x) = u(s,x)$. Since $v(t,x) \rightrightarrows 0$ as $x \to \infty$, we obtain by Corollary 8.2 that $v(t,x) = P_t f(x)$ that is,

$$u(t+s,x) = P_t f(x) = \int_M p_t(x,\cdot)\, u(s,\cdot)\, d\mu.$$

Let $\{\Omega_k\}_{k=1}^{\infty}$ be a compact exhaustion sequence in M. For any k, we have

$$\begin{aligned} u(t+s,x) - \int_{\Omega_k} p_t(x,\cdot)\, u(s,\cdot)\, d\mu &= \int_{M\setminus\Omega_k} p_t(x,\cdot)\, u(s,\cdot)\, d\mu \\ &\le \sup_{M\setminus\Omega_k} u(s,\cdot) \int_{M\setminus\Omega_k} p_t(x,\cdot)\, d\mu. \end{aligned}$$

Since the total integral of the heat kernel is bounded by 1 and

$$\sup_{s\in(0,T)} \sup_{z\in M\setminus\Omega_k} u(s,z) \to 0 \text{ as } k \to \infty,$$

we see that, for any $\varepsilon > 0$ there is k so big that, for all $s \in (0, T)$,

$$u(t+s, x) - \int_{\Omega_k} p_t(x, \cdot)\, u(s, \cdot)\, d\mu \leq \varepsilon.$$

Letting here $s \to 0$ and applying Lemma 9.4 in Ω_k with function $f = p_t(x, \cdot)$, we obtain

$$u(t, x) - p_t(x, y) \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude $u(t, x) \leq p_t(x, y)$, which was to be proved. $\square$

EXAMPLE 9.9. By Lemma 1.1, the Gauss-Weierstrass function

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) \tag{9.13}$$

is a regular fundamental solution of the heat equation in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is stochastically complete, we conclude by Corollary 9.6 that $p_t(x, y)$ is the heat kernel in $\mathbb{R}^n$. Alternatively, this can be concluded by Theorem 9.7 because $p_t(x, y) \to 0$ as $x \to \infty$ uniformly in t (cf. Exercise 1.5). Yet another proof of the fact that $p_t(x, y)$ is the heat kernel in $\mathbb{R}^n$ was given in Example 4.12.

EXAMPLE 9.10. For any weighted manifold $(M, \mathbf{g}, \mu)$ and an open set $\Omega \subset M$, denote as before by $p_t^\Omega(x, y)$ the heat kernel in $(\Omega, \mathbf{g}, \mu)$. Extend $p_t^\Omega(x, y)$ to all $x, y \in M$ by setting it to 0 if either x or y is outside Ω. Let us show that if Ω_1 and Ω_2 are two disjoint open sets and $\Omega = \Omega_1 \cup \Omega_2$ then

$$p_t^\Omega = p_t^{\Omega_1} + p_t^{\Omega_2}. \tag{9.14}$$

Indeed, if $y \in \Omega_1$ then $p_t^{\Omega_1}(x, y)$ is a fundamental solution not only in Ω_1 but also in Ω because, being identical zero in Ω_2, it satisfies the heat equation also in Ω_2. Therefore, by the minimality of p_t^Ω, we conclude $p_t^\Omega(x, y) \leq p_t^{\Omega_1}(x, y)$. On the other hand, since $\Omega_1 \subset \Omega$, we have the opposite inequality by Exercise 7.40. Hence,

$$p_t^\Omega(x, y) = p_t^{\Omega_1}(x, y),$$

which implies (9.14) because $p_t^{\Omega_2}(x, y) = 0$. Similarly, (9.14) holds if $y \in \Omega_2$.

The identity (9.14) implies that $p_t(x, y) = 0$ if the points x, y belong to different connected components of M (assuming, of course, that M is disconnected). Indeed, if x is contained in a connected component Ω_1 and $\Omega_2 := M \setminus \Omega_1$ then $y \in \Omega_2$ and, hence, $p_t^{\Omega_i}(x, y) = 0$ for $i = 1, 2$, whence

$$p_t(x, y) = p_t^{\Omega_1}(x, y) + p_t^{\Omega_2}(x, y) = 0.$$

9.2. Some examples

We give here some examples of application of the techniques developed in the previous sections.

9.2.1. Heat kernels on products. Let $(X, \mathbf{g}_X, \mu_X)$ and $(Y, \mathbf{g}_Y, \mu_Y)$ be two weighted manifold and $(M, \mathbf{g}, \mu)$ is their direct (see Section 3.8). Denote by p_t^X and p_t^Y the heat kernels on X and Y, respectively.

THEOREM 9.11. *Assume that $(M, \mathbf{g}, \mu)$ is stochastically complete. Then the heat kernel p_t on M satisfies the identity*

$$p_t\left((x,y),(x',y')\right) = p_t^X\left(x,x'\right) p_t^Y\left(y,y'\right), \tag{9.15}$$

for all $t>0$ and $x,x' \in X$, $y,y' \in Y$.

The statement is true without the hypothesis of stochastic completeness, but the proof of that requires a different argument (see Exercise 7.41).

PROOF. Denote by Δ_X and Δ_Y the Laplace operators on X and Y, respectively. Then the Laplace operator Δ_μ on M is given by

$$\Delta_\mu u = \Delta_X u + \Delta_Y u,$$

where $u(x,y)$ is a smooth function on M, and Δ_X acts on the variable x, Δ_Y acts on the variable y (see Section 3.8).

Fix $(x',y') \in M$ and prove that the function

$$u(t,(x,y)) = p_t^X\left(x,x'\right) p_t^Y\left(y,y'\right)$$

is a regular fundamental solution at (x',y'), which will imply (9.15) by Corollary 9.6.

Indeed, the heat equation for u is verified as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t} p_t^X\left(x,x'\right) p_t^Y\left(y,y'\right) + p_t^X\left(x,x'\right) \frac{\partial}{\partial t} p_t^Y\left(y,y'\right) \\ &= \Delta_X p_t^X\left(x,x'\right) p_t^Y\left(y,y'\right) + p_t^X\left(x,x'\right) \Delta_Y p_t^Y\left(y,y'\right) \\ &= \left(\Delta_X + \Delta_Y\right) u = \Delta_\mu u. \end{aligned}$$

The integral of u is evaluated by

$$\int_M u(t,\cdot)\,d\mu = \int_M p_t^X\left(\cdot,x'\right) d\mu_X \int_M p_t^Y\left(\cdot,y'\right) d\mu_Y \le 1.$$

To check the condition (9.3) of Lemma 9.2, it suffices to take the set $U \subset M$ in the form $U = V \times W$ where $V \subset X$ and $W \subset Y$. If $(x',y') \in U$ then $x' \in V$ and $y' \in W$, which implies

$$\int_U u(t,\cdot)\,d\mu = \int_V p_t^X\left(\cdot,x'\right) d\mu_X \int_W p_t^Y\left(\cdot,y'\right) d\mu_Y \to 1$$

as $t \to 0$. Hence, by Lemma 9.2, u is a regular fundamental solution at (x',y'). $\square$

9.2.2. Heat kernels and isometries. We use here the notion of isometry of weighted manifolds introduced in Section 3.12.

THEOREM 9.12. *Let $J : M \to M$ be an isometry of a weighted manifold (M, g, μ). Then the heat kernel of M is J-invariant, that is, for all $t > 0$ and $x, y \in M$,*

$$p_t(Jx, Jy) = p_t(x, y) \tag{9.16}$$

See also Exercise 7.24 for an alternative proof.

PROOF. Let us first show that the function $u(t, x) = p_t(Jx, Jy)$ is a fundamental solution at y. Indeed, by Lemma 3.27, for any smooth function f on M,

$$(\Delta_\mu f)(Jx) = \Delta_\mu (f(Jx)).$$

Applying this for $f = p_t(\cdot, Jy)$, we obtain

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} p_t(Jx, Jy) = (\Delta_\mu p_t)(Jx, Jy) = \Delta_\mu u,$$

so that u solves the heat equation.

By Lemma 3.27, we have the identity

$$\int_M f(Jx)\, d\mu(x) = \int_M f(x)\, d\mu(x), \tag{9.17}$$

for any integrable function f. Hence, for any $\varphi \in \mathcal{D}(M)$,

$$\begin{aligned} \int_M u(t, x) \varphi(x)\, d\mu(x) &= \int_M p_t(Jx, Jy) \varphi(x)\, d\mu(x) \\ &= \int_M p_t(x, Jy) \varphi\left(J^{-1}x\right) d\mu(x). \end{aligned}$$

Since $p_t(x, Jy)$ is a fundamental solution at Jy, the last integral converges as $t \to 0$ to

$$\left(\varphi \circ J^{-1}\right)(Jy) = \varphi(y),$$

which proves that $u(t, x) \xrightarrow{\mathcal{D}'} \delta_y$. Hence, $u(t, x)$ is a non-negative fundamental solution at y, which implies by Theorem 9.5

$$u(t, x) \geq p_t(x, y),$$

that is,

$$p_t(Jx, Jy) \geq p_t(x, y).$$

Applying the same argument to J^{-1} instead of J, we obtain the opposite inequality, which finishes the proof. □

EXAMPLE 9.13. By Exercise 3.46, for any four points $x, y, x', y' \in \mathbb{H}^n$ such that

$$d(x', y') = d(x, y),$$

there exists a Riemannian isometry $J : \mathbb{H}^n \to \mathbb{H}^n$ such that $Jx' = x$ and $Jy' = y$. By Theorem 9.12, we conclude

$$p_t(x', y') = p_t(x, y).$$

Hence, $p_t(x,y)$, as a function of x,y, depends only on the distance $d(x,y)$.

9.2.3. Heat kernel on model manifolds. Let $(M,\mathbf{g},\mu)$ be a weighed model as in Sections 3.10 and 8.4.3. That is, M is either a ball $B_{r_0} = \{|x| < r_0\}$ in $\mathbb{R}^n$ or $M = \mathbb{R}^n$ (in this case $r_0 = \infty$), and the metric $\mathbf{g}$ and the density function of μ depend only on the polar radius. Let $S(r)$ be the area function of $(M,\mathbf{g},\mu)$ and $p_t(x,y)$ be the heat kernel.

Let $(M,\widetilde{\mathbf{g}},\widetilde{\mu})$ be another weighted model based on the same smooth manifold M, and let $\widetilde{S}(r)$ and $\widetilde{p}_t(x,y)$ be its area function and heat kernel, respectively.

THEOREM 9.14. *If $S(r) \equiv \widetilde{S}(r)$ then $p_t(x,0) = \widetilde{p}_t(y,0)$ for all $x,y \in M$ such that $|x| = |y|$.*

Note that the area function $S(r)$ does not fully identify the structure of the weighted model unless the latter is a Riemannian model. Nevertheless, $p_t(x,0)$ is completely determined by this function.

PROOF. Let us first show that $p_t(x,0) = p_t(y,0)$ if $|x| = |y|$. Indeed, there is a rotation J of $\mathbb{R}^n$ such that $Jx = Jy$ and $J0 = 0$. Since J is an isometry of $(M,\mathbf{g},\mu)$, we obtain by Theorem 9.12 that p_t is J-invariant, which implies the claim.

By Lemma 9.2, the fact that a smooth non-negative function $u(t,x)$ on $\mathbb{R}_+ \times M$ is a regular fundamental solution at 0, is equivalent to the conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_\mu u, \\ \displaystyle\int_M u(t,x)\,d\mu(x) \le 1, \\ \displaystyle\int_{B_\varepsilon} u(t,x)\,d\mu(x) \to 1 \quad \text{as } t \to 0, \end{cases} \tag{9.18}$$

for all $0 < \varepsilon < r_0$. The heat kernel $p_t(x,0)$ is a regular fundamental solution on $(M,\mathbf{g},\mu)$ at the point 0, and it depends only on t and $r = |x|$ so that we can write $p_t(x,0) = u(t,r)$.

Using the fact that u does not depend on the polar angle, we obtain from (3.93)

$$\Delta_\mu u = \frac{\partial^2 u}{\partial r^2} + \frac{S'(r)}{S(r)}\frac{\partial u}{\partial r}.$$

For $0 < \varepsilon < r_0$, we have by (3.86), (3.88), (3.91)

$$\int_{B_\varepsilon} u\,d\mu = \frac{1}{\omega_n}\int_0^\varepsilon \int_{\mathbb{S}^{n-1}} u(t,r)\,S(r)\,d\theta dr = \int_0^\varepsilon u(t,r)\,S(r)\,dr.$$

Hence, we obtain the following equivalent form of (9.18):

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{S'(r)}{S(r)} \dfrac{\partial u}{\partial r}, \\ \displaystyle\int_0^{r_0} u(t,r) S(r)\, dr \leq 1, \\ \displaystyle\int_0^{\varepsilon} u(t,r) S(r)\, dr \to 1 \quad \text{as } t \to 0. \end{cases} \tag{9.19}$$

Since by hypothesis $S(r) = \widetilde{S}(r)$, the conditions (9.19) are satisfied also with S replaced by $\widetilde{S}$, which means that $u(t,r)$ is a regular fundamental solution at 0 also on the manifold $(M, \widetilde{\mathbf{g}}, \widetilde{\mu})$. By Theorem 9.5, we conclude that $u(t,|x|) \geq \widetilde{p}_t(x,0)$, that is,

$$p_t(x,0) \geq \widetilde{p}_t(x,0).$$

The opposite inequality follows in the same way by switching p_t and $\widetilde{p}_t$, which finishes the proof. $\square$

9.2.4. Heat kernel and change of measure. Let $(M, \mathbf{g}, h)$ be a weighted manifold. Any smooth positive function h on M determines a new measure $\widetilde{\mu}$ on M by

$$d\widetilde{\mu} = h^2 d\mu, \tag{9.20}$$

and, hence, a new weighted manifold $(M, \mathbf{g}, \widetilde{\mu})$. Denote by $\widetilde{P}_t$ and $\widetilde{p}_t$ respectively the heat semigroup and the heat kernel on $(M, \mathbf{g}, \widetilde{\mu})$.

THEOREM 9.15. *Let h be a smooth positive function on M that satisfies the equation*

$$\Delta_\mu h + \alpha h = 0, \tag{9.21}$$

where α is a real constant. Then the following identities holds

$$\Delta_{\widetilde{\mu}} = \frac{1}{h} \circ (\Delta_\mu + \alpha\, \mathrm{id}) \circ h, \tag{9.22}$$

$$\widetilde{P}_t = e^{\alpha t} \frac{1}{h} \circ P_t \circ h, \tag{9.23}$$

$$\widetilde{p}_t(x,y) = e^{\alpha t} \frac{p_t(x,y)}{h(x)\, h(y)}, \tag{9.24}$$

for all $t > 0$ and $x, y \in M$.

In (9.22) and (9.23), h and $\frac{1}{h}$ are regarded as multiplication operators, the domain of the operators in (9.22) is $C^\infty(M)$, and the domain of the operators in (9.23) is $L^2(M, \widetilde{\mu})$.

The change of measure (9.20) satisfying (9.21) and the associated change of operator (9.22) are referred to as Doob's *h-transform*.

PROOF. By the definition of the weighted Laplace operator (see Section 3.6), we obtain, for any smooth function f on M,

$$\begin{aligned}
\Delta_{\widetilde{\mu}} f &= \frac{1}{h^2}\operatorname{div}_\mu(h^2\nabla f) = \operatorname{div}_\mu(\nabla f) + \frac{1}{h^2}\langle\nabla h^2, \nabla f\rangle_{\mathrm{g}} \\
&= \Delta_\mu f + 2\langle\frac{\nabla h}{h}, \nabla f\rangle_{\mathrm{g}}. \qquad (9.25)
\end{aligned}$$

On the other hand, using the equation (9.21) and the product rule for Δ_μ (cf. Exercise 3.8) and (9.21), we obtain

$$\begin{aligned}
\frac{1}{h}\Delta_\mu(hf) &= \frac{1}{h}(h\Delta_\mu f + 2\langle\nabla h, \nabla f\rangle_{\mathrm{g}} + f\Delta_\mu h) \\
&= \Delta_\mu f + 2\langle\frac{\nabla h}{h}, \nabla f\rangle_{\mathrm{g}} + f\frac{\Delta_\mu h}{h} \\
&= \Delta_{\widetilde{\mu}} f - \alpha f.
\end{aligned}$$

Hence, we have the identity

$$\Delta_{\widetilde{\mu}} f = \frac{1}{h}\Delta_\mu(hf) + \alpha f, \qquad (9.26)$$

which is equivalent to (9.22).

Next, fix a point $y \in M$, set

$$u(t,x) = e^{\alpha t}\frac{p_t(x,y)}{h(x)h(y)}$$

and show that $u(t,x)$ is a fundamental solution on $(M, \mathrm{g}, \widetilde{\mu})$ at point y. Using (9.26), we obtain

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \alpha u + e^{\alpha t}\frac{\partial}{\partial t}\frac{p_t(x,y)}{h(x)h(y)} \\
&= \alpha u + \frac{e^{\alpha t}}{h(x)h(y)}\Delta_\mu p_t(x,y) \\
&= \alpha u + \frac{1}{h(x)}\Delta_\mu\left(h(x)e^{\alpha t}\frac{p_t(x,y)}{h(x)h(y)}\right) \\
&= \alpha u + \frac{1}{h}\Delta_\mu(hu) = \alpha u + \Delta_{\widetilde{\mu}} u - \alpha u = \Delta_{\widetilde{\mu}} u,
\end{aligned}$$

so that u solves the heat equation on $(M, \mathrm{g}, \widetilde{\mu})$.

For any function $\varphi \in \mathcal{D}(M)$, we have

$$\begin{aligned}
\int_M u(t,x)\varphi(x)\,d\widetilde{\mu}(x) &= \int_M e^{\alpha t}\frac{p_t(x,y)}{h(y)}h(x)\varphi(x)\,d\mu(x) \\
&= \frac{e^{\alpha t}}{h(y)}P_t(h\varphi)(y). \qquad (9.27)
\end{aligned}$$

Since $h(x)\varphi(x) \in \mathcal{D}(M)$, we have

$$P_t(h\varphi)(y) \to h\varphi(y) \quad \text{as } t \to 0,$$

whence it follows that

$$\int_M u(t,x)\,\varphi(x)\,d\widetilde{\mu}(x) \to \varphi(y),$$

and, hence, $u(t,\cdot) \xrightarrow{\mathcal{D}'} \delta_y$ on $(M, \mathbf{g}, \widetilde{\mu})$.

Therefore, u is a non-negative fundamental solution on $(M, \mathbf{g}, \widetilde{\mu})$ at point y and, by Theorem 9.5, we conclude that $u(t,x) \geq \widetilde{p}_t(x,y)$, that is,

$$e^{\alpha t}\frac{p_t(x,y)}{h(x)\,h(y)} \geq \widetilde{p}_t(x,y). \tag{9.28}$$

To prove the opposite inequality, observe that the function $\widetilde{h} := \frac{1}{h}$ satisfies the equation

$$\Delta_{\widetilde{\mu}}\widetilde{h} - \alpha\widetilde{h} = 0,$$

which follows from (9.26) because $h\widetilde{h} = 1$ and

$$\Delta_{\widetilde{\mu}}\widetilde{h} = \frac{1}{h}\Delta_\mu(h\widetilde{h}) + \alpha\widetilde{h} = \alpha\widetilde{h}.$$

Switching the roles of μ and $\widetilde{\mu}$, replacing α by $-\alpha$ and h by $\widetilde{h}$, we obtain by the above argument

$$e^{-\alpha t}\frac{\widetilde{p}_t(x,y)}{\widetilde{h}(x)\,\widetilde{h}(y)} \geq p_t(x,y),$$

which is exactly the opposite inequality in (9.28).

Finally, using (9.20) and (9.24), we obtain, for any $f \in L^2(M, \widetilde{\mu})$,

$$\begin{aligned}
\widetilde{P}_t f &= \int_M \widetilde{p}_t(x,y)\,f(y)\,d\widetilde{\mu}(y) \\
&= \int_M e^{\alpha t}\frac{p_t(x,y)}{h(x)\,h(y)} f(y)\,h^2(y)\,d\mu(y) \\
&= e^{\alpha t}\frac{1}{h}P_t(fh),
\end{aligned}$$

whence (9.23) follows. □

EXAMPLE 9.16. The heat kernel in $(\mathbb{R}^n, \mathbf{g}_{\mathbb{R}^n}, \mu)$ with the Lebesgue measure μ is given by

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{|x-y|^2}{4t}\right). \tag{9.29}$$

Let h be any positive smooth function on $\mathbb{R}^n$ that determines a new measure $\widetilde{\mu}$ on $\mathbb{R}^n$ by $d\widetilde{\mu} = h^2 d\mu$. Then we have $\Delta_\mu = \frac{d^2}{dx^2}$ and

$$\Delta_{\widetilde{\mu}} = \frac{d^2}{dx^2} + 2\frac{h'}{h}\frac{d}{dx} \tag{9.30}$$

(cf. (9.25)). The equation (9.21) becomes

$$h'' + \alpha h = 0,$$

which is satisfied, for example, if $h(x) = \cosh \beta x$ and $\alpha = -\beta^2$. In this case, we have by (9.30)

$$\Delta_{\widetilde{\mu}} = \frac{d^2}{dx^2} + 2\beta \coth \beta x \frac{d}{dx}.$$

By Theorem 9.15, we obtain

$$\begin{aligned}\widetilde{p}_t(x,y) &= e^{\alpha t}\frac{p_t(x,y)}{h(x)h(y)} \\ &= \frac{1}{(4\pi t)^{1/2}}\frac{1}{\cosh \beta x \cosh \beta y}\exp\left(-\frac{|x-y|^2}{4t} - \beta^2 t\right).\end{aligned}$$

EXAMPLE 9.17. Using the notation of the previous Example, observe that the heat kernel (9.29) is a regular fundamental solution also on $(M, \mathrm{g}_{\mathbb{R}^n}, \mu)$ where $M := \mathbb{R}^n \setminus \{0\}$. If $n \geq 2$ then by Exercise 8.11, M is stochastically complete, which implies by Corollary 9.6 that p_t is the heat kernel also in M.

Assuming $n > 2$, consider the following function

$$h(x) = |x|^{2-n},$$

that is harmonic in M (cf. Exercise 3.24). Hence, (9.21) holds for this function with $\alpha = 0$. Defining measure $\widetilde{\mu}$ by $d\widetilde{\mu} = h^2 d\mu$, we obtain by Theorem 9.15 that the heat kernel in $(M, \mathrm{g}, \widetilde{\mu})$ is given by

$$\widetilde{p}_t(x,y) = \frac{1}{(4\pi t)^{n/2}}|x|^{n-2}|y|^{n-2}\exp\left(-\frac{|x-y|^2}{4t}\right).$$

EXAMPLE 9.18. Consider in $\mathbb{R}^1$ measure μ is given by

$$d\mu = e^{x^2}dx,$$

where dx is the Lebesgue measure. Then, by (9.30) with $h = e^{\frac{1}{2}x^2}$,

$$\Delta_\mu = \frac{d^2}{dx^2} + 2x\frac{d}{dx}. \tag{9.31}$$

We claim that the heat kernel $p_t(x,y)$ of $(\mathbb{R}, \mathrm{g}_{\mathbb{R}}, \mu)$ is given by the explicit formula:

$$p_t(x,y) = \frac{1}{(2\pi \sinh 2t)^{1/2}}\exp\left(\frac{2xye^{-2t} - x^2 - y^2}{1 - e^{-4t}} - t\right), \tag{9.32}$$

which is a modification of the *Mehler kernel* (cf. Exercise 11.18). It is a matter of a routine (but hideous) computation to verify that the function (9.32) does solve the heat equation and satisfy the conditions of Lemma 9.2, which implies that is it a regular fundamental solution. It is easy to see that

$$p_t(x,y) \leq \frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{|x-y|^2}{4t} - t\right),$$

which implies that $p_t(x,y) \to 0$ as $x \to \infty$ uniformly in t (cf. Exercise 1.5). Hence, we conclude by Theorem 9.7 that $p_t(x,y)$ is indeed the heat kernel.

Alternatively, this follows from Corollary 9.6 provided we know that the manifold $(\mathbb{R}, g_{\mathbb{R}}, \mu)$ is stochastically complete. To prove the latter, observe that $(\mathbb{R}, g_{\mathbb{R}}, \mu)$ fits the description of "a model with two ends" of Exercise 8.16. Its area function is $S(r) = e^{r^2}$, and the volume function

$$V(r) = \int_{[0,r]} e^{x^2} dx$$

is even and satisfies the condition

$$\int^{+\infty} \frac{V(r)}{S(r)} dr = \infty,$$

because

$$\frac{V(r)}{S(r)} \sim \frac{1}{2r} \text{ as } r \to +\infty.$$

Hence, by Exercise 8.16, $(\mathbb{R}, g_{\mathbb{R}}, \mu)$ is stochastically complete. Alternatively, this conclusion follows also from Theorem 11.8, which will be proved in Chapter 11, because $\mathbb{R}$ is geodesically complete and, for large r,

$$V(r) \le \exp\left(Cr^2\right).$$

EXAMPLE 9.19. Continuing the previous example, it easily follows from (9.31) that function

$$h(x) = e^{-x^2}$$

satisfies the equation

$$\Delta_\mu h + 2h = 0.$$

Clearly, the change of measure $d\widetilde{\mu} = h^2 d\mu$ is equivalent to

$$d\widetilde{\mu} = e^{-x^2} dx.$$

By Theorem 9.15 and (9.32), we obtain that the heat kernel $\widetilde{p}_t$ of $(\mathbb{R}, g_{\mathbb{R}}, \widetilde{\mu})$ is given by

$$\begin{aligned} \widetilde{p}_t(x,y) &= e^{2t} \frac{p_t(x,y)}{h(x)\,h(y)} = p_t(x,y) \exp\left(x^2 + y^2 + 2t\right) \\ &= \frac{1}{(2\pi \sinh 2t)^{1/2}} \exp\left(\frac{2xye^{-2t} - \left(x^2+y^2\right)e^{-4t}}{1 - e^{-4t}} + t\right). \end{aligned}$$

9.2.5. Heat kernel in $\mathbb{H}^3$. As was shown in Example 9.13, the heat kernel $p_t(x,y)$ in the hyperbolic space $\mathbb{H}^n$ is a function of $r = d(x,y)$ and t. The following formulas for $p_t(x,y)$ are known: if $n = 2m+1$ then

$$p_t(x,y) = \frac{(-1)^m}{(2\pi)^m (4\pi t)^{1/2}} \left(\frac{1}{\sinh r} \frac{\partial}{\partial r}\right)^m e^{-m^2 t - \frac{r^2}{4t}}, \tag{9.33}$$

and if $n = 2m+2$ then

$$p_t(x,y) = \frac{(-1)^m \sqrt{2}}{(2\pi)^m (4\pi t)^{3/2}} e^{-\frac{(2m+1)^2}{4} t} \left(\frac{1}{\sinh r} \frac{\partial}{\partial r}\right)^m \int_r^\infty \frac{s e^{-\frac{s^2}{4t}} ds}{(\cosh s - \cosh r)^{\frac{1}{2}}}. \tag{9.34}$$

In particular, the heat kernel in $\mathbb{H}^2$ is given by

$$p_t(x,y) = \frac{\sqrt{2}}{(4\pi t)^{3/2}} e^{-\frac{1}{4}t} \int_r^\infty \frac{se^{-\frac{s^2}{4t}} ds}{(\cosh s - \cosh r)^{\frac{1}{2}}}, \tag{9.35}$$

and the heat kernel in $\mathbb{H}^3$ is expressed by a particularly simple formula

$$p_t(x,y) = \frac{1}{(4\pi t)^{3/2}} \frac{r}{\sinh r} \exp\left(-\frac{r^2}{4t} - t\right). \tag{9.36}$$

Of course, once the formula is known, one can prove it by checking that it is a regular fundamental solution, and then applying Corollary 9.6, because $\mathbb{H}^n$ is stochastically complete.

We will give here a non-computational proof of (9.36), which to some extend also explains why the heat kernel has this form. Rename y by o and let (r,θ) be the polar coordinates (r,θ) in $\mathbb{H}^3 \setminus \{o\}$. By means of the polar coordinates, $\mathbb{H}^3$ can be identified with $\mathbb{R}^3$ and considered as a model (see Sections 3.10 and 9.2.3). The area function of $\mathbb{H}^3$ is given by

$$S(r) = 4\pi \sinh^2 r,$$

and the Laplacian in the polar coordinates is as follows:

$$\Delta_{\mathbb{H}^3} = \frac{\partial^2}{\partial r^2} + 2\coth r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \Delta_{\mathbb{S}^2}. \tag{9.37}$$

Denote by μ the Riemannian measure of $\mathbb{H}^3$. For a smooth positive function h on $\mathbb{H}^3$, depending only on r, consider the weighted model $(\mathbb{H}^3, \widetilde{\mu})$ where $d\widetilde{\mu} = h^2 d\mu$. The area function of $(\mathbb{H}^3, \widetilde{\mu})$ is given by

$$\widetilde{S}(r) = h^2(r) S(r).$$

Choose function h as follows:

$$h(r) = \frac{r}{\sinh r},$$

so that $\widetilde{S}(r) = 4\pi r^2$ is equal to the area function of $\mathbb{R}^3$. By a miraculous coincidence, function h happens to satisfy in $\mathbb{H}^3 \setminus \{o\}$ the equation

$$\Delta_\mu h + h = 0, \tag{9.38}$$

which follows from (9.37) by a straightforward computation. The function h extends by continuity to the origin o by setting $h(o) = 1$. In fact, the extended function is smooth[1] in $\mathbb{H}^3$, which can be seen, for example, by representing h in another coordinate system (cf. Exercise 3.23).

Denoting by $\widetilde{p}_t$ the heat kernel of $(\mathbb{H}^3, \widetilde{\mu})$, we obtain by Theorem 9.15 that

$$\widetilde{p}_t(x,y) = \frac{e^t p_t(x,y)}{h(x)h(y)}. \tag{9.39}$$

[1]It is true for any weighted manifold of dimension $n \geq 2$ that any bounded solution to (9.38) in a punctured neighborhood of a point extends smoothly to this point, but we do not prove this result here.

Since the area functions of the weighted models $(\mathbb{H}^3, \widetilde{\mu})$ and $\mathbb{R}^3$ are the same, we conclude by Theorem 9.14 that their heat kernels at the origin are the same, that is

$$\widetilde{p}_t(x,o) = \frac{1}{(4\pi t)^{3/2}} \exp\left(-\frac{r^2}{4t}\right).$$

Combining with (9.39), we obtain

$$p_t(x,o) = e^{-t}\widetilde{p}_t(x,o)\,h(x)\,h(o) = \frac{1}{(4\pi t)^{3/2}} \frac{r}{\sinh r} \exp\left(-\frac{r^2}{4t} - t\right),$$

which was to be proved.

Exercises.

9.1. Let μ be a measure in $\mathbb{R}^n$ defined by

$$d\mu = \exp(2c\cdot x)\,dx$$

where dx is the Lebesgue measure and c is a constant vector from $\mathbb{R}^n$. Prove that the heat kernel of $(\mathbb{R}^n, \mathbf{g}_{\mathbb{R}^n}, \mu)$ is given by

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-c\cdot(x+y) - |c|^2 t - \frac{|x-y|^2}{4t}\right). \tag{9.40}$$

9.2. *(Heat kernel in half-space)* Let

$$M = \left\{(x^1,\dots,x^n) \in \mathbb{R}^n : x^n > 0\right\}.$$

Prove that the heat kernel of M with the canonical Euclidean metric and the Lebesgue measure is given by

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \left(\exp\left(-\frac{|x-y|^2}{4t}\right) - \exp\left(-\frac{|x-\overline{y}|^2}{4t}\right)\right) \tag{9.41}$$

where $\overline{y}$ is the reflection of y at the hyperplane $x^n = 0$, that is,

$$\overline{y} = \left(y^1,\dots,y^{n-1},-y^n\right).$$

9.3. *(Heat kernel in Weyl's chamber)* Let

$$M = \left\{(x^1,\dots,x^n) \in \mathbb{R}^n : x^1 < x^2 < \dots < x^n\right\}.$$

Prove that the heat kernel of M with the canonical Euclidean metric and the Lebesgue measure is given by

$$p_t(x,y) = \det\left(p_t^{\mathbb{R}^1}\left(x^i,y^j\right)\right)_{i,j=1}^n, \tag{9.42}$$

where $p_t^{\mathbb{R}^1}$ is the heat kernel in $\mathbb{R}^1$.

9.4. Let $(M, \mathbf{g}, \mu)$ be a weighted manifold, and let h be a smooth positive function on M satisfying the equation

$$-\Delta_\mu h + \Phi h = 0, \tag{9.43}$$

where Φ is a smooth function on M. Define measure $\widetilde{\mu}$ on M by $d\widetilde{\mu} = h^2 d\mu$.

(a) Prove that, for any $f \in C^\infty(M)$,

$$\Delta_\mu f - \Phi f = h\Delta_{\widetilde{\mu}}\left(h^{-1}f\right). \tag{9.44}$$

(b) Prove that, for any $f \in \mathcal{D}(M)$,

$$\int_M \left(|\nabla f|^2 + \Phi f^2\right) d\mu \geq 0. \tag{9.45}$$

9.5. Applying (9.45) in $\mathbb{R}^n \setminus \{0\}$ with suitable functions h and Φ, prove the *Hardy inequality*: for any $f \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$,

$$\int_{\mathbb{R}^n} |\nabla f|^2 \, dx \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{f^2}{|x|^2} dx. \tag{9.46}$$

9.3. Eternal solutions

In this section, we consider solutions to the heat equation defined for all $t \in (-\infty, +\infty)$, which, hence, are called *eternal* solutions.

Let $u(t,x)$ be a regular fundamental solution at a point $y \in M$. Let us extend $u(t,x)$ to $t \leq 0$ by setting $u(t,x) \equiv 0$. Since for any $t \in \mathbb{R}$,

$$\int_M u(t,x) \, d\mu(x) \leq 1, \tag{9.47}$$

we see that $u(t,x) \in L^1_{loc}(\mathbb{R} \times M)$. In particular, $u(t,x)$ can be regarded as a distribution on $\mathbb{R} \times M$.

THEOREM 9.20. *Let $u(t,x)$ be a regular fundamental solution of the heat equation at $y \in M$, extended to $t \leq 0$ by setting $u(t,x) \equiv 0$. Then $u(t,x)$ satisfies in $\mathbb{R} \times M$ the following equation*

$$\frac{\partial u}{\partial t} - \Delta_\mu u = \delta_{(0,y)}. \tag{9.48}$$

Here $\delta_{(0,y)}$ is the delta function at the point $(0,y)$ on the manifold $\mathbb{R} \times M$, defined by

$$\left(\delta_{(0,y)}, \varphi\right) = \varphi(0,y) \quad \text{for any } \varphi \in \mathcal{D}(\mathbb{R} \times M).$$

The equation (9.48) means that $u(t,x)$ is a fundamental solution of the operator $\frac{\partial}{\partial t} - \Delta_\mu$ in $\mathbb{R} \times M$.

PROOF. The equation (9.48) is equivalent to the identity

$$-\int_{\mathbb{R} \times M} (\partial_t \varphi + \Delta_\mu \varphi) \, u \, d\mu dt = \varphi(0,y), \tag{9.49}$$

which should be satisfied for any $\varphi \in \mathcal{D}(\mathbb{R} \times M)$. Since $u \equiv 0$ for $t \leq 0$, the integral in (9.49) is equal to

$$\int_0^\infty \int_M (\partial_t \varphi + \Delta_\mu \varphi) \, u \, d\mu dt = \lim_{\varepsilon \to 0+} \int_\varepsilon^\infty \int_M (\partial_t \varphi + \Delta_\mu \varphi) \, u \, d\mu dt.$$

Next, we have, for any $\varepsilon > 0$,

$$
\begin{aligned}
&\int_\varepsilon^\infty \int_M (\partial_t \varphi + \Delta_\mu \varphi)\, u \, d\mu dt \\
= &\int_\varepsilon^\infty \int_M (\partial_t (\varphi u) - \varphi \partial_t u)\, d\mu dt + \int_\varepsilon^\infty \left(\int_M \Delta_\mu \varphi \, u \, d\mu \right) dt \\
= &\int_\varepsilon^\infty \left(\partial_t \int_M \varphi u \, d\mu \right) dt - \int_\varepsilon^\infty \int_M \varphi \, \partial_t u \, d\mu dt + \int_\varepsilon^\infty \int_M \varphi \Delta_\mu u \, d\mu dt \\
= &-\int_M \varphi(\varepsilon, \cdot)\, u(\varepsilon, \cdot)\, d\mu, \qquad (9.50)
\end{aligned}
$$

where we have used the Green formula for Δ_μ and the fact that $u(t,x)$ satisfies the heat equation for $t > 0$. We are left to verify that the integral in (9.50) tends to $\varphi(0,y)$ when $\varepsilon \to 0$. By the definition (9.1) of a fundamental solution, we have

$$
\int_M \varphi(0, \cdot)\, u(\varepsilon, \cdot)\, d\mu \to \varphi(0,y) \quad \text{as } \varepsilon \to 0. \qquad (9.51)
$$

Using the regularity (9.2) of the fundamental solution, we obtain

$$
\begin{aligned}
&\left| \int_M \varphi(\varepsilon, \cdot)\, u(\varepsilon, \cdot)\, d\mu - \int_M \varphi(0, \cdot)\, u(\varepsilon, \cdot)\, d\mu \right| \\
= &\left| \int_M (\varphi(\varepsilon, \cdot) - \varphi(0, \cdot))\, u(\varepsilon, \cdot)\, d\mu \right| \\
\leq &\sup_{x \in M} |\varphi(\varepsilon, x) - \varphi(0, x)| \longrightarrow 0 \ \text{ as } \varepsilon \to 0.
\end{aligned}
$$

Together with (9.51), this proves that the integral in (9.50) tends to $\varphi(0,y)$, which was to be proved. □

Since the heat kernel $p_t(x,y)$ is a regular fundamental solution for any fixed y (or x), Theorem 9.20 can be applied to it as well. The next statement contains the ultimate result on the smoothness of the heat kernel jointly in t, x, y. Set

$$
\operatorname{diag} := \{(x,y) \in M \times M : x = y\},
$$

and denote by Δ_x, Δ_y the operator Δ_μ with respect to the variables x and y, respectively.

COROLLARY 9.21. *Let us extend $p_t(x,y)$ for $t \leq 0$ by setting $p_t(x,y) \equiv 0$. Then, as a function on $\mathbb{R} \times M \times M$, the heat kernel $p_t(x,y)$ is C^∞ smooth away from $\{0\} \times \operatorname{diag}$ (see Fig. 9.3), and it satisfies in this domain the equation*

$$
\frac{\partial p_t}{\partial t} = \Delta_x p_t = \Delta_y p_t. \qquad (9.52)
$$

PROOF. Let $N = M \times M$ be the product manifold with the product measure $d\nu = d\mu \, d\mu$. It follows from (9.47) that $p_t(x,y) \in L^1_{loc}(\mathbb{R} \times N)$. If we show that $p_t(x,y)$ satisfies in $\mathbb{R} \times N \setminus \{0\} \times \operatorname{diag}$ the equation (9.52) in

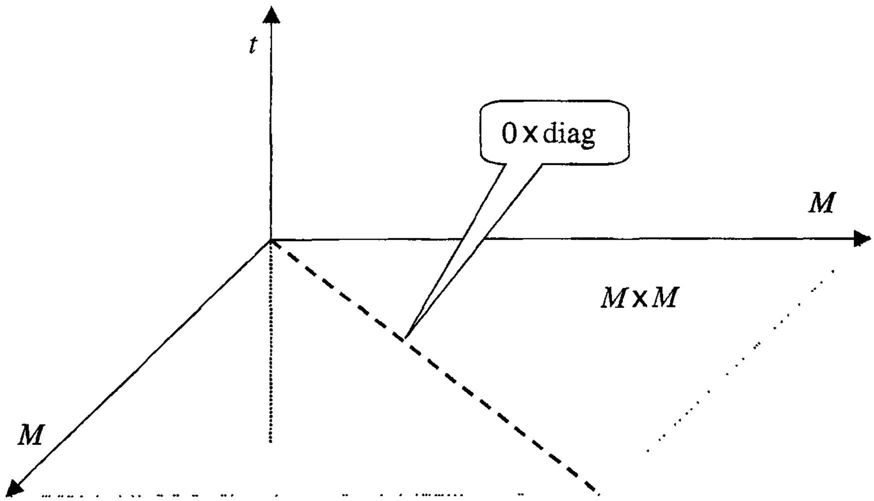

FIGURE 9.3. The heat kernel is singular at the set $0 \times$ diag

the distributional sense, then this will imply the C^∞-smoothness of $p_t(x,y)$ in this domain. Indeed, the Laplace operator Δ_ν on N can be represented as $\Delta_\nu = \Delta_x + \Delta_y$ (cf. the proof of Theorem 7.20), so that (9.52) implies

$$\partial_t p_t = \frac{1}{2}\Delta_\nu p_t.$$

Hence, the function $p_t(x,y)$ satisfies the heat equation in $t \in \mathbb{R}$ (up to the change $t \mapsto 2t$) and $(x,y) \in N$ (away from $\{0\} \times$ diag) which implies the C^∞-smoothness of $p_t(x,y)$ by Theorem 7.4.

Since $p_t(x,y)$ is symmetric in x, y, it suffices to prove the first of the equations (9.52). This equation is equivalent to the identity

$$\int_{\mathbb{R}\times M\times M} (\partial_t\varphi + \Delta_x\varphi)\, p_t(x,y)\, d\mu(x)\, d\mu(y)\, dt = 0, \tag{9.53}$$

which should be satisfied for any function $\varphi(t,x,y) \in \mathcal{D}(\mathbb{R}\times N)$ supported away from $\{0\}\times$ diag. Expanding the integral in (9.53) by Fubini's theorem with the external integration in y, we see that it suffices to prove that, for any $y \in M$,

$$\int_{\mathbb{R}\times M} (\partial_t\varphi + \Delta_x\varphi)\, p_t(x,y)\, d\mu(x) dt = 0. \tag{9.54}$$

Since y is fixed here, we obtain by Theorem 9.20 (more precisely, by (9.49)) that

$$\int_{\mathbb{R}\times M} (\partial_t\varphi + \Delta_x\varphi)\, p_t(x,y)\, d\mu(x) dt = -\varphi(0,x,y)\,|_{x=y} = -\varphi(0,x,y)\,.$$

By hypothesis, we have $\varphi(0,y,y) = 0$ whence (9.54) follows. □

REMARK 9.22. Let M be connected so that the heat kernel $p_t(x,y)$ is strictly positive for $t > 0$ (cf. Corollary 8.12). Considering function

$u(t,x) = p_t(x,y)$ in $\mathbb{R} \times (M \setminus \{y\})$, we obtain an example of a solution to the heat equation which is identical zero for $t \leq 0$ and strictly positive for $t > 0$. This example shows that, in the parabolic strong minimum principle (Theorem 8.11), the time direction is essential: the fact that a non-negative solution vanishes at a point does not imply that it will vanish in the future, although it does imply that it was identical zero in the past.

Exercises.

9.6. Prove that if u and v are two regular fundamental solutions at point $y \in M$ then the difference $u - v$ is a C^∞-smooth function on $\mathbb{R} \times M$ satisfying in $\mathbb{R} \times M$ the heat equation.

9.7. Let $\Omega \subset M$ be an open set. Prove that the function $u_t(x,y) := p_t(x,y) - p_t^\Omega(x,y)$ is C^∞ smooth jointly in $t \in \mathbb{R}$ and $x, y \in \Omega$.

9.8. Let a smooth function $u(t,x)$ on $\mathbb{R}_+ \times M$ satisfy the following conditions

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_\mu u & \text{in } \mathbb{R}_+ \times M, \\ u(t,\cdot) \xrightarrow{L^1_{loc}} f & \text{as } t \to 0, \end{cases} \tag{9.55}$$

where $f \in L^1_{loc}(M)$. Extend $u(t,x)$ to $t \leq 0$ by setting $u(t,x) \equiv 0$.

(a) Prove that the function $u(t,x)$ satisfies in $\mathbb{R} \times M$ the equation

$$\frac{\partial u}{\partial t} - \Delta_\mu u = F, \tag{9.56}$$

where F is a distribution on $\mathbb{R} \times M$ defined by

$$(F, \varphi) = \int_M \varphi(0,x) f(x) \, d\mu(x),$$

for any $\varphi \in \mathcal{D}(\mathbb{R} \times M)$.

(b) Prove that if in (9.55) $f \equiv 0$ in M then $u \in C^\infty(\mathbb{R} \times M)$.

(c) Prove that if $f \in C^\infty(M)$ then

$$u(t,\cdot) \xrightarrow{C^\infty(M)} f \text{ as } t \to 0+.$$

Consequently, the function

$$\widetilde{u}(t,x) = \begin{cases} u(t,x), & t > 0, \\ f(x), & t \leq 0, \end{cases}$$

belongs to $C^\infty(\mathbb{R} \times M)$.

HINT. Use Exercise 7.19.

9.9. Prove that, on any weighted manifold M, for any open set Ω, any compact set $K \subset \Omega$, and any $N > 0$,

$$\sup_{x \in K} \int_{\Omega^c} p_t(x,y) \, d\mu(y) = o\left(t^N\right) \text{ as } t \to 0. \tag{9.57}$$

9.10. Define the *resolvent kernel* $r_\alpha(x,y)$ by

$$r_\alpha(x,y) = \int_0^\infty e^{-\alpha t} p_t(x,y) \, dt. \tag{9.58}$$

Prove that, for any $\alpha > 0$, $r_\alpha(x,y)$ is a non-negative smooth function on $M \times M \setminus \text{diag}$. Furthermore, for any $y \in M$, $r_\alpha(\cdot,y)$ satisfies the equation

$$-\Delta_\mu r_\alpha + \alpha r_\alpha = \delta_y. \tag{9.59}$$

Notes

Theorem 9.5 was proved by Dodziuk [**108**].

Theorem 9.14 can be regarded as a simple model case for comparison theorems of [**58**] and [**104**].

The idea of changing the measure by $d\widetilde{\mu} = h^2 d\mu$, where h is a harmonic function, is widely used in the theory of stochastic processes where it is referred to as *Doob's h-transform* (probabilistically this means conditioning of the diffusion process to exit in the direction h on the Martin boundary).

The *Mehler kernel* is by definition the heat kernel of the operator $-\frac{d^2}{dx^2} + \left(x^2 - 1\right)$ that is the Hamiltonian of the quantum harmonic oscillator. The Mehler kernel is given explicitly by the *Mehler formula* (11.21) (see Exercise 11.18). The formula (9.32) for the heat kernel in $(\mathbb{R}, \mathbf{g}_{\mathbb{R}}, e^{x^2}dx)$ is equivalent to the Mehler formula. The proof of the latter can be found in [**92**].

The formula (9.35) for the heat kernel $p_t^{\mathbb{H}^2}$ in $\mathbb{H}^2$ was stated by McKean [**272**][2], and the formula (9.36) for $p_t^{\mathbb{H}^3}$ was stated in [**104**] (the derivation of (9.36) in Section 9.2.5 seems to be new). The formulas (9.33) and (9.34) for $p_t^{\mathbb{H}^n}$ follow then by the recursive relation between $p_t^{\mathbb{H}^n}$ and $p_t^{\mathbb{H}^{n-2}}$ (see [**51**], [**104**], [**96**]). A direct proof of (9.33) and (9.34), based on the reduction to the wave equation, can be found in [**175**]. Convenient explicit estimates of $p_t^{\mathbb{H}^n}$ can be found in [**102**].

The extension of the heat kernel to negative times is a standard procedure for evolution equations (see, for example, [**356**]).

[2]It was privately communicated to the author by Peter Laurence, that Henry McKean attributed (9.35) to the book [**311**, p.154].

CHAPTER 10

Spectral properties

Here we consider some spectral properties of the Dirichlet Laplace operator such as the discreetness of the spectrum, the positivity of the bottom of the spectrum, and others. The notion of the bottom of the spectrum will be essentially used in Chapters 13, 14, 15.

10.1. Spectra of operators in Hilbert spaces

We start with some basic properties of spectra of self-adjoint operators in a Hilbert space. The knowledge of the relevant material from Appendix A is assumed here.

10.1.1. General background. Let A be a densely defined self-adjoint operator in a Hilbert space $\mathcal{H}$. Denote by $\lambda_{\min}(A)$ the *bottom of the spectrum* of A, that is,

$$\lambda_{\min}(A) := \inf \operatorname{spec} A.$$

Since $\operatorname{spec} A$ is a closed subset of $\mathbb{R}$, $\lambda_{\min}(A)$ is the minimal point of $\operatorname{spec} A$ provided $\lambda_{\min}(A) > -\infty$. It is a general fact that $\lambda_{\min}(A)$ admits the following variational characterization:

$$\lambda_{\min}(A) = \inf_{x \in \operatorname{dom} A \setminus (0)} \frac{(Ax, x)}{\|x\|^2} \tag{10.1}$$

(cf. Exercise A.26).

DEFINITION 10.1. The *discrete spectrum* of A consists of all $\alpha \in \operatorname{spec} A$ such that

- α is an eigenvalue of A of a finite multiplicity;
- and α is an isolated point of $\operatorname{spec} A$, that is, for some $\varepsilon > 0$, the interval $(\alpha - \varepsilon, \alpha + \varepsilon)$ contains no other points from $\operatorname{spec} A$, except for α.

The *essential spectrum* of A is the complement in $\operatorname{spec} A$ of the discrete spectrum of A.

It easily follows from the definition that the discrete spectrum is at most countable, and any point of accumulation of the discrete spectrum belongs to the essential spectrum or is $\pm\infty$.

Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral resolution of A. For any Borel set $U \subset \mathbb{R}$, set

$$E_U := 1_U(A) = \int_U dE_\lambda = \int_{U\cap \operatorname{spec} A} dE_\lambda.$$

The operator E_U is a projector in $\mathcal{H}$ (cf. Exercise A.27). Moreover, if α is an eigenvalue of A then $E_{\{\alpha\}}$ is the projector onto the eigenspace $\ker(A - \alpha\,\mathrm{id})$ of α (cf. Exercise A.28).

LEMMA 10.2. *Let S be the essential spectrum of A. Then the space $(\operatorname{ran} E_S)^\perp$ (the orthogonal complement of $\operatorname{ran} E_S$ in $\mathcal{H}$) admits at most countable orthonormal basis $\{v_k\}_{k=1}^N$ such that each v_k is an eigenvector of A. Moreover, if λ_k is the eigenvalue of v_k then the sequence $\{\lambda_k\}_{k=1}^N$ consists of all the points of the discrete spectrum of A counted with multiplicities.*

In particular, if S is empty, that is, if the entire spectrum of A is discrete, then $\operatorname{ran} E_S = \{0\}$ and, hence, such a basis $\{v_k\}$ exists in the entire space $\mathcal{H}$. Assuming that $\dim \mathcal{H} = \infty$, we obtain that in this case the basis $\{v_k\}$ is countable, and $|\lambda_k| \to \infty$ as $k \to \infty$, because ∞ is the only possible accumulation point of the sequence $\{|\lambda_k|\}$.

PROOF. Let $\{\alpha_i\}$ be a sequence of all distinct points in the discrete spectrum of A, enumerated in some order. By the spectral theorem, we have

$$\mathrm{id} = \int_{\operatorname{spec} A} dE_\lambda = E_S + \sum_i E_{\{\alpha_i\}},$$

whence it follows that, for any $x \perp \operatorname{ran} E_S$,

$$x = \sum_i E_{\{\alpha_i\}} x \tag{10.2}$$

(a priori, the convergence of the series in (10.2) is weak, which, however, implies the strong convergence by Exercise A.4). Since $\operatorname{ran} E_{\{\alpha_i\}}$ is the eigenspace of the eigenvalue α_i, it admits an orthonormal basis that consists of the eigenvectors of A. Since the eigenspaces of different eigenvalues are orthogonal, merging the bases of the eigenspaces across all α_i, we obtain an orthonormal sequence, say $\{v_k\}$ (this sequence is at most countable because each eigenspace is finitely dimensional and the number of points α_i is at most countable). Since $E_{\{\alpha_i\}}x$ is a linear combination of some vectors v_k, it follows from (10.2) that every $x \in (\operatorname{ran} E_S)^\perp$ can be expanded into a series $\sum_k c_k v_k$, which means that $\{v_k\}$ is a basis in $(\operatorname{ran} E_S)^\perp$.

Let now λ_k be the eigenvalue of v_k. By construction, the number of the eigenvectors v_k with the given eigenvalue α_i is equal to $\dim \operatorname{ran} E_{\{\alpha_i\}}$, which is the multiplicity of α_i. Hence, each α_i is counted in the sequence $\{\lambda_k\}$ as many times as its multiplicity. □

10.1.2. Counting measure. The *counting measure* of the operator A is the following function defined on all Borel sets $U \subset \mathbb{R}$:

$$m(U) := \dim \operatorname{ran} E_U. \tag{10.3}$$

The value of $m(U)$ is a non-negative integer or $+\infty$.

As was mentioned above, if α is a real number then

$$\operatorname{ran} E_{\{\alpha\}} = \ker(A - \alpha \,\mathrm{id}),$$

which implies

$$m(\{\alpha\}) = \dim \ker(A - \alpha \,\mathrm{id}). \tag{10.4}$$

Hence, if α is an eigenvalue of A then $m(\{a\})$ is the multiplicity of the eigenvalue α; otherwise, $m(\{\alpha\}) = 0$.

THEOREM 10.3. *Let $m(U)$ be the counting measure of a self-adjoint operator A. Then the following is true:*

(*i*) *$m(U)$ is a Borel measure on $\mathbb{R}$.*
(*ii*) *For any open interval $U \subset \mathbb{R}$, $m(U) > 0$ if and only if the intersection $U \cap \operatorname{spec} A$ is non-empty.*
(*iii*) *A point $\alpha \in \operatorname{spec} A$ belongs to the discrete spectrum of A if and only if $m(U) < \infty$ for some open interval U containing α.*

PROOF. (*i*) The proof of this part consists of two claims.

CLAIM 1. *If U and V two disjoint Borel subsets of $\mathbb{R}$ then*

$$m(U \cup V) = m(U) + m(V). \tag{10.5}$$

Indeed, if U and V disjoint then $1_U 1_V = 0$ and hence $E_U E_V = 0$, that is, the ranges of E_U and E_V are orthogonal subspaces. On the other hand, $1_{U \cup V} = 1_U + 1_V$ whence

$$E_{U \cup V} = E_U + E_V.$$

Therefore, $E_{U \cup V}$ is the projector onto $\operatorname{ran} E_U \oplus \operatorname{ran} E_V$ whence

$$\dim \operatorname{ran} E_{U \cup V} = \dim \operatorname{ran} E_U + \dim \operatorname{ran} E_V,$$

which is equivalent to (10.5).

Consequently, we obtain from Claim 1 that if $U \subset V$ then $m(U) \leq m(V)$, because $m(V) = m(U) + m(V \setminus U)$.

CLAIM 2. *If $\{U_k\}_{k=1}^{\infty}$ is a sequence of disjoint Borel sets in $\mathbb{R}$ and $U = \bigcup_k U_k$ then*

$$m(U) = \sum_{k=1}^{\infty} m(U_k). \tag{10.6}$$

Consider first a particular case when $m(U_k) = 0$ for all k and show that $m(U) = 0$. The condition $m(U_k) = 0$ means that $E_{U_k} = 0$. Since

$$1_U = \sum_k 1_{U_k},$$

we obtain by Lemma 4.8 that, for any $x \in \mathcal{H}$,

$$E_U x = \int_{\mathbb{R}} \left(\sum_k 1_{U_k} \right) dE_\lambda x = \sum_k \int_{\mathbb{R}} 1_{U_k} dE_\lambda x = \sum_k E_{U_k} x = 0,$$

whence $E_U = 0$ and $m(U) = 0$.

In the general case, the previous Claim implies that

$$m(U) \geq \sum_k m(U_k).$$

If $\sum_k m(U_k) = \infty$ then this yields (10.6). If $\sum_k m(U_k) < \infty$ then only finitely many terms $m(U_k)$ are non-zero, say, for $k = 1, 2, ..., K$. Let

$$V = \bigcup_{k>K} U_k$$

so that $m(V) = 0$ by the first part of the proof. Since U is a disjoint union of V and $U_1, ..., U_K$, we obtain by the previous Claim, that

$$m(U) = \sum_{k=1}^{K} m(U_k) + m(V) = \sum_{k=1}^{K} m(U_k) = \sum_{k=1}^{\infty} m(U_k).$$

(*ii*) Let φ be any continuous function on $\mathbb{R}$ supported in U and such that $0 \leq \varphi \leq 1$ and $\varphi(\lambda) = 1$ for some $\lambda \in U \cap \operatorname{spec} A$. Then by (A.53)

$$\|\varphi(A)\| = \sup_{\operatorname{spec} A} |\varphi| = 1,$$

so that there is $x \in \mathcal{H} \setminus \{0\}$ such that $\varphi(A) x \neq 0$. Then we have by (A.50)

$$\begin{aligned} \|E_U x\|^2 &= \int_{\mathbb{R}} 1_U(\lambda)^2 \, d\|E_\lambda x\|^2 = \int_U d\|E_\lambda x\|^2 \\ &\geq \int_U \varphi(\lambda)^2 \, d\|E_\lambda x\|^2 = \|\varphi(A) x\|^2 > 0, \end{aligned}$$

whence it follows that $E_U \neq 0$ and, hence, $m(U) > 0$.

(*iii*) Let α belong to the discrete spectrum of A. Then there is an open interval U containing no spectrum of A except for α, whence it follows by parts (*i*), (*ii*) and (10.4), that

$$m(U) = m(\{a\}) = \dim \ker (A - \alpha \operatorname{id}) < \infty.$$

Conversely, assume that $m(U) < \infty$ for some open interval U containing α. Since m is a σ-additive measure, we have

$$m(\{\alpha\}) = \inf_{U \ni \alpha} m(U),$$

where the infimum is taken over all open intervals U containing α. By part (*ii*), we have $m(U) \geq 1$ for any such interval U, and by hypothesis, we have $m(U) < \infty$ for some interval U. Hence, we conclude that

$$1 \leq m(\{\alpha\}) < \infty,$$

which together with (10.4) implies that α is an eigenvalue of A of a finite multiplicity.

Let us show that α is an isolated point of the spectrum. If not, then there exists a sequence $\{\alpha_k\} \subset \operatorname{spec} A$ such that $\alpha_k \to \alpha$, all α_k are disjoint, and $\alpha_k \neq \alpha$. There exists a sequence $\{U_k\}$ of disjoint open intervals such that U_k contains α_k and $U_k \to \alpha$. Then any open interval U containing α, contains infinitely many of the intervals U_k, whence it follows by parts (i) and (ii) that

$$m(U) \geq \sum_{U_k \subset U} m(U_k) \geq \sum_{U_k \subset U} 1 = \infty,$$

thus, contradicting the hypothesis $m(U) < \infty$. □

Exercises.

10.1. Let (X, d) be a separable metric space and $S \subset X$ be a subset of X. Prove that if all points of S are isolated then S is at most countable.

10.1.3. Trace.

In this section, A is a densely defined self-adjoint operator in a Hilbert space $\mathcal{H}$ such that

$$\operatorname{spec} A \in [0, +\infty). \tag{10.7}$$

The condition (10.7) is equivalent to A being non-negative definite, that is, to

$$(Ax, x) \geq 0 \text{ for all } x \in \operatorname{dom} A$$

(cf. Exercise A.26).

Define the *trace* of such an operator by

$$\operatorname{trace} A = \int_{(0,+\infty)} \lambda dm(\lambda), \tag{10.8}$$

where m is the counting measure of A defined by (10.3). Note that the point 0 is excluded from the domain of integration in (10.8), and that trace A takes values in $[0, +\infty]$.

LEMMA 10.4. *If $\{v_k\}$ is an orthonormal basis in $\mathcal{H}$ such that all $v_k \in \operatorname{dom} A$ then*

$$\operatorname{trace} A = \sum_k (Av_k, v_k). \tag{10.9}$$

PROOF. We have by (A.49)

$$(Av_k, v_k) = \int_{\operatorname{spec} A} \lambda d(E_\lambda v_k, v_k) = \int_{[0,+\infty)} \lambda d\|E_\lambda v_k\|^2 = \int_{(0,+\infty)} \lambda d\|E_\lambda v_k\|^2. \tag{10.10}$$

Fix a Borel set $U \subset \mathbb{R}$ and let $\{u_i\}$ be an orthonormal basis in $\operatorname{ran} E_U$. Then

$$E_U v_k = \sum_i (v_k, u_i) u_i$$

and, applying twice the Parseval Identity, we obtain

$$\|E_U v_k\|^2 = \sum_i (v_k, u_i)^2$$

and

$$\begin{aligned}\sum_k \|E_U v_k\|^2 &= \sum_k \sum_i (v_k, u_i)^2 = \sum_i \sum_k (v_k, u_i)^2 \\ &= \sum_i \|u_i\|^2 = \sum_i 1 = \dim \operatorname{ran} E_U = m(U). \quad (10.11)\end{aligned}$$

Since the right hand side of (10.10) is a Lebesgue integral against the measure $U \mapsto \|E_U v_k\|^2$, when adding up in k we obtain a Lebesgue integral against the measure $m(U)$, that is,

$$\sum_k (Av_k, v_k) = \int_{(0,+\infty)} \lambda dm(\lambda) = \operatorname{trace} A,$$

which was to be proved. □

REMARK 10.5. The identity (10.9) can be used as the alternative definition of the trace. In this case, Lemma 10.4 means that the definition of the trace is independent of the choice of the basis $\{v_k\}$. For a direct proof of this fact see Exercise 10.4.

LEMMA 10.6. *For any non-negative Borel function φ on $[0, +\infty)$,*

$$\operatorname{trace} \varphi(A) = \int_S \varphi(\lambda)\, dm(\lambda), \tag{10.12}$$

where

$$S := \{\lambda \geq 0 : \varphi(\lambda) > 0\}. \tag{10.13}$$

PROOF. By (10.9), we have

$$\operatorname{trace} \varphi(A) = \sum_k (\varphi(A) v_k, v_k)$$

where $\{v_k\}$ is an orthonormal basis. Similarly, to (10.10), we have

$$(\varphi(A) v_k, v_k) = \int_{[0,+\infty)} \varphi(\lambda)\, d\|E_\lambda v_k\|^2 = \int_S \varphi(\lambda)\, d\|E_\lambda v_k\|^2.$$

Summing up in k and using (10.11), we obtain (10.12). □

LEMMA 10.7. *Let φ be a non-negative continuous function on $[0, +\infty)$.*

(i) *If* $\operatorname{trace} \varphi(A) < \infty$ *then the spectrum of A in the set S is discrete, where S is defined by (10.13).*

(ii) *If the spectrum of A in S is discrete then*

$$\operatorname{trace} \varphi(A) = \sum_k \varphi(\lambda_k), \tag{10.14}$$

where $\{\lambda_k\}$ is the sequence of all the eigenvalues of A in S counted with multiplicities.

PROOF. (*i*) Let $\alpha \in S$ be a point of the essential spectrum of A. By Theorem 10.3, for any open interval U containing α, we have $m(U) = \infty$. Since φ is continuous, the set S is open and, hence, there is a bounded open interval $U \subset S$ containing α. Furthermore, we can assume that the closed interval $\overline{U}$ is also contained in S. Then by (10.12)

$$\operatorname{trace} \varphi(A) \geq \int_U \varphi(\lambda)\, dm(\lambda) \geq \left(\inf_{\overline{U}} \varphi\right) m(U) = \infty,$$

because $\inf_{\overline{U}} \varphi > 0$ and $m(U) = \infty$.

(*ii*) By hypothesis, the set $S \cap \operatorname{spec} A$ consists of isolated eigenvalues of finite multiplicity. In particular, the set $S \cap \operatorname{spec} A$ is at most countable and, hence, can be enumerated as a sequence $\{\alpha_i\}$ (where each eigenvalue is counted once). The set $S \setminus \operatorname{spec} A$ is open and is outside the spectrum. It follows from Theorem 10.3, that measure m does not charge this set. Hence, measure m in S sits at the sequence $\{\alpha_i\}$ whence by (10.12)

$$\operatorname{trace} \varphi(A) = \int_{\{\alpha_i\}} \varphi(\lambda)\, dm(\lambda) = \sum_i \varphi(\alpha_i)\, m(\{\alpha_i\}).$$

Noticing that by (10.4) $m(\{\alpha_i\})$ is the multiplicity of the eigenvalue α_i, we obtain (10.14). $\square$

Exercises.

10.2. Prove that, for any Borel set U,

$$m(U) = \operatorname{trace} E_U.$$

10.3. Prove that if A is a non-negative definite self-adjoint operator with a finite trace then A is a compact operator.

10.4. For any non-negative definite operator A with $\operatorname{dom} A = \mathcal{H}$, define its trace by

$$\operatorname{trace} A = \sum_k (Av_k, v_k),$$

where $\{v_k\}$ is any orthonormal basis of $\mathcal{H}$. Prove that the trace does not depend on the choice of the basis $\{v_k\}$.

10.2. Bottom of the spectrum

The Dirichlet Laplace operator $\mathcal{L} = -\Delta_\mu|_{W_0^2(M,\mu)}$ constructed in Section 4.2, is a self-adjoint operator, and $\operatorname{spec} \mathcal{L} \subset [0, +\infty)$. Here we investigate further properties of the spectrum of $\mathcal{L}$. Denote by $\lambda_{\min}(M)$ the bottom of the spectrum of $\mathcal{L}$, that is

$$\lambda_{\min}(M) = \inf \operatorname{spec} \mathcal{L}.$$

This notation reflect the point of view that the spectral properties of $\mathcal{L}$ are regarded as the properties of manifold M itself.

For any non-zero function $f \in W^1(M)$, define its *Rayleigh quotient* by

$$\mathcal{R}(f) := \frac{\int_M |\nabla f|^2 \, d\mu}{\int_M f^2 d\mu}. \tag{10.15}$$

THEOREM 10.8. (The variational principle) *The following identity is true*

$$\lambda_{\min}(M) = \inf_{f \in \mathcal{T} \setminus \{0\}} \mathcal{R}(f), \tag{10.16}$$

where $\mathcal{T}$ is any class of functions such that

$$C_0^\infty(M) \subset \mathcal{T} \subset W_0^1(M). \tag{10.17}$$

Furthermore, the infimum in (10.16) *can be restricted to non-negative functions $f \in \mathcal{T}$.*

PROOF. It is obvious that the functional $\mathcal{R}$ is continuous on $W^1 \setminus \{0\}$. Since $C_0^\infty \subset W_0^1$ and that C_0^∞ is dense in W_0^1 in W^1-norm, the infimum in the right hand side of (10.16) is the same for any functional class $\mathcal{T}$ satisfying (10.17). Since $C_0^\infty \subset W_0^2 \subset W_0^1$, it suffices to verify (10.16) for $\mathcal{T} = W_0^2 = \operatorname{dom} \mathcal{L}$.

By (10.1), we have

$$\inf \operatorname{spec} \mathcal{L} = \inf_{f \in \operatorname{dom} \mathcal{L} \setminus \{0\}} \frac{(\mathcal{L}f, f)}{\|f\|^2}. \tag{10.18}$$

By Lemma 4.4, we obtain, for any $f \in \operatorname{dom} \mathcal{L}$,

$$(\mathcal{L}f, f)_{L^2} = -\int_M f \Delta_\mu f \, d\mu = \int_M |\nabla f|^2 \, d\mu$$

whence

$$\inf \operatorname{spec} \mathcal{L} = \inf_{f \in \operatorname{dom} \mathcal{L} \setminus \{0\}} \frac{\int_M |\nabla f|^2 \, d\mu}{\int_M f^2 d\mu},$$

which proves (10.16).

Let us show that the infimum in (10.16) can be restricted to *non-negative* functions, that is,

$$\lambda_{\min}(M) = \inf_{0 \le f \in \mathcal{T} \setminus \{0\}} \mathcal{R}(f). \tag{10.19}$$

It suffices to consider the borderline cases $\mathcal{T} = C_0^\infty$ and $\mathcal{T} = W_0^1$. By Lemma 5.4, for any non-negative function $f \in W_0^1$ there is a sequence $\{f_k\}$ of non-negative functions from C_0^∞ that converges to f in W^1. Therefore, the right hand side of (10.19) has the same value for $\mathcal{T} = C_0^\infty$ and $\mathcal{T} = W_0^1$.

Hence, it suffices to prove (10.19) in the case $\mathcal{T} = W_0^1$. For simplicity of notation, let us allow also $f = 0$ in (10.19) by setting $\mathcal{R}(0) = +\infty$. As follows from Lemma 5.2, for any $f \in W_0^1$, also the functions f_+ and f_- belong to W_0^1 and

$$\nabla f_+ = 1_{\{f>0\}} \nabla f \ \text{ and } \ \nabla f_- = -1_{\{f<0\}} \nabla f \tag{10.20}$$

(see (5.9) and (5.10)). Let us show that

$$\mathcal{R}(f) \geq \min\left(\mathcal{R}(f_+), \mathcal{R}(f_-)\right). \tag{10.21}$$

If $f_+ = 0$ or $f_- = 0$ then this is obvious. Otherwise, observe that, by (10.20), ∇f_+ and ∇f_- are orthogonal in $\vec{L}^2$. Since $f = f_+ - f_-$, we obtain

$$\mathcal{R}(f) = \frac{\|\nabla f\|^2}{\|f\|^2} = \frac{\|\nabla f_+\|^2 + \|\nabla f_-\|^2}{\|f_+\|^2 + \|f_-\|^2} \geq \min\left(\mathcal{R}(f_+), \mathcal{R}(f_-)\right).$$

Set $\mathcal{R}(0) = 0$ so that the infimum in (10.19) can be taken for all $f \in W_0^1$. It follows from (10.21) that

$$\inf_{f \in W_0^1} \mathcal{R}(f) \geq \inf_{f \in W_0^1} \mathcal{R}(f_+) = \inf_{0 \leq f \in W_0^1} \mathcal{R}(f),$$

whereas the opposite inequality

$$\inf_{f \in W_0^1} \mathcal{R}(f) \leq \inf_{0 \leq f \in W_0^1} \mathcal{R}(f)$$

is trivial. We conclude that

$$\lambda_{\min}(M) = \inf_{f \in W_0^1} \mathcal{R}(f) = \inf_{0 \leq f \in W_0^1} \mathcal{R}(f),$$

which finishes the proof. $\square$

EXAMPLE 10.9. Let us show that

$$\lambda_{\min}(\mathbb{R}^n) = 0. \tag{10.22}$$

Choose a non-zero function $\varphi \in C_0^\infty(\mathbb{R}^n)$ and set

$$\varphi_k(x) = \varphi(x/k), \quad k = 1, 2, \ldots.$$

Then we have

$$\int_{\mathbb{R}^n} \varphi_k^2(x)\, dx = k^n \int_{\mathbb{R}^n} \varphi^2(x)\, dx$$

and

$$\int_{\mathbb{R}^n} |\nabla \varphi_k|^2\, dx = k^{n-2} \int_{\mathbb{R}^n} |\nabla \varphi|^2\, dx$$

whence

$$\mathcal{R}(\varphi_k) = k^{-2} \mathcal{R}(\varphi).$$

Letting $k \to \infty$, we obtain (10.22).

It is possible to show that the spectrum of the Dirichlet Laplace operator in $\mathbb{R}^n$ is the interval $[0, +\infty)$.

THEOREM 10.10. *Assume that the infimum of $\mathcal{R}(f)$ in (10.16) is attained on a function $f \in W_0^1(M) \setminus \{0\}$. Then $f \in \operatorname{dom} \mathcal{L}$ and*

$$\mathcal{L} f = \lambda_{\min}(M)\, f.$$

PROOF. Denote for simplicity $\lambda = \lambda_{\min}(M)$ and observe that, for any $\varphi \in C_0^\infty(M)$ and real t, we have

$$\mathcal{R}(f + t\varphi) \geq \lambda = \mathcal{R}(f),$$

that is

$$\|\nabla(f + t\varphi)\|^2 - \lambda\|f + t\varphi\|^2 \geq 0 = \|\nabla f\|^2 - \lambda\|f\|^2.$$

We have

$$\|\nabla f + t\nabla\varphi\|^2 = \|\nabla f\|^2 + 2t(\nabla f, \nabla\varphi) + t^2\|\nabla\varphi\|^2$$

and

$$\|f + t\varphi\|^2 = \|f\|^2 + 2(f, \varphi) + t^2\|\varphi\|^2,$$

whence

$$\|\nabla(f + t\varphi)\|^2 - \lambda\|f + t\varphi\|^2 = 2t((\nabla f, \nabla\varphi) - \lambda(f, \varphi)) + t^2\left(\|\nabla\varphi\|^2 - \lambda\varphi^2\right).$$

Since the left hand side is non-negative for all real t, the linear in t term in the right hand side must vanish, that is

$$(\nabla f, \nabla\varphi) - \lambda(f, \varphi) = 0.$$

This implies

$$(f, \Delta_\mu\varphi)_{\mathcal{D}} + \lambda(f, \varphi)_{\mathcal{D}} = 0$$

whence it follows that $\Delta_\mu f + \lambda f = 0$ in the distributional sense. Therefore, $\Delta_\mu f \in L^2$, whence $f \in W_0^2 = \operatorname{dom}\mathcal{L}$ and $\mathcal{L}f = \lambda f$, which was to be proved. ☐

Exercises.

10.5. Prove that, for any $f \in L^2(M)$,

$$(P_t f, f) \leq \exp(-\lambda_{\min}(M)t)\|f\|_{L^2}^2.$$

10.6. Prove the following properties of $\lambda_{\min}$ for subsets of a weighted manifold M.

(a) If $\Omega_1 \subset \Omega_2$ are two open sets then

$$\lambda_{\min}(\Omega_1) \geq \lambda_{\min}(\Omega_2).$$

(b) If $\{\Omega_k\}$ is a finite or countable sequence of disjoint open sets and $\Omega = \bigcup_k \Omega_k$ then

$$\lambda_{\min}(\Omega) = \inf_k \lambda_{\min}(\Omega_k).$$

(c) If $\{\Omega_k\}_{k=1}^\infty$ is an increasing sequence of open sets and $\Omega = \bigcup_k \Omega_k$ then

$$\lambda_{\min}(\Omega) = \lim_{k\to\infty} \lambda_{\min}(\Omega_k).$$

10.7. Let $(M, \mathbf{g}, \mu)$ and $(M, \widetilde{\mathbf{g}}, \widetilde{\mu})$ be two weighted manifolds based on the same smooth manifold M of dimension n. Assume that they are quasi-isometric, that is, for some positive constant A and B,

$$A^{-1} \leq \frac{\widetilde{\mathbf{g}}}{\mathbf{g}} \leq A \quad \text{and} \quad B^{-1} \leq \frac{\widetilde{\Upsilon}}{\Upsilon} \leq B, \tag{10.23}$$

where Υ and $\widetilde{\Upsilon}$ are the density functions of measures μ and $\widetilde{\mu}$ respectively. Prove that

$$C^{-1}\lambda_{\min}(M) \leq \widetilde{\lambda}_{\min}(M) \leq C\lambda_{\min}(M) \tag{10.24}$$

where $C = C(A, B, n)$ is a positive constant, $\lambda_{\min}(M)$ is the bottom of the spectrum of the Dirichlet Laplacian on $(M, \mathbf{g}, \mu)$, and $\widetilde{\lambda}_{\min}(M)$ is the bottom of the spectrum of the Dirichlet Laplacian on $(M, \widetilde{\mathbf{g}}, \widetilde{\mu})$.

10.8. (*Cheeger's inequality*) The Cheeger constant of a manifold is defined by

$$h(M) := \inf_{f \in C_0^\infty(M)\setminus\{0\}} \frac{\int_M |\nabla f| \, d\mu}{\int_M |f| \, d\mu}. \tag{10.25}$$

Prove that

$$\lambda_{\min}(M) \geq \frac{1}{4} h^2(M). \tag{10.26}$$

10.3. The bottom eigenfunction

A non-zero function $f \in \operatorname{dom}(\mathcal{L})$ such that

$$\mathcal{L} f = \lambda_{\min}(M) f,$$

is called the *bottom eigenfunction* of $\mathcal{L}$.

The bottom eigenfunction does not always exist (for example, it does not exist in $\mathbb{R}^n$). Theorem 10.10 provides a sufficient condition for the existence of the bottom eigenfunction. The next theorem ensures that the bottom eigenfunction does not change the sign and, hence, can be assumed to be positive.

THEOREM 10.11. *If f is the bottom eigenfunction on a connected weighted manifold M then f never vanishes on M.*

The connectedness of M is essential. Indeed, if M is disconnected and contains a compact component Ω then the function $f = 1_\Omega$ is the bottom eigenfunction with the eigenvalue $\lambda_{\min}(M) = 0$, while f vanishes in $M \setminus \Omega$.

PROOF. Denote for simplicity $\lambda = \lambda_{\min}(M)$ so that $\mathcal{L} f = \lambda f$ in M. Then we have

$$\mathcal{R}(f) = \frac{(\mathcal{L} f, f)}{(f, f)} = \lambda. \tag{10.27}$$

Let us prove that either f_+ or f_- is identical zero. Assume from the contrary that the both functions f_+ and f_- are non-zero as elements of $L^2(M)$. Since $f \in W_0^1(M)$, by Lemma 5.2 both f_+ and f_- belong to $W_0^1(M)$ and

$$\nabla f_+ = 1_{\{f>0\}} \nabla f, \quad \nabla f_- = 1_{\{f<0\}} \nabla f \tag{10.28}$$

(cf. Example 5.3). By Theorem 10.8, we have

$$\mathcal{R}(f_+) \geq \lambda \ \text{ and } \ \mathcal{R}(f_-) \geq \lambda. \tag{10.29}$$

Assume that one of these two inequalities is strict, say the first one. Then we obtain by (10.28) and (10.29)

$$\int_{\{f>0\}} |\nabla f|^2 \, d\mu > \lambda \int_{\{f>0\}} f^2 d\mu$$

and

$$\int_{\{f<0\}} |\nabla f|^2 \, d\mu \geq \lambda \int_{\{f<0\}} f^2 d\mu.$$

Adding up these inequalities yields

$$\int_M |\nabla f|^2 \, d\mu > \lambda \int_M f^2 d\mu,$$

which contradicts (10.27).

Hence, the both inequalities in (10.29) turn to be equalities. By Theorem 10.10, we conclude that both functions f_+ and f_- are eigenfunctions of $\mathcal{L}$ with the eigenvalue λ, whence

$$\Delta_\mu f_\pm + \lambda f_\pm = 0. \tag{10.30}$$

By Corollary 7.3, f_+ and f_- are C^∞ smooth functions in M, and, by the strong minimum principle (cf. Corollary 8.14), they cannot vanish in M. However, this contradicts the fact that the inequalities $f_+(x) > 0$ and $f_-(x) > 0$ cannot occur at the same point x.

This proves that either f_+ or f_- is identical 0. Changing the sign of f, if necessary, we can assume $f \geq 0$. Since $f \not\equiv 0$, applying again the strong minimum principle, we conclude that $f > 0$ in M, which was to be proved. □

SECOND PROOF. Since $\inf \operatorname{spec} P_t = e^{-\lambda t}$, it follows that $\|P_t\|_{L^2 \to L^2} = e^{-\lambda t}$ and, hence,

$$\|P_t f\| \leq e^{-\lambda t} \|f\| . \tag{10.31}$$

Since f is the eigenfunction of $\mathcal{L}$ with the eigenvalue λ, f is also the eigenfunction of $P_t = e^{-t\mathcal{L}}$ with the eigenvalue $e^{-\lambda t}$, that is,

$$P_t f = e^{-\lambda t} f.$$

On the other hand, the identity

$$P_t f = P_t f_+ - P_t f_-$$

implies that

$$P_t f_+ \geq (P_t f)_+ = e^{-t\lambda} f_+.$$

The comparison with (10.31) shows that we have, in fact,

$$P_t f_+ = e^{-t\lambda} f_+.$$

A similar identity holds for f_-, so that we obtain (10.30). The proof is then finished in the same way as the previous proof. □

COROLLARY 10.12. *For any connected weighted manifold,*

$$\dim \ker (\mathcal{L} - \lambda_{\min} \mathrm{id}) \leq 1.$$

In other words, if the bottom eigenfunction exists then it is unique up to a constant multiple.

PROOF. Indeed, let f and g be two linearly independent bottom eigenfunctions. By Theorem 10.11, we can assume that both f and g are positive on M. Fix a point $x_0 \in M$ and choose a real constant c so that

$$f(x_0) + c g(x_0) = 0.$$

The function $h = f + cg$ is obviously contained in $\ker(\mathcal{L} - \lambda_{\min}\,\mathrm{id})$. However, h cannot be the bottom eigenfunction because it vanishes at point x_0. The only alternative left is that $h \equiv 0$, which contradicts the assumption of the linear independence of f, g. □

It follows from Theorem 10.11 and Corollary 10.12 that if the bottom eigenfunction f exists then f can be normalized to satisfy the conditions

$$\|f\|_{L^2} = 1 \text{ and } f > 0, \tag{10.32}$$

which determines f uniquely.

10.4. The heat kernel in relatively compact regions

Let (M, g, μ) be a weighted manifold. To simplify the terminology, we will call by the spectrum of M the spectrum of the Dirichlet Laplace operator $\mathcal{L} = -\Delta_\mu|_{W_0^2}$ on M, and the same convention applies to the eigenvalues and the eigenfunctions of $\mathcal{L}$.

The next statement is one of the main results of this chapter.

THEOREM 10.13. *Let Ω be a non-empty relatively compact open subset of a weighted manifold (M, g, μ). Then the following is true.*

(i) *The spectrum of Ω is discrete and consists of an increasing sequence $\{\lambda_k\}_{k=1}^\infty$ of non-negative eigenvalues (counted according to multiplicity) such that $\lim_{k\to\infty} \lambda_k = +\infty$. There is an orthonormal basis $\{\varphi_k\}_{k=1}^\infty$ in $L^2(\Omega)$ such that each function φ_k is an eigenfunction of Ω with the eigenvalue λ_k.*

(ii) *In any such basis $\{\varphi_k\}$, the heat kernel $p_t^\Omega(x, y)$ of Ω admits the following expansion*

$$p_t^\Omega(x, y) = \sum_{k=1}^\infty e^{-\lambda_k t}\varphi_k(x)\,\varphi_k(y). \tag{10.33}$$

The series in (10.33) converges absolutely and uniformly in the domain $t \geq \varepsilon$, $x, y \in \Omega$ for any $\varepsilon > 0$, as well as in the topology of $C^\infty(\mathbb{R}_+ \times \Omega \times \Omega)$.

Clearly, Theorem 10.13 applies when M is compact and $\Omega = M$.

In a more general context, the eigenvalue λ_k of Ω will be denoted by $\lambda_k(\Omega)$. One can consider $\lambda_k(\Omega)$ as a function of k and Ω, and this function is tightly linked to various analytic and geometric properties of the set Ω, the identity (10.33) being one of them.

Note that the first eigenvalue $\lambda_1(\Omega)$ is the bottom of the spectrum of Ω, that is,

$$\lambda_1(\Omega) = \lambda_{\min}(\Omega). \tag{10.34}$$

It is also worth mentioning that

$$\lambda_k(\Omega) = \mathcal{R}(\varphi_k). \tag{10.35}$$

Indeed, we have $\varphi_k \in W_0^2(\Omega)$ and

$$-\Delta_\mu \varphi_k = \mathcal{L}^\Omega \varphi_k = \lambda_k(\Omega)\varphi_k,$$

which implies by the Green formula (4.12)

$$\int_\Omega |\nabla \varphi_k|^2 d\mu = -\int_\Omega \varphi_k \Delta_\mu \varphi_k d\mu = \lambda_k(\Omega)\int_\Omega \varphi^2 d\mu,$$

whence (10.35) follows.

For the proof of Theorem 10.13, we need the following abstract lemma.

LEMMA 10.14. *Let (X,μ) be a measure space such that $L^2 = L^2(X,\mu)$ is a separable Hilbert space. Set $L^{2,2} := L^2(X \times X, \mu \times \mu)$ and consider a non-negative symmetric function $q(x,y) \in L^{2,2}$ and the operator Q defined on measurable functions on X by*

$$Qf(x) = \int_X q(x,y) f(y)\, d\mu(y), \tag{10.36}$$

whenever the right hand side of (10.36) make sense. Then Q is a bounded self-adjoint operator in L^2 and

$$\operatorname{trace} Q^2 = \|q\|^2_{L^{2,2}}. \tag{10.37}$$

PROOF. The fact that Q is bounded as an operator from L^2 to L^2 follows from the Cauchy-Schwarz inequality:

$$|Qf(x)|^2 \le \int_X q^2(x,y)\, d\mu(y)\, \|f\|_2^2$$

and

$$\int_X |Qf(x)|^2 d\mu(x) \le \int_X \int_X q^2(x,y)\, d\mu(y)\, d\mu(x)\, \|f\|^2_{L^2} = \|q\|^2_{L^{2,2}} \|f\|^2_{L^2}.$$

Let us show that the operator Q is symmetric. For all $f, g \in L^2$, we have by Fubini's theorem

$$(Qf,g) = \int_X Qf(x)\, g(x)\, d\mu(x) = \int_X \int_X q(x,y) f(y) g(x)\, d\mu(x)\, d\mu(y)$$

and similarly

$$(f,Qg) = \int_X \int_X q(x,y) f(x) g(y)\, d\mu(x)\, d\mu(y).$$

Switching x and y and using $q(x,y) = q(y,x)$, we obtain $(Qf,g) = (f,Qg)$.

The operator Q^2 is, hence, also bounded and self-adjoint. Besides, Q^2 is non-negative definite because for any $f \in L^2$,

$$\left(Q^2 f, f\right) = (Qf, Qf) \ge 0.$$

To prove (10.37), choose any orthonormal basis $\{v_k\}_{k=1}^\infty$ in L^2. Write (10.36) in the form

$$Qf(x) = (q_x, f),$$

where $q_x := q(x, \cdot)$. By Lemma 10.4 and (10.36), we have

$$\operatorname{trace} Q^2 = \sum_k \left(Q^2 v_k, v_k\right) = \sum_k \left(Q v_k, Q v_k\right) = \sum_k \int_X \left(q_x, v_k\right)^2 d\mu(x), \tag{10.38}$$

Expanding $q_x \in L^2$ in the basis $\{v_k\}$ we obtain

$$q_x = \sum_k \left(q_x, v_k\right) v_k \tag{10.39}$$

whence, by the Parseval identity,

$$\sum_k \left(q_x, v_k\right)^2 = \|q_x\|_{L^2}^2. \tag{10.40}$$

Hence, (10.38) and (10.40) yield

$$\operatorname{trace} Q^2 = \int_X \|q_x\|_{L^2}^2 d\mu(x) = \|q\|_{L^{2,2}}^2, \tag{10.41}$$

which was to be proved. □

PROOF OF THEOREM 10.13. (*i*) Since Ω is relatively compact, by the estimate (7.25) of Theorem 7.7 (cf. Theorem 7.6) we obtain

$$\sup_{x\in\Omega} \|p_{t,x}\|_{L^2} \le F_\Omega(t) := C\left(1 + t^{-\sigma}\right), \tag{10.42}$$

where σ is any integer larger than $n/4$ and C is a constant depending on Ω. Since $p_{t,x}^\Omega \le p_{t,x}$ (cf. Exercise 7.40 or Theorem 5.23), (10.42) implies

$$\sup_{x\in\Omega} \|p_{t,x}^\Omega\|_{L^2} \le F_\Omega(t) \tag{10.43}$$

and

$$\|p_t^\Omega\|_{L^{2,2}}^2 = \int_\Omega \|p_{t,x}^\Omega\|_{L^2}^2 d\mu \le F_\Omega^2(t)\, \mu(\Omega), \tag{10.44}$$

whence it follows that $\|p_t^\Omega\|_{L^{2,2}} < \infty$.

Applying Lemma 10.14 to the operator $Q = P_t^\Omega$ and noticing that $Q^2 = P_{2t}^\Omega$, we obtain that

$$\operatorname{trace} P_{2t}^\Omega = \|p_t^\Omega\|_{L^{2,2}}^2 < \infty. \tag{10.45}$$

Since $P_{2t}^\Omega = \exp\left(-2t\mathcal{L}^\Omega\right)$, we conclude by Lemma 10.7, that the spectrum of $\mathcal{L}^\Omega$ is discrete on the set where the function $\lambda \mapsto e^{-2t\lambda}$ is positive; hence, all the spectrum of $\mathcal{L}^\Omega$ is discrete.

By Lemma 10.2, there is an orthonormal basis $\{\varphi_k\}_{k=1}^\infty$ of eigenfunctions of $\mathcal{L}^\Omega$ in $L^2(\Omega)$ such that the sequence $\{\lambda_k\}$ of their eigenvalues consists of all eigenvalues of $\mathcal{L}^\Omega$ counted with multiplicity. Besides, we have $|\lambda_k| \to \infty$ and $\lambda_k \ge 0$, which implies that $\lambda_k \to +\infty$. Since any bounded interval contains only a finite number of terms λ_k, the sequence $\{\lambda_k\}$ can be renumbered in the increasing order.

(*ii*) Noticing that

$$\left(p_{t,x}^\Omega, \varphi_k\right)_{L^2} = P_t^\Omega \varphi_k(x) = e^{-t\mathcal{L}^\Omega} \varphi_k(x) = e^{-t\lambda_k} \varphi_k(x), \tag{10.46}$$

we obtain the following expansion of $p^{\Omega}_{t,x}$ in the basis $\{\varphi_k\}$:

$$p^{\Omega}_{t,x} = \sum_k e^{-t\lambda_k} \varphi_k(x) \varphi_k, \tag{10.47}$$

that is,

$$p^{\Omega}_t(x,y) = \sum_k e^{-t\lambda_k} \varphi_k(x) \varphi_k(y), \tag{10.48}$$

where the series converges in $L^2(\Omega)$ in variable y for any $x \in \Omega$ and $t > 0$.

Note that, by (10.14) and (10.45),

$$\sum_k e^{-2t\lambda_k} = \operatorname{trace} P^{\Omega}_{2t} = \|p^{\Omega}_t\|^2_{L^{2,2}} < \infty. \tag{10.49}$$

The sequence $\{\varphi_k(x)\varphi_k(y)\}_{k=1}^{\infty}$ is obviously orthonormal in $L^2(\Omega \times \Omega)$, which together with (10.49) implies that the series (10.48) converges in $L^2(\Omega \times \Omega)$.

To show the absolute and uniform convergence, observe that by (10.43), for any $f \in L^2(\Omega)$,

$$\sup_{x \in \Omega} \left| P^{\Omega}_t f(x) \right| = \sup_{x \in \Omega} \left| \left(p^{\Omega}_{t,x}, f\right)_{L^2} \right| \leq F_{\Omega}(t) \|f\|_{L^2}.$$

Applying this to $f = \varphi_k$ and using (10.46), we obtain

$$\sup_{x \in \Omega} \left| e^{-t\lambda_k} \varphi_k(x) \right| \leq F_{\Omega}(t), \tag{10.50}$$

whence

$$\sup_{x,y \in \Omega} \left| e^{-2t\lambda_k} \varphi_k(x) \varphi_k(y) \right| \leq F_{\Omega}(t)^2.$$

Since function $F_{\Omega}(t)$ is decreasing in t, we obtain, for any $\varepsilon > 0$,

$$\sum_k \sup_{\substack{x,y \in \Omega \\ t \geq \varepsilon}} \left| e^{-3t\lambda_k} \varphi_k(x) \varphi_k(y) \right| \leq F_{\Omega}(\varepsilon)^2 \sum_k e^{-\varepsilon\lambda_k}, \tag{10.51}$$

where the right hand side is finite by (10.49). Renaming $3t$ to t and 3ε to ε, we obtain that the series (10.48) converges absolutely and uniformly in the domain $t \geq \varepsilon$, $x, y \in \Omega$.

Finally, let us show that the series (10.48) converges in $C^{\infty}(\mathbb{R}_+ \times \Omega \times \Omega)$. The function $u(t,x,y) = p^{\Omega}_{2t}(x,y)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = (\Delta_x + \Delta_y) u$$

with respect to the Laplace operator $\Delta_x + \Delta_y$ of the manifold $\Omega \times \Omega$ (cf. the proof of Theorem 7.20), and its is straightforward to check that each function $u_k(t,x,y) = e^{-2t\lambda_k}\varphi_k(x)\varphi_k(y)$ also satisfies the same equation. It follows from the previous argument that the series $\sum_k u_k$ converges to u in $L^2_{loc}(\mathbb{R}_+ \times \Omega \times \Omega)$, which implies by Theorem 7.4 that it converges also in $C^{\infty}(\mathbb{R}_+ \times \Omega \times \Omega)$. □

REMARK 10.15. It follows from (10.51) that, for any $t > 0$ and $n \in \mathbb{N}$,

$$S_n(t) := \sum_{k=n}^{\infty} e^{-\lambda_k t} \sup_{x,y\in\Omega} |\varphi_k(x)\varphi_k(y)| < \infty.$$

Since $S_n(t)$ is a decreasing function of t, it follows that, for all $t \geq t_0 > 0$

$$S_n(t) \leq e^{-\lambda_n t/2} S_n(t/2) \leq e^{-\lambda_n t/2} S_n(t_0/2). \tag{10.52}$$

In particular, if $\lambda_n > 0$, then $S_n(t) \to 0$ as $t \to \infty$.

EXAMPLE 10.16. Let us show that if Ω is a non-empty relatively compact open subset of a weighted manifold, then, for all large enough k,

$$\lambda_k(\Omega) \geq ck^{1/(2\sigma)}, \tag{10.53}$$

where σ is the exponent from (10.42) and c is a positive constant depending on Ω (better estimates for $\lambda_k(\Omega)$ will be proved later in Corollary 15.12).

Write for simplicity $\lambda_k = \lambda_k(\Omega)$. Since the sequence $\{\lambda_k\}_{k=1}^{\infty}$ is increasing, we have, for any $k \geq 1$,

$$\sum_{k=1}^{\infty} e^{-2t\lambda_k} \geq ke^{-2t\lambda_k}.$$

It follows from (10.49) that

$$ke^{-2t\lambda_k} \leq \|p_t^\Omega\|^2_{L^{2,2}}$$

and, hence,

$$\lambda_k(\Omega) \geq \frac{1}{2t}\log\frac{k}{\|p_t^\Omega\|^2_{L^{2,2}}}. \tag{10.54}$$

Assuming $0 < t \leq 1$, we obtain from (10.42) and (10.44),

$$\|p_t\|^2_{L^{2,2}} \leq Ct^{-2\sigma},$$

for some constant C depending on Ω, whence

$$\lambda_k(\Omega) \geq \frac{1}{2t}\log\frac{kt^{2\sigma}}{C}. \tag{10.55}$$

Let us choose t from the condition $\frac{kt^{2\sigma}}{C} = e$, that is

$$t = \left(\frac{Ce}{k}\right)^{1/(2\sigma)}. \tag{10.56}$$

Since we want $t \leq 1$, this is only possible if $k \geq Ce$. Assuming that k is that large and substituting (10.56) into (10.55), we obtain (10.53).

EXAMPLE 10.17. It is easy to show that the eigenvalues of the circle $\mathbb{S}^1$ are given by the sequence $\{k^2\}_{k=0}^{\infty}$, all with multiplicity 1 (see Exercise 10.18). For the sphere $\mathbb{S}^n$, the distinct eigenvalues are given by

$$\alpha_k = k(k+n-1), \quad k = 0, 1, ...,$$

(see Exercise 10.19), where the multiplicity of α_0 is 1 and the multiplicity of α_k, $k \geq 1$, is equal to

$$\frac{(k+n-2)!}{(n-1)!k!}(2k+n-1).$$

Exercises.

10.9. In the setting of Lemma 10.14, prove that the integral operator Q is compact without using the trace.

10.10. Let M be a compact weighted manifold, which has a finite number m of connected components.

(a) Prove that $\lambda_1(M) = ... = \lambda_m(M) = 0$ and $\lambda_{m+1}(M) > 0$.

(b) Show that the estimate (10.53) holds for all $k \geq m+1$ and does not hold for $k \leq m$.

10.11. Let M be a compact connected weighted manifold. Prove that

$$p_t(x,y) \rightrightarrows \frac{1}{\mu(M)} \text{ as } t \to \infty,$$

where the convergence is uniform for all $x, y \in M$.

10.12. Let Ω be a non-empty relatively compact connected open subset of a weighted manifold M. Using the notation of Theorem 10.13, prove that, for all $x, y \in \Omega$,

$$p_t^\Omega(x,y) \sim e^{-\lambda_1 t}\varphi_1(x)\varphi_1(y) \text{ as } t \to \infty.$$

10.13. Prove that, under the conditions of Theorem 10.13,

$$\sup_{x\in\Omega}|\varphi_k(x)| \leq C(1+\lambda_k^\sigma), \text{ for all } k \geq 1, \tag{10.57}$$

where σ is the exponent from (10.42) and C is a constant that does not depend on k.

10.14. Let $(M, \mathbf{g}, \mu)$ be a weighted manifold with the discrete spectrum. Let $\{\varphi_k\}$ be an orthonormal basis in $L^2(M)$ that consists of the eigenfunctions of M, and let λ_k be the eigenvalue of φ_k.

(a) Prove that, for any $f \in L^2(M)$, if $f = \sum_k a_k\varphi_k$ is the expansion of f in the basis $\{\varphi_k\}$ in $L^2(M)$ then

$$P_t f = \sum_k e^{-\lambda_k t} a_k \varphi_k, \tag{10.58}$$

where the series converges in $L^2(M)$ for any $t > 0$. Show also that the series converges in $C^\infty(\mathbb{R}_+ \times M)$.

(b) Assume in addition that

$$\operatorname{trace} P_t = \sum_k e^{-\lambda_k t} < \infty$$

for all $t > 0$. Prove that

$$p_t(x,y) = \sum_k e^{-\lambda_k t}\varphi_k(x)\varphi_k(y), \tag{10.59}$$

where the series converges in $C^\infty(\mathbb{R}_+ \times M \times M)$.

10.15. On an arbitrary weighted manifold, consider the resolvent $R = (\mathrm{id} + \mathcal{L})^{-1}$ and its powers $R^s = (\mathrm{id} + \mathcal{L})^{-s}$, where $\mathcal{L}$ is the Dirichlet Laplace operator and $s > 0$.

(a) Prove that

$$\operatorname{trace} R^s = \int_0^\infty \frac{t^{s-1}}{\Gamma(s)} e^{-t} \operatorname{trace} P_t dt. \tag{10.60}$$

(*b*) Assuming in addition that $\mu(M) < \infty$ and

$$p_t(x,x) \leq Ct^{-\nu} \text{ for all } 0 < t < 1, \ x \in M,$$

where C and ν are positive constants, prove that trace R^s is finite for all $s > \nu$.

10.16. Let Ω be a relatively compact open subset of a weighted manifold M of dimension n. Let $\{\varphi_k\}$ be an orthonormal basis in $L^2(\Omega)$ that consists of the eigenfunctions of M, and let $\{\lambda_k\}$ be the sequence of the corresponding eigenvalues.

(*a*) Prove that if $s > s_0 = s_0(n)$ then

$$\sum_{k:\lambda_k>0}^{\infty} \lambda_k^{-s} < \infty. \tag{10.61}$$

(*b*) Prove that if $f \in C_0^\infty(\Omega)$ then the Fourier series

$$f = \sum_k c_k \varphi_k$$

of function f converges to f absolutely and uniformly in Ω.

10.17. Let $(M, \mathbf{g}, \mu)$ be a compact weighted manifold and $\{\varphi_k\}$ be an orthonormal basis in $L^2(M)$ that consists of the eigenfunctions of M. Prove that the set of all finite linear combinations of functions φ_k is dense in $C(M)$.

REMARK. This can be considered as a generalization of the classical Stone-Weierstrass theorem that any continuous 2π-periodic function on $\mathbb{R}$ can be uniformly approximated by trigonometric polynomials.

10.18. In this problem, the circle $\mathbb{S}^1$ is identified with $\mathbb{R}/2\pi\mathbb{Z}$.

(*i*) Prove that the heat kernel $p_t(x,y)$ of $\mathbb{S}^1$ is given by

$$p_t(x,y) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} e^{-k^2 t}\cos k(x-y). \tag{10.62}$$

(*ii*) Show that the heat kernel $p_t(x,y)$ of $\mathbb{S}^1$ can be obtained from the heat kernel $\widetilde{p}_t(x,y)$ of $\mathbb{R}^1$ by

$$p_t(x,y) = \sum_{n\in\mathbb{Z}} \widetilde{p}_t(x+2\pi n, y). \tag{10.63}$$

(*iii*) Prove the Poisson summation formula

$$\sum_{k\in\mathbb{Z}} e^{-k^2 t} = \sqrt{\frac{\pi}{t}} \sum_{n\in\mathbb{Z}} \exp\left(-\frac{\pi^2 n^2}{t}\right). \tag{10.64}$$

10.19. Let $P(x)$ be a homogeneous of degree k harmonic polynomial on $\mathbb{R}^{n+1}$. Prove that the function $f = P|_{\mathbb{S}^n}$ is an eigenfunction of the Laplacian of $\mathbb{S}^n$ with the eigenvalue $\alpha = k(k+n-1)$.

REMARK. It is possible to prove that such eigenfunctions exhaust all eigenfunctions on $\mathbb{S}^n$.

10.20. Consider the weighted manifold $(\mathbb{R}, \mathbf{g}_{\mathbb{R}}, \mu)$ where $d\mu = e^{-x^2}dx$. Prove that the spectrum of this manifold is discrete, its eigenvalues are $\lambda_k = 2k$, $k = 0, 1, \ldots$, and the eigenfunctions are $h_k(x)$ – the Hermite polynomials (see Exercise 3.10). Hence, show that the heat kernel of this manifold satisfies the identity

$$p_t(x,y) = \sum_{k=0}^{\infty} e^{-2kt} \frac{h_k(x)\, h_k(y)}{\sqrt{\pi} 2^k k!}. \tag{10.65}$$

REMARK. The same heat kernel is given by the formula

$$p_t(x,y) = \frac{1}{(2\pi \sinh 2t)^{1/2}} \exp\left(\frac{2xye^{-2t} - (x^2+y^2)e^{-4t}}{1-e^{-4t}} + t\right),$$

cf. Example 9.19.

10.5. Minimax principle

Let $(M, \mathbf{g}, \mu)$ be a weighted manifold with discrete spectrum, and let $\{\lambda_k(M)\}_{k=1}^{\infty}$ be the increasing sequence of all the eigenvalues of M, counted according to multiplicity. The following theorem generalizes the variational formula (10.1) for $\lambda_{\min}(M)$.

THEOREM 10.18. *If the spectrum of $(M, \mathbf{g}, \mu)$ is discrete then the following identities hold:*

$$\lambda_k(M) = \sup_{\dim E = k-1} \inf_{f \in E^{\perp} \setminus \{0\}} \mathcal{R}(f), \tag{10.66}$$

where the supremum is taken over all subspaces $E \subset W_0^1(M)$ of dimension $k-1$ and the infimum is taken over all non-zero functions f in the orthogonal complement of E in $W_0^1(M)$, and

$$\lambda_k(M) = \inf_{\dim F = k} \sup_{f \in F \setminus \{0\}} \mathcal{R}(f), \tag{10.67}$$

which is understood similarly.

For example, for $k = 1$ (10.66) and (10.67) yield

$$\lambda_1(M) = \inf_{f \in W_0^1 \setminus \{0\}} \mathcal{R}(f),$$

matching Theorem 10.8.

PROOF. Let $\{\varphi_k\}$ be an orthonormal basis in $L^2(M)$ such that φ_k is an eigenfunction of M with the eigenvalue $\lambda_k = \lambda_k(M)$ (cf. Lemma 10.2).

CLAIM 1. *For any $f \in W_0^1(M)$ and $i \geq 1$,*

$$(\nabla f, \nabla \varphi_i)_{L^2} = \lambda_i (f, \varphi_i)_{L^2}. \tag{10.68}$$

Indeed, since $f \in W_0^1(M)$ and

$$\Delta_\mu \varphi_i = -\lambda_i \varphi_i \in L^2(M),$$

the Green formula 4.12 yields

$$(\nabla f, \nabla \varphi_i)_{L^2} = -\int_M f \Delta_\mu \varphi_i d\mu = \lambda_i \int_M f \varphi_i d\mu,$$

which is equivalent to (10.68).

In particular, applying (10.68) to $f = \varphi_j$, we obtain

$$(\nabla \varphi_i, \nabla \varphi_j)_{L^2} = \begin{cases} \lambda_i, & i = j, \\ 0, & i \neq j. \end{cases} \tag{10.69}$$

CLAIM 2. *If E is a $(k-1)$-dimensional subspace of a Hilbert space $\mathcal{H}$ and F is a k-dimensional subspace of $\mathcal{H}$ then there exists a non-zero vector $v \in F \cap E^{\perp}$.*

Indeed, let $\{\varphi_1, ..., \varphi_k\}$ be a basis in F and let us look for v in the form

$$v = \sum_{i=1}^{k} c_i \varphi_i \ ,$$

where $c_1, ..., c_k$ are unknown reals. If $\{e_1, ...e_{k-1}\}$ is a basis in E then the condition $v \in E^{\perp}$ means $(v, e_j) = 0$ for all $j = 1, ..., k-1$, which amounts to a linear system for c_i:

$$\sum_{i=1}^{k} (\varphi_i, e_j)\, c_i = 0, \ \ j = 1, ..., k-1.$$

Since the number of the equations in this homogeneous system is less than the number of unknowns, there is a non-zero solution $\{c_i\}$, which determines to a non-zero vector $v \in F \cap E^{\perp}$.

Now we can prove (10.66) and (10.67). Consider the space

$$E = \operatorname{span}\{\varphi_1, ..., \varphi_{k-1}\},$$

which is a $(k-1)$-dimensional subspace of $W_0^1(M)$. Any function $f \in E^{\perp} \setminus \{0\}$ can represented in the form

$$f = \sum_{i \geq k} c_i \varphi_i,$$

whence we obtain, using (10.69),

$$\mathcal{R}(f) = \frac{(\nabla f, \nabla f)_{L^2}}{(f,f)_{L^2}} = \frac{\sum_{i,j \geq k} c_i c_j (\nabla \varphi_i, \nabla \varphi_j)_{L^2}}{\sum_{i,j \geq k} c_i c_j (\varphi_i, \varphi_j)_{L^2}} = \frac{\sum_{i \geq k} \lambda_i c_i^2}{\sum_{i \geq k} c_i^2} \geq \lambda_k.$$

Hence, we obtain, for this particular space E

$$\inf_{f \in E^{\perp} \setminus \{0\}} \mathcal{R}(f) \geq \lambda_k. \tag{10.70}$$

If F is any k-dimensional subspace of $W_0^1(M)$ then, by Claim 2, there exists a non-zero function $f \in F \cap E^{\perp}$, which implies that

$$\sup_{f \in F \setminus \{0\}} \mathcal{R}(f) \geq \lambda_k. \tag{10.71}$$

Hence, taking in (10.70) supremum over subspaces E and in (10.71) infimum over F, we obtain upper bounds for λ_k in (10.66) and (10.67), respectively.

To prove the lower bounds, consider the k-dimensional subspace

$$F = \operatorname{span}\{\varphi_1, .., \varphi_k\}.$$

Writing a function $f \in F \setminus \{0\}$ in the form

$$f = \sum_{i \leq k} c_i \varphi_i,$$

we obtain, similarly to the first part of the proof,

$$\mathcal{R}(f) = \frac{\sum_{i \le k} \lambda_i c_i^2}{\sum_{i \le k} c_i^2} \le \lambda_k.$$

Hence, for this particular space F,

$$\sup_{f \in F \setminus \{0\}} \mathcal{R}(f) \le \lambda_k. \tag{10.72}$$

If E is any $(k-1)$-dimensional subspace of $W_0^1(M)$ then, by Claim 2, there is a non-zero function $f \in F \cap E^\perp$, which implies that

$$\inf_{f \in E^\perp \setminus \{0\}} \mathcal{R}(f) \le \lambda_k. \tag{10.73}$$

Taking in (10.72) infimum over all subspaces F and in (10.73) supremum over E, we obtain the lower bounds for λ_k in (10.67) and (10.66), respectively, which finishes the proof. □

COROLLARY 10.19. *If Ω and Ω' are non-empty relatively compact open subsets of M and $\Omega' \subset \Omega$ then, for any $k \ge 1$,*

$$\lambda_k(\Omega') \ge \lambda_k(\Omega).$$

PROOF. Note that the space $W_0^1(\Omega')$ can be considered as a subspace of $W_0^1(\Omega)$ by identifying any function $f \in W_0^1(\Omega')$ with its trivial extension (cf. Section 5.5), and the trivial extension does not change $\mathcal{R}(f)$. Hence, any k-dimensional subspace F of $W_0^1(\Omega')$ is also that of $W_0^1(\Omega)$, and the value of the functional

$$\mathcal{R}(F) := \sup_{f \in F \setminus \{0\}} \mathcal{R}(f)$$

does not depend on whether F is considered as a subspace of $W_0^1(\Omega')$ or $W_0^1(\Omega)$. By (10.67), we obtain

$$\lambda_k(\Omega') = \inf_{F \subset W_0^1(\Omega')} \mathcal{R}(F) \ge \inf_{F \subset W_0^1(\Omega)} \mathcal{R}(F) = \lambda_k(\Omega),$$

which was to be proved. □

Exercises.

10.21. Let (M, g, μ) be a weighted manifold with discrete spectrum, and let $\{\varphi_k\}$ be an orthonormal basis in $L^2(M)$ of the eigenfunctions of M with eigenvalues $\{\lambda_k\}$.

(a) Prove that $\{\varphi_k\}$ is an orthogonal basis also in $W_0^1(M)$.

(b) Let $f \in L^2(M)$ and assume that $f = \sum_k a_k \varphi_k$ is its expansion in the basis $\{\varphi_k\}$ in $L^2(M)$. Prove that if, in addition, $f \in W_0^1(M)$ then

$$\nabla f = \sum_k a_k \nabla \varphi_k \quad \text{in } \vec{L}^2(M) \tag{10.74}$$

and

$$\int_M |\nabla f|^2 \, d\mu = \sum_k \lambda_k a_k^2. \tag{10.75}$$

(c) Prove that if $f \in W_0^2(M)$ then

$$-\Delta_\mu f = \sum_k \lambda_k a_k \varphi_k \quad \text{in } L^2(M) \tag{10.76}$$

and

$$\int_M (\Delta_\mu f)^2 \, d\mu = \sum_k \lambda_k^2 a_k^2. \tag{10.77}$$

10.22. Let manifold M admit k non-zero functions $f_1, ..., f_k \in W_0^1(M)$ with disjoint supports such that $\mathcal{R}(f_i) \leq a$ for all $i = 1, ..., k$ and some number a. Assuming that the spectrum of $\mathcal{L}$ is discrete, prove that $\lambda_k(M) \leq a$.

10.6. Discrete spectrum and compact embedding theorem

Recall that, on any weighted manifold $(M, \mathbf{g}, \mu)$, the identical mapping $W^1(M) \to L^2(M)$ is an embedding (cf. Section 4.1). In this section, we discuss the conditions when the embedding operator $W_0^1(M) \hookrightarrow L^2(M)$ is compact.

THEOREM 10.20. *Let $(M, \mathbf{g}, \mu)$ be a weighted manifold. Then the following conditions are equivalent.*

(a) *The spectrum of M is discrete.*
(b) *The embedding operator $W_0^1(M) \hookrightarrow L^2(M)$ is compact.*
(c) *The resolvent $R_\alpha = (\mathcal{L} + \alpha \,\mathrm{id})^{-1}$ is a compact operator in $L^2(M)$, for some/all $\alpha > 0$.*

PROOF. $(a) \Rightarrow (b)$. If the spectrum of the Dirichlet Laplace operator $\mathcal{L}$ is discrete, then, by Lemma 10.2, there exists an orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$ in $L^2(M)$ such that each φ_k is an eigenfunction of $\mathcal{L}$, and the corresponding eigenvalues λ_k tend to $+\infty$ as $k \to \infty$.

It follows from (10.68) that, for any $f \in W_0^1(M)$ and any $k \geq 1$,

$$(f, \varphi_k)_{W^1} = (1 + \lambda_k)(f, \varphi_k)_{L^2}. \tag{10.78}$$

In particular, (10.78) implies that

$$\|\varphi_k\|_{W^1}^2 = 1 + \lambda_k.$$

By Exercise 10.21, the sequence $\{\varphi_k\}$ forms an orthogonal basis in $W_0^1(M)$. Hence, any function $f \in L^2(M)$ can be expanded in the basis $\{\varphi_k\}$ as follows:

$$f = \sum_{k=1}^{\infty} a_k \varphi_k,$$

where $a_k = (f, \varphi_k)_{L^2}$, and if $f \in W_0^1(M)$ then the same series converges in $W_0^1(M)$. By the Parseval identity, we have

$$\|f\|_{L^2}^2 = \sum_{k=1}^{\infty} a_k^2$$

and

$$\|f\|_{W^1}^2 = \sum_{k=1}^{\infty} (1 + \lambda_k) a_k^2.$$

Now assume that we have a sequence $\{f_n\}$ in $W_0^1(M)$, which is bounded in the norm $W^1(M)$, and prove that there exists a subsequence that converges in $L^2(M)$, which will prove that the embedding $W_0^1(M) \hookrightarrow L^2(M)$ is compact. Set $a_{nk} = (f_n, \varphi_k)_{L^2}$ and observe that, by the boundedness of $\|f_n\|_{W^1}$, there exists a constant C such that, for all n,

$$\sum_{k=1}^{\infty} (1 + \lambda_k) a_{nk}^2 < C. \tag{10.79}$$

In particular, all the coefficients a_{nk} are uniformly bounded. Consider the infinite matrix

$$\begin{matrix} a_{11} & a_{21} & a_{31} & \dots & a_{n1} & \dots \\ a_{21} & a_{22} & a_{32} & \dots & a_{n2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & a_{k3} & \dots & a_{nk} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{matrix}$$

The boundedness of the entries implies that, in any row, there is a convergent subsequence. Using the diagonal process, choose a sequence of column indices $n_1, n_2, ... \to \infty$ such that the subsequence $\{a_{n_i k}\}_{i=1}^{\infty}$ converges for any k.

Let us show that the subsequence $\{f_{n_i}\}$ converges in $L^2(M)$. For simplicity of notation, renumber this sequence back to $\{f_n\}$. Then we have, for all indices n, m, K,

$$\|f_n - f_m\|_{L^2}^2 = \sum_{k=1}^{\infty} (a_{nk} - a_{mk})^2 = \sum_{k=1}^{K} (a_{nk} - a_{mk})^2 + \sum_{k=K+1}^{\infty} (a_{nk} - a_{mk})^2.$$

The condition (10.79) implies

$$\sum_{k=K+1}^{\infty} (a_{nk} - a_{mk})^2 \le 2 \sum_{k=K+1}^{\infty} a_{nk} + 2 \sum_{k=K+1}^{\infty} a_{mk} \le \frac{4C}{1 + \lambda_K}$$

whence

$$\|f_n - f_m\|_{L^2}^2 \le \sum_{k=1}^{K} (a_{nk} - a_{mk})^2 + \frac{4C}{1 + \lambda_K}.$$

Given $\varepsilon > 0$, choose K so big that

$$\frac{4C}{1 + \lambda_K} < \frac{\varepsilon}{2},$$

which is possible because $\lambda_K \to \infty$ as $K \to \infty$. For the already chosen K, we have

$$\sum_{k=1}^{K} (a_{nk} - a_{mk})^2 < \frac{\varepsilon}{2} \text{ for large enough } n, m,$$

because by construction the sequence $\{a_{nk}\}_{n=1}^{\infty}$ is Cauchy for any k. Hence, for large enough n, m,

$$\|f_n - f_m\|_{L^2}^2 < \varepsilon,$$

that is, $\{f_n\}$ is a Cauchy sequence in $L^2(M)$, which was to be proved.

$(b) \Rightarrow (c)$. Recall that, by Theorem 4.6, the resolvent $R_\alpha = (\mathcal{L} + \alpha\,\mathrm{id})^{-1}$ is a bounded self-adjoint operator in $L^2(M)$. For any $f \in L^2(M)$, we have

$$u := R_\alpha f \in \mathrm{dom}\,\mathcal{L} \subset W_0^1(M)$$

and

$$\mathcal{L}u + \alpha u = f$$

whence

$$(\nabla u, \nabla u)_{L^2} + \alpha (u, u)_{L^2} = (u, \mathcal{L}u)_{L^2} + (u, \alpha u)_{L^2} = (u, f)_{L^2}$$

(cf. (4.21)). Therefore,

$$\min(1, \alpha) \|u\|_{W^1}^2 \le \|u\|_{L^2} \|f\|_{L^2} \le \|u\|_{W^1} \|f\|_{L^2}$$

and

$$\|u\|_{W^1} \le \max\left(1, \alpha^{-1}\right) \|f\|_{L^2}. \tag{10.80}$$

Consider the operator $\widetilde{R}_\alpha : L^2(M) \to W_0^1(M)$ defined by $\widetilde{R}_\alpha f = R_\alpha f$ (the difference between R_a and $\widetilde{R}_\alpha$ is that they have different target spaces). By (10.80), the operator $\widetilde{R}_a$ is bounded. The resolvent $R_\alpha : L^2(M) \to L^2(M)$ is the composition of $\widetilde{R}_\alpha$ and the embedding operator, as follows:

$$L^2(M) \xrightarrow{\widetilde{R}_\alpha} W_0^1(M) \hookrightarrow L^2(M).$$

Since $\widetilde{R}_\alpha$ is bounded and the embedding operator $W_0^1(M) \hookrightarrow L^2(M)$ is compact, their composition is a compact operator.

$(c) \Rightarrow (a)$. Note that $\ker R_\alpha = \{0\}$ because $R_a f = 0$ implies $f = (\mathcal{L} + \alpha\,\mathrm{id})\,0 = 0$. By the Hilbert-Schmidt theorem, there is an orthonormal basis $\{\varphi_k\}$ in $L^2(M)$ that consists of the eigenfunctions φ_k of R_α with the eigenvalues $\rho_k \neq 0$ such that $\rho_k \to 0$ as $k \to \infty$. Since $\mathcal{L} = R_\alpha^{-1} - \alpha$, function φ_k is also an eigenfunction of $\mathcal{L}$ with the eigenvalue $\lambda_k = \rho_k^{-1} - \alpha$. Since $\lambda_k \to \infty$, there is no finite accumulation point of the sequence $\{\lambda_k\}$. Using this, the operator $(\mathcal{L} - \lambda\,\mathrm{id})^{-1}$ can be explicitly constructed for any $\lambda \neq \lambda_k$ as follows: if $f = \sum_k a_k \varphi_k$ in $L^2(M)$ then

$$(\mathcal{L} - \lambda\,\mathrm{id})^{-1} f = \sum_k \frac{a_k}{\lambda_k - \lambda} \varphi_k,$$

and this operator is bounded because $\inf_k |\lambda_k - \lambda| > 0$. Hence, the entire spectrum of $\mathcal{L}$ coincides with the sequence $\{\lambda_k\}$, which implies that the spectrum of $\mathcal{L}$ is discrete. $\square$

COROLLARY 10.21. (Compact embedding theorem) *If Ω is a non-empty relatively compact open subset of a weighted manifold M then the embedding operator $W_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.*

PROOF. By Theorem 10.13, the spectrum of $\mathcal{L}^{\Omega}$ is discrete, whence the result follows from Theorem 10.20. $\square$

SECOND PROOF. Let us present a more direct proof, without using Theorems 10.13 and 10.20. Instead, we assume that the compact embedding theorem is known for the case $M = \mathbb{R}^n$ (see Theorem 6.3).

Let us show that, for any bounded sequence $\{f_k\}$ in $W_0^1(\Omega)$, there is a subsequence $\{f_{k_i}\}$ that converges in $L^2(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $W_0^1(\Omega)$, we can assume without loss of generality that all functions f_k are in $C_0^\infty(\Omega)$. Since $\Omega \subset M$ is relatively compact, there is a *finite* family $\{U_j\}$ of small enough relatively compact charts such that

$$\overline{\Omega} \subset \bigcup_j U_j =: U.$$

By Theorem 3.5, there exist functions $\varphi_j \in C_0^\infty(U_j)$ such that $\sum_j \varphi_j \equiv 1$ in a neighborhood of $\overline{\Omega}$.

Fix j and observe that the sequence $\{f_k\varphi_j\}_{k=1}^{\infty}$ is bounded in W_0^1, because (suppressing indices k, j)

$$\begin{aligned}\|f\varphi\|^2_{W_0^1} &= \|f\varphi\|^2_{L^2} + \|\nabla(f\varphi)\|^2_{L^2} \\ &\le \|f\|^2_{L^2} + 2\|\nabla f\|^2_{L^2} + 2\sup|\nabla\varphi|\,\|f\|^2_{L^2} \\ &\le \text{const}.\end{aligned}$$

Since $W_0^1(U_j)$ embeds compactly into $L^2(U_j)$, there is a subsequence $\{f_{k_i}\varphi_j\}_{i=1}^{\infty}$ that converges in $L^2(U_j)$. Using the diagonal process, one can ensure that this subsequence converges in $L^2(U_j)$ for *any* j. Since $\sum_j \varphi_j \equiv 1$ in Ω, we conclude that $\{f_{k_i}\}$ converges in $L^2(\Omega)$, which finishes the proof. $\square$

Applying further Theorem 10.20, we obtain that the spectrum of $\mathcal{L}^{\Omega}$ is discrete, which is the main part of Theorem 10.13. Hence, this approach allows to prove Theorem 10.13 without Theorem 7.6. However, we use Theorem 7.6 also to prove the existence and smoothness of the heat kernel and, at the same token, it leads to a short proof of Theorem 10.13 via the properties of trace.

Yet another approach to the proof of Corollary 10.21 is presented in Exercise 7.47. That proof also uses the heat kernel, but in a more direct way.

Exercises.

10.23. Prove that if the spectrum of a weighted manifold $(M, \mathbf{g}, \mu)$ is discrete then also the spectrum of any non-empty open subset $\Omega \subset M$ is discrete.

10.24. Let $(M', \mathbf{g}', \mu')$ and $(M'', \mathbf{g}'', \mu'')$ be two weighted manifold with discrete spectra, whose eigenvalues are given by the sequences $\{\alpha_i\}$ and $\{\beta_j\}$, respectively (each eigenvalue is counted with multiplicity). Prove that the spectrum of the direct product $(M, \mathbf{g}, \mu)$ is also discrete, and the eigenvalues are given by the double sequence $\{\alpha_i + \beta_j\}$.

10.25. *(Compactness of the embedding $W^1_{loc} \hookrightarrow L^2_{loc}$)* Let $\{u_k\}$ be a sequence of functions from $W^1_{loc}(M)$ such that $\{u_k\}$ is bounded in $W^1(\Omega)$ for any relatively compact open set $\Omega \subset M$. Prove that there exists a subsequence $\{u_{k_i}\}$ that converges in $L^2_{loc}(M)$.

10.7. Positivity of λ_1

Throughout this section, Ω is a non-empty relatively compact open subset of a weighted manifold $(M, \mathbf{g}, \mu)$. Recall that, by Theorem 10.13, the spectrum of the Dirichlet Laplace operator $\mathcal{L}^{\Omega}$ is discrete and consists of a sequence $\{\lambda_k(\Omega)\}_{k=1}^{\infty}$ of non-negative eigenvalues such that $\lambda_k(\Omega) \to \infty$ as $k \to \infty$.

THEOREM 10.22. *Let $(M, \mathbf{g}, \mu)$ be a connected weighted manifold. If $\Omega \subset M$ is a non-empty relatively compact open set such that $M \setminus \overline{\Omega}$ is non-empty then $\lambda_1(\Omega) > 0$.*

Neither connectedness of M nor the fact that $\overline{\Omega} \neq M$ can be dropped. Indeed, if a disconnected manifold is allowed then let M consist of two disjoint copies of $\mathbb{S}^n$ and Ω be one of these copies. Obviously, function $\varphi \equiv 1$ is an eigenfunction in Ω and, hence, $\lambda_1(\Omega) = 0$. If $\overline{\Omega} = M$ is allowed then take $\Omega = M = \mathbb{S}^n$ with the same effect.

PROOF. Assume that $\lambda_1(\Omega) = 0$ so that there is an eigenfunction f of $\mathcal{L}^{\Omega}$ with the eigenvalue 0, that is, $\mathcal{L}^{\Omega} f = 0$. By Lemma 4.4, we have

$$(\nabla f, \nabla f)_{L^2(\Omega)} = \left(\mathcal{L}^{\Omega} f, f\right)_{L^2(\Omega)} = 0$$

so that $\nabla f = 0$ in Ω. By Corollary 7.3, $f \in C^{\infty}(\Omega)$. Hence, f is a constant on any connected component of Ω. Since $f \not\equiv 0$, there is a component of Ω where f is a non-zero constant, say, $f \equiv 1$. Denote this component again by Ω.

The set $\overline{\Omega}$ is closed and its complement is non-empty. Since M is connected, $\overline{\Omega}$ is not open, which implies that the boundary $\partial\Omega$ is not empty. Choose a point $x_0 \in \partial\Omega$ and let U be any connected open neighborhood of x_0. Consider the set $\Omega' = \Omega \cup U$, which is a connected open set. Note that, by construction, $\Omega' \setminus \overline{\Omega}$ is non-empty.

Since $f \in \operatorname{dom} \mathcal{L}^{\Omega} \subset W_0^1(\Omega)$, extending f to Ω' by setting $f = 0$ in $\Omega' \setminus \Omega$, we obtain a function from $W_0^1(\Omega')$ (see Section 5.5). Since $f = 0$ on $\Omega' \setminus \Omega$, by (5.11) we have $\nabla f = 0$ in $\Omega' \setminus \Omega$. Since also $\nabla f = 0$ in Ω, we conclude that $\nabla f = 0$ in Ω'. This implies, that, for any $\varphi \in \mathcal{D}(\Omega')$,

$$(\Delta_\mu f, \varphi)_{\mathcal{D}} = (f, \Delta_\mu \varphi)_{\mathcal{D}} = -(\nabla f, \nabla \varphi)_{\mathcal{D}} = 0.$$

Hence, we have $f \in W_0^1(\Omega')$ and $\Delta_\mu f = 0$ in Ω', which implies by Theorem 7.1 that $f \in C^{\infty}(\Omega')$. Since $\nabla f \equiv 0$ in Ω', we conclude that $f \equiv \text{const}$ in Ω', which contradicts to the construction that $f = 1$ in Ω and $f = 0$ in $\Omega' \setminus \overline{\Omega}$.

Theorem 10.22 is complemented by the following statement. □

THEOREM 10.23. *For any a weighted manifold $(M, \mathbf{g}, \mu)$ and any non-empty relatively compact connected open set $\Omega \subset M$,*

$$\lambda_2(\Omega) > \lambda_1(\Omega). \tag{10.81}$$

PROOF. Indeed, by Corollary 10.12, the eigenvalue $\lambda_1(\Omega)$ is simple whence (10.81) follows. □

10.8. Long time asymptotic of $\log p_t$

We will show here that the bottom of the spectrum $\lambda_{\min}(M)$ determines the long time behavior of the heat kernel.

THEOREM 10.24. *On any connected weighted manifold $(M, \mathbf{g}, \mu)$, we have, for all $x, y \in M$,*

$$\lim_{t \to \infty} \frac{\log p_t(x,y)}{t} = -\lambda_{\min}(M). \tag{10.82}$$

PROOF. Set $\lambda = \lambda_{\min}(M)$. Since the spectrum of operator $P_t = e^{-t\mathcal{L}}$ is bounded by $e^{-\lambda t}$, we obtain that $\|P_t\| \le e^{-\lambda t}$ and, hence, for any $f \in L^2$,

$$\|P_t f\|_{L^2} \le e^{-\lambda t} \|f\|_{L^2}. \tag{10.83}$$

Applying this to $f = p_{s,x}$ (where $s > 0$ and $x \in M$) and notices that $P_t p_{s,x} = p_{t+s,x}$, we obtain

$$\|p_{t+s,x}\|_{L^2} \le e^{-\lambda t} \|p_{s,x}\|_{L^2}, \tag{10.84}$$

whence

$$\limsup_{t \to \infty} \frac{\log \|p_{t,x}\|_{L^2}}{t} \le -\lambda.$$

It follows from (7.48) that

$$p_t(x,y) = \left(p_{t/2,x}, p_{t/2,y}\right) \le \|p_{t/2,x}\|_{L^2} \|p_{t/2,y}\|_{L^2},$$

whence

$$\limsup_{t \to \infty} \frac{\log p_t(x,y)}{t} \le -\lambda.$$

To prove the opposite inequality, take any connected relatively compact open set $\Omega \subset M$ and recall that, by Theorem 10.13, the spectrum of the Dirichlet Laplace operator $\mathcal{L}^\Omega$ is discrete, and the heat kernel p_t^Ω is given by the expansion (10.33). By Theorem 10.23, $\lambda_k(\Omega) > \lambda_1(\Omega)$ for any $k > 1$ and, by Theorem 10.11, the first eigenfunction $\varphi_1(x)$ of $\mathcal{L}^\Omega$ is strictly positive in Ω. Hence, the first term in expansion (10.33) is the leading one as $t \to \infty$, that is,

$$p_t^\Omega(x,y) \sim e^{-\lambda_1(\Omega)t} \varphi_1(x) \varphi_1(y) \quad \text{as } t \to \infty,$$

for all $x, y \in \Omega$, which implies

$$\lim_{t \to \infty} \frac{\log p_t^\Omega(x,y)}{t} = -\lambda_1(\Omega).$$

Since $p_t \ge p_t^\Omega$ (cf. Theorem 5.23 and Exercise 7.40), it follows that

$$\liminf_{t \to \infty} \frac{\log p_t(x,y)}{t} \ge -\lambda_{\min}(\Omega).$$

Exhausting M be such sets Ω and noticing that $\lambda_{\min}(\Omega) \to \lambda$ (see Exercise 10.6), we finish the proof. $\square$

Exercises.

10.26. Let $f \in C^2(M)$ be a non-negative function on a connected weighted manifold M that satisfies the inequality

$$\Delta_\mu f + \alpha f \le 0$$

with a real constant α. Prove that either $f \equiv 0$ or $\alpha \le \lambda_{\min}(M)$.

REMARK. The converse is also true, that is, for any $\alpha \ge \lambda_{\min}(M)$ there exists a positive solution to the equation $\Delta_\mu f + \alpha f = 0$. This will be proved later in Chapter 13 (cf. Theorem 13.16). Exercise 10.27 contains a partial result in this direction.

10.27. Let α be a real number.

(a) Prove that if $\alpha < \lambda_{\min}(M)$ then the operator $\mathcal{L} - \alpha\,\mathrm{id}$ has the inverse in $L^2(M)$ and

$$(\mathcal{L} - \alpha\,\mathrm{id})^{-1} = \int_0^\infty e^{\alpha t} P_t dt. \tag{10.85}$$

(b) Prove that if $\mu(M) < \infty$ and $\alpha < \lambda_{\min}(M)$ then the weak Dirichlet problem

$$\begin{cases} \Delta_\mu u + \alpha u = 0 \\ u \in 1 \mod W_0^1(M) \end{cases}$$

has a unique solution that is given by the formula

$$u = 1 + \alpha \int_0^\infty e^{\alpha t}(P_t 1)\,dt \tag{10.86}$$

Deduce that $u > 0$.

10.28. (*Maximum principle*) Let Ω be a non-empty relatively compact open set in a connected weighted manifold M such that $M \setminus \overline{\Omega}$ is non-empty. Prove that if $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a subharmonic function in Ω then

$$\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u. \tag{10.87}$$

REMARK. Of course, this statement follows from Corollary 8.16. Find another proof using Theorem 10.22 and Exercise 4.28.

10.29. Prove that, for all $x, y \in M$ and $t \ge s > 0$,

$$p_t(x,y) \le \sqrt{p_s(x,x)\,p_s(y,y)}\exp\left(-\lambda_{\min}(M)(t-s)\right).$$

Notes

Most of the material of this Chapter is an adaptation of the classical spectral theory, that is associated with the names of Rayleigh, Courant, Neumann, Weyl, to the particular case of the Dirichlet Laplacian.

The computation of the spectra of $\mathbb{S}^n$ and some other compact manifolds can be found in [**36**] (see also [**51**]).

CHAPTER 11

Distance function and completeness

Here we introduce the techniques of Lipschitz test functions (Section 11.2), which allow to relate the geodesic distance to the properties of solutions of the Laplace and heat equations. Apart from the applications within the present Chapter, this techniques will also be used in Chapters 12, 15, 16.

11.1. The notion of completeness

Let (M, g) be a Riemannian manifold and $d(x, y)$ be the geodesic distance on M (see Section 3.11 for the definition). The manifold (M, g) is said to be *metrically complete* if the metric space (M, d) is complete, that is, any Cauchy sequence in (M, d) converges.

A smooth path $\gamma(t) : (a, b) \to M$ is called a *geodesics* if, for any $t \in (a, b)$ and for all s close enough to t, the path $\gamma|_{[t,s]}$ is a shortest path between the points $\gamma(t)$ and $\gamma(s)$. A Riemannian manifold (M, g) is called *geodesically complete* if, for any $x \in M$ and $\xi \in T_xM \setminus \{0\}$, there is a geodesics $\gamma : [0, +\infty) \to M$ of infinite length such that $\gamma(0) = x$ and $\dot{\gamma}(0) = \xi$. It is known that, on a geodesically complete connected manifold, any two points can be connected by a shortest geodesics.

We state the following theorem without proof.

HOPF-RINOW THEOREM. *For a Riemannian manifold (M, g), the following conditions are equivalent:*

(a) *(M, g) is metrically complete.*
(b) *(M, g) is geodesically complete.*
(c) *All geodesic balls in M are relatively compact sets.*

This theorem will not be used, but it motivates us to give the following definition.

DEFINITION 11.1. A Riemannian manifold (M, g) is said to be *complete* if all the geodesic balls in M are relatively compact.

For example, any compact manifold is complete.

Exercises.

11.1. Let g be a metric in $\mathbb{R}^n$, which is given in the polar coordinates (r, θ) by

$$\mathrm{g} = dr^2 + \psi^2(r)\, \mathrm{g}_{\mathbb{S}^{n-1}}, \tag{11.1}$$

where $\psi(r)$ is a smooth positive function (cf. Sections 3.10 and 8.4.3). Prove that the Riemannian model $(\mathbb{R}^n, \mathrm{g})$ is complete.

11.2. Prove the implication (c) $\Rightarrow$ (a) of Hopf-Rinow Theorem, that is, if all geodesic balls are relatively compact then (M, d) is a complete metric space.

11.2. Lipschitz functions

Let d be the geodesic distance on a Riemannian manifold $(M, \mathbf{g})$. Let f be a function defined on a set $S \subset M$. We say that f is Lipschitz on S if there exists a finite constant C such that

$$|f(x) - f(y)| \leq C d(x, y) \quad \text{for all } x, y \in S.$$

The constant C is called the *Lipschitz constant* of f. The smallest possible value of C is called the *Lipschitz seminorm* of f and is denoted by $\|f\|_{Lip(S)}$; that is,

$$\|f\|_{Lip(S)} := \sup_{x,y \in S, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

The set of all Lipschitz functions on M is denoted by $Lip(M)$. It is obvious that $Lip(M)$ is a linear space (cf. Exercise 11.5). It follows from Lemma 3.24 that any Lipschitz function on M is continuous, that is, $Lip(M) \subset C(M)$.

A large variety of Lipschitz functions arise from the following construction. For any non-empty set $E \subset M$ and any point $x \in M$, define the distance from x to E by

$$d(x, E) := \inf \{d(x, z) : z \in E\}.$$

LEMMA 11.2. *If manifold M is connected then the function $x \mapsto d(x, E)$ is Lipschitz on M with the Lipschitz constant* 1.

PROOF. The connectedness of M ensures that $d(x, E)$ is finite. Let us show that, for any two points $x, y \in M$,

$$d(x, E) - d(y, E) \leq d(x, y), \tag{11.2}$$

which will imply the claim. For any $\varepsilon > 0$, there exists $z \in E$ such that

$$d(y, E) \geq d(y, z) - \varepsilon.$$

Then we have by the triangle inequality

$$d(x, E) - d(y, E) \leq d(x, z) - (d(y, z) - \varepsilon) \leq d(x, z) - d(y, z) + \varepsilon \leq d(x, y) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (11.2) follows. $\square$

It is important for applications that any Lipschitz function has the weak gradient as stated below.

THEOREM 11.3. *Let $(M, \mathbf{g}, \mu)$ be a weighted manifold. Then, for any $f \in Lip(M)$, the distributional gradient ∇f is an L^∞-vector field on M and*

$$\|\nabla f\|_{L^\infty} \leq \|f\|_{Lip}. \tag{11.3}$$

FIRST PROOF. Let U be a chart on M such that

$$d(x,y) \le C|x-y| \quad \text{for all } x,y \in U,$$

where $|x-y|$ is the Euclidean distance in the local coordinates in U. By Lemma 3.24, the manifold M can be covered by charts with this property. It follows that the function $f|_U$ is Lipschitz with respect to the Euclidean distance.

We will take without proof the following fact from the theory of functions of real variables.

RADEMACHER'S THEOREM. *Any Lipschitz function f in a an open set $U \subset \mathbb{R}^n$ is differentiable at almost all points $x \in U$ in the following sense: there exists a covector $u(x) \in \mathbb{R}^n$, such that*

$$f(x+\xi) - f(x) = \langle u(x), \xi\rangle + o(|\xi|), \quad \text{as } \xi \to 0. \tag{11.4}$$

Moreover, the components $u_i(x)$ of $u(x)$ coincide with the distributional derivatives $\frac{\partial f}{\partial x^i}$ of f.

The Lipschitz condition implies that all the components $u_i = \frac{\partial f}{\partial x^i}$ belong to $L^\infty(U)$. By Exercise 4.11, the vector field v with components

$$v^i = g^{ij}u_i$$

is the distributional gradient of f in U with respect to the Riemannian metric g. In particular, if $x \in U$ is a point where (11.4) holds then, for any vector $\xi \in T_xM$, we have

$$\langle v, \xi\rangle_{\mathrm{g}} = g_{ik}v^i\xi^k = g_{ik}g^{ij}u_i\xi^k = u_i\xi^i = \langle u, \xi\rangle. \tag{11.5}$$

Let us show that $|v|_{\mathrm{g}} \le C$ a.e. where $C = \|f\|_{Lip}$, which will prove (11.3). It suffices to show that, for any point $x \in U$ where (11.4) holds and for any tangent vector $\xi \in T_xM$,

$$\langle v, \xi\rangle_{\mathrm{g}} \le C|\xi|_{\mathrm{g}}\,. \tag{11.6}$$

Choose a smooth path γ in U such that $\gamma(0) = x$ and $\dot\gamma(0) = \xi$. Then

$$\gamma(t) - \gamma(0) = \xi t + o(t) \quad \text{as } t \to 0,$$

whence, by (11.4) and (11.5),

$$f(\gamma(t)) - f(\gamma(0)) = \langle u, \gamma(t)-\gamma(0)\rangle + o(t) = \langle u, \xi\rangle t + o(t) = \langle v, \xi\rangle_{\mathrm{g}} t + o(t).$$

On the other hand, the Lipschitz condition implies

$$|f(\gamma(t)) - f(\gamma(0))| \le C\ell\left(\gamma|_{[0,t]}\right) = C\int_0^t |\dot\gamma(s)|_{\mathrm{g}}\,ds = C|\xi|_{\mathrm{g}}\,t + o(t).$$

Comparing the above two lines and letting $t \to 0$, we obtain (11.6), which was to be proved. $\square$

SECOND PROOF. In this proof, we do not use Rademacher's theorem, but instead, we will use Exercise 2.23, which proves the statement of Theorem 11.3 in the case when M is an open set in $\mathbb{R}^n$ and the metric is Euclidean.

Let us first prove the following claim.

CLAIM. *For any point $p \in M$ and for any $C > 1$, there exists a chart $U \ni p$ such that for all $x \in U$, $\xi \in T_xM$, $\eta \in T_x^*M$,*

$$g_{ij}(x)\,\xi^i\xi^j \le C^2\left(\left(\xi^1\right)^2 + \ldots + \left(\xi^n\right)^2\right) \tag{11.7}$$

and

$$g^{ij}(x)\,\eta_i\eta_j \le C^2\left(\eta_1^2 + \ldots + \eta_n^2\right). \tag{11.8}$$

Let so far U be any chart containing p. Arguing as in the proof of Theorem 8.10, the coordinates $x^1, \ldots, x^n$ in U can be chosen so that $g_{ij}(p) =$ id. By continuity, the matrix $\widetilde{g}$ is close enough to id in a small enough neighborhood V of p. More precisely, by choosing V small enough, we can ensure that the matrices $\widetilde{g}_{ij}$ and $\widetilde{g}^{ij}$ satisfy the conditions (11.7) and (11.8), respectively. We are left to rename V to U, y^i to x^i, and $\widetilde{g}$ to g.

Shrinking further the chart U from the above Claim, we can assume that U is a ball in the coordinates $x^1, \ldots, x^n$ centered at p. Then, for any two points $x, y \in U$, the straight line segment between x, y is also contained in U. By (11.7), the Riemannian length of this segment is bounded by $C|x-y|$, which implies that

$$d(x,y) \le C|x-y|. \tag{11.9}$$

Let now f be a Lipschitz function on M with the Lipschitz constant K. In a chart U as above, we have

$$|f(x) - f(y)| \le Kd(x,y) \le CK|x-y|,$$

so that f is Lipschitz with a Lipschitz constant CK in the Euclidean metric in U. By Exercise 2.23, we conclude that f has the distributional partial derivatives $\frac{\partial f}{\partial x^i} \in L^\infty(U)$ such that

$$\sum_{i=1}^n \left(\frac{\partial f}{\partial x^i}\right)^2 \le (CK)^2 \text{ a.e.}. \tag{11.10}$$

By Exercise 4.11, the Riemannian distributional gradient $\nabla_{\mathbf{g}} f$ is given by

$$(\nabla_{\mathbf{g}} f)^k = g^{ki}\frac{\partial f}{\partial x^i},$$

and

$$|\nabla_{\mathbf{g}} f|^2 = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}.$$

It follows (11.8) and (11.10) that

$$|\nabla_{\mathbf{g}} f|^2 \le C^2 \sum_{i=1}^n \left(\frac{\partial f}{\partial x^i}\right)^2 \le C^2 (CK)^2 \text{ a.e.}$$

that is, in U,

$$|\nabla_{\mathbf{g}} f| \leq C^2 K \text{ a.e..} \tag{11.11}$$

Since M can be covered by a countable family of such charts U, (11.11) holds also in M. Finally, since $C > 1$ was arbitrary, we obtain $|\nabla_{\mathbf{g}} f| \leq K$ a.e., which finishes the proof. □

Denote by $Lip_0(M)$ the set of all Lipschitz functions on M with compact support. It is obvious that

$$Lip_0(M) \subset L^p(M),$$

for all $1 \leq p \leq \infty$.

COROLLARY 11.4. *We have the following inclusions:*

$$C_0^1(M) \subset Lip_0(M) \subset W_0^1(M). \tag{11.12}$$

PROOF. By Theorem 11.3, any function $f \in Lip_0(M)$ has distributional gradient $\nabla f \in \vec{L}^\infty(M)$. Since $\operatorname{supp} f$ is compact, it follows that $f \in L^2(M)$ and $\nabla f \in \vec{L}^2(M)$, that is, $f \in W^1(M)$. By Lemma 5.5, the compactness of $\operatorname{supp} f$ implies $f \in W_0^1(M)$, which proves the second inclusion in (11.12).

Let now $f \in C_0^1(M)$. Set

$$C := \sup_M |\nabla f| < \infty$$

and show that, for any two points $x, y \in M$,

$$|f(x) - f(y)| \leq C d(x, y), \tag{11.13}$$

which will prove the first inclusion in (11.12). If the points $x, y \in M$ cannot be connected by a smooth path then $d(x, y) = \infty$ and (11.13) holds. Let $\gamma(t) : [a, b] \to M$ be a smooth path such that $\gamma(a) = x$ and $\gamma(b) = y$. Then

$$f(y) - f(x) = \int_a^b \frac{d}{dt} f(\gamma(t))\, dt = \int_a^b \langle df, \dot{\gamma} \rangle dt = \int_a^b \langle \nabla f, \dot{\gamma} \rangle_{\mathbf{g}} dt,$$

whence

$$|f(y) - f(x)| \leq \int_a^b |\nabla f| |\dot{\gamma}|\, dt \leq C \int_a^b |\dot{\gamma}|\, dt = C \ell(\gamma).$$

Minimizing over all γ, we obtain (11.13). □

A function f on M is said to be *locally Lipschitz* if f is Lipschitz on any compact subset of M. The class of all locally Lipschitz functions is denoted by $Lip_{loc}(M)$, so that we have

$$Lip_0(M) \subset Lip(M) \subset Lip_{loc}(M).$$

Some additional properties of Lipschitz and locally Lipschitz functions are stated in the following Exercises.

Exercises.

11.3. Prove that a function $f \in C^1(M)$ is Lipschitz if and only if $|\nabla f|$ is bounded, and

$$\|f\|_{Lip} = \sup_M |\nabla f|.$$

11.4. Prove the following properties of Lipschitz functions.

(a) Let $f_1, ..., f_m \in Lip(M)$ and let $I_k = f_k(M)$ be the range of f_k. Let φ be a Lipschitz function on the set $I_1 \times ... \times I_m \subset \mathbb{R}^m$. Then the composite function

$$\Phi(x) := \varphi(f_1(x), ..., f_m(x))$$

is Lipschitz on M and

$$\|\Phi\|_{Lip} \leq \|\varphi\|_{Lip} \left(\sum_{k=1}^m \|f_k\|_{Lip}^2\right)^{1/2}. \tag{11.14}$$

(b) If $f \in Lip_0(M)$ and $\varphi \in Lip(\mathbb{R})$ is such that $\varphi(0) = 0$ then $\varphi \circ f \in Lip_0(M)$.

11.5. Prove that $f, g \in Lip(M)$ then also the functions $f + g$, $\max(f, g)$, $\min(f, g)$ are Lipschitz; moreover, fg is also Lipschitz provided one of the functions f, g is bounded on the support of the other.

Hence show, that if $f, g \in Lip_0(M)$ then also the functions $f + g$, fg, $\max(f, g)$, $\min(f, g)$ belong to $Lip_0(M)$.

11.6. Prove that for any open set $\Omega \subset M$ and any compact set $K \subset \Omega$ there is a function $f \in Lip_0(\Omega)$ such that $0 \leq f \leq 1$ in Ω, $f|_K \equiv 1$, and $\|f\|_{Lip} \leq \frac{2}{d(K, \Omega^c)}$.

REMARK. A function f with the above properties is called a *Lipschitz cutoff function* of K in Ω.

11.7. Let f be a real valued function on a Riemannian manifold M.

(a) Prove that if $\{U_\alpha\}$ is a countable family of open sets covering the manifold M such that

$$C := \sup_\alpha \|f\|_{Lip(U_\alpha)} < \infty,$$

then $f \in Lip(M)$ and $\|f\|_{Lip(M)} \leq C$.

(b) Prove that if E_1, E_2 are two closed sets in M such that $E_1 \cup E_2 = M$ and f is Lipschitz in each set E_1, E_2 with the Lipschitz constant C, then f is also Lipschitz in M with the Lipschitz constant C.

11.8. Prove that

$$C^1(M) \subset Lip_{loc}(M) \subset W^1_{loc}(M).$$

11.9. Prove that the set of functions from $Lip_{loc}(M)$ with compact support is identical to $Lip_0(M)$.

11.10. Prove that if $f_1, ..., f_m \in Lip_{loc}(M)$ and $\varphi \in Lip_{loc}(\mathbb{R}^m)$ then the composite function $\Phi(x) := \varphi(f_1(x), ..., f_m(x))$ is locally Lipschitz on M.

11.11. Prove that if $f, g \in Lip_{loc}(M)$ then the functions $f + g$, fg, $\max(f, g)$, $\min(f, g)$ are also in $Lip_{loc}(M)$.

11.12. Prove that if $f \in Lip_{loc}(M)$ then the distributional gradient ∇f belongs to $\vec{L}^\infty_{loc}(M)$.

11.13. *(Product rule for Lipschitz functions)*

(a) Prove that, for all $f, g \in Lip_{loc}(M)$,

$$\nabla(fg) = f\nabla g + g\nabla f. \tag{11.15}$$

(b) Prove that if $f \in Lip(M) \cap L^\infty(M)$ and $g \in W^1_0(M)$ then $fg \in W^1_0(M)$ and (11.15) holds.

(c) Prove that if $f \in Lip_0(M)$ and $g \in W^1_{loc}(M)$ then $fg \in W^1_0(M)$ and (11.15) holds.

11.14. *(Chain rule for Lipschitz functions)* Prove that if $f \in Lip_{loc}(M)$ and $\psi \in C^1(\mathbb{R})$, then $\psi(f) \in Lip_{loc}(M)$ and

$$\nabla\psi(f) = \psi'(f)\nabla f.$$

11.3. Essential self-adjointness

Apart from the Dirichlet Laplace operator, the operator $\Delta_\mu|_{\mathcal{D}}$ may have other self-adjoint extensions related to other boundary conditions. A densely defined operator in L^2 is said to be *essentially* self-adjoint if it admits a unique self-adjoint extension.

THEOREM 11.5. *If the weighted manifold $(M, \mathbf{g}, \mu)$ is complete then the operator $\Delta_\mu|_{\mathcal{D}}$ is essentially self-adjoint in $L^2(M)$.*

We precede the proof by lemmas of independent interest.

LEMMA 11.6. *Let $(M, \mathbf{g}, \mu)$ be a complete weighted manifold. If a function $u \in L^2(M)$ satisfies the equation $\Delta_\mu u - \lambda u = 0$ with a constant $\lambda \geq 0$, then $u \equiv$ const on each connected component of M. If in addition $\lambda > 0$ then $u \equiv 0$.*

PROOF. By Theorem 7.1, we have $u \in C^\infty(M)$. Let $f \in Lip_0(M)$, that is, f is a Lipschitz function on M with compact support. Then also $uf^2 \in Lip_0(M)$ and, hence, $uf^2 \in W^1_0(M)$ (cf. Corollary 11.4). Multiplying the equation $\Delta_\mu u = \lambda u$ by uf^2, we obtain $uf^2\Delta_\mu u \geq 0$. Integrating this inequality and using the Green formula (4.12), we obtain

$$\begin{aligned} 0 &\geq -\int_M uf^2\Delta_\mu u\, d\mu = \int_M \langle\nabla\left(uf^2\right), \nabla u\rangle_{\mathbf{g}} d\mu \\ &= \int_M |\nabla u|^2 f^2 d\mu + 2\int_M \langle\nabla u, \nabla f\rangle_{\mathbf{g}} u\, f\, d\mu, \end{aligned}$$

whence

$$\begin{aligned} \int_M |\nabla u|^2 f^2 d\mu &\leq -2\int_M \langle\nabla u, \nabla f\rangle_{\mathbf{g}} u\, f\, d\mu \\ &\leq \frac{1}{2}\int_M |\nabla u|^2 f^2 d\mu + 2\int_M |\nabla f|^2 u^2 d\mu \end{aligned}$$

and

$$\int_M |\nabla u|^2 f^2 d\mu \leq 4\int_M |\nabla f|^2 u^2 d\mu. \tag{11.16}$$

Fix a point $o \in M$, numbers $R > r > 0$, and specify f as follows:

$$f(x) = (R - d(x, o))_+ .$$

Alternatively, this function can be defined by $f = \varphi \circ d(\cdot, o)$ where $\varphi(s) = (R - s)_+$. Since both functions φ and $d(\cdot, o)$ are Lipschitz function on $\mathbb{R}$ and M, respectively, with Lipschitz constants 1, the function f is also Lipschitz with the Lipschitz constant 1 (cf. Exercise 11.4). Obviously, supp f coincides with the closed geodesic ball $\overline{B(o,R)}$. By hypotheses, all the geodesic balls on M are relatively compact, whence it follows that $\operatorname{supp} f$ is compact. Hence, $f \in Lip_0(M)$, and (11.16) holds with this f.

Since $f \geq R - r$ on $B(o,r)$ and, by Theorem 11.3, $|\nabla f| \leq 1$ a.e., we obtain from (11.16)

$$\int_{B(o,r)} |\nabla u|^2 \, d\mu \leq \frac{4}{(R-r)^2} \int_M u^2 d\mu.$$

Letting $R \to \infty$ and using $u \in L^2(M)$, we obtain

$$\int_{B(o,r)} |\nabla u|^2 \, d\mu = 0.$$

Since r is arbitrary, we conclude $\nabla u \equiv 0$ and hence $u \equiv \text{const}$ on any connected component of M. In the case $\lambda > 0$ it follows that $u \equiv 0$ because 0 is the only constant that satisfies the equation $\Delta_\mu u - \lambda u = 0$. □

LEMMA 11.7. *On a complete weighted manifold, if $u \in L^2(M)$ and $\Delta_\mu u \in L^2(M)$ then $u \in W_0^2(M)$.*

PROOF. By Theorem 4.5, the equation $-\Delta_\mu v + v = f$ has a solution $v = R_1 f \in W_0^2(M)$ for any $f \in L^2(M)$. Set $f = -\Delta_\mu u + u$ and observe that, for the function $v = R_1 f$, we have

$$-\Delta_\mu (u - v) + (u - v) = 0.$$

Since $u - v \in L^2(M)$ we conclude by Lemma 11.6 that $u - v = 0$ whence $u \in W_0^2(M)$. □

PROOF OF THEOREM 11.5. Let $\mathcal{L} = -\Delta_\mu|_{W_0^2}$ and $\mathcal{L}_0 = -\Delta_\mu|_{\mathcal{D}}$. The inclusion $\mathcal{L}_0 \subset \mathcal{L}$ implies $\mathcal{L} = \mathcal{L}^* \subset \mathcal{L}_0^*$. By (4.10), we have

$$\operatorname{dom} \mathcal{L}_0^* = \left\{ u \in L^2(M) : \Delta_\mu u \in L^2(M) \right\},$$

whence by Lemma 11.7,

$$\operatorname{dom} \mathcal{L}_0^* \subset W_0^2(M) = \operatorname{dom} \mathcal{L},$$

which implies $\mathcal{L}_0^* = \mathcal{L}$.

If $\mathcal{L}_1$ is another self-adjoint extension of $\mathcal{L}_0$ then $\mathcal{L}_0 \subset \mathcal{L}_1$ implies $\mathcal{L}_1 = \mathcal{L}_1^* \subset \mathcal{L}_0^*$ and, hence, $\mathcal{L}_1 \subset \mathcal{L}$. In turn, this implies $\mathcal{L}^* \subset \mathcal{L}_1^*$ whence $\mathcal{L} = \mathcal{L}_1$. □

Exercises.

11.15. Prove that if (M, g, μ) is a complete weighted manifold then $W_0^1(M) = W^1(M)$.

11.16. Let (M, g, μ) be a complete weighted manifold.

(a) Let $\{u_k\}_{k=1}^\infty$ be a sequence from $W^1(M)$ such that, for all $\varphi \in C_0^\infty(M)$,

$$(u_k, \varphi)_{W^1} \to (u, \varphi)_{W^1} \tag{11.17}$$

for some $u \in W^1$, and

$$(u_k, \varphi)_{L^2} \to (v, \varphi)_{L^2}, \tag{11.18}$$

for some $v \in L^2(M)$. Prove that $u = v$.

(b) Show that without the hypothesis of completeness, the claim of (a) is not true in general.

11.17. Let (M, g, μ) be a complete weighted manifold, and let h be a smooth positive function on M satisfying (9.43). Set $d\widetilde{\mu} = h^2 d\mu$.

(a) Let $\widetilde{\mathcal{L}} = -\Delta_{\widetilde{\mu}}|_{W_0^2}$ be the Dirichlet Laplace operator of $(M, \mathrm{g}, \widetilde{\mu})$. Prove that the operator $-\Delta_\mu + \Phi|_{\mathcal{D}}$ is essentially self-adjoint in $L^2(M,\mu)$, and its unique self-adjoint extension, denoted by $\mathcal{L}^\Phi$, is given by

$$\mathcal{L}^\Phi = J\widetilde{\mathcal{L}}J^{-1}, \tag{11.19}$$

where J is a bijection $L^2(M,\widetilde{\mu}) \to L^2(M,\mu)$ defined by $Jf = hf$.

(b) Prove that the heat semigroup $e^{-t\mathcal{L}^\Phi}$ of the operator $\mathcal{L}^\Phi$ in $L^2(M,\mu)$ has the integral kernel $p_t^\Phi(x,y)$, given by

$$p_t^\Phi(x,y) = h(x)\,h(y)\,\widetilde{p}_t(x,y). \tag{11.20}$$

11.18. Consider in $\mathbb{R}$ the function $\Phi(x) = x^2 - 1$. Verify that the function $h(x) = e^{-\frac{1}{2}x^2}$ satisfies (9.43) with this function. Hence, prove that

$$p_t^\Phi(x,y) = \frac{e^t}{(2\pi\sinh 2t)^{1/2}} \exp\left(-\frac{(x-y)^2}{2\sinh 2t} - \frac{x^2+y^2}{2}\tanh t\right). \tag{11.21}$$

Remark. The function (11.21) is called *the Mehler kernel.*

Hint. Use Example 9.19.

11.4. Stochastic completeness and the volume growth

Define the *volume function* $V(x,r)$ of a weighted manifold (M, g, μ) by

$$V(x,r) := \mu(B(x,r)),$$

where $B(x,r)$ is the geodesic ball. Note that $V(x,r) < \infty$ for all $x \in M$ and $r > 0$ provided M is complete.

Recall that a manifold M is *stochastically* complete, if the heat kernel $p_t(x,y)$ satisfies the identity

$$\int_M p_t(x,y)\,d\mu(y) = 1,$$

for all $x \in M$ and $t > 0$ (see Section 8.4.1). The result of this section is the following volume test for the stochastic completeness.

Theorem 11.8. *Let (M, g, μ) be a complete connected weighted manifold. If, for some point $x_0 \in M$,*

$$\int^\infty \frac{r\,dr}{\log V(x_0,r)} = \infty, \tag{11.22}$$

then M is stochastically complete.

Condition (11.22) holds, in particular, if

$$V(x_0, r) \leq \exp\left(Cr^2\right) \tag{11.23}$$

for all r large enough or even if

$$V(x_0, r_k) \leq \exp\left(Cr_k^2\right), \tag{11.24}$$

for a sequence $r_k \to \infty$ as $k \to \infty$ (cf. Exercise 11.19). This provides yet another proof of the stochastic completeness of $\mathbb{R}^n$ and $\mathbb{H}^n$. See Exercise 12.4 for an alternative proof of the stochastic completeness of M under the condition (11.24).

Fix $0 < T \leq \infty$, set $I = (0, T)$ and consider the following Cauchy problem in $I \times M$

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_\mu u & \text{in } I \times M, \\ u|_{t=0} = 0. \end{cases} \tag{11.25}$$

A solution is sought in the class $u \in C^\infty(I \times M)$, and the initial condition means that $u(t,x) \to 0$ locally uniformly in $x \in M$ as $t \to 0$ (cf. Section 8.4.1). By Theorem 8.18, the stochastic completeness of M is equivalent to the uniqueness property of the Cauchy problem in the class of bounded solutions. In other words, in order to prove Theorem 11.8, it suffices to verify that the only bounded solution to (11.25) is $u \equiv 0$.

The assertion will follow from the following more general fact.

THEOREM 11.9. *Let (M, g, μ) be a complete connected weighted manifold, and let $u(x,t)$ be a solution to the Cauchy problem* (11.25). *Assume that, for some $x_0 \in M$ and for all $R > 0$,*

$$\int_0^T \int_{B(x_0,R)} u^2(x,t)\, d\mu(x) dt \leq \exp\left(f(R)\right), \tag{11.26}$$

where $f(r)$ is a positive increasing function on $(0, +\infty)$ such that

$$\int^\infty \frac{rdr}{f(r)} = \infty. \tag{11.27}$$

Then $u \equiv 0$ in $I \times M$.

Theorem 11.9 provides the uniqueness class (11.26) for the Cauchy problem. The condition (11.27) holds if, for example, $f(r) = Cr^2$, but fails for $f(r) = Cr^{2+\varepsilon}$ when $\varepsilon > 0$.

Before we embark on the proof, let us mention the following consequence.

COROLLARY 11.10. *If $M = \mathbb{R}^n$ and $u(t,x)$ be a solution to* (11.25) *satisfying the condition*

$$|u(t,x)| \leq C \exp\left(C|x|^2\right) \quad \text{for all } t \in I,\ x \in \mathbb{R}^n, \tag{11.28}$$

then $u \equiv 0$. Moreover, the same is true if u satisfies instead of (11.28) *the condition*

$$|u(t,x)| \leq C \exp\left(f(|x|)\right) \quad \text{for all } t \in I,\ x \in \mathbb{R}^n, \tag{11.29}$$

where $f(r)$ is a convex increasing function on $(0, +\infty)$ satisfying (11.27).

PROOF. Since (11.28) is a particular case of (11.29) for the function $f(r) = Cr^2$, it suffices to treat the condition (11.29). In $\mathbb{R}^n$ we have $V(x, r) = cr^n$. Therefore, (11.29) implies that

$$\int_0^T \int_{B(0,R)} u^2(x,t)\, d\mu(x)dt \le CR^n \exp\left(f(R)\right) = C\exp(\widetilde{f}(R)),$$

where $\widetilde{f}(r) := f(r) + n\log r$. The convexity of f implies that $\log r \le Cf(r)$ for large enough r. Hence, $\widetilde{f}(r) \le Cf(r)$ and function $\widetilde{f}$ also satisfies the condition (11.27). By Theorem 11.9, we conclude $u \equiv 0$. □

The class of functions u satisfying (11.28) is called the *Tikhonov class*, and the conditions (11.29) and (11.27) define the *Täcklind class*. The uniqueness of the Cauchy problem in $\mathbb{R}^n$ in each of these classes is a classical result, generalizing Theorem 1.7.

PROOF OF THEOREM 11.8. By Theorem 8.18, it suffices to verify that the only bounded solution to the Cauchy value problem (11.25) is $u \equiv 0$. Indeed, if u is a bounded solution of (11.25), then setting

$$S := \sup|u| < \infty$$

we obtain

$$\int_0^T \int_{B(x_0,R)} u^2(t,x)d\mu(x) \le S^2TV(x_0, R) = \exp\left(f(R)\right),$$

where

$$f(r) := \log\left(S^2TV(x_0, r)\right).$$

It follows from the hypothesis (11.22) that the function f satisfies (11.27). Hence, by Theorem 11.9, we obtain $u \equiv 0$. □

PROOF OF THEOREM 11.9. Denote for simplicity $B_r = B(x_0, r)$. The main technical part of the proof is the following claim.

CLAIM. *Let $u(t,x)$ solve the heat equation in $(b,a) \times M$ where $b < a$ are reals, and assume that $u(t,x)$ extends to a continuous function in $[b,a] \times M$. Assume also that, for all $R > 0$,*

$$\int_a^b \int_{B_R} u^2(x,t)\, d\mu(x)dt \le \exp\left(f(R)\right),$$

where f is a function as in Theorem 11.8. *Then, for any $R > 0$ satisfying the condition*

$$a - b \le \frac{R^2}{8f(4R)}, \tag{11.30}$$

the following inequality holds:

$$\int_{B_R} u^2(a,\cdot)d\mu \le \int_{B_{4R}} u^2(b,\cdot)d\mu + \frac{4}{R^2}. \tag{11.31}$$

Let us first show how this Claim allows to prove that any solution u to (11.25), satisfying (11.26), is identical 0. Extend $u(t,x)$ to $t=0$ by setting $u(0,x)=0$ so that u is continuous in $[0,T)\times M$. Fix $R>0$ and $t\in(0,T)$. For any non-negative integer k, set

$$R_k = 4^k R$$

and, for any $k\geq 1$, choose (so far arbitrarily) a number τ_k to satisfy the condition

$$0<\tau_k\leq c\frac{R_k^2}{f(R_k)}, \tag{11.32}$$

where $c=\frac{1}{128}$. Then define a decreasing sequence of times $\{t_k\}$ inductively by $t_0=t$ and $t_k=t_{k-1}-\tau_k$ (see Fig. 11.1).

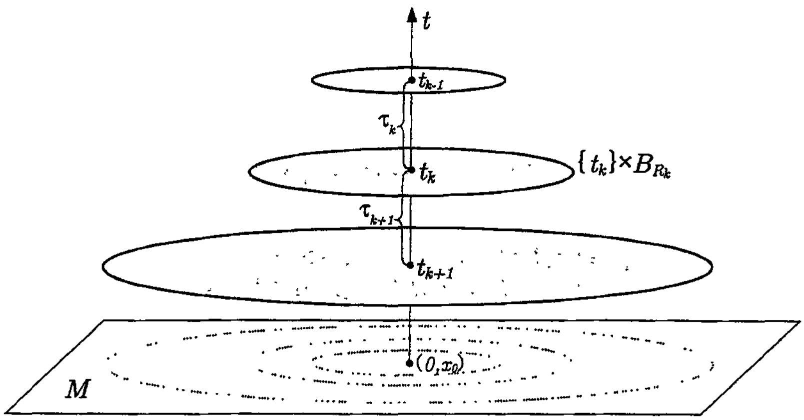

FIGURE 11.1. The sequence of the balls B_{R_k} and the time moments t_k.

If $t_k\geq 0$ then function u satisfies all the conditions of the Claim with $a=t_{k-1}$ and $b=t_k$, and we obtain from (11.31)

$$\int_{B_{R_{k-1}}} u^2(t_{k-1},\cdot)d\mu \leq \int_{B_{R_k}} u^2(t_k,\cdot)d\mu + \frac{4}{R_{k-1}^2}, \tag{11.33}$$

which implies by induction that

$$\int_{B_R} u^2(t,\cdot)d\mu \leq \int_{B_{R_k}} u^2(t_k,\cdot)d\mu + \sum_{i=1}^{k}\frac{4}{R_{i-1}^2}. \tag{11.34}$$

If it happens that $t_k=0$ for some k then, by the initial condition in (11.25),

$$\int_{B_{R_k}} u^2(t_k,\cdot)d\mu = 0.$$

In this case, it follows from (11.34) that

$$\int_{B_R} u^2(t,\cdot)d\mu \le \sum_{i=1}^{\infty} \frac{4}{R_{i-1}^2} = \frac{C}{R^2},$$

which implies by letting $R \to \infty$ that $u(\cdot,t) \equiv 0$ (here we use the connectedness of M).

Hence, to finish the proof, it suffices to construct, for any $R > 0$ and $t \in (0,T)$, a sequence $\{t_k\}$ as above that vanishes at a finite k. The condition $t_k = 0$ is equivalent to

$$t = \tau_1 + \tau_2 + \dots + \tau_k\,. \tag{11.35}$$

The only restriction on τ_k is the inequality (11.32). The hypothesis that $f(r)$ is an increasing function implies that

$$\int_R^\infty \frac{rdr}{f(r)} \le \sum_{k=0}^{\infty} \int_{R_k}^{R_{k+1}} \frac{rdr}{f(r)} \le \sum_{k=0}^{\infty} \frac{R_{k+1}^2}{f(R_k)},$$

which together with (11.27) yields

$$\sum_{k=1}^{\infty} \frac{R_k^2}{f(R_k)} = \infty.$$

Therefore, the sequence $\{\tau_k\}_{k=1}^\infty$ can be chosen to satisfy simultaneously (11.32) and

$$\sum_{k=1}^{\infty} \tau_k = \infty.$$

By diminishing some of τ_k, we can achieve (11.35) for any finite t, which finishes the proof.

Now we prove the above Claim. Since the both integrals in (11.31) are continuous with respect to a and b, we can slightly reduce a and slightly increase b; hence, we can assume that $u(t,x)$ is not only continuous in $[b,a] \times M$ but also smooth.

Let $\rho(x)$ be a Lipschitz function on M (to be specified below) with the Lipschitz constant 1. Fix a real $s \notin [b,a]$ (also to be specified below) and consider the following the function

$$\xi(t,x) := \frac{\rho^2(x)}{4(t-s)},$$

which is defined on $\mathbb{R} \times M$ except for $t = s$, in particular, on $[b,a] \times M$. By Theorem 11.3, the distributional gradient $\nabla\rho$ is in $L^\infty(M)$ and satisfies the inequality $|\nabla\rho| \le 1$, which implies, for any $t \neq s$,

$$|\nabla\xi(t,x)| \le \frac{\rho(x)}{2(t-s)}.$$

Since

$$\frac{\partial\xi}{\partial t} = -\frac{\rho^2(x)}{4(t-s)^2},$$

we obtain

$$\frac{\partial \xi}{\partial t} + |\nabla \xi|^2 \le 0. \tag{11.36}$$

For a given $R > 0$, define a function $\varphi(x)$ by

$$\varphi(x) = \min\left(\left(3 - \frac{d(x,x_0)}{R}\right)_+, 1\right)$$

(see Fig. 11.2). Obviously, we have $0 \le \varphi \le 1$ on M, $\varphi \equiv 1$ in B_{2R}, and $\varphi \equiv 0$ outside B_{3R}. Since the function $d(\cdot, x_0)$ is Lipschitz w. ı the Lipschitz constant 1, we obtain that φ is Lipschitz with the Lipschitz constant $1/R$. By Theorem 11.3, we have $|\nabla \varphi| \le 1/R$. By the completeness of M, all the balls in M are relatively compact sets, which implies $\varphi \in Lip_0(M)$.

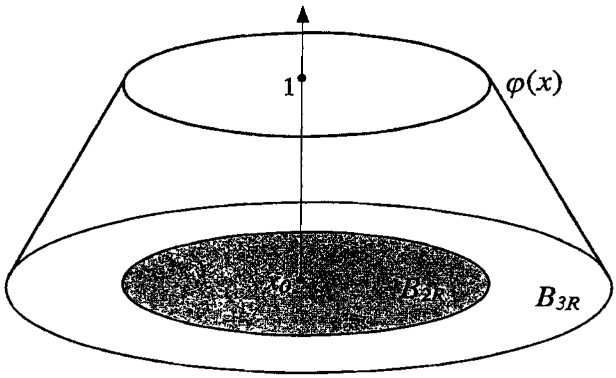

FIGURE 11.2. Function $\varphi(x)$

Consider the function $u\varphi^2 e^\xi$ as a function of x for any fixed $t \in [b,a]$. Since it is obtained from locally Lipschitz functions by taking product and composition, this function is locally Lipschitz on M (cf. Exercise 11.11). Since this function has a compact support, it belongs to $Lip_0(M)$, whence by Corollary 11.4

$$u\varphi^2 e^\xi \in W_c^1(M).$$

Multiplying the heat equation

$$\frac{\partial u}{\partial t} = \Delta_\mu u$$

by $u\varphi^2 e^\xi$ and integrating it over $[b,a] \times M$, we obtain

$$\int_b^a \int_M \frac{\partial u}{\partial t} u\varphi^2 e^\xi d\mu dt = \int_b^a \int_M (\Delta_\mu u) u\varphi^2 e^\xi d\mu dt. \tag{11.37}$$

Since both functions u and ξ are smooth in $t \in [b, a]$, the time integral on the left hand side can be computed as follows:

$$\frac{1}{2}\int_b^a \frac{\partial(u^2)}{\partial t}\varphi^2 e^\xi dt = \frac{1}{2}\left[u^2\varphi^2 e^\xi\right]_b^a - \frac{1}{2}\int_b^a \frac{\partial \xi}{\partial t} u^2 \varphi^2 e^\xi dt. \tag{11.38}$$

Using the Green formula (4.12) (cf. Exercise 5.9) to evaluate the spatial integral on the right hand side of (11.37), we obtain

$$\int_M (\Delta_\mu u)\, u\varphi^2 e^\xi d\mu = -\int_M \langle \nabla u, \nabla(u\varphi^2 e^\xi)\rangle d\mu.$$

Applying the product rule and the chain rule to compute $\nabla(u\varphi^2 e^\xi)$ (cf. Exercises 11.13 and 11.14), we obtain

$$\begin{aligned}
-\langle \nabla u, \nabla(u\varphi^2 e^\xi)\rangle &= -|\nabla u|^2 \varphi^2 e^\xi - \langle \nabla u, \nabla \xi\rangle u\varphi^2 e^\xi - 2\langle \nabla u, \nabla\varphi\rangle u\varphi e^\xi \\
&\le -|\nabla u|^2 \varphi^2 e^\xi + |\nabla u|\,|\nabla \xi|\,|u|\,\varphi^2 e^\xi \\
&\quad + \left(\frac{1}{2}|\nabla u|^2 \varphi^2 + 2|\nabla\varphi|^2 u^2\right) e^\xi \\
&= \left(-\frac{1}{2}|\nabla u|^2 + |\nabla u|\,|\nabla\xi|\,|u|\right)\varphi^2 e^\xi + 2|\nabla\varphi|^2 u^2 e^\xi.
\end{aligned}$$

Combining with (11.37), (11.38), and using (11.36), we obtain

$$\begin{aligned}
\left[\int_M u^2\varphi^2 e^\xi d\mu\right]_b^a &= \int_b^a\int_M \frac{\partial\xi}{\partial t} u^2\varphi^2 e^\xi\, d\mu dt + 2\int_b^a\int_M (\Delta_\mu u)\, u\varphi^2 e^\xi d\mu dt \\
&\le \int_b^a\int_M \left(-|\nabla\xi|^2 u^2 - |\nabla u|^2 + 2|\nabla u|\,|\nabla\xi|\,|u|\right)\varphi^2 e^\xi d\mu dt \\
&\quad +4\int_b^a\int_M |\nabla\varphi|^2 u^2 e^\xi d\mu dt \\
&= -\int_b^a\int_M \left(|\nabla\xi|\,|u| - |\nabla u|\right)^2 \varphi^2 e^\xi d\mu dt \\
&\quad +4\int_b^a\int_M |\nabla\varphi|^2 u^2 e^\xi d\mu dt
\end{aligned}$$

whence

$$\left[\int_M u^2\varphi^2 e^\xi d\mu\right]_b^a \le 4\int_b^a\int_M |\nabla\varphi|^2 u^2 e^\xi d\mu dt. \tag{11.39}$$

Using the properties of function $\varphi(x)$, in particular, $|\nabla\varphi| \leq 1/R$, we obtain from (11.39)

$$\int_{B_R} u^2(a,\cdot)e^{\xi(a,\cdot)}d\mu \leq \int_{B_{4R}} u^2(b,\cdot)e^{\xi(b,\cdot)}d\mu + \frac{4}{R^2}\int_b^a \int_{B_{4R}\setminus B_{2R}} u^2 e^{\xi} d\mu dt. \tag{11.40}$$

Let us now specify $\rho(x)$ and s. Set $\rho(x)$ to be the distance function from the ball B_R, that is,

$$\rho(x) = (d(x,x_0) - R)_+$$

(see Fig. 11.3).

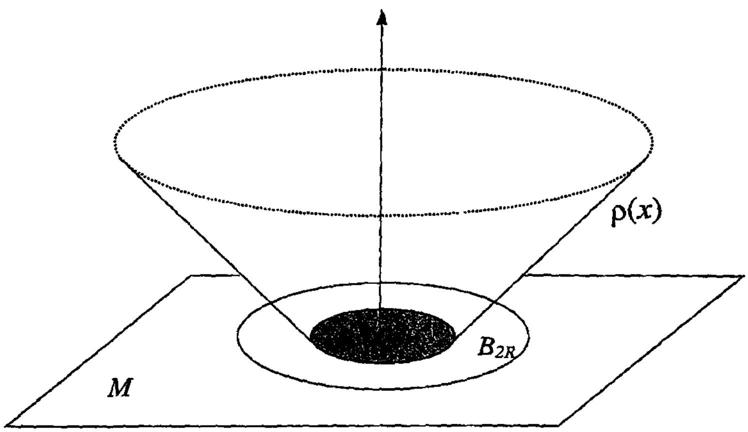

FIGURE 11.3. Function $\rho(x)$.

Set $s = 2a - b$ so that, for all $t \in [b,a]$,

$$a - b \leq s - t \leq 2(a-b),$$

whence

$$\xi(t,x) = -\frac{\rho^2(x)}{4(s-t)} \leq -\frac{\rho^2(x)}{8(a-b)} \leq 0. \tag{11.41}$$

Consequently, we can drop the factor e^{ξ} on the left hand side of (11.40) because $\xi = 0$ in B_R, and drop the factor e^{ξ} in the first integral on the right hand side of (11.40) because $\xi \leq 0$. Clearly, if $x \in B_{4R}\setminus B_{2R}$ then $\rho(x) \geq R$, which together with (11.41) implies that

$$\xi(t,x) \leq -\frac{R^2}{8(a-b)} \quad \text{in } [b,a] \times B_{4R}\setminus B_{2R}.$$

Hence, we obtain from (11.40)

$$\int_{B_R} u^2(a,\cdot)d\mu \leq \int_{B_{4R}} u^2(b,\cdot)d\mu + \frac{4}{R^2}\exp\left(-\frac{R^2}{8(a-b)}\right)\int_b^a \int_{B_{4R}} u^2 d\mu dt.$$

By (11.26) we have

$$\int_b^a \int_{B_{4R}} u^2 d\mu dt \leq \exp\left(f(4R)\right)$$

whence

$$\int_{B_R} u^2(a,\cdot)d\mu \leq \int_{B_{4R}} u^2(b,\cdot)d\mu + \frac{4}{R^2}\exp\left(-\frac{R^2}{8\,(a-b)} + f(4R)\right).$$

Finally, applying the hypothesis (11.30), we obtain (11.31). □

EXAMPLE 11.11. The hypothesis

$$\int^{\infty} \frac{rdr}{\log V(x_0,r)} = \infty \tag{11.42}$$

of Theorem 11.8 is sufficient for the stochastic completeness of M but not necessary as one can see from Example 8.25. Nevertheless, let us show that the condition (11.42) is sharp in the following sense: if $f(r)$ is a smooth positive convex function on $(0,+\infty)$ with $f'(r) > 0$ and such that

$$\int^{\infty} \frac{rdr}{f(r)} < \infty, \tag{11.43}$$

then there exists a complete but stochastically incomplete weighted manifold M such that

$$\log V(x_0,r) = f(r),$$

for some $x_0 \in M$ and large enough r. Indeed, let M be a weighted model as in Section 8.4.3. Note that M is complete by Exercise 11.1. Define its volume function $V(r)$ for large r by

$$V(r) = \exp\left(f(r)\right)$$

so that

$$\frac{V(r)}{V'(r)} = \frac{1}{f'(r)}. \tag{11.44}$$

Let us show that, for all $r \geq 1$,

$$\frac{1}{f'(r)} \leq c\frac{r}{f(r)}, \tag{11.45}$$

where

$$c = \min\left(\frac{f'(1)}{f(1)}, 1\right) > 0.$$

Indeed, the function

$$h(r) = rf'(r) - cf(r)$$

is non-negative for $r = 1$ and its derivative is

$$h'(r) = rf''(r) + (1-c)f'(r) \geq 0.$$

Hence, h is increasing and $h(r) \geq 0$ for $r \geq 1$, whence (11.45) follows.

Combining (11.44), (11.45), and (11.43), we obtain

$$\int^{\infty} \frac{V(r)}{V'(r)} dr < \infty,$$

which implies by Theorem 8.24 the stochastic incompleteness of M.

EXAMPLE 11.12. We say that a weighted manifold $(M, \mathbf{g}, \mu)$ has *bounded geometry* if there exists $\varepsilon > 0$ such that all the geodesic balls $B(x, \varepsilon)$ are *uniformly* quasi-isometric to the Euclidean ball B_ε; that is, there is a constant C and, for any $x \in M$, a diffeomorphism $\varphi_x : B(x, \varepsilon) \to B_\varepsilon$ such that φ_x changes the Riemannian metric and the measure at most by the factor C (see Fig. 11.4).

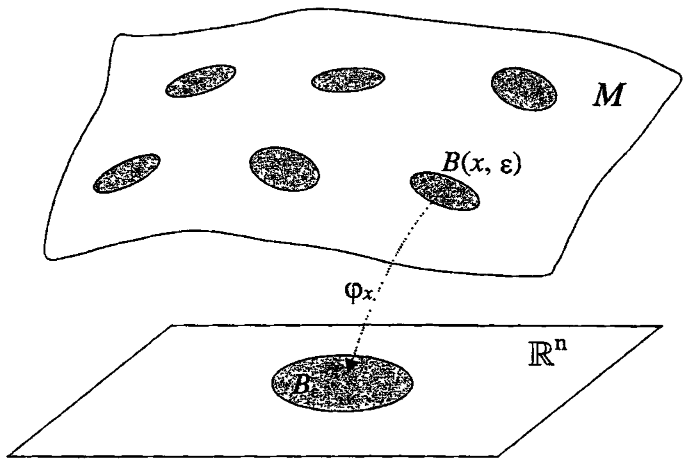

FIGURE 11.4. A manifold of bounded geometry is "patched" by uniformly distorted Euclidean balls.

For example, $\mathbb{R}^n$ and $\mathbb{H}^n$ have bounded geometry. Any manifold of bounded geometry is stochastically complete, which follows from the fact that it is complete and its volume function satisfies the estimate

$$V(x, r) \leq \exp(Cr),$$

for all $x \in M$ and large r (see Exercise 11.20 for the details).

Exercises.

11.19. Let $f(r)$ be a positive increasing function on $(0, +\infty)$ and assume that there exists a sequence $\{r_k\} \to \infty$ such that

$$f(r_k) \leq C r_k^2 \text{ for all } k.$$

Prove that

$$\int^{\infty} \frac{r dr}{f(r)} = \infty.$$

11.20. Let M be a connected manifold with bounded geometry as in Example 11.12.

(a) Prove that there is a constant N such that for any $x \in M$, the ball $B(x, \varepsilon)$ can be covered by at most N balls of radius $\varepsilon/2$.

(b) Prove that for any $x \in M$ and integer $k > 1$, the ball $B(x, k\varepsilon/2)$ can be covered by at most N^{k-1} balls of radii $\varepsilon/2$.
(c) Prove that any geodesic ball on M is relatively compact.
(d) Prove that, $V(x,r) \leq \exp(Cr)$ for all $x \in M$ and $r \geq 1$. Conclude that M is stochastically complete.

11.21. Let (M, μ) be a complete connected weighted manifold with $\mu(M) < \infty$. Prove that, for all $x, y \in M$,

$$p_t(x,y) \to \frac{1}{\mu(M)} \quad \text{as } t \to \infty. \tag{11.46}$$

11.22. Let (M, μ) be a complete connected weighted manifold and let h be a positive harmonic function on M such that, for some $x_0 \in M$, the function

$$v(r) := \int_{B(x_0,r)} h^2 d\mu$$

satisfies the condition

$$\int^{\infty} \frac{rdr}{\log v(r)} = \infty. \tag{11.47}$$

Prove that $P_t h = h$.

11.5. Parabolic manifolds

DEFINITION 11.13. A weighted manifold (M, g, μ) is called *parabolic* if any positive superharmonic function on M is constant.

THEOREM 11.14. *Let (M, g, μ) be a complete connected weighted manifold. If, for some point $x_0 \in M$,*

$$\int^{\infty} \frac{rdr}{V(x_0, r)} = \infty, \tag{11.48}$$

then M is parabolic.

For example, (11.48) holds if $V(x_0, r) \leq Cr^2$ for all r large enough or even if

$$V(x_0, r_k) \leq Cr_k^2, \tag{11.49}$$

for a sequence $r_k \to \infty$ as $k \to \infty$ (cf. Exercise 11.19).

PROOF. Let $u \in C^2(M)$ be a positive superharmonic function on M. Choose any Lipschitz function v on M with compact support. Multiplying the inequality $\Delta_\mu u \leq 0$ by $\frac{v^2}{u}$ and integrating using the Green formula (4.12) (note that $v^2 \in W_0^1(M)$), we obtain

$$\begin{aligned} \int_M \frac{|\nabla u|^2}{u^2} v^2 d\mu &\leq 2 \int_M \frac{\langle \nabla u, \nabla v \rangle}{u} v \, d\mu \\ &\leq 2 \left(\int_M \frac{|\nabla u|^2}{u^2} v^2 d\mu \right)^{1/2} \left(\int_M |\nabla v|^2 d\mu \right)^{1/2}, \end{aligned}$$

whence it follows that

$$\int_M \frac{|\nabla u|^2}{u^2} v^2 d\mu \leq 4 \int_M |\nabla v|^2 d\mu. \tag{11.50}$$

Set $\rho(x) = d(x, x_0)$ and choose $v(x)$ in the form $v(x) = \varphi(\rho(x))$ where φ is a function on $[0, +\infty)$ to be defined. Denote for simplicity $V(r) = V(x_0, r)$ and $B_r = B(x_0, r)$. Fix a finite sequence

$$0 < r_0 < r_1 < \ldots < r_k < \infty$$

and define function φ by the conditions that it is continuous and piecewise linear on $[0, +\infty)$,

$$\varphi(r) = 1 \ \text{ if } 0 \leq r \leq r_0, \quad \varphi(r) = 0 \ \text{ if } r \geq r_k, \tag{11.51}$$

and, for any $i = 1, \ldots, k$,

$$\varphi'(r) = -a \frac{r_i - r_{i-1}}{V(r_i)} \ \text{ if } r_{i-1} < r < r_i, \tag{11.52}$$

where

$$a = \left(\sum_{i=1}^{k} \frac{(r_i - r_{i-1})^2}{V(r_i)} \right)^{-1}$$

(see Fig. 11.5).

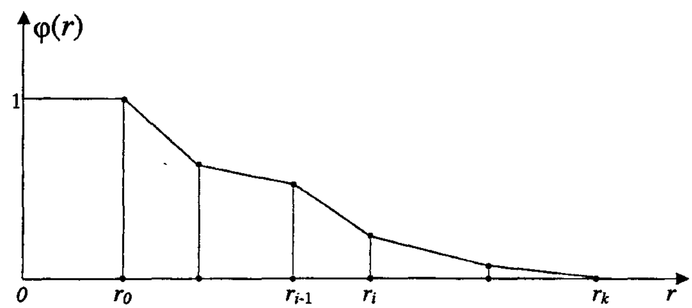

FIGURE 11.5. Function $\varphi(r)$.

For this value of a, we have

$$\int_{r_0}^{r_k} \varphi'(r)\, dr = \sum_{i=1}^{k} \int_{r_{i-1}}^{r_i} \varphi'(r)\, dr = -a \sum_{i=1}^{k} \frac{(r_i - r_{i-1})^2}{V(r_i)} = -1,$$

which makes the conditions (11.51) and (11.52) compatible.

Clearly, $\varphi(r)$ is a Lipschitz function, which implies that $v = \varphi \circ \rho$ is Lipschitz on M. By (11.51), $\operatorname{supp} v \subset \overline{B}_{r_k}$ and, since the balls are relatively compact, $v \in Lip_0(M)$. Obviously, $\nabla \varphi = 0$ in B_{r_0} and outside B_{r_k}. Since $|\nabla \rho| \le 1$ a.e., in each annulus $B_{r_i} \setminus B_{r_{i-1}}$ we have[1]

$$|\nabla v| \le a\frac{r_i - r_{i-1}}{V(r_i)} \text{ a.e.,} \tag{11.53}$$

which implies

$$\int_M |\nabla v|^2 d\mu = \sum_{i=1}^{k} \int_{B_{r_i}\setminus B_{r_{i-1}}} |\nabla v|^2 d\mu \le a^2 \sum_{i=1}^{k} \frac{(r_i - r_{i-1})^2}{V(r_i)^2} V(r_i) = a. \tag{11.54}$$

On the other hand, using the monotonicity of $V(r)$, we obtain

$$\int_{r_1}^{r_k} \frac{rdr}{V(r)} = \sum_{i=1}^{k-1} \int_{r_i}^{r_{i+1}} \frac{rdr}{V(r)} \le \sum_{i=1}^{k-1} \frac{1}{V(r_i)} \int_{r_i}^{r_{i+1}} rdr = \frac{1}{2} \sum_{i=1}^{k-1} \frac{r_{i+1}^2 - r_i^2}{V(r_i)}.$$

Specifying $\{r_i\}$ to be a geometric sequence with $r_i = 2r_{i-1}$, we obtain

$$r_{i+1}^2 - r_i^2 = 3r_i^2 = 12(r_i - r_{i-1})^2,$$

which implies

$$\int_{r_1}^{r_k} \frac{rdr}{V(r)} \le 6 \sum_{i=1}^{k-1} \frac{(r_i - r_{i-1})^2}{V(r_i)} \le 6a^{-1}.$$

Comparing with (11.54), we conclude that

$$\int_M |\nabla v|^2 d\mu \le 6 \left(\int_{r_1}^{r_k} \frac{rdr}{V(r)} \right)^{-1}.$$

Returning to (11.50) and using the fact that $v = 1$ on B_{r_0}, we obtain

$$\int_{B_{r_0}} \frac{|\nabla u|^2}{u^2} d\mu \le 24 \left(\int_{r_1}^{r_k} \frac{rdr}{V(r)} \right)^{-1}.$$

We can still choose r_0 and k. By the hypothesis (11.48), for any $r_0 > 0$ and $\varepsilon > 0$, there exists k so big that

$$\int_{r_1}^{r_k} \frac{rdr}{V(r)} > \varepsilon^{-1},$$

---

[1] Strictly speaking, we can apply the chain rule $\nabla v = \varphi'(\rho) \nabla \rho$ and, hence, obtain (11.53) only in the open set $B_{r_i} \setminus \overline{B}_{r_{i-1}}$. Then (11.53) in $B_{r_i} \setminus B_{r_{i-1}}$ follows from the fact that the boundary of any geodesic ball has measure zero. However, the proof of this fact requires more Riemannian geometry than we would like to use here. Without this fact, one can argue as follows. The volume function $V(r)$ is monotone and, hence, the set S of the points of discontinuity of $V(r)$ is at most countable. We can choose the sequence $\{r_i\}$ to avoid S, which implies that

$$\mu(\partial B_{r_i}) = \lim_{\varepsilon \to 0} (V(r_i + \varepsilon) - V(r_i)) = 0.$$

which implies

$$\int_{B_{r_0}} \frac{|\nabla u|^2}{u^2} d\mu \le 24\varepsilon.$$

Since r_0 and ε are arbitrary, we conclude $\nabla u \equiv 0$ and $u = \text{const}$, which was to be proved. □

REMARK 11.15. Assume that the volume function $V(r)$ belongs to $C^1(\mathbb{R}_+)$ and $V'(r) > 0$. Then one can choose function $\varphi(r)$ is a simpler manner. Namely, for fixed $0 < r_0 < R$, define $\varphi(r)$ by

$$\varphi(r) = 1 \text{ if } 0 \le r \le r_0, \quad \varphi(r) = 0 \text{ if } r \ge R,$$

and

$$\varphi(r) = -\frac{b}{V'(r)} \quad \text{if } r_0 < r < R,$$

where

$$b = \left(\int_{r_0}^{R} \frac{dr}{V'(r)}\right)^{-1}.$$

Then we have

$$\int_M |\nabla v|^2 d\mu \le \int_{B_R \setminus B_{r_0}} \frac{b^2}{V'(\rho)^2} d\mu = \int_{r_0}^{R} \frac{b^2}{V'(r)^2} dV(r) = b^2 \int_{r_0}^{R} \frac{dr}{V'(r)} = b,$$

whence it follows that

$$\int_{B_{r_0}} |\nabla u|^2 d\mu \le 4\left(\int_{r_0}^{R} \frac{dr}{V'(r)}\right)^{-1}.$$

Letting $R \to \infty$ and $r_0 \to \infty$, we obtain that $u = \text{const}$ provided the following condition holds:

$$\int^{\infty} \frac{dr}{V'(r)} = \infty. \tag{11.55}$$

Note that (11.48) implies (11.55) by Exercise 11.23.

EXAMPLE 11.16. Set $M = \mathbb{R}^n$ and let (M, g, μ) be a weighted model as in Section 8.4.3. Let $V(r)$ be the volume function of M, that is, $V(r) = V(0, r)$. Let us show that in this case the condition (11.55) is not only sufficient but is also necessary for the parabolicity of M. Denoting as in Section 8.4.3 $S(r) = V'(r)$ and assuming that

$$\int^{\infty} \frac{dr}{S(r)} < \infty, \tag{11.56}$$

consider the function $u(R)$ from the proof of Theorem 8.24 defined by (8.48), that is,

$$u(R) = \int_R^{\infty} \frac{dr}{S(r)} \int_0^r S(t) f(t)\, dt,$$

where $f \in C_0^\infty(1, 2)$ is a non-negative non-zero function. It was shown in the proof of Theorem 8.24 that u extends to a smooth function on M and

$\Delta_\mu u = -f$ on M so that u is a positive superharmonic function on M. Since $u \not\equiv$ const, we conclude that M is non-parabolic.

The non-parabolicity test (11.56) implies that $\mathbb{R}^n$ is non-parabolic if ana only if $n > 2$, and $\mathbb{H}^n$ is non-parabolic for any $n \geq 2$.

Exercises.

11.23. Let $f(r)$ be a C^1-function on $(0, +\infty)$ such that $f'(r) > 0$. Prove that

$$\int^\infty \frac{rdr}{f(r)} = \infty \implies \int^\infty \frac{dr}{f'(r)} = \infty.$$

11.24. Prove that any parabolic manifold is stochastically complete.

11.6. Spectrum and the distance function

We present here some estimates of $\lambda_{\min}(M)$ using the geodesic distance.

THEOREM 11.17. *Assume that, on a weighted manifold (M, g, μ), there exists a Lipschitz function ρ with the Lipschitz constant* 1 *such that*

$$\Delta_\mu \rho \geq \alpha,$$

where α is a positive constant and the inequality is understood in the distributional sense. Then

$$\lambda_{\min}(M) \geq \frac{\alpha^2}{4}. \tag{11.57}$$

PROOF. For any function $\varphi \in \mathcal{D}(M)$, we have by hypothesis

$$\left(\Delta_\mu \rho, \varphi^2\right) \geq \alpha \int_M \varphi^2 d\mu. \tag{11.58}$$

By Theorem 11.3, $\nabla \rho \in \vec{L}^\infty(M)$ and $|\nabla \rho| \leq 1$ so that

$$\begin{aligned}\left(\Delta_\mu \rho, \varphi^2\right) &= \left(\operatorname{div}_\mu(\nabla \rho), \varphi^2\right) = -\left(\nabla \rho, \nabla \varphi^2\right) \\ &= -2\int_M \langle \nabla \rho, \nabla \varphi \rangle_{\mathrm{g}} \varphi \, d\mu \leq 2\left(\int_M |\nabla \varphi|^2 d\mu\right)^{1/2}\left(\int_M \varphi^2 d\mu\right)^{1/2}.\end{aligned}$$

Combining with (11.58), we obtain

$$a\left(\int_M \varphi^2 d\mu\right)^{1/2} \leq 2\left(\int_M |\nabla \varphi|^2 d\mu\right)^{1/2},$$

which implies (11.57) by Theorem 10.8. ∎

EXAMPLE 11.18. Let o be the origin of the polar coordinate system in $\mathbb{H}^n$, and set $\rho(x) = d(o, x)$. Function ρ is Lipschitz with the Lipschitz constant 1 (cf. Lemma 11.2). Evaluating by (3.85) its Laplacian away from o and noticing that, in the polar coordinates (r, θ), $\rho(x) = r$, we obtain

$$\Delta_{\mathbb{H}^n} \rho = \frac{\partial^2 \rho}{\partial r^2} + (n-1)\coth r \frac{\partial \rho}{\partial r} = (n-1)\coth r \geq n-1.$$

Therefore, for any open set $\Omega \subset \mathbb{H}^n$ not containing o, we obtain by Theorem 11.17

$$\lambda_{\min}(\Omega) \geq \frac{(n-1)^2}{4}. \tag{11.59}$$

Observe that the origin o may be taken to be any point of $\mathbb{H}^n$ (cf. Exercise 3.46), which implies that (11.59) holds for any open subset $\Omega \subset \mathbb{H}^n$ with non-empty complement. Finally, applying (11.59) to an exhaustion sequence $\{\Omega_k\}$ and using Exercise 10.6, we obtain

$$\lambda_{\min}(\mathbb{H}^n) \geq \frac{(n-1)^2}{4}. \tag{11.60}$$

Alternatively, by Exercise 11.26, (11.60) follows from (11.59) with $\Omega = \mathbb{H}^n \setminus \{o\}$.

THEOREM 11.19. *Assume that, on a weighted manifold $(M, \mathbf{g}, \mu)$, there exists a Lipschitz function ρ with the Lipschitz constant* 1 *such that $\rho(x) \to +\infty$ as $x \to \infty$ and $e^{-\beta\rho} \in L^1(M)$ for some $\beta > 0$. Then*

$$\lambda_{\min}(M) \leq \frac{\beta^2}{4}. \tag{11.61}$$

PROOF. Set $f(x) = e^{-\frac{1}{2}\beta\rho(x)}$ so that $f \in L^2(M)$ and notice that, by Exercise 11.14,

$$\nabla f = \frac{1}{2}\beta f \nabla\rho,$$

whence

$$\int_M |\nabla f|^2 d\mu = \frac{\beta^2}{4}\int_M f^2 |\nabla \rho|^2 d\mu \leq \frac{\beta^2}{4}\int_M f^2 d\mu.$$

In particular, we see that $f \in W^1(M)$ and $\mathcal{R}(f) \leq \beta^2/4$. The hypothesis $\rho(x) \to +\infty$ implies $f(x) \to 0$ as $x \to \infty$, whence $f(x) \to 0$ as $x \to \infty$. By Exercise 5.7, we obtain that $f \in W_0^1(M)$. Hence, (11.61) follows from Theorem 10.8. □

EXAMPLE 11.20. Consider again $\mathbb{H}^n$, and let ρ be the same function as in Example 11.18. Using the area function

$$S(r) = \omega_n \sinh^{n-1} r$$

of $\mathbb{H}^n$ (see Section 3.10), we obtain

$$\|e^{-\beta\rho}\|_1 = \int_{\mathbb{H}^n} e^{-\beta\rho} d\mu = \int_0^\infty e^{-\beta r} S(r)\, dr.$$

Since $S(r) \sim \text{const}\, e^{(n-1)r}$, the above integral converges for any $\beta > n-1$, which implies by Theorem 11.19 that

$$\lambda_{\min}(\mathbb{H}^n) \leq \frac{(n-1)^2}{4}.$$

Comparing to (11.60), we obtain

$$\lambda_{\min}\left(\mathbb{H}^n\right)=\frac{(n-1)^2}{4}.$$

It is possible to show that the spectrum of the Dirichlet Laplace operator in $\mathbb{H}^n$ is the full interval $[\frac{(n-1)^2}{4},+\infty)$.

Exercises.

11.25. Prove that, for any bounded open set $\Omega\subset\mathbb{R}^n$,

$$\lambda_{\min}(\Omega)\geq\frac{1}{n\,(\operatorname{diam}\Omega)^2}. \tag{11.62}$$

Hence or otherwise show that there exists a constant $c_n>0$ such that, for any ball $B_r\subset\mathbb{R}^n$,

$$\lambda_{\min}(B_r)=c_n r^{-2}.$$

11.26. Let (M,g,μ) be a weighted manifold of dimension $n\geq 2$, and o be a point in M.

(a) Prove that, for any open neighborhood U of o and for any $\varepsilon>0$, there exists a cutoff function ψ of $\{o\}$ in U such that

$$\int_U |\nabla\psi|^2\,d\mu<\varepsilon.$$

(b) Prove that

$$\lambda_{\min}(M\setminus\{o\})=\lambda_{\min}(M). \tag{11.63}$$

(c) Show that (11.63) fails if $n=1$.

11.27. Let (M,g,μ) be a complete weighted manifold. Fix a point $x_0\in M$ and set

$$\alpha=\limsup_{r\to\infty}\frac{1}{r}\log\mu\left(B(x_0,r)\right). \tag{11.64}$$

Prove that

$$\lambda_{\min}(M)\leq\frac{\alpha^2}{4}.$$

11.28. Let (M,g,μ) be a weighted model based on $\mathbb{R}^n$ as in Sections 3.10 and 8.4.3, and let $S(r)$ be the area function of this model. Set

$$\alpha'=\inf_{r>0}\frac{S'(r)}{S(r)}\quad\text{and}\quad\alpha=\limsup_{r\to\infty}\frac{S'(r)}{S(r)}. \tag{11.65}$$

Prove that

$$\frac{(\alpha')^2}{4}\leq\lambda_{\min}(M)\leq\frac{\alpha^2}{4}.$$

Notes

The proof of the Hopf-Rinow theorem can be found in the most of standard courses on Riemannian geometry (see for example [**227**], [**299**]). The proof of Rademacher's theorem, that was used in the first proof of Theorem 11.3, can be found in [**119**, p.281].

The essential self-adjointness of the Dirichlet Laplacian on a complete manifold was proved by Gaffney [**126**] (see also [**63**] and [**316**]). The proof presented here is due to R.Strichartz [**330**]. The key ingredient of the proof – Lemma 11.6, was proved by S.-T. Yau [**363**].

A statement that any harmonic function from a function class S on a manifold M is identical constant is called the *S-Liouville theorem*. Lemma 11.6 is a particular case of a more general result of [**363**] that the L^p-Liouville theorem holds on any complete manifold for any $p \in (1, +\infty)$. Although L^∞-Liouville theorem does hold in $\mathbb{R}^n$ by the classical Liouville theorem (cf. Exercise 13.23), on an arbitrary complete manifold the L^∞- and L^1-Liouville theorems are not necessarily true (see **[72]**, **[142]**, **[155]**, **[240]**, **[246]**).

The uniqueness class (11.28) for the Cauchy problem in $\mathbb{R}^n$ was obtained by Tikhonov **[347]** and (11.29) – by Täcklind **[342]**. Similar integrated uniqueness classes for parabolic equations in unbounded domains in $\mathbb{R}^d$ were introduced by Oleinik and Radkevich **[298]** and by Gushchin **[192]**, using different methods.

The fact that the condition $V(x_0, r) \leq \exp(Cr^2)$ on complete manifolds and other similar settings implies the stochastic completeness was proved by various methods in **[97]**, **[209]**, **[222]**, **[338]**, **[343]**. Historically, the first result in this direction is due to Gaffney **[127]** who obtained the stochastic completeness under a stronger assumption $\log V(x, r) = o(r)$. Theorems 11.8 and 11.9 in the present form were proved in **[137]** (see also **[143]**, **[155]**).

Let M be a geodesically complete manifold with bounded below Ricci curvature, and let μ be its Riemannian measure. It follows from the Bishop-Gromov volume comparison theorem that

$$V(x, r) \leq \exp(Cr) \tag{11.66}$$

(see for example **[48]**) so that M is stochastically complete. The stochastic completeness for Riemannian manifolds with bounded below Ricci curvature was first proved by S.-T.Yau **[364]** (see also **[155]**, **[209]**, **[212]**, **[286]**, **[352]** for extensions of this result).

It was proved earlier by Azencott **[16]** that a *Cartan-Hadamard* manifold with bounded below sectional curvature is stochastically complete. Azencott also gave the first example of a geodesically complete manifold that is stochastically incomplete. Note that Theorem 8.24 provides plenty of examples of such manifolds (cf. Example 11.11). It was shown by T.Lyons **[266]** that the stochastic completeness is in general *not* stable under quasi-isometry. It is also worth mentioning that on manifolds of bounded geometry not only a regular fundamental solution is unique but also any positive fundamental solution is unique that hence coincides with the heat kernel (see **[214]**, **[231]**, **[286]**).

The condition (11.49) for the parabolicity of a complete manifold is due to Cheng and Yau **[62]**. Theorem 11.14 was proved in **[134]**, **[136]**, **[221]**, **[352]** (see also **[110]**, **[155]**, **[211]**, **[206]**, **[274]** for related results). The stability of the parabolicity under quasi-isometry was proved in **[136]** using the capacity criterion (see also **[155]**).

Theorem 11.17 was proved in **[362]**, Theorem 11.19 is due to R.Brooks **[47]**. The fact that the spectrum of $\mathbb{H}^2$ fills the interval $[1/4, +\infty)$ was proved in **[272]**.

CHAPTER 12

Gaussian estimates in the integrated form

As one can see from explicit examples of heat kernels (9.13), (9.32), (9.36), (9.40), the dependence of the heat kernel $p_t(x,y)$ on the points x,y is frequently given by the term $\exp\left(-c\frac{d^2(x,y)}{t}\right)$ that is called the *Gaussian factor*. The Gaussian pointwise upper bounds of the heat kernel, that is, the estimates containing the Gaussian factor, will be obtained in Chapters 15 and 16 after introduction of the necessary techniques. These bounds require additional hypotheses on the manifolds in question.

On the contrary, it is relatively straightforward to obtain the integrated upper bounds of the heat kernel, which is the main topic of this Chapter. From the previous Chapters, we use general the properties of solutions of the heat equation, including those of the heat semigroup P_t, as well as the properties of Lipschitz functions from Section 11.2.

The results of this Chapters are used in the subsequent chapters as follows:

- Theorem 12.1 (the integrated maximum principle) – in Chapters 15 and 16.
- Theorem 12.3 (the Davies-Gaffney inequality) and depending on it Lemma 12.7 – in Chapter 13 from Section 13.3 onwards. Theorem 12.3 is also used in the proof of Theorem 16.2 in Chapter 16.

The results of Sections 12.3 and 12.5 do not have applications within this book.

12.1. The integrated maximum principle

Recall that, by Theorem 11.3, any function $f \in Lip_{loc}(M)$ has the distributional gradient $\nabla f \in \vec{L}^{\infty}_{loc}(M)$.

THEOREM 12.1. (The integrated maximum principle) *Let $\xi(t,x)$ be a continuous function on $I \times M$, where $I \subset [0,+\infty)$ is an interval. Assume that, for any $t \in I$, $\xi(t,x)$ is locally Lipschitz in $x \in M$, the partial derivative $\frac{\partial \xi}{\partial t}$ exists and is continuous in $I \times M$, and the following inequality holds on $I \times M$:*

$$\frac{\partial \xi}{\partial t} + \frac{1}{2}|\nabla \xi|^2 \leq 0. \tag{12.1}$$

Then, for any function $f \in L^2(M)$, the function

$$J(t) := \int_M (P_t f)^2(x)\, e^{\xi(t,x)} d\mu(x) \tag{12.2}$$

is non-increasing in $t \in I$. Furthermore, for all $t, t_0 \in I$, if $t > t_0$ then

$$J(t) \le J(t_0)\, e^{-2\lambda_{\min}(M)(t-t_0)}. \tag{12.3}$$

REMARK 12.2. Let $d(x)$ be a Lipschitz function on M with the Lipschitz constant 1. By Theorem 11.3, we have $|\nabla d| \le 1$. It follows that the following functions satisfy (12.1):

$$\xi(t,x) = \frac{d^2(x)}{2t}$$

and

$$\xi(t,x) = ad(x) - \frac{a^2}{2}t,$$

where a is a real constant. In applications $d(x)$ is normally chosen to be the distance from x to some set (cf. Lemma 11.2).

PROOF. Let us first reduce the problem to the case of non-negative f. Indeed, if f is signed then set $g = |P_{t_0} f|$ and notice that

$$|P_t f| = |P_{t-t_0} P_{t_0} f| \le P_{t-t_0} g.$$

Assuming that Theorem 12.1 has been already proved for function g, we obtain

$$\begin{aligned}
\int_M (P_t f)^2 e^{\xi(t,\cdot)} d\mu &\le \int_M (P_{t-t_0} g)^2 e^{\xi(t,\cdot)} d\mu \\
&\le e^{-2\lambda_{\min}(t-t_0)} \int_M g^2 e^{\xi(t_0,\cdot)} d\mu \\
&= e^{-2\lambda_{\min}(t-t_0)} \int_M (P_{t_0} f)^2 e^{\xi(t_0,\cdot)} d\mu.
\end{aligned}$$

Hence, we can assume in the sequel that $f \ge 0$. In the view of Theorem 5.23, it suffices to prove that, for any relatively compact open set $\Omega \subset M$, the function

$$J_\Omega(t) := \int_\Omega \left(P_t^\Omega f\right)^2(x)\, e^{\xi(t,x)} d\mu(x)$$

is non-increasing in $t \in I$. Since $u(t,\cdot) := P_t^\Omega f \in L^2(\Omega)$ and $\xi(t,\cdot)$ is bounded in Ω, the function $J_\Omega(t)$ is finite (unlike $J(t)$ that a priori may be equal to ∞). Note also that $J_\Omega(t)$ is continuous in $t \in I$. Indeed, by Theorem 4.9 the path $t \mapsto u(t,\cdot)$ is continuous in $t \in [0,+\infty)$ in $L^2(\Omega)$ and the path $t \mapsto e^{\frac{1}{2}\xi(t,\cdot)}$ is obviously continuous in $t \in I$ in the sup-norm in $C_b(\Omega)$, which implies that the path $t \mapsto u(t,\cdot)\, e^{\frac{1}{2}\xi(t,\cdot)}$ is continuous in $t \in I$ in $L^2(\Omega)$ (cf. Exercise 4.46).

To prove that $J_\Omega(t)$ is non-increasing in I it suffices to show that the derivative $\frac{dJ_\Omega}{dt}$ exists and is non-positive for all $t \in I_0 := I \setminus \{0\}$. Fix some $t \in I_0$. Since the functions $\xi(t,\cdot)$ and $\frac{\partial \xi}{\partial t}(t,\cdot)$ are continuous and bounded in $\overline{\Omega}$, they both belong to $C_b(\Omega)$. Therefore, the partial derivative $\frac{\partial \xi}{\partial t}$ is at the same time the strong derivative $\frac{d\xi}{dt}$ in $C_b(\Omega)$ (cf. Exercise 4.47). In the same way, the function $e^{\xi(t,\cdot)}$ is strongly differentiable in $C_b(\Omega)$ and

$$\frac{de^{\xi}}{dt} = \frac{\partial e^{\xi}}{\partial t} = e^{\xi}\frac{\partial \xi}{\partial t}. \tag{12.4}$$

By Theorem 4.9, the function $u(t,\cdot)$ is strongly differentiable in $L^2(\Omega)$ and its strong derivative $\frac{du}{dt}$ in $L^2(\Omega)$ is given by

$$\frac{du}{dt} = \Delta_\mu u. \tag{12.5}$$

Using the product rules for strong derivatives (see Exercise 4.46), we conclude that ue^{ξ} is strongly differentiable in $L^2(\Omega)$ and

$$\frac{d}{dt}\left(ue^{\xi}\right) = \frac{du}{dt}e^{\xi} + u\frac{de^{\xi}}{dt}. \tag{12.6}$$

It follows that the inner product $\left(u, ue^{\xi}\right) = J_\Omega(t)$ is differentiable as a real valued function of t and, by the product rule and by (12.4), (12.5), (12.6),

$$\begin{aligned}\frac{dJ_\Omega}{dt} &= \left(\frac{du}{dt}, ue^{\xi}\right) + \left(u, \frac{d\left(ue^{\xi}\right)}{dt}\right)\\ &= 2\left(\frac{du}{dt}, ue^{\xi}\right) + \left(u^2, \frac{de^{\xi}}{dt}\right)\\ &= 2\left(\Delta_\mu u, ue^{\xi}\right) + \left(u^2, \frac{\partial \xi}{\partial t}e^{\xi}\right).\end{aligned} \tag{12.7}$$

By the chain rule for Lipschitz functions (see Exercise 11.14), we have $e^{\xi(t,\cdot)} \in Lip_{loc}(M)$. Since the function $e^{\xi(t,\cdot)}$ is bounded and Lipschitz in Ω and $u(t,\cdot) \in W_0^1(\Omega)$, we obtain by Exercise 11.13 that $ue^{\xi} \in W_0^1(\Omega)$. By the Green formula of Lemma 4.4, we obtain

$$2\left(\Delta_\mu u, ue^{\xi}\right) = -2\int_\Omega \langle \nabla u, \nabla\left(ue^{\xi}\right)\rangle d\mu.$$

Since both functions u and $e^{\xi(t,\cdot)}$ are locally Lipschitz, the product rule and the chain rule apply for expanding $\nabla\left(ue^{\xi}\right)$ (cf. Exercises 11.13, 11.14). Substituting the result into (12.7) and using (12.1), we obtain

$$\begin{aligned}\frac{dJ_\Omega}{dt} &\le -2\int_\Omega \left(|\nabla u|^2 e^{\xi} + ue^{\xi}\langle \nabla u, \nabla \xi\rangle + \frac{1}{4}u^2 |\nabla \xi|^2 e^{\xi}\right) d\mu\\ &= -2\int_\Omega \left(\nabla u + \frac{1}{2}u\nabla \xi\right)^2 e^{\xi} d\mu,\end{aligned} \tag{12.8}$$

whence $\frac{dJ_\Omega}{dt} \le 0$. To prove (12.3), observe that

$$\left(\nabla u + \frac{1}{2}u\nabla \xi\right)e^{\xi/2} = \nabla(ue^{\xi/2}).$$

Since $ue^{\xi/2} \in W_0^1(\Omega)$, we can apply the variational principle (Theorem 10.8) which yields

$$\begin{aligned}\int_\Omega \left(\nabla u + \frac{1}{2}u\nabla\xi\right)^2 e^\xi d\mu &= \int_\Omega |\nabla(ue^{\xi/2})|^2 d\mu \\ &\geq \lambda_{\min}(\Omega)\int_\Omega |ue^{\xi/2}|^2 d\mu = \lambda_{\min}(\Omega)\, J_\Omega(t). \quad (12.9)\end{aligned}$$

Hence, (12.8) yields

$$\frac{dJ_\Omega}{dt} \leq -2\lambda_{\min}(\Omega)\, J_\Omega(t),$$

whence (12.3) follows. □

Exercises.

12.1. Let Φ be a C^2-function in $I := [0, +\infty)$ such that $\Phi, \Phi', \Phi'' \geq 0$ and

$$\Phi''\Phi \geq \delta(\Phi')^2, \tag{12.10}$$

for some $\delta > 0$. Let $\xi(t,x)$ be a continuous function on $I \times M$ and assume that $\xi(t,x)$ is locally Lipschitz in $x \in M$ for any $t \in I$, $\frac{\partial\xi}{\partial t}$ exists and is continuous on $I \times M$, and the following inequality holds on $I \times M$:

$$\frac{\partial\xi}{\partial t} + \frac{1}{4\delta}|\nabla\xi|^2 \leq 0.$$

Prove that the quantity

$$J(t) := \int_M \Phi(P_t f)\, e^{\xi(t,\cdot)} d\mu$$

is non-increasing in $t \in I$ for any non-negative $f \in L^2(M)$.

12.2. The Davies-Gaffney inequality

For any set A on a weighted manifold M and any $r > 0$, denote by A_r the r-neighborhood of A, that is,

$$A_r = \{x \in M : d(x, A) < r\}.$$

Write also $A_r^c = (A_r)^c = M \setminus A_r$.

THEOREM 12.3. *Let A be a measurable subset of a weighted manifold M. Then, for any function $f \in L^2(M)$ and for all positive r, t,*

$$\int_{A_r^c} (P_t f)^2 d\mu \leq \int_{A^c} f^2 d\mu + \exp\left(-\frac{r^2}{2t} - 2\lambda t\right)\int_A f^2 d\mu, \tag{12.11}$$

where $\lambda = \lambda_{\min}(M)$. In particular, if $f \in L^2(A)$ then

$$\int_{A_r^c} (P_t f)^2 d\mu \leq \|f\|_2^2 \exp\left(-\frac{r^2}{2t} - 2\lambda t\right) \tag{12.12}$$

(see Fig. 12.1).

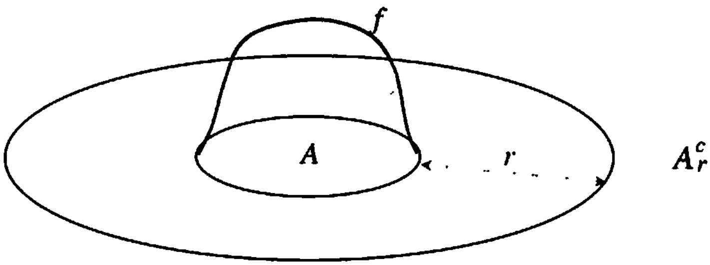

FIGURE 12.1. Sets A and A_r^c

PROOF. Fix some $s > t$ and consider the function

$$\xi(\tau, x) = \frac{d^2(x, A_r^c)}{2(\tau - s)},$$

defined for $x \in M$ and $\tau \in [0, s)$. Set also

$$J(\tau) := \int_M (P_\tau f)^2 e^{\xi(\tau,\cdot)} d\mu.$$

Since the function ξ satisfies the condition

$$\frac{\partial \xi}{\partial \tau} + \frac{1}{2} |\nabla \xi|^2 \leq 0,$$

we obtain by Theorem 12.1 that

$$J(t) \leq J(0) \exp(-2\lambda t). \tag{12.13}$$

Since $\xi(\tau, x) = 0$ for $x \in A_r^c$, we have

$$J(t) \geq \int_{A_r^c} (P_t f)^2 d\mu. \tag{12.14}$$

On the other hand, using the fact that $\xi(0, x) \leq 0$ for all x and

$$\xi(0, x) \leq -\frac{r^2}{2s} \text{ for all } x \in A,$$

we obtain

$$J(0) \leq \int_{A^c} f^2 d\mu + \exp\left(-\frac{r^2}{2s}\right) \int_A f^2 d\mu. \tag{12.15}$$

Combining together (12.13), (12.14), (12.15) and letting $s \to t+$, we obtain (12.11).

The inequality (12.12) trivially follows from (12.11) and the observation that $\int_{A^c} f^2 d\mu = 0$. $\square$

COROLLARY 12.4. (The Davies-Gaffney inequality). *If A and B are two disjoint measurable subsets of M and $f \in L^2(A)$, $g \in L^2(B)$, then, for all $t > 0$,*

$$|(P_t f, g)| \le \|f\|_2 \|g\|_2 \exp\left(-\frac{d^2(A,B)}{4t} - \lambda t\right) \tag{12.16}$$

(see Fig. 12.2*).*

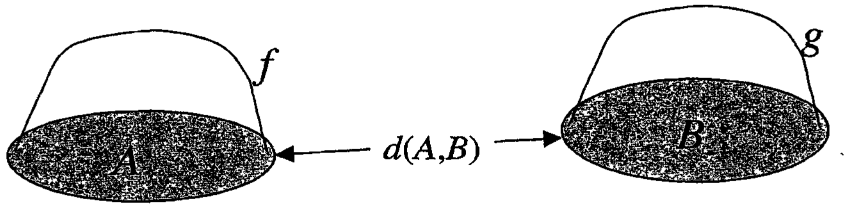

FIGURE 12.2. Sets A and B

PROOF. Set $r = d(A,B)$. Then $B \subset A_r^c$ and by (12.12)

$$\int_B (P_t f)^2 \, d\mu \le \|f\|_2^2 \exp\left(-\frac{r^2}{2t} - 2\lambda t\right).$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |(P_t f, g)| &\le \left(\int_B (P_t f)^2 \, d\mu\right)^{1/2} \|g\|_2 \\ &\le \|f\|_2 \|g\|_2 \exp\left(-\frac{r^2}{4t} - \lambda t\right), \end{aligned}$$

which was to be proved. □

Note that (12.16) is in fact equivalent to (12.12) since the latter follows from (12.16) by dividing by $\|g\|_2$ and taking sup in all $g \in L^2(B)$ with $B = A_c^r$.

Assuming that the sets A and B in (12.16) have finite measures and setting $f = 1_A$ and $g = 1_B$, we obtain from (12.16)

$$(P_t 1_A, 1_B) \le \sqrt{\mu(A)\mu(B)} \exp\left(-\frac{d^2(A,B)}{4t} - \lambda t\right),$$

or, in terms of the heat kernel,

$$\int_A \int_B p_t(x,y) d\mu(x) d\mu(y) \le \sqrt{\mu(A)\mu(B)} \exp\left(-\frac{d^2(A,B)}{4t} - \lambda t\right). \tag{12.17}$$

This can be considered as an integrated form of the Gaussian upper bound of the heat kernel. Note that, unlike the pointwise bounds, the estimate (12.17) holds on an *arbitrary* manifold.

Exercises.

12.2. Give an alternative proof of (12.12) applying Theorem 12.1 with the function

$$\xi(t,x) := \alpha d(x,A) - \frac{\alpha^2}{2}t,$$

where α is an arbitrary real parameter.

12.3. The purpose of this question is to prove the following enhanced version of (12.16): if f and g are two functions from $L^2(M)$ such that

$$d(\operatorname{supp} f, \operatorname{supp} g) \geq r,$$

where $r \geq 0$, then, for all $t > 0$,

$$|(P_t f, g)| \leq \|f\|_2 \|g\|_2 \int_r^\infty \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{s^2}{4t}\right) ds. \tag{12.18}$$

(a) (*Finite propagation speed for the wave equation*) Let $u(t,x)$ be a C^∞ function on $\mathbb{R} \times M$ that solves in $\mathbb{R} \times M$ the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta_\mu u.$$

Set $K_t = \operatorname{supp} u(t,\cdot)$. Prove that K_t is contained in the closed $|t|$-neighborhood of K_0.

(b) Prove (12.18) using part (a) and the transmutation formula of Exercise 4.52.

REMARK. See Exercise 13.25 concerning the additional factor $e^{-\lambda t}$ in (12.18).

12.3. Upper bounds of higher eigenvalues

We give here an application of Corollary 12.4 to eigenvalue estimates on a compact weighted manifold M. Recall that by Theorem 10.13 the spectrum of the Dirichlet Laplace operator $\mathcal{L}$ on M is discrete. As before, denote by $\lambda_k(M)$ be the k-th smallest eigenvalue of $\mathcal{L}$ counted with the multiplicity. Recall that $\lambda_k(M) \geq 0$ and $\lambda_1(M) = 0$ (cf. Exercise 10.10).

THEOREM 12.5. *Let M be a connected compact weighted manifold. Let $A_1, A_2, ..., A_k$ be $k \geq 2$ disjoint measurable sets on M, and set*

$$\delta := \min_{i \neq j} d(A_i, A_j).$$

Then

$$\lambda_k(M) \leq \frac{4}{\delta^2} \max_{i\neq j}\left(\log \frac{2\mu(M)}{\sqrt{\mu(A_i)\mu(A_j)}}\right)^2. \tag{12.19}$$

In particular, if we have two sets $A_1 = A$ and $A_2 = B$ then (12.19) becomes

$$\lambda_2(M) \leq \frac{4}{\delta^2}\left(\log \frac{2\mu(M)}{\sqrt{\mu(A)\mu(B)}}\right)^2, \tag{12.20}$$

where $\delta := d(A,B)$.

PROOF. We first prove (12.20). Let $\{\varphi_k\}_{k=1}^{\infty}$ be an orthonormal basis in $L^2(M,\mu)$ that consists of the eigenfunctions of $\mathcal{L}$, so that φ_k has the eigenvalue $\lambda_k = \lambda_k(M)$. By the eigenfunction expansion (10.33), we have for any $t > 0$

$$\begin{aligned}\iint_{AB} p_t(x,y)d\mu(x)d\mu(y) &= \sum_{i=1}^{\infty} e^{-t\lambda_i}\int_A \varphi_i(x)d\mu(x)\int_B \varphi_i(y)d\mu(y) \\ &= \sum_{i=1}^{\infty} e^{-t\lambda_i}a_ib_i, \qquad (12.21)\end{aligned}$$

where

$$a_i = (1_A,\varphi_i) \quad \text{and} \quad b_i = (1_B,\varphi_i).$$

By the Parseval identity

$$\sum_{i=1}^{\infty} a_i^2 = \|1_A\|_2^2 = \mu(A) \quad \text{and} \quad \sum_{i=1}^{\infty} b_i^2 = \|1_B\|_2^2 = \mu(B).$$

Since $\lambda_1 = 0$, the first eigenfunction φ_1 is identical constant. By the normalization condition $\|\varphi_1\|_2 = 1$ we obtain $\varphi_1 \equiv 1/\sqrt{\mu(M)}$, which implies

$$a_1 = (1_A,\varphi_1) = \frac{\mu(A)}{\sqrt{\mu(M)}} \quad \text{and} \quad b_1 = (1_B,\varphi_1) = \frac{\mu(B)}{\sqrt{\mu(M)}}.$$

Therefore, (12.21) yields

$$\begin{aligned}\iint_{AB} p_t(x,y)d\mu(x)d\mu(y) &= a_1b_1 + \sum_{i=2}^{\infty} e^{-t\lambda_i}a_ib_i \\ &\ge a_1b_1 - e^{-t\lambda_2}\left(\sum_{i=2}^{\infty} a_i^2\right)^{1/2}\left(\sum_{i=2}^{\infty} b_i^2\right)^{1/2} \\ &\ge \frac{\mu(A)\mu(B)}{\mu(M)} - e^{-t\lambda_2}\sqrt{\mu(A)\mu(B)}.\end{aligned}$$

Comparing with (12.17), we obtain

$$\sqrt{\mu(A)\mu(B)}e^{-\frac{\delta^2}{4t}} \ge \frac{\mu(A)\mu(B)}{\mu(M)} - e^{-t\lambda_2}\sqrt{\mu(A)\mu(B)},$$

whence

$$e^{-t\lambda_2} \ge \frac{\sqrt{\mu(A)\mu(B)}}{\mu(M)} - e^{-\frac{\delta^2}{4t}}$$

Choosing t from the identity

$$e^{-\frac{\delta^2}{4t}} = \frac{1}{2}\frac{\sqrt{\mu(A)\mu(B)}}{\mu(M)},$$

we conclude

$$\lambda_2 \le \frac{1}{t}\log\frac{2\mu(M)}{\sqrt{\mu(A)\mu(B)}} = \frac{4}{\delta^2}\left(\log\frac{2\mu(M)}{\sqrt{\mu(A)\mu(B)}}\right)^2,$$

which was to be proved.

Let us now turn to the general case $k > 2$. Consider the following integrals

$$J_{lm} := \int_{A_l} \int_{A_m} p(t, x, y) d\mu(x) d\mu(y)$$

and set

$$a_i^{(l)} := (1_{A_l}, \varphi_i).$$

Exactly as above, we have

$$\begin{aligned} J_{lm} &= \sum_{i=1}^{\infty} e^{-t\lambda_i} a_i^{(l)} a_i^{(m)} \\ &= \frac{\mu(A_l)\mu(A_m)}{\mu(M)} + \sum_{i=k}^{\infty} e^{-\lambda_i t} a_i^{(l)} a_i^{(m)} + \sum_{i=2}^{k-1} e^{-\lambda_i t} a_i^{(l)} a_i^{(m)} \\ &\geq \frac{\mu(A_l)\mu(A_m)}{\mu(M)} - e^{-\lambda_k t} \sqrt{\mu(A_l)\mu(A_m)} \\ &\quad + \sum_{i=2}^{k-1} e^{-\lambda_i t} a_i^{(l)} a_i^{(m)}. \end{aligned} \tag{12.22}$$

On the other hand, by (12.17)

$$J_{lm} \leq \sqrt{\mu(A_l)\mu(A_m)} e^{-\frac{\delta^2}{4t}}. \tag{12.23}$$

Therefore, we can further argue as in the case $k = 2$ provided the term in (12.22) can be discarded, which the case when

$$\sum_{i=2}^{k-1} e^{-\lambda_i t} a_i^{(l)} a_i^{(m)} \geq 0. \tag{12.24}$$

Let us show that (12.24) can be achieved by *choosing* l, m. To that end, let us interpret the sequence

$$a^{(j)} := (a_2^{(j)}, a_3^{(j)}, ..., a_{k-1}^{(j)})$$

as a $(k-2)$-dimensional vector in $\mathbb{R}^{k-2}$. Here j ranges from 1 to k so that we have k vectors $a^{(j)}$ in $\mathbb{R}^{k-2}$. Let us introduce the inner product of two vectors $u = (u_2, ..., u_{k-1})$ and $v = (v_2, ..., v_{k-1})$ in $\mathbb{R}^{k-2}$ by

$$\langle u, v \rangle_t := \sum_{i=2}^{k-1} e^{-\lambda_i t} u_i v_i \tag{12.25}$$

and apply the following elementary fact:

LEMMA 12.6. *From any $n+2$ vectors in a n-dimensional Euclidean space, it is possible to choose two vectors with non-negative inner product.*

Note that $n+2$ is the smallest number for which the statement of Lemma 12.6 is true. Indeed, choose an orthonormal basis $e_1, e_2, ..., e_n$ in the given Euclidean space and consider the vector

$$v := -e_1 - e_2 - ... - e_n.$$

Then any two of the following $n+1$ vectors

$$e_1 + \varepsilon v,\ e_2 + \varepsilon v,\ ....,\ e_n + \varepsilon v,\ v$$

have a negative inner product, provided $\varepsilon > 0$ is small enough.

Lemma 12.6 is easily proved by induction in n. The inductive basis for $n = 1$ is trivial. The inductive step is shown on Fig. 12.3. Indeed, assume that the $n+2$ vectors v_1, v_2, ..., v_{n+2} in $\mathbb{R}^n$ have pairwise obtuse angles. Denote by E the orthogonal complement of v_{n+2} in $\mathbb{R}^n$ and by v_i' the orthogonal projection of v_i onto E.

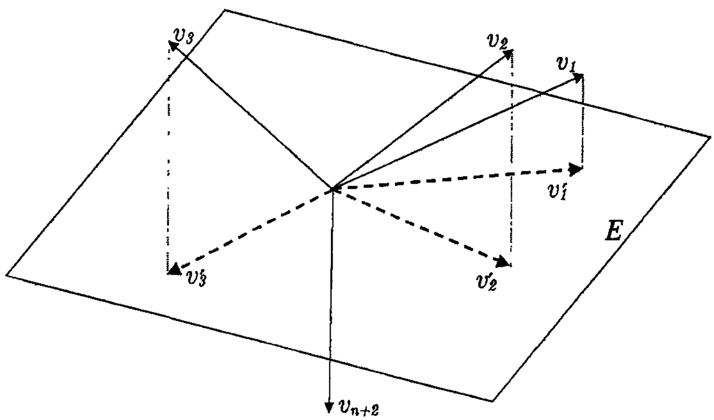

FIGURE 12.3. The vectors v_i' are the orthognal projections of v_i onto E.

For any $i \leq n+1$, the vector v_i can be represented as

$$v_i = v_i' - \varepsilon_i v_{n+2}\,,$$

where

$$\varepsilon_i = -\langle v_i, v_{n+2}\rangle > 0.$$

Therefore, we have

$$\langle v_i, v_j\rangle = \langle v_i', v_j'\rangle + \varepsilon_i \varepsilon_j \left|v_{n+2}\right|^2.$$

By the inductive hypothesis, we have $\langle v_i', v_j'\rangle \geq 0$ for some i, j, which implies $\langle v_i, v_j\rangle \geq 0$, contradicting the assumption.

Now we can finish the proof of Theorem 12.5. Fix some $t > 0$. By Lemma 12.6, we can find l, m so that $\langle a^{(l)}, a^{(m)}\rangle_t \geq 0$; that is (12.24) holds.

Then (12.22) and (12.23) yield

$$e^{-t\lambda_k} \geq \frac{\sqrt{\mu(A_l)\mu(A_m)}}{\mu(M)} - e^{-\frac{\delta^2}{4t}},$$

and we are left to choose t. However, t should not depend on l, m because we use t to define the inner product (12.25) *before* choosing l, m. So, we first write

$$e^{-t\lambda_k} \geq \min_{i,j} \frac{\sqrt{\mu(A_i)\mu(A_j)}}{\mu(M)} - e^{-\frac{\delta^2}{4t}}$$

and then define t by

$$e^{-\frac{\delta^2}{4t}} = \frac{1}{2} \min_{i,j} \frac{\sqrt{\mu(A_i)\mu(A_j)}}{\mu(M)},$$

whence (12.19) follows. □

12.4. Semigroup solutions with a harmonic initial function

The next statement can be viewed as an example of application of Theorem 12.3. On the other hand, it will be used in Section 13.3 in the proof of Theorem 13.9.

LEMMA 12.7. *Let V be an exterior of a compact subset of M and let f be a function from $W_0^1(M)$ such that $\Delta_\mu f = 0$ in V. Then, for any open set $U \subset V$ such that $\overline{U} \subset V$, the following holds:*

$$\lim_{t \to 0} \left\| \frac{P_t f - f}{t} \right\|_{L^2(U)} = 0. \tag{12.26}$$

REMARK 12.8. Since the function $P_t f$ satisfies the heat equation in $\mathbb{R}_+ \times V$ with the initial function $f \in C^\infty(V)$, by Exercise 9.8 the function

$$u(t, \cdot) = \begin{cases} P_t f, & t > 0, \\ f, & t \leq 0 \end{cases}$$

is C^∞ smooth in $\mathbb{R} \times V$. Since f is harmonic in V, it follows that u satisfies the heat equation in $\mathbb{R} \times V$. Hence, we have in V as $t \to 0+$

$$\frac{P_t f - f}{t} = \frac{u(t, \cdot) - u(0, \cdot)}{t} \to \left. \frac{\partial u}{\partial t} \right|_{t=0} = \Delta_\mu u(0, \cdot) = \Delta_\mu f = 0,$$

where the convergence is local uniform in V. If U is relatively compact then it follows that also

$$\left\| \frac{P_t f - f}{t} \right\|_{L^2(U)} \to 0 \text{ as } t \to 0.$$

However, this argument does not work in the general case when $\overline{U}$ is noncompact, and the latter case requires a different argument as below.

PROOF. Let us first prove (12.26) in the case when f is a function from $L^2(M)$ such that $f = 0$ in V. Noticing that $r := d\left(\overline{U}, V^c\right) > 0$ and applying the inequality (12.12) of Theorem 12.3 with $A = V^c$, we obtain

$$\|P_t f\|_{L^2(U)} \leq \|f\|_{L^2}^2 \exp\left(-\frac{r^2}{2t}\right) = o(t) \text{ as } t \to 0.$$

Together with $\|f\|_{L^2(U)} = 0$, this yields (12.26).

Let us now prove (12.26) in the case when $f \in W_0^2(M)$ and $\Delta_\mu f = 0$ in U. By Exercise 4.41, we have

$$\frac{P_t f - f}{t} \overset{L^2(M)}{\longrightarrow} \Delta_\mu f \text{ as } t \to 0,$$

whence it follows that

$$\frac{P_t f - f}{t} \overset{L^2(U)}{\longrightarrow} 0 \text{ as } t \to 0,$$

which is equivalent to (12.26).

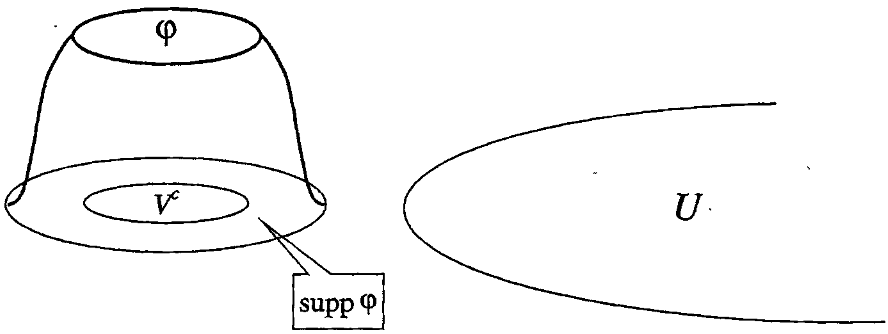

FIGURE 12.4. Illustration to the proof of Lemma 12.7

Finally, consider the general case, when $f \in W_0^1(M)$ and $\Delta_\mu f = 0$ in V. Let φ be a cutoff function of the compact set V^c in the open set $\overline{U}^c$ (see Fig. 12.4). Since $\varphi f = 0$ in $V' = (\operatorname{supp}\varphi)^c$ and $\overline{U} \subset V'$, we conclude by the first of the above cases that

$$\frac{P_t(\varphi f) - \varphi f}{t} \overset{L^2(U)}{\longrightarrow} 0 \text{ as } t \to 0. \tag{12.27}$$

Next, we claim that the function

$$g = (1-\varphi) f$$

belongs to $W_0^2(M)$. Indeed, by Exercise 4.21 (or 11.13), we have $\varphi f \in W_0^1$ whence $g \in W_0^1$. In a neighborhood of V^c, where $\varphi \equiv 1$, we have $g = 0$ and, hence, $\Delta_\mu g = 0$. On the other hand, using the hypothesis $\Delta_\mu f = 0$ in V, we obtain that the following identity holds in V:

$$\Delta_\mu g = (1-\varphi)\Delta_\mu f - 2\nabla\varphi\nabla f - (\Delta_\mu\varphi) f = -2\nabla\varphi\nabla f - (\Delta_\mu\varphi) f \tag{12.28}$$

(cf. Exercise 3.8) Since $\nabla\varphi$ and $\Delta_\mu\varphi$ are bounded, while f and ∇f belong to L^2, we obtain that $\Delta_\mu g \in L^2(V)$. It follows that $\Delta_\mu g \in L^2(M)$ and, hence, $g \in W_0^2(M)$. Since in U we have $\varphi \equiv 0$, which implies by (12.28) that $\Delta_\mu g = 0$, we obtain by the second of the above cases, that

$$\frac{P_t g - g}{t} \overset{L^2(U)}{\longrightarrow} 0 \text{ as } t \to 0. \tag{12.29}$$

Since $f = \varphi f + g$, adding up (12.27) and (12.29), we obtain (12.26). $\square$

12.5. Takeda's inequality

Similarly to Theorem 12.3, the next theorem provides a certain L^2-estimate for a solution to the heat equation. However, the setting and the estimate are essentially different.

THEOREM 12.9. *Let A, B be two relatively compact open subsets of a weighted manifold M such that $A \Subset B$ and let $R = d(A, B^c)$. Let $u(t,x)$ be a non-negative bounded C^2-function in $(0,T) \times B$ such that*

- $\frac{\partial u}{\partial t} - \Delta_\mu u \leq 0$ *in* $(0,T) \times B$,
- *and* $u(t,\cdot) \overset{L^2(B)}{\longrightarrow} 0$ *as* $t \to 0$ *(see Fig. 12.5).*

Then, for any $t \in (0,T)$,

$$\int_A u^2(t,\cdot)\, d\mu \leq \mu(B \setminus A)\, \|u\|^2_{L^\infty} \max\left(\frac{R^2}{2t}, \frac{2t}{R^2}\right) \exp\left(-\frac{R^2}{2t} + 1\right). \tag{12.30}$$

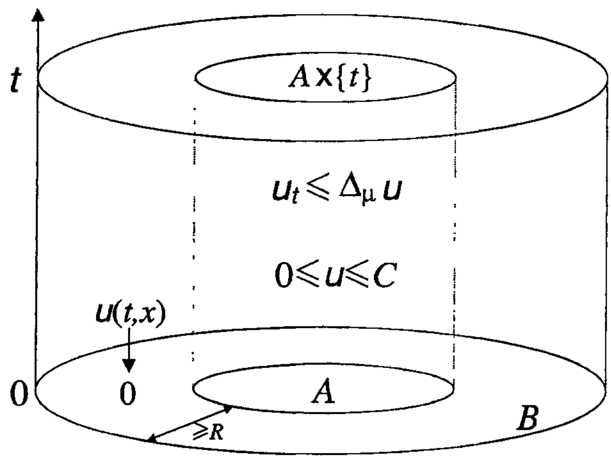

FIGURE 12.5. The function $u(t,x)$ in $(0,T) \times B$.

REMARK. The hypotheses of Theorem 12.9 are in particular satisfied if $u(t,\cdot) = P_t f$ where f is a non-negative function from $L^\infty(B^c)$ (see Exercise

12.5). Let us mention for comparison that Theorem 12.3 yields in this case the following estimate

$$\int_A u^2(t,\cdot)\,d\mu \le \|f\|_2^2 \exp\left(-\frac{R^2}{2t}\right),$$

assuming that $f \in L^2(B^c)$. The advantage of (12.30) is that it can be applied to functions like $f = 1_{B^c}$ that are bounded but are not necessarily in L^2. There are also applications of Theorem 12.9 for solutions u that do not have the form $P_t f$ (see Exercise 12.4).

PROOF. Without loss of generality, we can assume throughout that $0 \le u \le 1$. Let $\xi(t,x)$ be a continuous function on $[0,T] \times \overline{B}$ such that $\xi(t,x)$ is Lipschitz in x, continuously differentiable in t, and the following inequality holds almost everywhere on $[0,T] \times B$:

$$\xi' + \frac{\alpha}{2}|\nabla \xi|^2 \le 0 \tag{12.31}$$

for some $\alpha > 1$, where $\xi' \equiv \frac{\partial \xi}{\partial t}$. We claim that the following inequality is true for any $t \in (0,T)$ and any $\varphi \in W_0^1(B)$:

$$\int_B u(t,\cdot)^2 \varphi^2 e^{\xi(t,\cdot)}\,d\mu \le \frac{2\alpha}{\alpha-1}\int_0^t\int_B |\nabla\varphi|^2 e^{\xi(s,\cdot)}\,d\mu ds \tag{12.32}$$

(cf. the inequality (11.39) from the proof of Theorem 11.9). Since the functions u, ξ and φ are uniformly bounded in the domain of integration, the both sides of (12.32) are continuous as functionals of φ in W^1-norm. Hence, it suffices to prove (12.32) for $\varphi \in C_0^\infty(B)$, which will be assumed in the sequel.

Let us differentiate in t the left hand side of (12.32). Note that the time derivative $\frac{d}{dt}$ and $\int_B$ are interchangeable because the function under the integral is continuous differentiable in time and the integration can be restricted to a compact set $\operatorname{supp}\varphi$. We obtain

$$\begin{aligned}
\frac{d}{dt}\int_B u^2\varphi^2 e^\xi d\mu &= 2\int_B u'u\varphi^2 e^\xi d\mu + \int_B u^2\varphi^2\xi' e^\xi d\mu \\
&\le 2\int_B (\Delta_\mu u)\,u\varphi^2 e^\xi d\mu + \int_B u^2\varphi^2\xi' e^\xi d\mu \\
&= -2\int_B \left(|\nabla u|^2\varphi^2 e^\xi + \langle\nabla u,\nabla\xi\rangle u\varphi^2 e^\xi + 2\langle\nabla u,\nabla\varphi\rangle u\varphi e^\xi\right) \\
&\quad + \int_B u^2\xi'\varphi^2 e^\xi d\mu.
\end{aligned}$$

Here we have applied the Green formula (4.12) using that $u \in W^2_{loc}(B)$ and $u\varphi^2 e^\xi \in W_c^1(B)$ (cf. Exercise 5.9), and the product and chain rules for Lipschitz functions to evaluate $\nabla\left(u\varphi^2 e^\xi\right)$ (cf. Exercises 11.13 and 11.14). Applying the inequalities

$$\langle\nabla u,\nabla\xi\rangle u \ge -|\nabla u|\,|\nabla\xi|\,u$$

and

$$2\langle\nabla u,\nabla\varphi\rangle u\varphi\geq-\left(\frac{1}{\varepsilon}u^2\left|\nabla\varphi\right|^2+\varepsilon\left|\nabla u\right|^2\varphi^2\right)$$

where $\varepsilon\in(0,1)$ is to be specified later, we obtain

$$\begin{aligned}\frac{d}{dt}\int_B u^2\varphi^2e^{\xi}d\mu &\leq \frac{2}{\varepsilon}\int_B u^2\left|\nabla\varphi\right|^2e^{\xi}d\mu\\ &\quad-2\int_B\left((1-\varepsilon)\left|\nabla u\right|^2-\left|\nabla u\right|\left|\nabla\xi\right|u-\frac{1}{2}u^2\xi'\right)\varphi^2\,e^{\xi}d\mu.\end{aligned}$$

Using (12.31) we see that the expression in brackets in the last integral above is bounded below by

$$(1-\varepsilon)\left|\nabla u\right|^2-\left|\nabla u\right|\left|\nabla\xi\right|u+\frac{\alpha}{4}u^2\left|\nabla\xi\right|^2,$$

which is identical to a complete square

$$\left(\sqrt{1-\varepsilon}\left|\nabla u\right|-\sqrt{\alpha/4}u\left|\nabla\xi\right|\right)^2$$

provided

$$(1-\varepsilon)\,\alpha=1.$$

Choosing ε to satisfy this condition, that is, $\varepsilon=1-\alpha^{-1}$, we obtain

$$\frac{d}{dt}\int_B u^2e^{\xi}\varphi^2\,d\mu\leq\frac{2}{\varepsilon}\int_B u^2\left|\nabla\varphi\right|^2e^{\xi}d\mu.$$

Integrating this inequality against dt from 0 to t and using the hypotheses $u(t,\cdot)\overset{L^2(B)}{\to}0$ as $t\to0$ and $u^2\leq1$, we obtain (12.32).

Now we will specify the functions φ and ξ in (12.32). In all cases, we will have $\varphi\equiv1$ on A, whence also $|\nabla\varphi|=0$ on A, so that (12.32) implies

$$\int_A u(t,\cdot)^2\,e^{\xi(t,\cdot)}d\mu\leq\frac{2\alpha}{\alpha-1}\int_{B\setminus A}\left|\nabla\varphi\right|^2\left(\int_0^t e^{\xi(s,\cdot)}\,ds\right)d\mu.\qquad(12.33)$$

In order to prove (12.30) for $R=d(A,B^c)$, it suffices to prove (12.30) for any $R<d(A,B^c)$. Fix $R<d(A,B^c)$, $t\in(0,T)$, set

$$\rho(x)=d(x,A),$$

and consider the function

$$\varphi(x)=\psi(\rho(x))$$

where $\psi(r)$ is a Lipschitz function on $[0,+\infty)$ such that

$$\psi(0)=1\text{ and }\psi(r)=0\text{ if }r\geq R$$

(see Fig. 12.6).

This ensures that $\varphi\in Lip_0(B)\subset W_0^1(B)$ (cf. Corollary 11.4), and $\varphi\equiv1$ on A. The function ψ will be chosen to be smooth in $(0,R)$. Then by Exercise 11.14 we have

$$\nabla\varphi=\psi'(\rho)\,\nabla\rho,$$

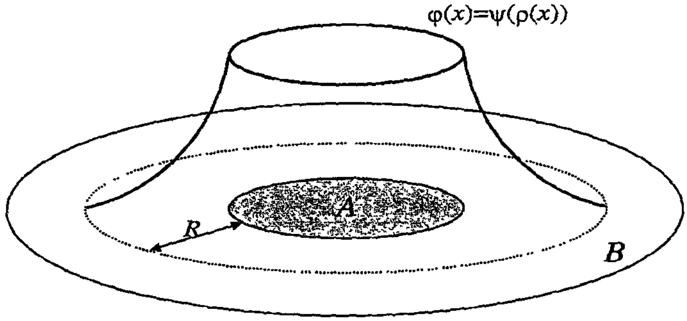

FIGURE 12.6. Function $\varphi(x)$

and since $\|\nabla \rho\|_{L^\infty} \le 1$ (see Theorem 11.3), it follows that

$$|\nabla \varphi(x)| \le |\psi'(\rho(x))| \text{ for almost all } x \in B \setminus A. \tag{12.34}$$

To specify further ψ and ξ, consider two cases.

Case 1. Let

$$\frac{R^2}{2t} \le 1,$$

then set $\xi \equiv 0$ and

$$\psi(r) = \frac{(R-r)_+}{R}.$$

By (12.34) we have $|\nabla \varphi| \le \frac{1}{R}$, and it follows from (12.33) that

$$\int_A u^2(t,\cdot)\, d\mu \le \frac{2\alpha}{\alpha - 1} \frac{t}{R^2} \mu(B \setminus A).$$

Letting $\alpha \to \infty$, we obtain

$$\int_A u^2(t,\cdot)\, d\mu \le \frac{2t}{R^2} \mu(B \setminus A) \le \frac{2t}{R^2} e^{-\frac{R^2}{2t}+1} \mu(B \setminus A). \tag{12.35}$$

Case 2. Let

$$\frac{R^2}{2t} > 1.$$

In this case, set

$$\xi(s,x) = -2a\rho(x) - bs,$$

where a and b are positive constants to be chosen below. Clearly, ξ satisfies (12.31) provided

$$b = 2a^2\alpha.$$

Note also that

$$\int_0^t e^{\xi(s,x)}\, ds = \frac{1 - e^{-bt}}{b} e^{-2a\rho(x)}. \tag{12.36}$$

Next, define ψ as follows:

$$\psi(r) = \frac{\left(e^{aR} - e^{ar}\right)_+}{e^{aR} - 1}.$$

Then we have

$$\psi'(r) = -ce^{ar} \text{ for } r \in (0, R),$$

where

$$c := \frac{a}{e^{aR} - 1},$$

whence it follows that

$$|\nabla \varphi(x)|^2 \le c^2 e^{2a\rho(x)} \tag{12.37}$$

for almost all $x \in B \setminus A$. Substituting (12.36) and (12.37) into (12.33) and observing that $\xi|_A = -bt$, we obtain

$$\begin{aligned}
\int_A u(t,\cdot)^2 \, d\mu &= e^{bt} \int_A u(t,\cdot)^2 e^{\xi(t,\cdot)} d\mu \\
&\le \frac{2\alpha}{\alpha - 1} e^{bt} \int_{B\setminus A} |\nabla\varphi|^2 \left(\int_0^t e^{\xi(s,\cdot)} \, ds\right) d\mu \\
&\le \frac{2\alpha}{\alpha - 1} \frac{e^{bt} - 1}{b} c^2 \mu(B \setminus A) \\
&= \frac{1}{\alpha - 1} \frac{e^{2a^2\alpha t} - 1}{(e^{aR} - 1)^2} \mu(B \setminus A).
\end{aligned}$$

Setting further

$$\delta = \frac{R^2}{2t} - 1, \quad \alpha = \frac{\delta + 1}{\delta} \text{ and } a = \frac{R}{2t\alpha},$$

we obtain the identities

$$2a^2\alpha t = aR = \frac{R^2}{2t\alpha} = \delta = \frac{1}{\alpha - 1},$$

whence

$$\int_A u(t,\cdot)^2 \, d\mu \le \frac{\delta}{e^\delta - 1} \mu(B \setminus A).$$

Since $e^\delta \ge 1 + \delta$, we have

$$\frac{e^\delta}{e^\delta - 1} \le \frac{1 + \delta}{(1 + \delta) - 1} = \frac{1 + \delta}{\delta},$$

whence

$$\frac{\delta}{e^\delta - 1} \le (1 + \delta) e^{-\delta}$$

and

$$\int_A u(t,\cdot)^2 \, d\mu \le \frac{R^2}{2t} e^{-\frac{R^2}{2t} + 1} \mu(B \setminus A). \tag{12.38}$$

Combining (12.35) and (12.38), we obtain (12.30). $\square$

REMARK 12.10. As one can see from the proof, if u satisfies the heat equation $\frac{\partial u}{\partial t} = \Delta_\mu u$ in $(0,T)\times B$ then the assumption $u \geq 0$ can be dropped.

COROLLARY 12.11. *Under the conditions of* Theorem 12.9, *the following inequalities are satisfied:*

$$\int_A u^2(t,\cdot)\,d\mu \leq \mu(B)\,\|u\|^2_{L^\infty} \max\left(\frac{R^2}{2t},1\right)\exp\left(-\frac{R^2}{2t}+1\right) \tag{12.39}$$

and

$$\int_A u(t,\cdot)\,d\mu \leq \sqrt{\mu(A)\,\mu(B)}\|u\|_{L^\infty} \max\left(\frac{R}{\sqrt{2t}},1\right)\exp\left(-\frac{R^2}{4t}+\frac{1}{2}\right). \tag{12.40}$$

PROOF. If $R^2/2t \geq 1$ then (12.39) trivially follows from (12.30). If $R^2/2t \leq 1$ then

$$\int_A u^2(t,\cdot)\,d\mu \leq \mu(A)\,\|u\|^2_{L^\infty} \leq \mu(B)\,\|u\|^2_{L^\infty}\exp\left(-\frac{R^2}{2t}+1\right),$$

which implies (12.39).

Inequality (12.40) follows from (12.39) and the Cauchy-Schwarz inequality. ☐

In fact, the following inequality is true:

$$\int_A u(t,\cdot)\,d\mu \leq 16\mu(B)\,\|u\|_{L^\infty}\int_R^\infty \frac{1}{(4\pi t)^{1/2}}\exp\left(-\frac{r^2}{4t}\right)dr, \tag{12.41}$$

which is called *Takeda's inequality*. Estimating in a certain way the integral in the right hand side, one obtains

$$\int_A u(t,\cdot)\,d\mu \leq \frac{16}{\sqrt{\pi}}\mu(B)\,\|u\|_{L^\infty}\frac{\sqrt{t}}{R}\exp\left(-\frac{R^2}{4t}\right).$$

For large ratios $\frac{R}{\sqrt{t}}$, this inequality is somewhat better than (12.40). The inequalities (12.30) and (12.39) can be considered as L^2 versions of Takeda's inequality.

For applications of Theorem 12.9 and Corollary 12.11 see Exercises 12.4, 12.5, 15.1, 9.9.

Exercises.

12.4. Using Corollary 12.11, prove that if the weighted manifold M is geodesically complete and, for some point $x \in M$, a constant $C > 0$, and a sequence $\{r_k\} \to \infty$,

$$\mu(B(x,r_k)) \leq \exp\left(Cr_k^2\right) \tag{12.42}$$

then M is stochastically complete.

REMARK. Of course, this follows from Theorem 11.8 but the purpose of this Exercise is to give an alternative proof.

12.5. Let A and B be sets as in Theorem 12.9.

(a) Prove that, for any function $f \in L^{\infty}(B^c)$,

$$\int_A (P_t f)^2 \, d\mu \le \mu(B) \|f\|_{L^\infty}^2 \max\left(\frac{R^2}{2t}, 1\right) e^{-\frac{R^2}{2t}+1}. \tag{12.43}$$

(b) Prove that

$$\int_A \int_{B^c} p_t(x,y) d\mu(y) d\mu(x) \le C\sqrt{\mu(A)\mu(B)} \max\left(\frac{R}{\sqrt{t}}, 1\right) e^{-\frac{R^2}{4t}}, \tag{12.44}$$

where $C = \sqrt{e/2}$.

Notes

The integrated maximum principle goes back to Aronson [**10**], [**9**]. A good account of it in the context of parabolic equations in $\mathbb{R}^n$ can be found in [**306**]. Here we follow [**147**], [**146**] and [**154**]. For the integrated maximum principle in a discrete setting see [**85**].

The Davies-Gaffney inequality was proved by B.Davies [**97**] with reference to [**127**]. The present proof is taken from [**147**]. A somewhat sharper version of (12.16),

$$|(P_t f, g)| \le \|f\|_2 \|g\|_2 \int_{d(A,B)}^{\infty} \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{s^2}{4t}\right) ds,$$

was proved in [**154**], using the finite propagation speed approach of [**57**].

The upper bounds of eigenvalues of Section 12.3 were proved in [**65**], [**66**]. See also [**43**] for further results.

A slightly weaker version of Takeda's inequality (12.41) was first proved by M.Takeda [**343**] using a probabilistic argument. It was improved and brought to the form (12.41) by T.Lyons [**265**]. An analytic proof of an L^2 version of Takeda's inequality (12.30) was obtained in [**147**]. Here we reproduce the proof of [**147**] with some simplifications. An interesting application of Takeda's inequality to the escape rate of the Brownian motion on M can be found in [**152**].

CHAPTER 13

Green function and Green operator

Here we introduce the Green function and apply it to prove the local Harnack inequality, which requires a substantial use of the heat kernel. The results will not be used in the subsequent chapters.

13.1. The Green operator

By definition, the *Green operator* G acts on non-negative measurable functions f on a weighted manifold M as follows:

$$Gf(x) = \int_0^\infty P_t f(x)\, dt. \tag{13.1}$$

More generally, Gf is defined whenever the right hand side of (13.1) makes sense. If Ω is an open subset of M then set

$$G^\Omega f = \int_0^\infty P_t^\Omega f(x)\, dt.$$

LEMMA 13.1. *Let f be a non-negative function from $L^2_{loc}(M)$ such that $Gf \in L^2_{loc}(M)$. Then the function $u = Gf$ is the minimal non-negative solution in $L^2_{loc}(M)$ of the equation $-\Delta_\mu u = f$ considered in the distributional sense. If in addition $f \in C^\infty$ then also $u \in C^\infty$.*

PROOF. Let us use the resolvent operator R_α, $\alpha > 0$, as it was defined in Section 8.2, that is

$$R_\alpha f(x) = \int_0^\infty \int_M e^{-\alpha t} p_t(x,y) f(y)\, d\mu(y)\, dt.$$

If $f \geq 0$ then by the monotone convergence theorem, $R_\alpha f(x) \uparrow Gf(x)$ as $\alpha \downarrow 0$.

If $f \geq 0$, $f \in L^2_{loc}$, and $Gf \in L^2_{loc}$ then also $R_\alpha f \in L^2_{loc}$ and, by Theorem 8.4, the function $u_\alpha = R_\alpha f$ satisfies the equation

$$-\Delta_\mu u_\alpha + \alpha u_\alpha = f.$$

Passing to the limit as $\alpha \to 0$ and noticing that $u_\alpha \xrightarrow{\mathcal{D}'} u$, we obtain $-\Delta_\mu u = f$.

If $v \in L^2_{loc}$ is another non-negative solution to the equation $-\Delta_\mu v = f$ then, for any $\alpha > 0$,

$$-\Delta_\mu v + \alpha v = f + \alpha v \in L^2_{loc}.$$

Therefore, by Theorem 8.4,

$$v \geq R_\alpha (f + \alpha v) \geq R_\alpha f.$$

Letting $\alpha \to 0$, we obtain $v \geq u$.

If $f \in C^\infty$ then $u_\alpha \in C^\infty$ by Theorem 8.7. As $\alpha \downarrow 0$, the sequence $u_\alpha (x)$ increases and converges to $u(x)$ pointwise. By Exercise 7.13 we conclude that $u \in C^\infty$. $\square$

It follows from (13.1) that, for any non-negative measurable function f on M,

$$\begin{aligned} Gf(x) &= \int_0^\infty \int_M p_t(x,y) f(y)\, d\mu(y)\, dt \\ &= \int_M g(x,y) f(y)\, d\mu(y) \end{aligned} \tag{13.2}$$

where the function

$$g(x,y) = \int_0^\infty p_t(x,y)\, dt \tag{13.3}$$

is called the *Green function* of M. Note that g takes values in $[0, +\infty]$. The Green function is called *finite* if $g(x,y) < \infty$ for all *distinct* $x, y \in M$.

If Ω is an open subset of M then define the Green function of Ω by similarly

$$g^\Omega(x,y) = \int_0^\infty p_t^\Omega(x,y)\, dt. \tag{13.4}$$

EXAMPLE 13.2. Applying (13.3) with the Gauss-Weierstrass heat kernel (2.50) and using the identity (A.60) from Solution to Exercise 5.14, we obtain the following formulas for the Green function in $\mathbb{R}^n$:

$$g(x,y) = \begin{cases} c_n |x-y|^{2-n}, & n > 2, \\ +\infty, & n \leq 2, \end{cases} \tag{13.5}$$

where

$$c_n = \frac{\Gamma(n/2 - 1)}{4\pi^{n/2}} = \frac{\Gamma(n/2)}{2\pi^{n/2}(n-2)} = \frac{1}{(n-2)\,\omega_n}$$

(cf. (3.94)). Hence, the Green function in $\mathbb{R}^n$ is finite if and only if $n > 2$.

DEFINITION 13.3. A function $h \in L^1_{loc}(M)$ is called a *fundamental solution* of the Laplace operator at a point $x \in M$ if $-\Delta_\mu h = \delta_x$.

In particular, a fundamental solution h is harmonic away from x and, hence, is smooth in $M \setminus \{x\}$ (cf. Theorem 7.4 and Exercise 7.10).

THEOREM 13.4. *If $\lambda_{\min}(M) > 0$, then the following is true.*

(i) The Green function g of M is finite and, for any $x \in M$, $g(x,\cdot)$ is a fundamental solution of Δ_μ at x.

(ii) The Green operator G maps $L^2(M)$ into itself. Moreover, it is a bounded self-adjoint operator in $L^2(M)$, and $G = \mathcal{L}^{-1}$, where $\mathcal{L} = -\Delta_\mu|_{W_0^2}$ is the Dirichlet Laplace operator.

(*iii*) *If* $f \in C_0^\infty(M)$ *then* $Gf \in C^\infty(M)$.

PROOF. Let us first show that the integral in (13.3) converges for distinct x, y. Indeed, the convergence at 0 follows from the fact that $p_t(x,y) \to 0$ as $t \to 0$ (cf. Corollary 9.21), and the convergence at ∞ follows from $\lambda := \lambda_{\min}(M) > 0$ and the inequality

$$p_t(x,y) \leq \sqrt{p_s(x,x)\, p_s(y,y)} \exp(-\lambda(t-s)) \tag{13.6}$$

(cf. Exercise 10.29). Fix $s > 0$, a compact set $K \subset M$, and set

$$C = \sup_{y \in K} \sqrt{p_s(y,y)},$$

so that, for all $x \in M$, $y \in K$, and $t \geq s$, the following inequality takes place:

$$p_t(x,y) \leq C\sqrt{p_s(x,x)}e^{-\lambda(t-s)}. \tag{13.7}$$

Using (7.50), (13.3), and (13.7), we obtain

$$\begin{aligned} \int_K g(x,y)\, d\mu(y) &= \int_0^s \int_K p_t(x,y)\, d\mu(y)\, dt + \int_K \int_s^\infty p_t(x,y)\, dt\, d\mu(y) \\ &\leq s + \frac{C}{\lambda}\mu(K)\sqrt{p_s(x,x)}. \end{aligned}$$

Hence, the integral

$$\int_K g(x,y)\, d\mu(y)$$

is finite and, moreover, it is locally bounded as a function of x. It follows that $g(x,\cdot) \in L^1_{loc}(M)$. Let us also mention the following consequence of the above estimate and of the symmetry of $g(x,y)$: for any function $f \in L^1(M)$ with compact support, we have

$$\int_K |Gf|\, d\mu \leq \int_M \left(\int_K g(x,y)\, d\mu(x)\right) |f(y)|\, d\mu(y) < \infty$$

that is, $Gf \in L^1_{loc}(M)$.

The spectrum of $\mathcal{L}$ is contained in $[\lambda, +\infty)$ and hence, $\mathcal{L}^{-1}$ exists as a bounded operator, and $\|\mathcal{L}^{-1}\| \leq \lambda^{-1}$. By the functional calculus, we have

$$\int_0^\infty \left(e^{-t\mathcal{L}} f\right) dt = \mathcal{L}^{-1} f \tag{13.8}$$

for any $f \in L^2(M)$. Comparing (13.8) with (13.1), we see that the left hand side here coincides with Gf. Hence, the Green operator G maps $L^2(M)$ into itself and coincides in $L^2(M)$ with $\mathcal{L}^{-1}$. Consequently, for any $f \in L^2(M)$ there is a unique solution $u \in W_0^2(M)$ to the equation

$$-\Delta_\mu u = f, \tag{13.9}$$

and this solution is given by $u = Gf$.

The fact that $f \in C_0^\infty(M)$ implies $Gf \in C^\infty(M)$ follows from Lemma 13.1 if $f \geq 0$. If f is signed then it can be represented as a difference of two non-negative functions from $C_0^\infty(M)$ (cf. Exercise 4.6), which settles the claim.

Finally, let us prove that $g(x,\cdot)$ is a fundamental solution at x, that is,

$$-\Delta_\mu g(x,\cdot)=\delta_x. \tag{13.10}$$

We need to verify that, for any $u\in C_0^\infty(M)$,

$$-\int_M g(x,y)\,\Delta_\mu u(y)\,d\mu(y)=u(x). \tag{13.11}$$

Indeed, any such function u satisfies the equation (13.9) with the right hand side $f:=-\Delta_\mu u$. Hence, by the uniqueness of the solution in $W_0^2(M)$, functions u and Gf coincide as L^2-functions. Since both functions $u(x)$ and $Gf(x)$ are C^∞ (the latter being true because $f\in C_0^\infty(M)$), it follows that they coincide pointwise, which proves (13.11). □

As one can see from (13.5), the Green function $g(x,\cdot)$ does not have to belong to $L^2(M)$. Indeed, in $\mathbb{R}^4$ the integral of $g^2(x,\cdot)$ diverges both at x and at ∞. The following statement shows that, in a restricted setting, a "cut-down" Green function belongs even to $W_0^1(M)$. This is a technical result that has many applications.

Lemma 13.5. *Assume that $\lambda_{\min}(M)>0$ and $\mu(M)<\infty$. Let $\psi(s)$ be a C^∞-function on $[0,+\infty)$ such that, for some constant $C>0$,*

$$\psi(0)=0,\quad 0\le\psi\le C,\quad 0\le\psi'\le C \tag{13.12}$$

Then, for any $x_0\in M$, the function $u=\psi(g(x_0,\cdot))$ belongs to $W_0^1(M)$ and

$$\|\nabla u\|_{L^2}^2\le\int_0^\infty\left|\psi'(s)\right|^2ds$$

Proof. Define a new function φ on $[0,+\infty)$ by

$$\varphi(r)=\int_0^r\left|\psi'(s)\right|^2ds. \tag{13.13}$$

Clearly, φ also satisfies the conditions (13.12) with the constant C^2 instead of C. Extending oddly ψ and φ to $(-\infty,0)$, we obtain that φ and ψ are smooth Lipschitz functions on $\mathbb{R}$ that vanish at 0.

Since ψ is bounded, the function $u=\psi(g(x_0,\cdot))$ is also bounded. Since $\mu(M)<\infty$, it follows that $u\in L^2$ (this is the only place where the finiteness of $\mu(M)$ is used). The main difficulty lies in the proof of the fact that $u\in W_0^1$. By Theorem 13.4, the Green operator is bounded in L^2 and coincides with the inverse of $\mathcal{L}$. Consider for any $t>0$ the function

$$g_t=Gp_t(x_0,\cdot),$$

which is hence in $L^2(M)$ and, moreover,

$$g_t\in\operatorname{dom}(\mathcal{L})=W_0^2\subset W_0^1.$$

It follows that

$$-\Delta_\mu g_t=p_t(x_0,\cdot). \tag{13.14}$$

On the other hand, we have

$$\begin{aligned} g_t(x) &= \int_M g(x,y)\, p_t(x_0,y)\, d\mu(y) \\ &= \int_M \left(\int_0^\infty p_s(x,y)\, ds\right) p_t(x_0,y)\, d\mu(y) \\ &= \int_0^\infty \left(\int_M p_s(x,y)\, p_t(x_0,y)\, d\mu(y)\right) ds \\ &= \int_0^\infty p_{t+s}(x,x_0)\, ds \\ &= \int_t^\infty p_s(x,x_0)\, ds. \end{aligned} \tag{13.15}$$

It follows from (13.15) that g_t is increasing as t decreases, and

$$g_t(x) \uparrow g(x_0,x) \quad \text{as} \quad t \downarrow 0. \tag{13.16}$$

By Lemma 5.1, the functions $\psi(g_t)$ and $\varphi(g_t)$ belong to W_0^1. Using the chain rule, the Green formula of Lemma 4.4, (13.13) and (13.14), we obtain

$$\begin{aligned} \int_M |\nabla \psi(g_t)|^2\, d\mu &= \int_M |\psi'(g_t)|^2\, |\nabla g_t|^2\, d\mu \\ &= \int_M \varphi'(g_t)\, |\nabla g_t|^2 \\ &= \int_M \langle \nabla\varphi(g_t), \nabla g_t\rangle\, d\mu \\ &= -\int_M \varphi(g_t)\, \Delta_\mu g_t\, d\mu \\ &= \int_M \varphi(g_t)\, p_t(x_0,\cdot)\, d\mu \\ &= P_t\varphi(g_t)\,|_{x_0}. \end{aligned} \tag{13.17}$$

It follows that, for all $t > 0$,

$$\int_M |\nabla\psi(g_t)|^2\, d\mu \le \sup\varphi < \infty. \tag{13.18}$$

In particular, the integral in (13.18) remains bounded as $t \to 0$.

By (13.16), the sequence $\psi(g_t)$ increases and converges pointwise to $\psi(g)$ as $t \downarrow 0$, where we write for simplicity $g = g(x_0,\cdot)$. Using the uniform bound (13.18) and that $\psi(g) \in L^2$, we conclude by Exercise 4.18 that $\psi(g) \in W^1$, $\psi(g_t) \xrightarrow{W^1} \psi(g)$, and

$$\int_M |\nabla\psi(g)|^2\, d\mu \le \sup\varphi = \int_0^\infty |\psi'(s)|^2\, ds.$$

Finally, since $\psi(g_t) \in W_0^1$ and W_0^1 is weakly closed in W^1 (cf. Exercise A.5), it follows that also $\psi(g) \in W_0^1$, which finishes the proof. □

COROLLARY 13.6. *Assume that* $\lambda_{\min}(M) > 0$ *and* $\mu(M) < \infty$. *Then, for any constant* $c > 0$ *and any* $x_0 \in M$, *the function*

$$u = \min\left(g\left(x_0, \cdot\right), c\right)$$

belongs to $W_0^1(M)$ *and* $\|\nabla u\|_{L^2}^2 \leq c$.

PROOF. Clearly, we have $u = \psi\left(g\left(x_0, \cdot\right)\right)$ where

$$\psi\left(s\right) = \min\left(s, c\right).$$

It is easy to see that there is a sequence $\{\psi_k\}_{k=1}^{\infty}$ of smooth non-negative functions on $[0, +\infty)$ such that $0 \leq \psi_k' \leq 1$ and

$$\psi_k\left(s\right) \uparrow \psi\left(s\right) \text{ as } k \to \infty.$$

(see Fig. 13.1).

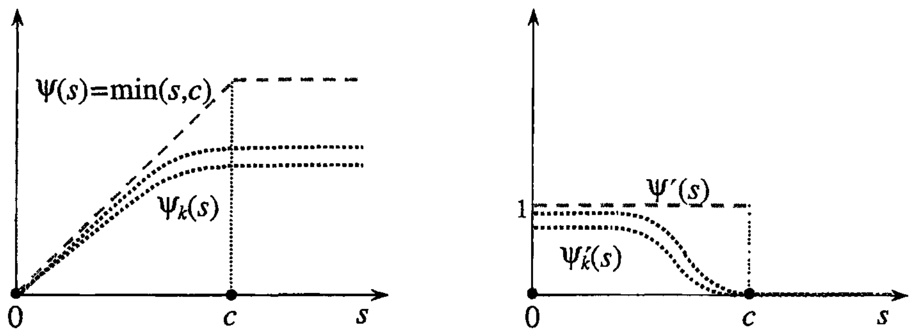

FIGURE 13.1. Construction of functions $\psi_k\left(s\right)$ via their derivatives

Since each ψ_k satisfies the hypothesis of Lemma 13.5, we conclude that $\psi_k\left(g\right) \in W_0^1$ and

$$\int_M \left|\nabla \psi_k\left(g\right)\right|^2 d\mu \leq \int_0^{\infty} \left|\psi_k'\left(s\right)\right|^2 ds \leq \int_0^{\infty} \psi_k'\left(s\right) ds = \sup \psi_k \leq c.$$

Letting $k \to \infty$, we conclude by Exercise 4.18 that $\psi\left(g\right) \in W_0^1$ and

$$\int_M \left|\nabla \psi\left(g\right)\right|^2 d\mu \leq c.$$

$\square$

REMARK 13.7. The hypotheses $\lambda_{\min}(M) > 0$ and $\mu(M) < \infty$ in Corollary 13.6 (as well as in Lemma 13.5) can be dropped but the conclusion will change as follows: $u \in W_{loc}^1$ and $\|\nabla u\|_{L^2}^2 \leq c$ – see Exercise 13.12.

Exercises.

13.1. Prove that if M is a compact manifold then

(a) $g(x,y) \equiv \infty$;

(b) there is no fundamental solution of the Laplace operator on M.

13.2. Let M be a weighted model (cf. Section 3.10) and $S(r)$ be the area function of M.

(a) Prove that, for any positive real R that is smaller than the radius of M, the following function
$$h(x) = \int_{|x|}^{R} \frac{dr}{S(r)}$$
is a fundamental solution in B_R of the Laplace operator at the pole o.

(b) Using (a), evaluate the fundamental solutions on $\mathbb{R}^n$ and $\mathbb{H}^n$.

13.3. Prove that if the manifold M is connected then $g(x,y) > 0$ for all $x, y \in M$.

13.4. Prove that if the Green function g is finite then the following identity takes place for all $t > 0$ and $x_0 \in M$:
$$P_t g(x_0, \cdot) = G p_t(x_0, \cdot).$$

13.5. Prove that if $\lambda_{\min}(M) > 0$ then the Green function $g(x,y)$ is C^∞ smooth jointly in x, y in $M \times M \setminus \text{diag}$.

13.6. Prove that if $\lambda_{\min}(M) > 0$ then
$$\|G\|_{L^2 \to L^2} \le \frac{1}{\lambda_{\min}(M)}. \tag{13.19}$$

13.7. Prove that if $\lambda_{\min}(M) > 0$ and $\mu(M) < \infty$ then $g(x,y) \in L^1(M \times M)$.

13.8. Prove that if $\{\Omega_k\}$ is any exhaustion sequence in M then, for all $x, y \in M$,
$$g^{\Omega_k}(x,y) \uparrow g(x,y) \quad \text{as } k \to \infty.$$

13.9. Let Ω be an open subset of a weighted manifold M. Prove that, for any compact set $K \subset \Omega$ and for any non-negative function $f \in L^2(M)$,
$$Gf \le G^\Omega f + \operatorname*{esup}_{M \setminus K} Gf. \tag{13.20}$$

13.10. Let Ω be a non-empty relatively compact open subset of a connected manifold M such that $M \setminus \overline{\Omega}$ is non-empty. Fix a point $x_0 \in \Omega$.

(a) Let φ be a cutoff function of $\{x_0\}$ in Ω. Prove that
$$(1-\varphi)\, g^\Omega(x_0, \cdot) \in W_0^1(\Omega).$$

(b) Prove that for any open set $U \subset \Omega$, containing x_0,
$$g^\Omega(x_0, \cdot) - g^U(x_0, \cdot) \in W_0^1(\Omega).$$

13.11. Assume that $\lambda_{\min}(M) > 0$ and $\mu(M) < \infty$. Prove that, for all $0 \le a < b$ and any $x_0 \in M$, the function
$$v(x) = \begin{cases} g(x_0, x) & \text{if } g(x_0,x) \in [a,b], \\ a, & \text{if } g(x_0,x) < a, \\ b, & \text{if } g(x_0,x) > b, \end{cases}$$
belongs to $W^1(M)$ and
$$\|\nabla v\|_{L^2}^2 \le b - a.$$

13.12. Prove that, for any weighted manifold M and for all $c > 0$, $x_0 \in M$, the function $u = \min(g(x_0, \cdot), c)$ belongs to $W_{loc}^1(M)$ and
$$\|\nabla u\|_{L^2}^2 \le c.$$

13.13. Let Ω be a non-empty relatively compact connected open subset of a weighted manifold M. Prove that

$$\sup_{x\in\Omega}\int_\Omega g^\Omega(x,y)\,d\mu(y)\geq\frac{1}{\lambda_{\min}(\Omega)}. \tag{13.21}$$

13.14. Let M be a connected weighted manifold and Ω be a relatively compact open subset of M such that $M\setminus\overline{\Omega}$ is non-empty. Let $\{\varphi_k\}_{k=1}^\infty$ be an orthonormal basis in $L^2(\Omega)$ of eigenfunctions of Ω and $\{\lambda_k\}$ be the corresponding sequence of eigenfunctions. Prove the identity

$$g^\Omega(x,y)=\sum_{k=1}^\infty\frac{1}{\lambda_k}\varphi_k(x)\,\varphi_k(y),$$

where the series converges in $\mathcal{D}'(\Omega\times\Omega)$.

13.2. Superaveraging functions

We say that a function f on M is *superaveraging* if $f\in L^1_{loc}(M)$, $f\geq 0$, and

$$P_tf\leq f \text{ for all } t>0.$$

By Exercise 7.30, if f is superaveraging then P_tf is a smooth solution of the heat equation, which is decreasing in t, and $P_tf\xrightarrow{L^1_{loc}}f$ as $t\to 0$; besides, $\Delta_\mu f\leq 0$ in the distributional sense. Furthermore, if $f\in W^1_{loc}(M)$ and $f\geq 0$ then f is superaveraging if and only if $\Delta_\mu f\leq 0$ (cf. Exercise 7.29). In particular, any non-negative superharmonic function is superaveraging.

LEMMA 13.8. *Let U be an open subset of M such that $\lambda_{\min}(U)>0$ and U^c is compact. Fix a function $f\in W^1(M)$, a cutoff function ψ of U^c in M, and let $u\in W^1(U)$ be the solution to the weak Dirichlet problem in U:*

$$\begin{cases}\Delta_\mu u=0,\\ u=f\psi \mod W^1_0(U).\end{cases} \tag{13.22}$$

Define the function $\widetilde{f}$ on M by

$$\widetilde{f}=\begin{cases} f & \text{in } U^c,\\ u & \text{in } U\end{cases}$$

(see Fig. 13.2).

(a) *Then $\widetilde{f}\in W^1_0(M)$.*

(b) *If in addition f is superaveraging then also $\widetilde{f}$ is superaveraging and $0\leq\widetilde{f}\leq f$.*

PROOF. (a) By Corollary 5.6, we have $f\psi\in W^1_0(M)$, and by (13.22) $v:=f\psi-u\in W^1_0(U)$. Extending v to M by setting $v=0$ in U, we obtain $v\in W^1_0(M)$. Observe that

$$\widetilde{f}=f\psi-v \quad\text{in } M. \tag{13.23}$$

Indeed, in U^c we have

$$\widetilde{f}=f=f\psi-v$$

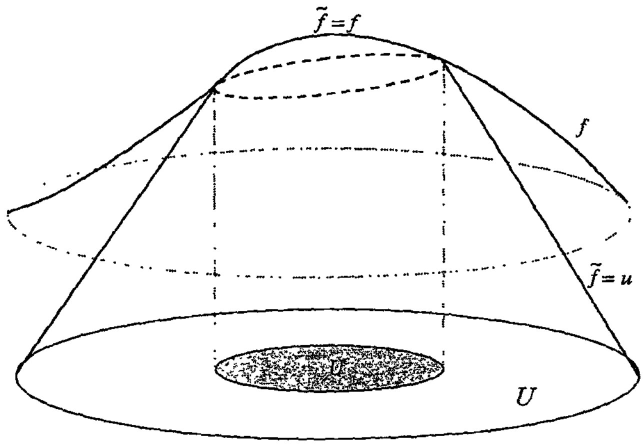

FIGURE 13.2. Function $\widetilde{f}$ in Lemma 13.8

because $\psi \equiv 1$ and $v \equiv 0$ in U^c, and in U we have

$$\widetilde{f} = u = f\psi - v$$

by the definition of v. It follows from (13.23) that $\widetilde{f} \in W_0^1(M)$.

(b) Since $f\psi \geq 0$ and $\lambda_{\min}(U) > 0$, by Theorem 5.13 we obtain from (13.22) that $u \geq 0$. Hence, $\widetilde{f} \geq 0$. Since f is superaveraging, we have in U

$$-\Delta_\mu (u - f) = -\Delta_\mu u + \Delta_\mu f \leq 0$$

and

$$u - f \leq u - f\psi = 0 \mod W_0^1(U).$$

Hence, by Theorem 5.13, $u - f \leq 0$ in U. It follows that $\widetilde{f} \leq f$ in M. In particular, $\widetilde{f} \in L^2(M)$.

We are left to prove that $P_t\widetilde{f} \leq \widetilde{f}$. In U^c, we have

$$P_t\widetilde{f} \leq P_t f \leq f = \widetilde{f}.$$

To prove that $P_t\widetilde{f} \leq \widetilde{f}$ in U, observe that the functions $w_1(t,\cdot) = P_t\widetilde{f}$ and $w_2(t,\cdot) = \widetilde{f}$ as paths in $W^1(M)$ satisfy the conditions

$$\begin{cases} \dfrac{dw_1}{dt} - \Delta_\mu w_1 = 0 \leq \dfrac{dw_2}{dt} - \Delta_\mu w_2, \\ w_1 \leq w_2 \mod W_0^1(M), \\ \lim\limits_{t\to 0} w_1(t,\cdot) = \lim\limits_{t\to 0} w_2(t,\cdot) = \widetilde{f}. \end{cases}$$

Hence, by Theorem 5.16, $w_1 \leq w_2$, which finishes the proof. $\square$

Exercises.

13.15. Prove the following properties of superaveraging functions.

(a) If $\{f_k\}_{k=1}^{\infty}$ is an increasing sequence of superaveraging functions and $f_k \to f \in L^1_{loc}$ then f is also superaveraging.

(a) If $\{f_i\}_{i\in I}$ is a family of superaveraging functions depending on a parameter i then the function

$$f=\inf_{i\in I} f_i$$

is also superaveraging.

13.16. Let M be a connected, stochastically complete weighted manifold, and let f be a non-negative continuous superaveraging function on M.

(a) Prove that the inequality $P_t f \le f$ is satisfied pointwise and that $P_t f \to f$ as $t \to 0$ pointwise.
(b) (*Strong minimum principle*) Prove that if $f(x) = \inf f$ at some point $x \in M$ then $f \equiv \mathrm{const}$ on M.
(b) (*Minimum principle*) Let Ω be a relatively compact open subset of M with non-empty boundary. Prove that

$$\inf_{\overline{\Omega}} f = \inf_{\partial\Omega} f.$$

13.17. Prove that if the Green function is finite then it is superaveraging with respect to each of its arguments.

13.18. Let Ω be a relatively compact open subset of M such that $\lambda_{\min}(\Omega) > 0$. Let u be a solution of the following weak Dirichlet problem in Ω

$$\begin{cases} \Delta_\mu u = 0, \\ u = f \mod W_0^1(\Omega), \end{cases} \tag{13.24}$$

where $f \in W^1(M)$, and set

$$\widetilde{f} = \begin{cases} f & \text{in } \Omega^c, \\ u & \text{in } \Omega, \end{cases}$$

(see Fig. *13.3*).

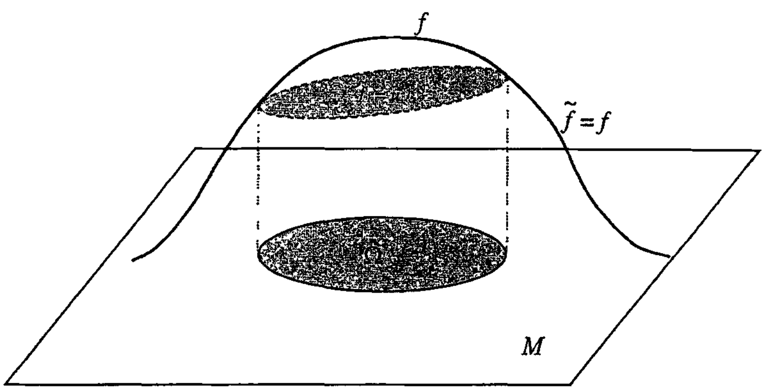

FIGURE 13.3. Function $\widetilde{f}$ in Exercise 13.18

(a) Prove that if $f \in W_0^1(M)$ then also $\widetilde{f} \in W_0^1(M)$.
(b) Prove that if f is superaveraging then also $\widetilde{f}$ is superaveraging and $0 \le \widetilde{f} \le f$.

13.19. Let f and h be two superaveraging functions from $W_0^1(M)$. Then, for any $t>0$,

$$(-\Delta_\mu P_t f, h) \le (\nabla f, \nabla h). \tag{13.25}$$

13.20. Let $f \in W_0^1(M)$ and $\{\Omega_k\}$ be a compact exhaustion sequence in M. Let $u_k \in W^1(\Omega_k)$ solve in Ω_k the weak Dirichlet problem problem

$$\begin{cases} \Delta_\mu u_k = 0, \\ u_k = f \mod W_0^1(\Omega_k). \end{cases}$$

Then

$$\|\nabla u_k\|_{L^2} \to 0 \text{ as } k \to \infty.$$

13.21. Let f and h be two superaveraging functions from $W_0^1(M)$. If $\{\Omega_k\}_{k=1}^\infty$ is a compact exhaustion sequence such that $\lambda_{\min}(\Omega_k) > 0$ for any k, then

$$\sup_{t>0} \int_{M \setminus \Omega_k} (-\Delta_\mu P_t f)\, h\, d\mu \to 0 \text{ as } k \to \infty.$$

13.3. Local Harnack inequality

The next statement contains a useful technical result, which will be then used to prove the local Harnack inequality in Theorem 13.10.

THEOREM 13.9. *Let M be a weighted manifold with $\lambda_{\min}(M) > 0$, and let $\Omega_0 \Subset \Omega_1 \Subset \Omega_2$ be relatively compact open subsets of M. Then, for any non-negative harmonic function $f \in W^1(M)$, we have*

$$\sup_{\Omega_0} f \leq C \inf_{\Omega_0} f \tag{13.26}$$

where

$$C = \sup_{\substack{x, x' \in \Omega_0 \\ y \in \Omega_2 \setminus \Omega_1}} \frac{g(x,y)}{g(x',y)} \tag{13.27}$$

and g is the Green function of M.

REMARK. Note that, due to the hypothesis $\lambda_{\min}(M) > 0$, the Green function g is finite (cf. Theorem 13.4).

The hypotheses $\lambda_{\min}(M) > 0$ and $f \in W^1(M)$ are not restrictive because this theorem is normally applied when M is a relatively compact open subset of another manifold.

PROOF. Choose an open set Ω such that $\Omega_1 \Subset \Omega \Subset \Omega_2$, and let ψ be a cutoff function of $\overline{\Omega}$ in Ω_2. Let u be the solution to the following weak Dirichlet problem in $\overline{\Omega}^c$:

$$\begin{cases} \Delta_\mu u = 0, \\ u = f\psi \mod W_0^1\left(\overline{\Omega}^c\right). \end{cases}$$

Consider the function

$$\widetilde{f} = \begin{cases} f & \text{in } \overline{\Omega}, \\ u & \text{in } \overline{\Omega}^c. \end{cases}$$

Since $f = \widetilde{f}$ in $\Omega \supset \Omega_0$, it suffices to prove (13.26) for $\widetilde{f}$ instead of f (see Fig. 13.4).

Let us mention the following properties of $\widetilde{f}$, which will be used. Since f is harmonic and, hence, superaveraging, we conclude by Lemma 13.8, that

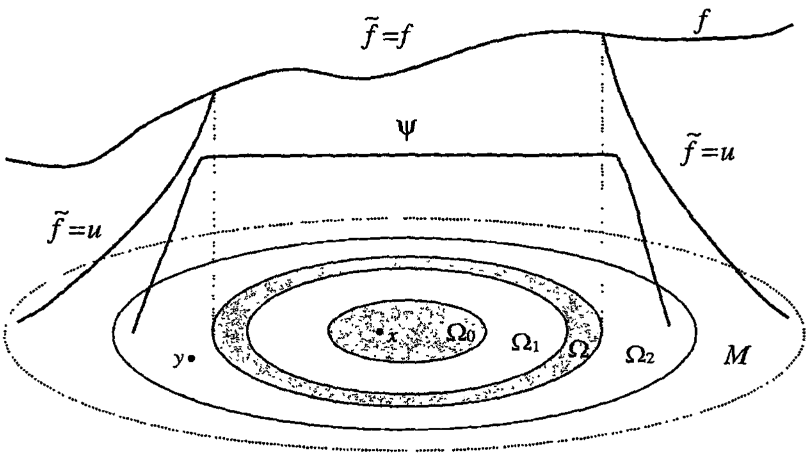

FIGURE 13.4. Illustration to the proof of Theorem 13.9

$\widetilde{f} \in W_0^1(M)$ and $\widetilde{f}$ is superaveraging. Since the both functions f and u are harmonic in their domains, we obtain that $\Delta_\mu \widetilde{f} = 0$ outside $\partial\Omega$. Renaming for simplicity $\widetilde{f}$ back to f, we have by (13.1)

$$\begin{aligned} G(f - P_t f) &= \int_0^\infty P_s (f - P_t f)\, ds \\ &= \int_0^\infty P_s f\, ds - \int_0^\infty P_s (P_t f)\, ds \\ &= \int_0^\infty P_t f\, ds - \int_0^\infty P_{s+t} f\, ds \\ &= \int_0^T P_t f\, ds - \int_t^\infty P_s f\, ds \\ &= \int_0^t P_t f\, ds. \end{aligned}$$

Since $P_t f \xrightarrow{L^2} f$ as $t \to 0$, we obtain that

$$G\left(\frac{f - P_t f}{t}\right) \xrightarrow{L^2} f \quad \text{as } t \to 0.$$

Observe that function f satisfies the hypotheses of Lemma 12.7 with $V = M \setminus \partial\Omega$ and $U = \Omega_1 \cup \overline{\Omega}_2^c$. Hence, we conclude by Lemma 12.7 that

$$\frac{f - P_t f}{t} \xrightarrow{L^2(U)} 0 \quad \text{as } t \to 0$$

Since by Theorem 13.4 the Green operator is bounded in L^2, it follows that

$$G\left(\frac{f - P_t f}{t} 1_U\right) \xrightarrow{L^2} 0 \quad \text{as } t \to 0,$$

whence

$$h_t := G\left(\frac{f - P_t f}{t} 1_{U^c}\right) \xrightarrow{L^2} f \quad \text{as } t \to 0.$$

Noticing that $U^c = \overline{\Omega}_2 \setminus \Omega_1$, we can write

$$h_t(x) = \int_{\overline{\Omega}_2 \setminus \Omega_1} g(x,y) \frac{f - P_t f}{t}(y)\, d\mu(y).$$

Since $f - P_t f \geq 0$, it follows from (13.27) that

$$\sup_{x \in \Omega_0} h_t(x) \leq C \inf_{x \in \Omega_0} h_t(x).$$

Since $h_t \xrightarrow{L^2} f$ as $t \to 0$, the same inequality holds for f. Indeed, there is a sequence $t_k \to 0$ such that $h_{t_k} \xrightarrow{\text{a.e.}} f$ as $k \to \infty$. It follows that

$$\operatorname*{esup}_{\Omega_0} f \leq C \operatorname*{einf}_{\Omega_0} f.$$

However, since f is continuous in Ω_0, esup and einf can be replaced by sup and inf, respectively, which yields (13.26). $\square$

Now we can prove the main result of this section.

THEOREM 13.10. (The local Harnack inequality) *Let M be an arbitrary connected weighted manifold and K be a compact subset of M. Then there is a constant $C = C(K)$ such that, for any non-negative harmonic function f on M,*

$$\sup_K f \leq C \inf_K f. \tag{13.28}$$

PROOF. If M is compact then any harmonic function f on M is constant because in this case $f \in C_0^\infty(M)$ and, hence,

$$\int_M |\nabla f|^2 d\mu = -\int_M f \Delta_\mu f \, d\mu = 0.$$

Therefore, (13.28) is satisfied with $C = 1$.

Assuming in the sequel that M is non-compact, choose a sequence $\Omega_0 \Subset \Omega_1 \Subset \Omega_2 \Subset \Omega$ of relatively compact open subsets of M such that $K \subset \Omega_0$. By Theorem 10.22, we have $\lambda_{\min}(\Omega) > 0$. Applying Theorem 13.9 to the manifold Ω and noticing that $f \in W^1(\Omega)$, we obtain that (13.28) holds with the constant C defined by (13.27) with g^Ω instead of g, that is,

$$C = \sup_{\substack{x,x' \in \Omega_0 \\ y \in \Omega_2 \setminus \Omega_1}} F(x, x', y),$$

where

$$F(x, x', y) = \frac{g^\Omega(x,y)}{g^\Omega(x',y)}.$$

We have still to make sure that $C < \infty$. For that, it suffices to show that function F is finite and continuous in the compact domain

$$(x, x', y) \in \overline{\Omega}_0 \times \overline{\Omega}_0 \times \left(\overline{\Omega}_2 \setminus \Omega_1\right). \tag{13.29}$$

By Theorem 13.4, $g^{\Omega}(x,y) < \infty$ because $x \neq y$. Choosing the set Ω to be connected, we obtain by Exercise 13.3 that $g^{\Omega}(x',y) > 0$, whence F is finite in the domain (13.29). Finally, by Exercise 13.5, $g^{\Omega}(x,y)$ is continuous jointly in x, y away from the diagonal, which implies that F is continuous in (13.29). ☐

The next theorem extends the local Harnack inequality to α-harmonic functions.

THEOREM 13.11. *Let M be an arbitrary connected weighted manifold and assume that there is a positive function h on M such that*

$$-\Delta_{\mu} h + \alpha h = 0,$$

where α is a real constant. Then, for any compact $K \subset M$ and for any $\beta \geq \alpha$, there is a constant $C = C(K,\beta)$ such that, for any non-negative β-harmonic function f on M,

$$\sup_K f \leq C \inf_K f. \tag{13.30}$$

Moreover, the constant $C(K,\beta)$ as a function of $\beta \in [\alpha,+\infty)$ is uniformly bounded on any bounded interval.

PROOF. Let us first prove the statement in the case $\beta = \alpha$, that is, when f is α-harmonic. By Corollary 7.3, we have $h \in C^{\infty}$. Consider a new measure $\widetilde{\mu}$ on M defined by

$$d\widetilde{\mu} = h^2 d\mu.$$

By (9.26), we have, for any smooth function function u on M,

$$\Delta_{\widetilde{\mu}} u = \frac{1}{h}\Delta_{\mu}(hu) - \alpha u. \tag{13.31}$$

Setting $u = f/h$, we obtain

$$\Delta_{\widetilde{\mu}} u = \frac{1}{h}\Delta_{\mu} f - \frac{1}{h}\alpha f = 0.$$

Hence, u is a non-negative harmonic function on the weighted manifold $(M, \mathrm{g}, \widetilde{\mu})$. By Theorem 13.10, we have

$$\sup_K u \leq C_K \inf_K u,$$

for some constant C_K. Then (13.30) holds with the constant

$$C = C_{K,\alpha} := C_K \frac{\sup_K h}{\inf_K h}.$$

To handle the case $\beta > \alpha$, fix a compact $K \subset M$, a relatively compact connected open set Ω containing K, and construct a positive β-harmonic function h_{β} on Ω. Consider the weak Dirichlet problem on the weighted manifold $(\Omega, \mathrm{g}, \widetilde{\mu})$:

$$\begin{cases} -\Delta_{\widetilde{\mu}} u + (\beta - \alpha) u = 0, \\ u = 1 \mod W_0^1(\Omega), \end{cases} \tag{13.32}$$

which has a unique solution $u \in W^1(\Omega)$ by Exercise 4.29. Since the constant function $v \equiv 1$ satisfies the inequality

$$-\Delta_{\widetilde{\mu}} v + (\beta - \alpha) v \geq 0,$$

we obtain by the comparison principle of Corollary 5.14 that $u \leq 1$. Similarly, we have $u \geq 0$. Moreover, by the strong minimum principle (cf. Corollary 8.14), we conclude that $u > 0$ in Ω.

Observe that the function u decreases when the parameter β increases. Indeed, if function u' solves the problem (13.32) with β' instead of β and if $\beta' > \beta$ then

$$-\Delta_{\widetilde{\mu}} u' + (\beta - \alpha) u' \leq 0,$$

which implies by Corollary 5.14 that $u' \leq u$.

The function $h_\beta := hu$ is positive in Ω and is β-harmonic in $(\Omega, \mathbf{g}, \mu)$, which easily follows from (13.31). By the first part of the proof, we conclude that, for any positive β-harmonic function f in Ω, (13.30) holds with the constant

$$C = C_{K,\beta} := C_K \frac{\sup_K h_\beta}{\inf_K h_\beta}.$$

We are left to verify that $C_{K,\beta}$ is uniformly bounded from above if β is bounded. By the monotonicity of u in β mentioned above, we have that h_β decreases when β increases. Therefore, if β varies in an interval $[\beta_1, \beta_2]$ where $\beta_1 < \beta_2$ then

$$C_{K,\beta} \leq C_k \frac{\sup_K h_{\beta_1}}{\inf_K h_{\beta_2}},$$

whence the uniform boundedness of $C_{K,\beta}$ follows. □

Exercises.

13.22. Prove the classical Harnack inequality: if $f(x)$ is a positive harmonic function in a ball $B(x,r)$ in $\mathbb{R}^n$ then

$$\sup_{B(x,r/2)} f \leq C_n \inf_{B(x,r/2)} f, \tag{13.33}$$

where the constant C_n depends only on n.

13.23. (*The Liouville theorem*) Prove that any positive harmonic function in $\mathbb{R}^n$ is identical constant.

13.4. Convergence of sequences of α-harmonic functions

THEOREM 13.12. (The compactness principle) *Let $\{u_k\}$ be a sequence of non-negative harmonic functions on a connected weighted manifold M. If the sequence $\{u_k(x)\}$ is bounded at some point $x \in M$ then there is a subsequence $\{u_{k_i}\}$ that converges to a harmonic function u on M in the sense of $C^\infty(M)$.*

PROOF. By the Harnack inequality of Theorem 13.10, if the sequence $\{u_k(x)\}$ is bounded at some point x then it is uniformly bounded in any compact set $K \ni x$. Hence, the sequence $\{u_k\}$ is locally uniformly bounded on M. In particular, this sequence is uniformly bounded in $L^2(\Omega)$ for any relatively compact open set $\Omega \subset M$. Using Exercise 7.9 and $\Delta_\mu u_k = 0$, we conclude that the sequence $\{u_k\}$ is uniformly bounded in $W^1(\Omega)$, also for any relatively compact open set $\Omega \subset M$. By Exercise 10.25, there is a subsequence $\{u_{k_i}\}$ that converges in $L^2_{loc}(M)$. By Exercise 7.11, the limit function u of the sequence $\{u_{k_i}\}$ is also harmonic and $u_{k_i} \xrightarrow{C^\infty} u$. □

COROLLARY 13.13. (Harnack's principle) *Let $\{u_k\}$ be a monotone sequence of harmonic functions on a connected weighted manifold M. If $\lim_{k\to\infty} u_k(x)$ is finite at some point $x \in M$ then it is finite at all points $x \in M$. Moreover, the function*

$$u(x) = \lim_{k\to\infty} u_k(x)$$

is harmonic and

$$u_k \overset{C^\infty(M)}{\longrightarrow} u \text{ as } k \to \infty.$$

PROOF. Assume for certainty that $\{u_k\}$ is monotone increasing. Replacing u_k by $u_k - u_1$, we can assume that $u_k \geq 0$. By Theorem 13.12, there is a subsequence $\{u_{k_i}\}$ that converges locally uniformly. Since the sequence $\{u_k\}$ is monotone increasing, the entire sequence $\{u_k\}$ must converge locally uniformly as well. Then the convergence is in C^∞ by Exercise 7.11. □

The following theorem extends the compactness principle to α-harmonic functions.

THEOREM 13.14. *Let $\{u_k\}_{k=1}^\infty$ be a sequence of non-negative functions on a connected weighted manifold M such that u_k is α_k-harmonic for some real α_k. Assume that the sequence $\{\alpha_k\}$ is bounded and $\{u_k(x)\}$ is bounded for some $x \in M$. Then there is a subsequence $\{u_{k_i}\}$ that converges to an α-harmonic function u on M in the sense of $C^\infty(M)$, for some real α.*

PROOF. Passing to a convergent subsequence of $\{\alpha_k\}$, we can assume that $\{\alpha_k\}$ converges and set $\alpha = \lim_{k\to\infty} \alpha_k$. Fix a connected relatively compact open subset $\Omega \subset M$ such that $\lambda_{\min}(\Omega) \geq \sup_k(-\alpha_k)$, that is,

$$\alpha_k \geq -\lambda_{\min}(\Omega) \text{ for all } k \geq 1 \tag{13.34}$$

(the question of existence of such sets will be addressed below). Set $\alpha_0 = -\lambda_{\min}(\Omega)$ and observe that, by Theorem 10.11, there is a positive α_0-harmonic function on Ω (namely, the first eigenfunction of $\mathcal{L}^\Omega$). Therefore, by Theorem 13.11, we conclude that the Harnack inequality (13.30) holds for any compact $K \subset \Omega$ and any positive β-harmonic function f in Ω. Moreover, the constant C in (13.30) can be taken to be the same for any bounded range of β. In particular, if β takes only the values α and α_k, $k = 1, 2, \dots$ then the constant C can be assumed to be the same.

Now we can argue as in the proof of Theorem 13.12. By the Harnack inequality (13.30) if the sequence $\{u_k(x)\}$ is bounded at some point $x \in \Omega$ then it is uniformly bounded in any compact set $K \ni x$. In particular, the sequence $\{u_k\}$ is uniformly bounded in $L^2(\Omega')$ for any relatively compact open set $\Omega' \Subset \Omega$. Using Exercise 7.9 and $\Delta_\mu u_k = \alpha_k u_k$, we conclude that the sequence $\{u_k\}$ is uniformly bounded in $W^1(\Omega')$, also for any relatively compact open set $\Omega' \Subset \Omega$. By Exercise 10.25, there is a subsequence $\{u_{k_i}\}$ that converges in $L^2_{loc}(\Omega)$. By Exercise 7.12, the limit function u of the sequence $\{u_{k_i}\}$ is α-harmonic and $u_{k_i} \overset{C^\infty(\Omega)}{\longrightarrow} u$.

To ensure the convergence of $\{u_{k_i}\}$ on M, let us observe that, for any point $x \in M$, there is a connected relatively compact open subset $\Omega \subset M$ such that $x \in \Omega$ and $\lambda_{\min}(\Omega)$ is arbitrarily large. Indeed, choose first a chart containing x and then take Ω to be a little Euclidean ball in this chart centered at x. By Exercise 11.25, the bottom eigenvalue of Ω in the Euclidean metric is $c_n r^{-2}$ where c_n is a positive constant depending only on the dimension $n = \dim M$ and r is the radius of the ball. Taking r small enough, we can get $c_n r^{-2}$ arbitrarily large. At the same time, in a small neighborhood of x, the ratio of the Riemannian metric $\mathbf{g}$ and the Euclidean metric remains uniformly bounded, and so is the ratio of the measure μ and the Lebesgue measure. This implies by Exercise 10.7 that the eigenvalue $\lambda_{\min}(\Omega, \mathbf{g}, \mu)$ is also large enough.

Hence, for any point $x \in M$ there is a set Ω as above and such that (13.34) is satisfied. Choose a cover of M be a countable sequence $\{\Omega_j\}_{j=1}^\infty$ of such sets and order then so that any two consecutive sets overlap, which is possible by the connectedness of M. We can assume also that Ω_1 contains the point x where the sequence $\{u_k(x)\}$ is bounded. Then by the above argument there is a subsequence that converges in Ω_1. Since this subsequence converges in $\Omega_1 \cap \Omega_2$, there is a sub-subsequence that converges in Ω_2, etc. Applying the diagonal process, we obtain finally a subsequence that converges on M. $\square$

Exercises.

13.24. Let M be a connected weighted manifold. Prove that if $g(x,y) < \infty$ for some couple $x, y \in M$ then $g(x,y)$ is finite, that is, $g(x,y) < \infty$ for all distinct points $x, y \in M$.

REMARK. Hence, the following dichotomy takes places: either $g(x,y) \equiv \infty$ for all $x, y \in M$ or $g(x,y) < \infty$ for all distinct $x, y \in M$.

13.5. The positive spectrum

DEFINITION 13.15. The *positive spectrum* of (the Laplace operator on) a weighted manifold M is the set of all real α such that the equation

$$\Delta_\mu u + \alpha u = 0 \tag{13.35}$$

has a positive solution u on M.

THEOREM 13.16. *For any connected weighted manifold M, the positive spectrum coincides with the interval $(-\infty, \lambda_{\min}(M)]$.*

In particular, there is always a positive solution on M of the equation

$$\Delta_\mu u + \lambda_{\min}(M)\, u = 0,$$

which is called the *ground state* of the manifold M. For comparison, let us recall that, by the definition of $\lambda_{\min}(M)$, the spectrum of the Dirichlet Laplace operator is contained in $[\lambda_{\min}(M), +\infty)$. Hence, $\lambda_{\min}(M)$ is the only common point of the L^2-spectrum and the positive spectrum of the Laplacian.

In terms of α-harmonic functions, Theorem 13.16 can be stated as follows: a positive α-harmonic function exists if and only is $\alpha \geq -\lambda_{\min}(M)$.

PROOF. That any α from the positive spectrum satisfies $\alpha \leq \lambda_{\min}(M)$ follows from Exercise 10.26. We need to prove the converse, that is, if $\alpha \leq \lambda_{\min}(M)$ then there is a positive solution of (13.35) on M. Choose a compact exhaustion sequence $\{\Omega_k\}$ in M such that all Ω_k are connected. Since by Exercise 10.6

$$\lambda_{\min}(\Omega_k) \uparrow \lambda_{\min}(M) \text{ as } k \to \infty,$$

there is a sequence $\{\alpha_k\}$ such that $\alpha_k < \lambda_{\min}(\Omega_k)$ for any k and $\alpha_k \uparrow \alpha$ as $k \to \infty$. By Exercise 4.29, the weak Dirichlet problem in Ω_k

$$\begin{cases} \Delta_\mu u_k + \alpha_k u_k = 0, \\ u_k = 1 \mod W_0^1(\Omega_k), \end{cases}$$

has a unique solution u_k. Moreover, we have $u_k > 0$ by Theorem 5.13 and Corollary 8.14. Alternatively, the solution of this problem is given explicitly by the formula

$$u_k = 1 + \alpha_k \int_0^\infty \left(P_t^{\Omega_k} 1\right) e^{\alpha_k t} dt$$

(see Exercise 10.27).

Select some point $x_0 \in \Omega_1$ and consider functions

$$v_k = \frac{u_k}{u_k(x_0)}$$

so that $v_k(x_0) = 1$. Using Theorem 13.14, we conclude that there is a subsequence $\{v_{k_i}\}$ that converges to function v on M, that satisfies (13.35). This function is clearly non-negative; moreover, since $v(x_0) = 1$, it is strictly positive by Corollary 8.14. $\square$

Exercises.

13.25. Prove the following improved version of (12.18): if f and g are two functions from $L^2(M)$ such that

$$d(\operatorname{supp} f, \operatorname{supp} g) \geq r,$$

where $r \geq 0$, then, for all $t > 0$,

$$|(P_t f, g)| \leq \|f\|_2 \|g\|_2 e^{-\lambda_{\min}(M)t} \int_r^\infty \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{s^2}{4t}\right) ds. \tag{13.36}$$

13.6. Green function as a fundamental solution

THEOREM 13.17. *Let M be a connected weighted manifold and x_0 be a point of M.*

(a) *If the Green function g of M is finite, then $g(x_0,\cdot)$ is a positive fundamental solution of the Laplace operator at x_0.*

(b) *If $h(x)$ is a positive fundamental solution at x_0 then $g(x_0,x) \leq h(x)$ for all $x \neq x_0$.*

This can be equivalently stated as follows: $g(x_0,\cdot)$ is the infimum of all positive fundamental solutions at x_0 (using the convention that the infimum of an empty set is ∞).

PROOF. If M is compact then all is settled by Exercise 13.1, which says that $g \equiv \infty$ and there is no fundamental solution. Assume that M is non-compact, and let $\{\Omega_k\}_{k=1}^{\infty}$ be a compact exhaustion sequence in M. Then all Ω_k are relatively compact and, hence, $M \setminus \overline{\Omega_k} \neq \emptyset$. By Theorem 10.22 we have $\lambda_{\min}(\Omega_k) > 0$, and by Theorem 13.4 the weighted manifold Ω_k has the Green function $g^{\Omega_k} \in L^1_{loc}(\Omega_k)$ satisfying the equation

$$-\Delta_\mu g^{\Omega_k} = \delta_{x_0} \text{ in } \Omega_k. \tag{13.37}$$

By Exercise 13.8, the sequence $\left\{g^{\Omega_k}\right\}$ increases and converges pointwise to g as $k \to \infty$.

(a) The identity (13.37) implies that, for all $k > m$,

$$\Delta_\mu \left(g^{\Omega_k} - g^{\Omega_m}\right) = 0 \text{ in } \Omega_m.$$

It follows from Corollary 7.5 that the function $g^{\Omega_k} - g^{\Omega_m}$ is smooth (and hence harmonic) in Ω_m; more precisely, this function being a priori smooth in $\Omega_m \setminus \{x_0\}$, can be extended to x_0 to become $C^\infty(\Omega_m)$. By the Harnack principle of Corollary 13.13, the function

$$g - g^{\Omega_m} = \lim_{k\to\infty} \left(g^{\Omega_k} - g^{\Omega_m}\right)$$

is also harmonic in Ω_m, and the convergence takes places in $C^\infty(\Omega_m)$. Hence, $g \in L^1_{loc}(\Omega_m)$ and g satisfies (13.10) in Ω_m. Letting $m \to \infty$ we conclude that $g \in L^1_{loc}(M)$ and g satisfies (13.10) in M, that is, g is a fundamental solution. The positivity of g follows from Exercise 13.3.

(b) Let us first observe that, for any open set U such that $x_0 \in U \Subset \Omega_k$ (see Fig. 13.5),

$$g^{\Omega_k} \in L^\infty \cup W^1\left(\Omega_k \setminus \overline{U}\right). \tag{13.38}$$

Indeed, we have $g^{\Omega_k} \leq g^{\Omega_{k+1}}$, and the function $g^{\Omega_{k+1}}$ is smooth in $\Omega_{k+1} \setminus \{x_0\}$. It follows that $g^{\Omega_{k+1}}$ is bounded on $\overline{\Omega}_k \setminus U$, whence the boundedness of g^{Ω_k} follows. Next, set

$$C := \sup_{\Omega_k \setminus \overline{U}} g^{\Omega_k}$$

and notice that $g^{\Omega_k} = \min\left(g^{\Omega_k}, C\right)$ in $\Omega_k \setminus \overline{U}$. Since by Corollary 13.6 the function $\min\left(g^{\Omega_k}, C\right)$ belongs to $W^1(\Omega_k)$, it follows that its restriction to

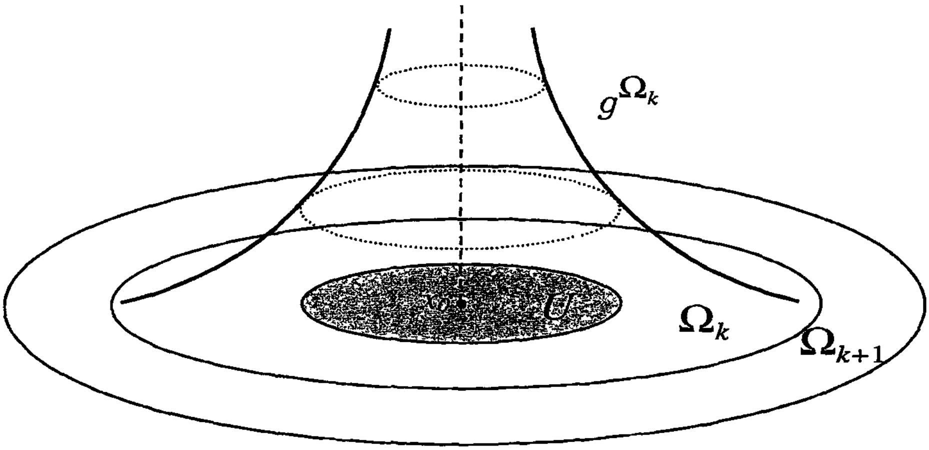

FIGURE 13.5. The Green function g^{Ω_k}

$\Omega_k \setminus \overline{U}$, that is g^{Ω_k}, belongs to $W^1\left(\Omega_k \setminus \overline{U}\right)$, which finishes the proof of (13.38).

Let h be a positive fundamental solution at x_0, that is, $-\Delta_\mu h = \delta_{x_0}$. We need to verify that $h \geq g$, and it suffices to show that $h \geq g^{\Omega_k}$ for all k. For any index k consider a function

$$u_k = g^{\Omega_k} - h.$$

Clearly, $\Delta_\mu u_k = 0$ on Ω_k so that u_k is a smooth harmonic function in Ω_k. We need to prove that $u_k \leq 0$, and we will do it in four steps.

1. The function u_k belongs to $W^1(\Omega_k)$, that is, u_k and $|\nabla u_k|$ belong to $L^2(\Omega_k)$. Indeed, let U be an open set as above. Then $u_k, |\nabla u_k| \in L^2(U)$ just by the smoothness of u_k in Ω_k, while $u_k, |\nabla u_k| \in L^2\left(\Omega_k \setminus \overline{U}\right)$ because both functions g^{Ω_k} and h belong to $W^1\left(\Omega_k \setminus \overline{U}\right)$.

2. The function u_k is bounded from above in Ω_k. Indeed, u_k is bounded in $\overline{U}$ by continuity, and is bounded in $\Omega_k \setminus \overline{U}$ because $u_k \leq g^{\Omega_k}$ and g^{Ω_k} is bounded in $\Omega_k \setminus \overline{U}$.

3. We have

$$u_k \leq 0 \mod W_0^1(\Omega_k). \tag{13.39}$$

Set $C = \sup_{\Omega_k} u_k$. Then the following inequality holds in Ω_k:

$$u_k \leq \min\left(g^{\Omega_k}, C\right) =: v_k.$$

By Corollary 13.6, the function v_k belongs to $W_0^1(\Omega_k)$, whence (13.39) follows.

4. Since the function u_k belongs to $W^1(\Omega_k)$, satisfies in Ω_k the Laplace equation $\Delta_\mu u_k = 0$ and the boundary condition (13.39), we conclude by Theorem 5.13 that $u_k \leq 0$. $\square$

Exercises.

13.26. Let M be a connected non-compact manifold and Ω be a relatively compact open subset of M.

(a) Prove that, for any $p \in [1, +\infty]$, G^Ω is a bounded operator from $L^p(\Omega)$ to $L^p(\Omega)$.

(b) Prove that the function $u = G^\Omega f$ satisfies the equation $-\Delta_\mu u = f$ for any $f \in L^p(\Omega)$.

13.27. Let M be a connected weighted manifold and let $f \in L^1_{loc}(M)$ and $f \geq 0$. Prove that if $Gf(x)$ is finite then Gf belongs to L^1_{loc} and $-\Delta_\mu (Gf) = f$.

13.28. Let M be a connected weighted manifold with a finite Green function $g(x,y)$. Fix a point $x_0 \in M$ and a compact set $K \subset M$. Prove that if u is a harmonic function on M and

$$u(x) \leq g(x, x_0) \text{ for all } x \in M \setminus K,$$

then $u(x) \leq 0$ for all $x \in M$.

13.29. Let M be a connected weighted manifold. Prove that if $h(x)$ is a fundamental solution of the Laplace operator at a point $x_0 \in M$ such that $h(x) \to 0$ as $x \to \infty$, then $h(x) = g(x, x_0)$.

13.30. Prove that, on an arbitrary connected weighted manifold M, the following conditions are equivalent:

(i) the Green function is finite;

(ii) there exists a positive non-constant superharmonic function (that is, M is non-parabolic);

(iii) there exists a positive non-constant superaveraging function.

13.31. Let M be a connected weighted manifold and Ω be a non-empty relatively compact open subset of M such that $M \setminus \overline{\Omega}$ is non-empty. Prove that, for all $x \in M$, $y \in \Omega$,

$$g(x,y) \leq g^\Omega(x,y) + \sup_{z \in \partial\Omega} g(z,y). \tag{13.40}$$

Here we set $g^\Omega(x,y) = 0$ if $x \notin \Omega$ or $y \notin \Omega$.

13.32. Prove that a fundamental solution of the Laplace operator exists on any non-compact connected weighted manifold.

13.33. Prove that if, for some $x \in M$ and a compact set $K \subset M$,

$$\int_{M \setminus K} g(x,y)\, d\mu(y) < \infty \tag{13.41}$$

then M is stochastically incomplete.

13.34. Let M be a weighted model of dimension $n \geq 2$, and $S(r)$ be its boundary area function (cf. Section 3.10). Prove that the Green function of the central ball B_R satisfies the identity

$$g^{B_R}(x, o) = \int_r^R \frac{ds}{S(s)}, \tag{13.42}$$

where $r = |x|$. Deduce that the Green function of M satisfies the identity

$$g(x, o) = \int_r^\infty \frac{ds}{S(s)}. \tag{13.43}$$

Hence or otherwise give an example of a complete manifold M where the Green function belongs to $L^1(M)$.

13.35. Prove that the Green function of the ball $B = B_R(0)$ in $\mathbb{R}^n$ is given by the following formulas, for all $x, y \in B$:

(a) If $n > 2$ then

$$g^B(x,y) = \frac{1}{\omega_n(n-2)}\left(\frac{1}{|x-y|^{n-2}} - \left(\frac{R}{|y|}\right)^{n-2}\frac{1}{|x-y_*|^{n-2}}\right), \tag{13.44}$$

where y^* is the inversion of y with respect to the ball B, that is

$$y^* = \frac{y}{|y|^2}R^2.$$

(b) If $n = 2$ then

$$g^B(x,y) = \frac{1}{2\pi}\log\frac{|x-y^*|\,|y|}{|x-y|\,R}.$$

(c) If $n = 1$ then

$$g^B(x,y) = \frac{1}{2}|x-y| - \frac{1}{2R}xy + \frac{R}{2}.$$

13.36. Let $F(t)$ be a positive monotone increasing function on $\mathbb{R}_+$ and assume that

$$p_t(x,y) \le \frac{1}{F(\sqrt{t})}\exp\left(-c\frac{r^2}{t}\right)$$

for some $x, y \in M$ and all $t > 0$, where $r = d(x,y)$ and $c > 0$. Prove that if F satisfies the doubling property

$$F(2s) \le AF(s) \quad \text{for all } s > 0, \tag{13.45}$$

then

$$g(x,y) \le C\int_r^\infty \frac{sds}{F(s)}, \tag{13.46}$$

where $C = C(A,c)$.

If in addition F satisfies the condition

$$\frac{F(s)}{F(s')} \ge a\left(\frac{s}{s'}\right)^\alpha, \quad \text{for all } s > s' > 0, \tag{13.47}$$

where $a > 0$ and $\alpha > 2$ then

$$g(x,y) \le C\frac{r^2}{F(r)}, \tag{13.48}$$

where $C = C(A,a,\alpha,c)$.

Notes

The present account of the Green function is somewhat different from the traditional approach (cf. [155]). Some proofs would have been simpler, had we used the fact that the Green function $g^\Omega(x,y)$ in a relatively compact open set Ω with smooth boundary vanishes at every point $x \in \partial\Omega$ while $y \in \Omega$. For example, the proof of the minimality of g in Theorem 13.17(b) would be as short as this: since $h(x) - g^\Omega(x,x_0)$ is a harmonic function in Ω that takes non-negative value on $\partial\Omega$, by the classical maximum principle this function is non-negative in Ω, that is, $h(x) \ge g^\Omega(x,x_0)$; letting $\Omega \to M$, we obtain $h(x) \ge g(x,x_0)$.

However, following the general approach adopted in this book, we avoid using the boundary regularity of solutions and employ instead other methods, based on the Sobolev space $W_0^1(\Omega)$. Despite of technical complications, we feel that this strategy has good prospects for the future applications in more singular settings.

The idea to use the Green function for the proof of the local Harnack inequality (Section 13.3) goes back to A.Boukricha [46] and W.Hansen [197]. However, the present implementation of this idea is entirely new. This approach allows us to avoid at this stage the technically involved proofs of the *uniform* Harnack inequalities, although at expense of loosing the uniformity of the Harnack constant. However, the local Harnack inequality is sufficient to prove the convergence properties of sequences of harmonic functions as we do in Section 13.4.

The treatment of the positive spectrum in Section 13.5 follows S.-T.Yau [**362**] and D.Sullivan [**340**]. We wanted necessarily to demonstrate how the convergence properties allow to prove the existence of the ground state. The latter is an important tool that is used in many applications (cf. Section 9.2.5).

CHAPTER 14

Ultracontractive estimates and eigenvalues

In the Chapter we study the problem of obtaining the uniform on-diagonal upper bounds of the heat kernel of the form

$$p_t(x,x) \le \frac{1}{\gamma(t)} \tag{14.1}$$

with some increasing function $\gamma(t)$. If $\lambda = \lambda_{\min}(M) > 0$ then by Exercise 10.29 the heat kernel $p_t(x,y)$ decays as $\exp(-\lambda t)$ when $t \to \infty$. However, if $\lambda = 0$ then we do not get any decay of $p_t(x,y)$ from the spectral theory, and more subtle methods are required. As we will see below, the function $\gamma(t)$ in (14.1) can be determined by a lower bound of $\lambda_{\min}(\Omega)$ via $\mu(\Omega)$, which shows the rate of $\lambda_{\min}(\Omega)$ approaching 0 when Ω exhausts M.

14.1. Ultracontractivity and heat kernel bounds

By Theorems 4.9 and 7.19 the heat semigroup $\{P_t\}$ on any weighted manifold $(M, \mathbf{g}, \mu)$ admits the estimates

$$\|P_t\|_{2\to 2} \le 1, \quad \|P_t\|_{1\to 1} \le 1,$$

so that P_t is a contraction in L^2 and L^1. In fact, by Exercises 7.33 and 7.36,

$$\|P_t\|_{r\to r} \le 1$$

for any $r \in [1, +\infty]$. Here we consider some estimates of $\|P_t\|_{p\to q}$ with $p < q$.

Definition 14.1. Let $1 \le p < q \le +\infty$. We say that the semigroup $\{P_t\}$ is $L^p \to L^q$ *ultracontractive* if there exists a positive function $\theta(t)$ on $(0, +\infty)$ such that, for all $f \in L^p \cap L^2$ and $t > 0$, we have $P_t f \in L^q$ and

$$\|P_t f\|_q \le \theta(t) \|f\|_p.$$

We write in this case

$$\|P_t\|_{p\to q} \le \theta(t).$$

The function θ is called the *rate function* of ultracontractivity.

For any $r \in [1, +\infty]$, denote by r^* its Hölder conjugate, that is

$$\frac{1}{r} + \frac{1}{r^*} = 1.$$

For example, $2^* = 2$, $1^* = +\infty$, and $+\infty^* = 1$.

Theorem 14.2. *Let the heat semigroup $\{P_t\}$ be $L^p \to L^q$ ultracontractive with the rate function $\theta(t)$. Then $\{P_t\}$ is also $L^{q^*} \to L^{p^*}$ ultracontractive with the same rate function.*

PROOF. By the hypothesis, we have for any $g \in L^p \cap L^2$ and $t > 0$

$$\|P_t g\|_q \le \theta(t)\|g\|_p.$$

Then for any $f \in L^{q^*} \cap L^2$ we obtain by the Hölder inequality

$$(P_t f, g) = (f, P_t g) \le \|f\|_{q^*}\|P_t g\|_q \le \theta(t)\|f\|_{q^*}\|g\|_p.$$

Therefore,

$$\|P_t f\|_{p^*} = \sup_{g \in L^p \cap L^2 \setminus \{0\}} \frac{(P_t f, g)}{\|g\|_p} \le \theta(t)\|f\|_{q^*},$$

whence the claim follows. □

COROLLARY 14.3. *The semigroup $\{P_t\}$ is $L^1 \to L^2$ ultracontractive if and only if it is $L^2 \to L^\infty$ ultracontractive, with the same rate function.*

The following statement elucidates the importance of the notion of ultracontractivity.

THEOREM 14.4. *The heat semigroup $\{P_t\}$ is $L^1 \to L^2$ ultracontractive with the rate function $\theta(t)$ if and only if the heat kernel satisfies estimate*

$$p_{2t}(x,x) \le \theta^2(t), \tag{14.2}$$

for all $t > 0$ and $x \in M$.

PROOF. By Theorem 14.2, the hypothesis that $\{P_t\}$ is $L^1 \to L^2$ ultracontractive with the rate function $\theta(t)$ is equivalent to the fact that $L^2 \to L^\infty$ is ultracontractive with the same rate function; that is, for all $f \in L^2$ and $t > 0$

$$\|P_t f\|_\infty \le \theta(t)\|f\|_2. \tag{14.3}$$

Substitute in (14.3) $f = p_t(x,\cdot)$ for some fixed $t > 0$ and $x \in M$. Then, using the properties of the heat kernel from Theorem 7.13, we obtain

$$P_t f(x) = \int_M p_t(x,z)\, p_t(x,z)\, d\mu(z) = p_{2t}(x,x)$$

and

$$\|f\|_2^2 = p_{2t}(x,x),$$

whence by (14.3)

$$p_{2t}(x,x) \le \theta(t)\sqrt{p_{2t}(x,x)},$$

which proves (14.2).

Conversely, if the heat kernel satisfies (14.2) then, for all $t > 0$ and $x \in M$,

$$\begin{aligned} |P_t f(x)| &= \left|\int_M p_t(x,y) f(y) d\mu(y)\right| \\ &\le \left(\int_M p_t^2(x,y)\, d\mu(y)\right)^{1/2} \|f\|_2 \\ &= p_{2t}(x,x)^{1/2}\|f\|_2 \end{aligned}$$

whence

$$\|P_t f(x)\|_\infty \le \theta(t)\|f\|_2,$$

which proves (14.3). $\square$

REMARK 14.5. Using the inequality

$$p_t(x,y) \le \sqrt{p_t(x,x)\, p_t(y,y)}$$

(see Exercise 7.21), we obtain the following version of Theorem 14.4: the heat semigroup $\{P_t\}$ is $L^1 \to L^2$ ultracontractive with the rate function $\theta(t)$ if and only if the heat kernel satisfies estimate

$$p_{2t}(x,y) \le \theta^2(t),$$

for all $t > 0$ and $x, y \in M$.

Exercises.

14.1. Prove that if the heat semigroup $\{P_t\}$ is $L^p \to L^2$ ultracontractive with the rate function $\theta(t)$ where $1 \le p < 2$ then $\{P_t\}$ is also $L^p \to L^{p^*}$ ultracontractive with the rate function $\theta^2(t/2)$.

14.2. Faber-Krahn inequalities

Given a non-negative non-increasing function Λ on $(0, +\infty)$, we say that a weighted manifold $(M, \mathbf{g}, \mu)$ satisfies the *Faber-Krahn inequality* with function Λ if, for any non-empty relatively compact open set $\Omega \subset M$,

$$\lambda_{\min}(\Omega) \ge \Lambda(\mu(\Omega)). \tag{14.4}$$

Of course, since the spectrum of the Dirichlet Laplacian $\mathcal{L}^\Omega$ is discrete in Ω, we can replace here $\lambda_{\min}(\Omega)$ by $\lambda_1(\Omega)$. However, for most applications we do not need to use the fact that $\lambda_{\min}(\Omega)$ is an eigenvalue.

If Ω is an open subset of $\mathbb{R}^n$ then, by the *Faber-Krahn theorem*,

$$\lambda_1(\Omega) \ge \lambda_1(\Omega^*),$$

where Ω^* is a ball of the same volume as Ω. If the radius of Ω^* is r then $\lambda_1(\Omega^*) = \frac{c_n}{r^2}$ with some positive constant c_n (see Exercise 11.25). Since by (3.90)

$$\mu(\Omega) = \mu(\Omega^*) = \frac{\omega_n}{n} r^n,$$

it follows that

$$\lambda_1(\Omega) \ge a\mu(\Omega)^{-2/n}, \tag{14.5}$$

where $a = a(n) > 0$. Therefore, (14.4) holds with the function $\Lambda(v) = av^{-2/n}$.

An alternative proof of (14.5) (although with a non-sharp constant a) can be found below in Example 14.31.

REMARK 14.6. It is known that the Faber-Krahn inequality (14.5) with some constant $a > 0$ holds on the following two classes of n-dimensional Riemannian manifolds:

(1) Cartan-Hadamard manifolds, that is, complete simply connected manifolds of non-positive sectional curvature. This class includes, in particular, $\mathbb{R}^n$ and $\mathbb{H}^n$.
(2) Minimal submanifolds of $\mathbb{R}^N$.

See the Notes at the end of this Chapter for bibliographic references.

Exercises.

14.2. Prove that if (14.4) holds for all relatively compact open sets Ω then it holds also for all open sets Ω with $\mu(\Omega) < \infty$.

14.3. The Nash inequality

LEMMA 14.7. (The generalized Nash inequality) *Let (M, g, μ) a weighted manifold satisfying the Faber-Krahn inequality with a function $\Lambda : (0, +\infty) \to [0, +\infty)$ that is monotone decreasing and right continuous. Then, for any $0 < \varepsilon < 1$ and for any function $u \in L^1 \cap W_0^1(M) \setminus \{0\}$, the following inequality holds*

$$\int_M |\nabla u|^2 \, d\mu \geq (1 - \varepsilon) \|u\|_2^2 \Lambda\left(\frac{2}{\varepsilon} \frac{\|u\|_1^2}{\|u\|_2^2}\right). \tag{14.6}$$

For example, for the function

$$\Lambda(v) = a v^{-2/n}$$

we obtain from (14.6)

$$\int_M |\nabla u|^2 \, d\mu \geq c \left(\int_M |u| \, d\mu\right)^{-4/n} \left(\int_M u^2 d\mu\right)^{1+2/n}, \tag{14.7}$$

where $c = c(a, n) > 0$. In particular, (14.7) holds in $\mathbb{R}^n$ where it is referred to as the (classical) *Nash inequality*.

PROOF. It suffices to consider non-negative u since by (5.13) $|\nabla u| = |\nabla |u||$. Consider first the case when u is in addition continuous. For any $s > 0$, consider the open set

$$\Omega_s = \{x \in M : u(x) > s\}$$

and observe by Exercise 5.22 $(u - s)_+ \in W_0^1(\Omega_s)$, and by (5.12)

$$\int_{\Omega_s} |\nabla (u - s)_+|^2 \, d\mu \leq \int_M |\nabla u|^2 \, d\mu. \tag{14.8}$$

By Theorem 10.8, we have

$$\lambda_{\min}(\Omega_s) \int_{\Omega_s} (u - s)_+^2 \, d\mu \leq \int_{\Omega_s} |\nabla (u - s)_+|^2 \, d\mu. \tag{14.9}$$

Set for simplicity

$$A = \|u\|_1 \quad \text{and} \quad B = \|u\|_2^2.$$

Integrating the obvious inequality,

$$u^2 - 2su \le (u-s)_+^2,$$

we obtain

$$B - 2sA \le \int_M (u-s)_+^2 d\mu, \tag{14.10}$$

which together with (14.8) and (14.9) yields

$$\lambda_{\min}(\Omega_s)(B - 2sA) \le \int_M |\nabla u|^2 d\mu. \tag{14.11}$$

On the other hand, by the definition of Ω, we have

$$\mu(\Omega_s) \le \frac{1}{s}\int_M u\, d\mu = \frac{1}{s}A,$$

whence by the Faber-Krahn inequality

$$\lambda_{\min}(\Omega_s) \ge \Lambda(\mu(\Omega_s)) \ge \Lambda(\frac{1}{s}A), \tag{14.12}$$

where we have used the hypotheses that Λ is monotone decreasing. Combining (14.11) and (14.12), we obtain

$$\Lambda(\frac{1}{s}A)(B - 2sA) \le \int_M |\nabla u|^2 d\mu,$$

whence (14.6) follows upon setting here $s = \frac{\varepsilon B}{2A}$.

To treat the general case $u \in L^1 \cap W_0^1(M)$, we will use the following observation.

CLAIM. *If $\{w_k\}$ is a sequence of functions from $L^1 \cap W_0^1(M)$ such that*

$$\|w_k - u\|_1 \to 0, \quad \|w_k - u\|_2 \to 0, \quad \|\nabla w_k - \nabla u\|_2 \to 0, \tag{14.13}$$

as $k \to \infty$ and if (14.6) *holds for each function w_k then* (14.6) *holds also for u.*

Indeed, it follows from the hypotheses that the function Λ is lower semicontinuous that is, for any convergent sequence $\{r_k\}$ of positive reals,

$$\liminf_{k\to\infty} \Lambda(r_k) \ge \Lambda(\lim_{k\to\infty} r_k). \tag{14.14}$$

Hence, using (14.13), we can pass to the limit in the inequality (14.6) for w_k and obtain (14.6) for u.

Consider now the case when u is a non-negative function from $W_c^1(M)$. Let Ω be a relatively compact open neighborhood of $\operatorname{supp} u$. Since by Lemma 5.5 $u \in W_0^1(\Omega)$, and $C_0^\infty(\Omega)$ is dense in $W_0^1(\Omega)$, there exists a sequence $\{u_k\} \subset C_0^\infty(M)$ such that

$$\|u_k - u\|_2 \to 0 \quad \text{and} \quad \|\nabla u_k - \nabla u\|_2 \to 0. \tag{14.15}$$

By the Cauchy-Schwarz inequality, we have $u, u_k \in L^1(\Omega)$ and

$$\|u_k - u\|_1 \le \sqrt{\mu(\Omega)}\|u_k - u\|_2 \to 0. \tag{14.16}$$

Since (14.6) holds for each function u_k by the first part of the proof, the above Claim applies, and we obtain (14.6) for function u.

Finally, let u be an arbitrary non-negative function from $L^1 \cap W_0^1(M)$. As above, there is a sequence $\{u_k\} \subset C_0^\infty(M)$ such that (14.15) holds. By Lemma 5.4, we can assume that $u_k \geq 0$. Let Ω_k be any relatively compact open set that contains $\operatorname{supp} u_k$. Consider the function

$$w_k := \min(u, u_k) = u - (u - u_k)_+$$

and observe that $w_k \in W_0^1(M)$ (cf. Example 5.3 and Exercise 5.3). Since $\operatorname{supp} w_k \subset \operatorname{supp} u_k$, we conclude by the previous part of the proof that (14.6) holds for the function w_k. We are left to prove that a subsequence of $\{w_k\}$ satisfies (14.13). We have

$$\begin{aligned}
\|\nabla w_k - \nabla u\|_2^2 &= \left(\int_{\{u_k \leq u\}} + \int_{\{u_k > u\}}\right) |\nabla (w_k - u)|^2 \, d\mu \\
&= \int_{\{u_k \leq u\}} |\nabla (u_k - u)|^2 \, d\mu \\
&\leq \|\nabla u_k - \nabla u\|_2^2,
\end{aligned}$$

because on the set $\{u_k > u\}$ we have $w_k = u$ and, hence, $\nabla(w_k - u) = 0$ (cf. (5.11)) and on the set $\{u_k \leq u\}$ we have $w_k = u_k$ and, hence, $\nabla(w_k - u) = \nabla(u_k - u)$. It follows that

$$\|\nabla w_k - \nabla u\|_2 \to 0 \text{ as } k \to \infty,$$

and similarly one proves that

$$\|w_k - u\|_2 \to 0 \text{ as } k \to \infty.$$

Therefore, there is a subsequence $\{w_{k_i}\}$ such that $w_{k_i} \to u$ almost everywhere. Since $0 \leq w_{k_i} \leq u$ and $u \in L^1$, the dominated convergence theorem yields

$$\|w_{k_i} - u\|_1 \to 0 \text{ as } k \to \infty.$$

Hence, the subsequence $\{w_{k_i}\}$ satisfies all the conditions of the Claim, which finishes the proof. □

Exercises.

14.3. Assume that the following Nash inequality holds:

$$\int_M |\nabla u|^2 \, d\mu \geq \|u\|_2^2 \Lambda\left(\frac{\|u\|_1^2}{\|u\|_2^2}\right),$$

for any non-zero function $u \in C_0^\infty(M)$, where Λ is a decreasing function on $[0, +\infty)$. Prove the Faber-Krahn inequality

$$\lambda_{\min}(\Omega) \geq \Lambda(\mu(\Omega)),$$

for any open set $\Omega \subset M$ with finite measure.

14.4. Give an example of a manifold where the Faber-Krahn inequality can holds only with function $\Lambda(v) \equiv 0$.

14.5. Prove that the Faber-Krahn inequality with function

$$\Lambda(v) = av^{-2/\nu} \tag{14.17}$$

where a and ν are positive constants, implies that, for any relatively compact ball $B(x,r)$,

$$\mu(B(x,r)) \geq ca^{\nu/2}r^{\nu}, \tag{14.18}$$

where $c = c(\nu) > 0$.

HINT. First prove that

$$\mu(B(x,r)) \geq c\left(ar^2\right)^{\frac{\nu}{\nu+2}} \mu(B(x,r/2))^{\frac{\nu}{\nu+2}}$$

and then iterate this inequality.

14.6. Prove that the Faber-Krahn inequality with function (14.17) with $\nu > 2$ is equivalent to the Sobolev inequality:

$$\int_M |\nabla u|^2 \, d\mu \geq c\left(\int_M |u|^{\frac{2\nu}{\nu-2}} \, d\mu\right)^{\frac{\nu-2}{\nu}} \tag{14.19}$$

for any $u \in W_0^1(M)$, where $c = c(a,\nu) > 0$.

14.7. Prove that the Sobolev inequality (14.19) implies the following inequality, for any $u \in C_0^\infty(M)$:

$$\int_M |\nabla u|^2 \, d\mu \geq c\left(\int_M |u|^{\alpha} \, d\mu\right)^{-a} \left(\int_M |u|^{\beta} \, d\mu\right)^{b} \tag{14.20}$$

for any set of positive reals α, β, a, b that satisfy the following conditions:

$$\alpha < \beta < \frac{2\nu}{\nu-2} \tag{14.21}$$

and

$$\begin{cases} b - a = 1 - \frac{2}{\nu}, \\ \beta b - \alpha a = 2. \end{cases} \tag{14.22}$$

REMARK. Under the conditions (14.21), the numbers a, b solving (14.22) always exist and are positive. For example, if $\alpha = 1$ and $\beta = 2$ then $a = 4/\nu$ and $b = 1 + 2/\nu$, so that (14.20) coincides with the Nash inequality (14.7). If $\alpha = 2$ and $\beta = 2 + 4/\nu$ then $a = 2/\nu$ and $b = 1$, and we obtain the *Moser inequality*

$$\int_M |\nabla u|^2 \, d\mu \geq c\left(\int_M |u|^2 \, d\mu\right)^{-2/\nu} \left(\int_M |u|^{2+4/\nu} \, d\mu\right).$$

14.4. The function classes $\mathbf{L}$ and Γ

We consider here a certain ordinary differential equation, which will be used in Section 14.5. The main results are Lemmas 14.10 and 14.18. This section can be skipped at first reading and be consulted in case of need.

DEFINITION 14.8. We say that a function $\Lambda : (0, +\infty) \to \mathbb{R}$ belongs to the class $\mathbf{L}$ if

(*i*) Λ is non-negative, monotone decreasing, and right continuous;
(*ii*) Λ is positive in a right neighborhood of 0 and

$$\int_0 \frac{dv}{v\Lambda(v)} < \infty. \tag{14.23}$$

For example, if a function Λ satisfies (i) and $\Lambda(v) = v^{-\alpha}$ $(\alpha > 0)$ for small v, then Λ belongs to $\mathbf{L}$, while if $\Lambda(v) \equiv \text{const}$ then $\Lambda \notin \mathbf{L}$.

DEFINITION 14.9. We say that a function $\gamma : (0, +\infty) \to \mathbb{R}$ belongs to the class Γ if γ is positive, monotone increasing, log-concave, and $\gamma(0+) = 0$.

Consequently, for any $\gamma \in \Gamma$, the function $\log \gamma$ is monotone increasing and concave. Hence, $\log \gamma$ is absolutely continuous, and its derivative $(\log \gamma)'$ exists almost everywhere and is monotone decreasing. Taking the right continuous version of $(\log \gamma)'$, we can assume that $(\log \gamma)'$ is defined pointwise. We see that $\gamma \in \Gamma$ if and only if

(i) the function γ is positive, monotone increasing, absolutely continuous, and $\gamma(0+) = 0$;
(ii) the function γ'/γ is monotone decreasing.

For example, the functions $\gamma(t) = t^{\alpha}$ and $\gamma(t) = \exp(-t^{-\alpha})$ belong to Γ for any $\alpha > 0$.

LEMMA 14.10. *For any function $\Lambda \in \mathbf{L}$, the following Cauchy problem on $(0, +\infty)$*

$$\frac{d\gamma}{dt} = \gamma\Lambda(\gamma), \quad \gamma(0+) = 0 \tag{14.24}$$

has a unique positive absolutely continuous solution $\gamma(t)$. This solution belongs to Γ and can be determined by

$$\begin{cases} t = \displaystyle\int_0^{\gamma(t)} \frac{dv}{v\Lambda(v)}, & 0 < t < t_0, \\ \gamma(t) = v_0, & t \geq t_0, \end{cases} \tag{14.25}$$

where

$$v_0 = \sup\{v : \Lambda(v) > 0\} \quad \text{and} \quad t_0 = \int_0^{v_0} \frac{dv}{v\Lambda(v)}. \tag{14.26}$$

Conversely, for any function $\gamma \in \Gamma$, there exists a unique non-negative, monotone decreasing, right continuous function Λ satisfying (14.25). This function belongs to $\mathbf{L}$ and can be determined by

$$\begin{cases} \Lambda(\gamma(t)) = \dfrac{\gamma'}{\gamma}(t), & t > 0, \\ \Lambda(v) = 0, & v \geq \sup \gamma. \end{cases} \tag{14.27}$$

Hence, the equation (14.24) (and each of the identities (14.25) and (14.27)) can be considered as the definition of a bijective mapping from $\mathbf{L}$ to Γ and back.

DEFINITION 14.11. For any $\Lambda \in \mathbf{L}$, the function γ, defined by (14.25), is called the Γ-*transform* of Λ. For any $\gamma \in \Gamma$, the function Λ, defined by (14.27), is called the $\mathbf{L}$-*transform* of γ.

PROOF OF LEMMA 14.10. *Uniqueness of* γ. Let $(0, v_0)$ be the maximal interval of positivity of Λ, that is v_0 is defined by (14.26). By (14.24) we have $\gamma' \geq 0$ so that γ is monotone increasing. Let $(0, t_0)$ be the maximal interval where $\gamma(t) < v_0$, that is

$$t_0 := \sup\{t : \gamma(t) < v_0\}.$$

For any $t \in (0, t_0)$, we have $\Lambda(\gamma(t)) > 0$ so that (14.24) implies upon integration

$$\int_0^t \frac{\gamma' dt}{\gamma\Lambda(\gamma)} = t.$$

Changing $v = \gamma(t)$ we arrive at the identity

$$t = \int_0^{\gamma(t)} \frac{dv}{v\Lambda(v)}, \quad \text{for all } 0 < t < t_0. \tag{14.28}$$

By continuity, (14.28) holds also for $t = t_0$. By the definition of t_0, we have $\gamma(t_0) \leq v_0$. Let us prove that in fact $\gamma(t_0) = v_0$, that is t_0 satisfies (14.26). Indeed, if $\gamma(t_0) < v_0$ then (14.28) and (14.23) imply

$$t_0 = \int_0^{\gamma(t_0)} \frac{dv}{v\Lambda(v)} < \infty.$$

However, for a finite t_0, we have $\gamma(t_0) = v_0$ just by continuity.

Hence, we have proved that the function γ is determined for $t \leq t_0$ by (14.28) where t_0 is determined by (14.26). For $t > t_0$ we have $\gamma(t) \geq v_0$ whence $\Lambda(\gamma(t)) = 0$. Therefore, (14.24) implies $\gamma'(t) \equiv 0$ and

$$\gamma(t) \equiv v_0 \quad \text{for all} \quad t \geq t_0, \tag{14.29}$$

which finishes the proof of the uniqueness of γ.

It follows directly from (14.24) that $\gamma \in \Gamma$ (the fact that γ'/γ is decreasing follows from $\gamma'/\gamma = \Lambda(\gamma)$ and the monotonicity properties of Λ and γ).

Existence of γ. Define $\gamma(t)$ by (14.25), where v_0 and t_0 are defined by (14.26). Observe that if t_0 is finite then also v_0 is finite. Indeed, if $v_0 = \infty$ then we obtain from (14.26) and the monotonicity of Λ

$$t_0 = \int_0^\infty \frac{dv}{v\Lambda(v)} \geq \int_1^\infty \frac{dv}{v\Lambda(v)} \geq \frac{1}{\Lambda(1)} \int_1^\infty \frac{dv}{v} = \infty.$$

Hence, (14.25) defines a positive absolutely continuous function γ on $(0, +\infty)$. It is straightforward to check that γ solves (14.24).

Uniqueness of Λ. Set $v_0 := \sup \gamma$ and let $(0, t_0)$ be the maximal interval where $\gamma(t) < v_0$, that is

$$t_0 := \sup\{t : \gamma(t) < v_0\}.$$

We claim that $\frac{\gamma'}{\gamma} > 0$ on the interval $(0, t_0)$. Indeed, if $\frac{\gamma'}{\gamma}(t_1) = 0$ for some $0 < t_1 < t_0$ then by the monotonicity $\frac{\gamma'}{\gamma}(t) = 0$ for all $t \geq t_1$. Therefore, γ attains its maximum at $t = t_1$, which cannot be the case because $\gamma(t_1) < v_0$.

Hence, $\gamma' > 0$ on $(0, t_0)$, and the function γ is strictly monotone on this interval and has the range $(0, v_0)$. It follows from (14.24) that

$$\Lambda(\gamma(t)) = \frac{\gamma'}{\gamma}(t), \quad \text{for all } 0 < t < t_0, \tag{14.30}$$

which uniquely determines $\Lambda(v)$ on the interval $(0, v_0)$. If $v_0 = \infty$ then (14.30) proves the uniqueness of Λ.

Assume now $v_0 < \infty$ and show that in this case $\Lambda(v_0) = 0$; this would imply by monotonicity that

$$\Lambda(v) \equiv 0 \quad \text{for all } v \geq v_0 \tag{14.31}$$

and prove the uniqueness of Λ in this case. Indeed, if $t_0 < \infty$ then for all $t \geq t_0$ we have $\gamma(t) \equiv v_0$ and hence $\gamma'(t) = 0$, which implies by (14.24) $\Lambda(v_0) = 0$.

Assume now $t_0 = \infty$ and show that

$$\lim_{t\to\infty} \frac{\gamma'}{\gamma}(t) = 0.$$

Indeed, the function γ'/γ is monotone decreasing and has a non-negative limit at ∞; denote it by c. If $c > 0$ then $\gamma'/\gamma \geq c$ implies that $\gamma(t)$ grows at least exponentially as $t \to \infty$, which contradicts the assumption $\sup \gamma = v_0 < \infty$. Hence, we conclude $c = 0$, which implies by (14.24)

$$\lim_{t\to\infty} \Lambda(\gamma(t)) = 0.$$

It follows by the monotonicity of Λ that $\Lambda(v_0) = 0$.

Finally, let us verify (14.23) that would prove $\Lambda \in \mathbf{L}$. Indeed, dividing (14.30) by the left hand side and integrating it as above we obtain again the identity (14.28), for any $t \in (0, t_0)$, whence (14.23) follows.

Existence of Λ. Define Λ by (14.27). Set $v_0 = \sup \gamma$ and observe that the first line (14.27) defines $\Lambda(v)$ for all v in the range of γ, which is either $(0, v_0)$ or $(0, v_0]$. If $v_0 = \infty$ then the second line in (14.27) is void. If $v_0 < \infty$ and the range of γ is $(0, v_0)$ then the second line in (14.27) extends Λ to be 0 in $[v_0, +\infty)$. If the range of γ is $(0, v_0]$ then γ attains its supremum; therefore at a point of the maximum of γ we have $\gamma'(t+) = 0$ and hence $\Lambda(v_0) = 0$, which is compatible with the second line in (14.27).

It is obvious that this function Λ satisfies (14.24). □

EXAMPLE 14.12. For all $\alpha, c > 0$, the function

$$\Lambda(v) = cv^{-\alpha}$$

belongs to $\mathbf{L}$ and its Γ-transform is

$$\gamma(t) = (c\alpha t)^{1/\alpha}.$$

In the next examples, let us always assume that $\Lambda \in \mathbf{L}$ and

$$\Lambda(v) = c_0 v^{-\alpha} \text{ for } v < 1,$$

where $c_0, \alpha > 0$. Set

$$t_0 = \int_0^1 \frac{dv}{v\Lambda(v)} = (c_0\alpha)^{-1}.$$

Let $\gamma(t)$ be the Γ-transform of Λ. For all $t < t_0$, we obtain by (14.25)

$$\gamma(t) = (c_0\alpha t)^{1/\alpha}.$$

Let

$$\Lambda(v) = cv^{-\beta} \text{ for } v \geq 1,$$

where $\beta, c > 0$ and $c \leq c_0$. Then the identity

$$t - t_0 = \int_1^{\gamma(t)} \frac{dv}{v\Lambda(v)} \tag{14.32}$$

implies that

$$\gamma(t) = \left(c\beta t + c'\right)^{1/\beta} \text{ for } t \geq t_0$$

where $c' = 1 - \frac{c\beta}{c_0\alpha}$. It follows that in this case

$$\gamma(t) \simeq \begin{cases} t^{1/\alpha}, & t < 1, \\ t^{1/\beta}, & t \geq 1. \end{cases}$$

Let

$$\Lambda(v) \equiv c \text{ for } v \geq 1,$$

where $0 < c \leq c_0$. Then by (14.32)

$$\gamma(t) = \exp(c(t - t_0)) \text{ for } t \geq t_0. \tag{14.33}$$

Let

$$\Lambda(v) \equiv 0 \text{ for } v \geq 1.$$

Then by (14.25) we obtain

$$\gamma(t) \equiv 1 \text{ for } t \geq t_0.$$

Formally this follows also from (14.33) with $c = 0$.

Let

$$\Lambda(v) = c\log^{-\beta} v \text{ for } v \geq 2.$$

Then it follows from (14.32) that, for large enough t,

$$\gamma(t) = \exp\left(\left(c't + c''\right)^{\frac{1}{\beta+1}}\right),$$

where $c' = c(\beta + 1)$ and c'' is a real number.

LEMMA 14.13. *Let $\Lambda \in \mathbf{L}$ and γ be its Γ-transform. If $f(t)$ is a positive absolutely continuous function satisfying on an interval $(0, T)$ the inequality*

$$f' \geq f\Lambda(f), \tag{14.34}$$

then

$$f(t) \geq \gamma(t) \quad \textit{for all } 0 < t < T.$$

PROOF. Let $(0, v_0)$ be the maximal interval of positivity of $\Lambda(v)$, and let $(0,\tau)$ be the maximal interval where $f(t) < v_0$. For all $t \in (0,\tau)$ we obtain from (14.34)

$$\int_0^t \frac{f'dt}{f\Lambda(f)} \geq t,$$

whence

$$\int_{f(0)}^{f(t)} \frac{dv}{v\Lambda(v)} \geq t.$$

Comparing with (14.25), we obtain $f(t) \geq \gamma(t)$ for all $t \in (0,\tau)$. If $\tau = T$ then this finishes the proof.

If $\tau < T$ (which includes also the case $\tau = 0$) then for all $t \in (\tau, T)$ we have $f(t) \geq v_0$, which implies $f(t) \geq \gamma(t)$ simply because $\gamma(t) \leq v_0$. □

DEFINITION 14.14. Fix $\delta \in (0,1)$ and let γ be a function from class Γ and Λ be its $\mathbf{L}$-transform. We say that γ belongs to the class Γ_δ (and Λ belongs to the class $\mathbf{L}_\delta$) if, for all $t > 0$,

$$\frac{\gamma'}{\gamma}(2t) - \delta\frac{\gamma'}{\gamma}(t) + \frac{\delta^{-1}}{(1+t)^{1+\delta}} \geq 0. \tag{14.35}$$

We say that γ belongs to the class $\widetilde{\Gamma}_\delta$ (and Λ belongs to the class $\widetilde{\mathbf{L}}_\delta$) if, for all $t > 0$,

$$\frac{\gamma'}{\gamma}(2t) \geq \delta\frac{\gamma'}{\gamma}(t). \tag{14.36}$$

Clearly, (14.36) implies (14.35) so that $\widetilde{\Gamma}_\delta \subset \Gamma_\delta$. Observe also that the parameter δ occurs in (14.35) three times. One could have three different parameters instead but the fact that the left hand side of (14.35) is monotone decreasing with δ allows to manage with a single parameter. This also implies that the class Γ_δ increases when δ decreases.

It is obvious that if $\gamma \in \Gamma_\delta$ or $\gamma \in \widetilde{\Gamma}_\delta$ then the function $a\gamma(bt)$ belongs to the same class for any positive constants a, b.

Recall that, for any $\gamma \in \Gamma$, the function γ'/γ is monotone decreasing. The condition $\gamma \in \widetilde{\Gamma}_\delta$ means that the rate of decay of γ'/γ is at most polynomial. For example, the function $\gamma(t) = t^\alpha$, $\alpha > 0$, satisfies (14.36) with $\delta = \frac{1}{2}$ and, hence, belongs to $\widetilde{\Gamma}_{1/2}$.

To have more examples, let us first prove the following lemma that helps in checking that $\gamma \in \Gamma_\delta$ or $\gamma \in \widetilde{\Gamma}_\delta$.

LEMMA 14.15. *Let* $0 < a < b < \infty$.

(a) *If* (14.35) *holds for all* $t < a$ *and* $t \geq b$, *then* $\gamma \in \Gamma_{\delta'}$ *for some* $\delta' > 0$.

(b) *If* (14.36) *holds for all* $t < a$ *and* $t \geq b$ *and* $\gamma'(2b) > 0$ *then* $\gamma \in \widetilde{\Gamma}_{\delta'}$ *for some* $\delta' > 0$.

PROOF. Denote $f = \frac{\gamma'}{\gamma}$ and recall that f is non-increasing.

(a) Then, for all $t \in [a, b)$, we have

$$f(t) \le C := f(a)$$

and, for a small enough δ',

$$\left(f(2t) - \delta' f(t)\right)_- \le C \le \frac{(\delta')^{-1}}{(1+t)^{1+\delta'}}.$$

Hence, for $t \in [a, b)$, (14.35) is satisfied with $\delta = \delta'$. Since for t outside $[a, b)$ (14.35) is satisfied by hypothesis, we conclude that $\gamma \in \Gamma_{\delta'}$.

(b) If $t \in [a, b)$ then

$$f(2t) \ge f(2b) = \varepsilon f(a) \ge \varepsilon f(t),$$

where $\varepsilon = \frac{f(2b)}{f(a)} > 0$. Hence, replacing δ in (14.36) by $\min(\delta, \varepsilon)$, we obtain that (14.36) holds for all $t > 0$. $\square$

EXAMPLE 14.16. 1. The function

$$\gamma(t) = \frac{1}{1+t^{-\alpha}}$$

belongs to $\widetilde{\Gamma}_\delta$ with $\delta = 2^{-\alpha-1}$.

2. The function

$$\gamma(t) = \begin{cases} t^\alpha, & t < 1, \\ t^\beta, & t \ge 1, \end{cases}$$

where $\alpha, \beta > 0$, belongs to some $\widetilde{\Gamma}_\delta$. Indeed, each of the function $\gamma(t) = t^\alpha$ and $\gamma(t) = t^\beta$ belong to $\widetilde{\Gamma}_{1/2}$, and the claim follows by Lemma 14.15.

3. In the same way, the functions

$$\gamma(t) = \begin{cases} t^\alpha, & t < 1, \\ \exp\left(t^\beta - 1\right), & t \ge 1 \end{cases}$$

and

$$\gamma(t) = \begin{cases} t^\alpha, & t < 1, \\ 1 + \log^\beta t, & t \ge 1 \end{cases}$$

belong to some $\widetilde{\Gamma}_\delta$.

4. The function

$$\gamma(t) = \begin{cases} t^\alpha, & t < 1, \\ 1, & t \ge 1 \end{cases} \tag{14.37}$$

obviously satisfies (14.36) if $t < \frac{1}{2}$ or $t \ge 1$. By Lemma 14.15, $\gamma \in \Gamma_\delta$ for some δ. Note that $\gamma \notin \widetilde{\Gamma}_\delta$ because in the range $t \in [\frac{1}{2}, 1)$ we have $\gamma'(2t) = 0$ while $\gamma'(t) > 0$, and (14.36) fails for any $\delta > 0$.

LEMMA 14.17. *If $\gamma \in \Gamma_\delta$ with $\delta \le \frac{1}{5}$ then there exists a smooth function g on $[0, +\infty)$ such that*

$$1 \le g \le e^{\delta^{-3}},\ g' > 0, \tag{14.38}$$

and such that the function $\widetilde{\gamma} := \gamma g$ belongs to $\widetilde{\Gamma}_\delta$.

PROOF. Define function $g(t)$ to be the solution of the Cauchy problem

$$\frac{g'}{g}(t) = \frac{\delta^{-2}}{(1+t)^{1+\delta}},\ g(0) = 1,$$

that is,

$$g(t) = \exp\left(\delta^{-2}\int_0^t \frac{ds}{(1+s)^{1+\delta}}\right).$$

The properties (14.38) are obvious. Set $\widetilde{\gamma} = \gamma g$ so that

$$\frac{\widetilde{\gamma}'}{\widetilde{\gamma}} = \frac{\gamma'}{\gamma} + \frac{g'}{g}$$

whence

$$\begin{aligned} \frac{\widetilde{\gamma}'}{\widetilde{\gamma}}(2t) - \delta\frac{\widetilde{\gamma}'}{\widetilde{\gamma}}(t) &= \left[\frac{\gamma'}{\gamma}(2t) - \delta\frac{\gamma'}{\gamma}(t)\right] + \left[\frac{g'}{g}(2t) - \delta\frac{g'}{g}(t)\right] \\ &\geq -\frac{\delta^{-1}}{(1+t)^{1+\delta}} + \frac{\delta^{-2}}{(1+2t)^{1+\delta}} - \frac{\delta^{-1}}{(1+t)^{1+\delta}} \\ &= \frac{\delta^{-2}}{(1+2t)^{1+\delta}} - \frac{2\delta^{-1}}{(1+t)^{1+\delta}}. \end{aligned} \tag{14.39}$$

We are left to verify that the right hand side of (14.39) is non-negative, which is true provided $\delta \leq \frac{1}{5}$ because

$$\frac{(1+2t)^{1+\delta}}{(1+t)^{1+\delta}} \leq 2^{1+\delta} < \frac{5}{2} \leq \tfrac{1}{2}\delta^{-1}.$$

$\square$

To state the next result, we use the notation $\log_+ s := (\log s)_+$.

LEMMA 14.18. *If $\gamma \in \Gamma_\delta$ then, for any $v > 0$,*

$$\sup_{t>0} \frac{1}{t}\log_+ \frac{\gamma(t)}{v} \geq \frac{\delta}{2}\Lambda(C_\delta v), \tag{14.40}$$

where Λ is the $\mathbf{L}$-transform of γ and $C_\delta \geq 1$ is a constant that depends only on δ.

If $\gamma \in \widetilde{\Gamma}_\delta$ then (14.40) is true with $C_\delta = 1$.

PROOF. Assume first that $\gamma \in \widetilde{\Gamma}_\delta$. If $v \geq \sup\gamma$ then $\Lambda(v) = 0$ and (14.40) is trivially satisfied. If $v < \sup\gamma$ then there exists $t > 0$ such that $v = \gamma(t/2)$. Using the concavity of $\log\gamma$ we obtain

$$\frac{2}{t}\log_+ \frac{\gamma(t)}{v} \geq \frac{\log\gamma(t) - \log\gamma(t/2)}{t/2} \geq (\log\gamma)'(t) = \frac{\gamma'}{\gamma}(t).$$

By (14.36) and (14.24) we obtain

$$\frac{\gamma'}{\gamma}(t) \geq \delta\frac{\gamma'}{\gamma}(t/2) = \delta\Lambda(\gamma(t/2)) = \delta\Lambda(v),$$

whence

$$\sup_{t>0} \frac{1}{t} \log_+ \frac{\gamma(t)}{v} \geq \frac{\delta}{2}\Lambda(v). \tag{14.41}$$

Let now $\gamma \in \Gamma_\delta$. Without loss of generality, we can assume $\delta \leq \frac{1}{5}$. Let g be the function from Lemma 14.17 so that $\widetilde{\gamma} := \gamma g \in \widetilde{\Gamma}_\delta$. Taking a multiple of g, we can assume that

$$c_\delta \leq g \leq 1,$$

where $c_\delta > 0$. Therefore, $\widetilde{\gamma} \leq \gamma$ and, by the first part of the proof,

$$\sup_{t>0} \frac{1}{t} \log_+ \frac{\gamma(t)}{v} \geq \sup_{t>0} \frac{1}{t} \log_+ \frac{\widetilde{\gamma}(t)}{v} \geq \frac{\delta}{2}\widetilde{\Lambda}(v),$$

where $\widetilde{\Lambda}$ is is the $\mathbf{L}$-transform of $\widetilde{\gamma}$.

If $v < \sup \widetilde{\gamma}$ then $v = \widetilde{\gamma}(t)$ for some t. Using $g' \geq 0$ and $\gamma \leq c_\delta^{-1}\widetilde{\gamma}$, we obtain

$$\widetilde{\Lambda}(v) = \widetilde{\Lambda}(\widetilde{\gamma}) = \frac{\widetilde{\gamma}'}{\widetilde{\gamma}} = \frac{\gamma'}{\gamma} + \frac{g'}{g} \geq \frac{\gamma'}{\gamma} = \Lambda(\gamma) \geq \Lambda\left(c_\delta^{-1}\widetilde{\gamma}\right) = \Lambda(C_\delta v),$$

where $C_\delta = c_\delta^{-1}$. Combining the previous two lines, we obtain (14.40).

If $v \geq \sup \widetilde{\gamma}$ then $C_\delta v \geq \sup \gamma$ and $\Lambda(C_\delta v) = 0$ so that (14.40) is trivially satisfied. $\square$

Exercises.

14.8. Prove that if Λ_1, Λ_2 are two functions of class $\mathbf{L}$ then also $\Lambda_1 + \Lambda_1$ and $\max(\Lambda_1, \Lambda_2)$ belong to $\mathbf{L}$.

14.9. Let Λ be a function of class $\mathbf{L}$ such that

$$\Lambda(v) = \begin{cases} c_1 v^{-\alpha_1}, & v \leq v_1, \\ c_2 v^{-\alpha_2}, & v \geq v_2, \end{cases}$$

where $\alpha_1, c_1, v_1 > 0$, $\alpha_2, c_2 \geq 0$, and $v_2 > v_1$. Prove that $\Lambda \in \mathbf{L}_\delta$ for some $\delta > 0$.

14.10. For any function $\gamma \in \Gamma$, denote by Λ_γ the $\mathbf{L}$-transform of γ, and for any function $\Lambda \in \mathbf{L}$, denote by γ_Λ the Γ-transform of Λ. Let a, b be positive constants.

(a) Set $\widetilde{\Lambda}(v) = a\Lambda(bv)$. Prove that

$$\gamma_{\widetilde{\Lambda}}(t) = b^{-1}\gamma_\Lambda(at).$$

(b) Set $\widetilde{\gamma}(t) = a\gamma(bt)$. Prove that

$$\Lambda_{\widetilde{\gamma}}(v) = b\Lambda_\gamma\left(a^{-1}v\right).$$

(c) Prove that if Λ_1 and Λ_2 are two functions from $\mathbf{L}$ and $\Lambda_1 \leq \Lambda_2$ then $\gamma_{\Lambda_1} \leq \gamma_{\Lambda_2}$.

14.11. Prove that the product of two functions from $\widetilde{\Gamma}_\delta$ belongs to $\widetilde{\Gamma}_\delta$, and the product of two functions from Γ_δ belongs to $\Gamma_{\delta/2}$.

14.12. Show that there is a function $\gamma \in \Gamma$ that does not belong to any class Γ_δ.

14.13. Let $F(s)$ be a positive function of class C^2 on $[0, +\infty)$ such that $F'(s)$ does not vanish for large s. Assume that

$$\int_0^\infty \frac{ds}{F(s)} = \infty$$

and

$$c := \lim_{s \to \infty} \frac{F''F}{(F')^2}(s) \neq 0.$$

Prove that

$$\int_0^t \frac{ds}{F(s)} \sim -\frac{c^{-1}}{F'(t)} \text{ as } t \to \infty.$$

14.14. Let Λ be a function of class $\mathbf{L}$ such that

$$\Lambda(v) = \exp\left(-v^{\beta}\right) \quad \text{for } v \geq 1,$$

where $\beta > 0$. Evaluate the asymptotic of its Γ-transform $\gamma(t)$ as $t \to \infty$.

14.5. Faber-Krahn implies ultracontractivity

As before, denote by P_t the heat semigroup of a weighted manifold $(M, \mathbf{g}, \mu)$ and by P_t^{Ω} the heat semigroup of Ω, for any open set $\Omega \subset M$. We use in this section the functional classes $\mathbf{L}$ and Γ defined in Section 14.4.

THEOREM 14.19. *Assume that the Faber-Krahn inequality holds on a weighted manifold $(M, \mathbf{g}, \mu)$ with a function $\Lambda \in \mathbf{L}$. Then, for any function $f \in L^1 \cap L^2(M, \mu)$ and for all $t > 0$,*

$$\|P_t f\|_2^2 \leq \frac{4}{\gamma(t)} \|f\|_1^2, \tag{14.42}$$

where $\gamma(t)$ is the Γ-transform of Λ.

Consequently, for all $t > 0$ and $x, y \in M$,

$$p_t(x, y) \leq \frac{4}{\gamma(t/2)}. \tag{14.43}$$

PROOF. Without loss of generality, we can also assume that

$$\|f\|_1 = 1. \tag{14.44}$$

For any $t \geq 0$, set $u(t, \cdot) = P_t f$ and consider the function

$$J(t) := \|u(t, \cdot)\|_2^2 = \|P_t f\|_2^2. \tag{14.45}$$

By Theorem 4.9(iv), we have for any $t > 0$

$$u(t, \cdot) \in \operatorname{dom} \mathcal{L} \subset W_0^1(M)$$

and

$$\frac{du}{dt} = -\mathcal{L} u \in L^2(M),$$

where $\frac{d}{dt}$ is the strong derivative in L^2 and $\mathcal{L}$ is the Dirichlet Laplace operator. It follows that function $J(t)$ differentiable in $(0, +\infty)$ and

$$\frac{dJ}{dt} = \frac{d}{dt}(u, u)_{L^2} = 2\left(\frac{du}{dt}, u\right)_{L^2} = -2(\mathcal{L} u, u)_{L^2}.$$

On the other hand, by the Green formula (4.12),

$$(\mathcal{L} u, u) = -(\Delta_\mu u, u) = \int_M |\nabla u|^2 d\mu,$$

whence we obtain

$$\frac{dJ}{dt} = -2 \int_M |\nabla u|^2 d\mu. \tag{14.46}$$

In particular, function $J(t)$ is monotone decreasing. Let $(0,T)$ be the maximal interval where $J(t) > 0$. For any $t \geq T$ we have $J(t) = 0$, and (14.42) is trivially satisfied. Assuming in the sequel that $t \in (0,T)$ and applying Lemma 14.7 (with $\varepsilon = \frac{1}{2}$), we obtain

$$\int_M |\nabla u|^2 \, d\mu \geq \frac{1}{2} \|u\|_2^2 \Lambda \left(4 \frac{\|u\|_1^2}{\|u\|_2^2}\right). \tag{14.47}$$

Theorem 7.19 and condition (14.44) imply

$$\|u\|_1 \leq 1. \tag{14.48}$$

Combining (14.45), (14.46), (14.47), and (14.48), we obtain the following differential inequality for J on the interval $(0,T)$:

$$\frac{dJ}{dt} \leq -J\Lambda\left(\frac{4}{J}\right). \tag{14.49}$$

Consequently, the function

$$f(t) := \frac{4}{J(t)}$$

satisfies on $(0,T)$ the inequality

$$f' \geq f\Lambda(f).$$

Resolving this inequality by Lemma 14.13, we conclude $f(t) \geq \gamma(t)$ whence $J(t) \leq 4/\gamma(t)$, which is equivalent to (14.42).

The estimate (14.43) follows from (14.42) by Theorem 14.4 and Remark 14.5. □

Exercises.

14.15. Prove that the claim of Theorem 14.19 remains true for any $f \in L^1(M)$.

14.6. Ultracontractivity implies a Faber-Krahn inequality

Here we prove a theorem which is "almost" converse to 14.19.

THEOREM 14.20. *Let $(M, \mathbf{g}, \mu)$ be a weighted manifold, and assume that the heat kernel satisfies the estimate*

$$p_t(x,x) \leq \frac{1}{\gamma(t)}$$

for all $t > 0$ and $x \in M$, where $\gamma(t)$ is a positive function on $(0,+\infty)$. Then M satisfies the Faber-Krahn inequality with the function $\widetilde{\Lambda}(v)$ defined by

$$\widetilde{\Lambda}(v) = \sup_{t>0} \frac{1}{t} \log_+ \frac{\gamma(t)}{v}. \tag{14.50}$$

If in addition $\gamma \in \Gamma_\delta$ then M satisfies the Faber-Krahn inequality with the function $\frac{\delta}{2}\Lambda(C_\delta v)$ where Λ is the $\mathbf{L}$-transform of γ and C_δ is the constant from Lemma 14.18.

The proof of Theorem 14.20 will be preceded by a lemma.

LEMMA 14.21. *For any function $f \in W_0^1(M)$ such that $\|f\|_2 = 1$ and for any $t \geq 0$, the following inequality holds*

$$\exp\left(-t\int_M |\nabla f|^2\, d\mu\right) \leq \|P_t f\|_2. \tag{14.51}$$

PROOF. Let $\{E_\lambda\}$ be the spectral resolution of the Dirichlet Laplacian $\mathcal{L}$. Then, for any $f \in \operatorname{dom}(\mathcal{L}) = W_0^2$ such that $\|f\|_2 = 1$ we have

$$1 = \|f\|_2^2 = \int_0^\infty d\|E_\lambda f\|^2$$

and

$$\int_M |\nabla f|^2\, d\mu = -\int_M (\Delta_\mu f)\, f\, d\mu = (\mathcal{L}f, f) = \int_0^\infty \lambda d\|E_\lambda f\|^2 \tag{14.52}$$

(cf. Exercise 4.24). Since the measure $d\|E_\lambda f\|^2$ has the total mass 1, we can apply Jensen's inequality which yields

$$\exp\left(-\int_0^\infty 2t\lambda\, d\|E_\lambda f\|^2\right) \leq \int_0^\infty \exp(-2t\lambda)\, d\|E_\lambda f\|^2.$$

Using (14.52) and the identity

$$\|P_t f\|_2^2 = \int_0^\infty \exp(-2t\lambda)\, d\|E_\lambda f\|^2,$$

we obtain

$$\exp\left(-2t\int_M |\nabla f|^2\, d\mu\right) \leq \|P_t f\|_2^2,$$

which coincides with (14.51).

Assume now that $f \in W_0^1$ and $\|f\|_2 = 1$. Since C_0^∞ is dense W_0^1, there is a sequence $\{f_k\} \subset C_0^\infty$ such that $f_k \xrightarrow{W^1} f$. Then $c_k := \|f_k\|_2 \to 1$, which implies that also $c_k^{-1} f_k \xrightarrow{W^1} f$. The inequality (14.51) holds for each function $c_k^{-1} f_k$. Since both sides of (14.51) survive when passing to the limit in the norm $\|\cdot\|_{W^1}$, we obtain (14.51) for f. □

See Exercise 4.36 for an alternative proof.

PROOF OF THEOREM 14.20. By Theorem 14.4, the heat semigroup P_t on M is $L^1 \to L^2$ ultracontractive with the rate function $\sqrt{1/\gamma(2t)}$, that is

$$\|P_{t/2}\|_{1\to 2}^2 \leq \frac{1}{\gamma(t)} \tag{14.53}$$

Let $\Omega \subset M$ be an open set with finite measure and $f \in W_0^1(\Omega)$ be a function such that $\|f\|_2 = 1$. Then also $f \in W_0^1(M)$ and we obtain by Lemma 14.21 and (14.53)

$$\exp\left(-t\int_M |\nabla f|^2\, d\mu\right) \leq \|P_{t/2} f\|_2^2 \leq \frac{1}{\gamma(t)}\|f\|_1^2.$$

Since by the Cauchy-Schwarz inequality $\|f\|_1^2 \leq \mu(\Omega)$, it follows that

$$\int_M |\nabla f|^2 \, d\mu \geq \frac{1}{t} \log_+ \frac{\gamma(t)}{\mu(\Omega)}.$$

Taking the infimum in f and the supremum in t, we obtain

$$\lambda_{\min}(\Omega) \geq \widetilde{\Lambda}(\mu(\Omega)), \tag{14.54}$$

which was to be proved.

If $\gamma \in \Gamma_\delta$ then by Lemma 14.18

$$\widetilde{\Lambda}(v) \geq \frac{\delta}{2} \Lambda(C_\delta v),$$

which proves the second claim. □

REMARK 14.22. It follows from (14.50) that $\widetilde{\Lambda}(v) > 0$ provided $\sup \gamma = \infty$. By (14.54), we conclude that $\lambda_{\min}(\Omega) > 0$ for any open set Ω with finite measure.

Recall that by Theorem 14.19 if M satisfies the Faber-Krahn inequality with a function $\Lambda \in \mathbf{L}$ then the heat kernel satisfies the estimate

$$p_t(x,x) \leq \frac{4}{\gamma(t/2)},$$

where γ is the Γ-transform of Λ. Hence, putting together Theorems 14.19 and 14.20 and assuming $\gamma \in \Gamma_\delta$ (which is equivalent to $\Lambda \in \mathbf{L}_\delta$), we obtain essentially the equivalence

$$p_t(x,x) \leq \frac{1}{\gamma(t)} \qquad \Longleftrightarrow \qquad \lambda_{\min}(\Omega) \geq \Lambda(\mu(\Omega)) \tag{14.55}$$

where all the constants factors are discarded.

COROLLARY 14.23. *For any weighted manifold and any $n > 0$, the following conditions are equivalent:*

(a) *The on-diagonal estimate $p_t(x,x) \leq Ct^{-n/2}$, for all $t > 0$ and $x \in M$.*
(b) *The Faber-Krahn inequality with function $\Lambda(v) = cv^{-2/n}$ where $c > 0$.*
(c) *The Nash inequality* (14.7).
(d) *The Sobolev inequality* (14.19), *provided $n > 2$.*

PROOF. Indeed, the equivalence $(a) \Leftrightarrow (b)$ follows from the above Remark 14.22 because the Λ-transform of the function $\gamma(t) = C^{-1}t^{n/2}$ is $\Lambda(v) = cv^{-2/n}$. The equivalence $(b) \Leftrightarrow (c)$ holds by Lemma 14.7 and Exercise 14.3, and $(b) \Leftrightarrow (d)$ holds by Exercise 14.6. □

REMARK 14.24. As it follows from Corollary 14.23 and Remark 14.6, all the equivalent conditions $(a) - (d)$ are satisfied on Cartan-Hadamard manifolds and minimal submanifolds of $\mathbb{R}^N$.

14.7. Lower bounds of higher eigenvalues

Here we prove a remarkable consequence of the equivalence (14.55) that the Faber-Krahn inequality, that is, the lower bound $\lambda_1(\Omega) \geq \Lambda(\mu(\Omega))$ for the bottom eigenvalue for any relatively compact open subsets $\Omega \subset M$, implies a similar estimate for the higher eigenvalues

$$\lambda_k(\Omega) \geq c\Lambda\left(C\frac{\mu(\Omega)}{k}\right),$$

subject to a mild restriction on the function Λ (see Corollary 14.28).

THEOREM 14.25. *Assume that the heat kernel on a weighted manifold M satisfies for all $t > 0$ the following estimate*

$$\int_M p_t(x,x)\,d\mu(x) \leq \rho(t), \tag{14.56}$$

where ρ is a positive function on $(0,+\infty)$. Then the spectrum of the Dirichlet Laplace operator $\mathcal{L}$ is discrete, and its k-th smallest eigenvalue $\lambda_k(M)$ satisfies for all $k = 1, 2, \ldots$ the inequality

$$\lambda_k(M) \geq \sup_{t>0} \frac{1}{t} \log_+ \frac{k}{\rho(t)}. \tag{14.57}$$

PROOF. Note that

$$\int_M p_t(x,x)\,d\mu(x) = \int_M \int_M p_{t/2}(x,y)^2\,d\mu(x)\,d\mu(y) = \|p_{t/2}\|^2_{L^{2,2}}.$$

By Lemma 10.14 and (14.56), we have

$$\operatorname{trace} e^{-t\mathcal{L}} = \operatorname{trace} P^2_{t/2} = \|p_{t/2}\|^2_{L^{2,2}} \leq \rho(t),$$

and by Lemma 10.7 we conclude that the spectrum of $\mathcal{L}$ is discrete. Furthermore, by (10.14) we have

$$\sum_{k=1}^{\infty} e^{-t\lambda_k} = \operatorname{trace} e^{-t\mathcal{L}} \leq \rho(t). \tag{14.58}$$

Since the sequence $\{\lambda_k\}_{k=1}^{\infty}$ is arranged in the increasing order, the left hand side of (14.58) is bounded below by $ke^{-t\lambda_k}$ for any index k. Therefore,

$$ke^{-t\lambda_k} \leq \rho(t),$$

which implies

$$\lambda_k \geq \frac{1}{t} \log \frac{k}{\rho(t)}.$$

Since $\lambda_k \geq 0$, the function log can be replaced by $\log_+$; since $t > 0$ is arbitrary, we obtain (14.57) by taking the supremum in t. $\square$

COROLLARY 14.26. *If the heat kernel on M satisfies the estimate*

$$p_t(x,x) \leq \frac{1}{\gamma(t)} \tag{14.59}$$

for all $t > 0$ and $x \in M$, where γ is a function of the class Γ_δ then, for any open set $\Omega \subset M$ with finite measure the spectrum of the Dirichlet Laplace operator $\mathcal{L}^\Omega$ is discrete and, for any $k \geq 1$,

$$\lambda_k(\Omega) \geq \frac{\delta}{2}\Lambda\left(C_\delta \frac{\mu(\Omega)}{k}\right), \tag{14.60}$$

where Λ is the $\mathbf{L}$*-transform of γ and C_δ is the constant from* Lemma 14.18.

Applying (14.60) for $k = 1$, we obtain the alternative proof of Theorem 14.20.

PROOF. We have

$$\int_\Omega p_t^\Omega(x, x)\, d\mu(x) \leq \frac{\mu(\Omega)}{\gamma(t)}$$

so that Ω as a manifold satisfies the hypotheses of Theorem 14.25 with function

$$\rho(t) = \frac{\mu(\Omega)}{\gamma(t)}.$$

By Theorem 14.25, we obtain that the spectrum of $\mathcal{L}^\Omega$ is discrete and

$$\lambda_k(\Omega) \geq \sup_{t>0} \frac{1}{t} \log_+ \frac{k\gamma(t)}{\mu(\Omega)}.$$

If $\gamma \in \Gamma_\delta$ then by Lemma 14.18 we obtain that

$$\sup_{t>0} \frac{1}{t} \log_+ \frac{k\gamma(t)}{\mu(\Omega)} \geq \frac{\delta}{2}\Lambda\left(C_\delta \frac{\mu(\Omega)}{k}\right)$$

which finishes the proof. $\square$

EXAMPLE 14.27. Let us show that, for any weighted n-dimensional manifold M and for any relatively compact open set $\Omega \subset M$, there exist constant $c, C > 0$ such that

$$\lambda_k(U) \geq c\left(\frac{k}{\mu(U)}\right)^{1/(2\sigma)} \tag{14.61}$$

for all open $U \subset \Omega$ and $k \geq C\mu(U)$, where σ is the same as in Theorem 7.6, that is, the smallest integer larger than $n/4$ (cf. Example 10.16).

Indeed, it follows from Theorem 7.6, that, for any $f \in L^2(\Omega)$,

$$\|P_t^\Omega f\|_\infty \leq \theta(t)\|f\|_2$$

where $\theta(t) = C(1 + t^{-\sigma})$ and $C = C(\Omega)$. Hence, the semigroup $\{P_t^\Omega\}$ is $L^2 \to L^\infty$ ultracontractive with the rate function $\theta(t)$, which implies by Theorem 14.4 that

$$p_t^\Omega(x, x) \leq \theta^2(t/2) \leq \frac{1}{\gamma(t)},$$

where

$$\gamma(t) = C' \begin{cases} t^{2\sigma}, & t < 1, \\ 1, & t \geq 1. \end{cases}$$

As was shown in Example 14.16, $\gamma \in \Gamma_\delta$. Evaluating Λ from (14.27), we obtain

$$\Lambda(v) \geq cv^{-1/(2\sigma)} \text{ for } v \leq v_0,$$

for some $c, v_0 > 0$. By Corollary 14.26 we obtain that (14.61) is true whenever $C_\delta \frac{\mu(U)}{k} \leq v_0$, which is equivalent to $k \geq C\mu(U)$ with $C = C_\delta v_0^{-1}$.

As we will see later, (14.61) holds with $\sigma = n/4$ (see Corollary 15.12).

COROLLARY 14.28. *Assume that M satisfies the Faber-Krahn inequality with a function $\Lambda \in \mathbf{L}_\delta$. Then, for any open set $\Omega \subset M$ with finite measure, the spectrum of the Dirichlet Laplace operator $\mathcal{L}^\Omega$ is discrete and satisfies for all $k \geq 1$ the estimate*

$$\lambda_k(\Omega) \geq \frac{\delta}{4}\Lambda\left(C_\delta \frac{\mu(\Omega)}{k}\right), \tag{14.62}$$

where $C_\delta > 0$ depends only on δ.

PROOF. By Theorem 14.19, we have the estimate

$$p_t(x,x) \leq \frac{4}{\gamma(t/2)},$$

for all $t > 0$ and $x \in M$, where γ is the Γ-transform of Λ. Since the function $\widetilde{\gamma}(t) = \frac{1}{4}\gamma(t/2)$ belongs to Γ_δ, we obtain by Corollary 14.26 the estimate (14.60) with function Λ that is the $\mathbf{L}$-transform of $\widetilde{\gamma}$, whence (14.62) follows (cf. Exercise 14.10). □

14.8. Faber-Krahn inequality on direct products

Here we give another example of application of the equivalence (14.55).

THEOREM 14.29. *If X and Y are two weighted manifolds satisfying the Faber-Krahn inequalities with functions $\Lambda_1, \Lambda_2 \in \mathbf{L}_\delta$, respectively, then the product manifold $M = X \times Y$ satisfies the Faber-Krahn inequality with the function*

$$v \mapsto \frac{\delta}{4}\Lambda(C_\delta v)$$

where

$$\Lambda(v) := \inf_{uw=v}(\Lambda_1(u) + \Lambda_2(w)) \tag{14.63}$$

and $C_\delta > 0$ depends only on δ.

PROOF. By Theorem 14.19, the heat kernels on X and Y admit the estimates

$$p_t^X(x_1,x_2) \leq \frac{4}{\gamma_1(t/2)} \quad \text{and} \quad p_t^Y(y_1,y_2) \leq \frac{4}{\gamma_2(t/2)},$$

for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, where γ_1 and γ_2 are the Γ-transforms of Λ_1 and Λ_2, respectively.

The heat kernel p_t on M is the product of p_t^X and p_t^Y in the following sense: if $z_i = (x_i, y_i) \in M_1 \times M_2$ where $i = 1, 2$ then

$$p_t(z_1, z_2) = p_t^X(x_1, x_2) p_t^Y(y_1, y_2).$$

(see Theorem 9.11 and Exercise 7.41). Hence, we obtain for all $z_1, z_2 \in M$,

$$p_t(z_1, z_2) \leq \frac{16}{\gamma_1(t/2)\gamma_2(t/2)} = \frac{16}{\gamma(t/2)},$$

where $\gamma = \gamma_1\gamma_2$. By hypothesis, the functions γ_1 and γ_2 are in the class Γ_δ; then γ is in $\Gamma_{\delta/2}$ by Exercise 14.11. Therefore, by Theorem 14.20, M satisfies the Faber-Krahn inequality with the function

$$v \mapsto \frac{\delta}{4}\widetilde{\Lambda}(C_\delta v),$$

where $\widetilde{\Lambda}$ is the **L**-transform of γ. We are left to show that $\widetilde{\Lambda}(v) \geq \Lambda(v)$ where Λ is defined by (14.63). If $v < \sup \gamma$ and, hence, $v = \gamma(t)$ for some t then by (14.27) and (14.63)

$$\widetilde{\Lambda}(v) = \widetilde{\Lambda}(\gamma(t)) = \frac{\gamma'}{\gamma} = \frac{\gamma_1'}{\gamma_1} + \frac{\gamma_2'}{\gamma_2} = \Lambda_1(\gamma_1(t)) + \Lambda_2(\gamma_2(t)) \geq \Lambda(v).$$

If $v \geq \sup \gamma$ then we can choose numbers $u \geq \sup \gamma_1$ and $w \geq \sup \gamma_2$ such that $uw = v$. Since $\Lambda_1(u) = \Lambda_2(w) = 0$, it follows from (14.63) that $\Lambda(v) = 0$. Hence, in the both cases, $\widetilde{\Lambda}(v) \geq \Lambda(v)$, which finishes the proof. □

EXAMPLE 14.30. Let $\Lambda_1(u) = c_1 u^{-2/n}$ and $\Lambda_2(w) = c_2 w^{-2/m}$. Then (14.63) gives

$$\Lambda(v) = \inf_{uw=v}\left(c_1 u^{-2/n} + c_2 w^{-2/m}\right) = c v^{-2/(n+m)},$$

where $c = c(c_1, c_2, n, m) > 0$. Since both Λ_1, Λ_2 belong to $\mathbf{L}_\delta$ for some $\delta > 0$, we conclude that $X \times Y$ satisfies the Faber-Krahn inequality with the function $\operatorname{const} v^{-2/(n+m)}$.

EXAMPLE 14.31. Let us show how the above example allows to prove the Faber-Krahn inequality in $\mathbb{R}^n$ with function $\Lambda(v) = c_n v^{-2/n}$ by induction in n. In $\mathbb{R}^1$ any open set Ω is a disjoint union of open intervals $\{I_k\}$. Set $r_k = \mu(I_k)$, where μ is the Lebesgue measure. Then we have, for some $c > 0$,

$$\lambda_{\min}(\Omega) \geq \inf_k \lambda_{\min}(I_k) = \inf_k \frac{c}{r_k^2} \geq c\mu(\Omega)^{-2}$$

(cf. Exercises 10.6 and 11.25), which proves the above claim in the case $n = 1$. Assuming that the Faber-Krahn inequality in $\mathbb{R}^n$ holds with function $\Lambda(v) = c_n v^{-2/n}$, we obtain by Example 14.30 that it holds in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}^1$ with function $\Lambda(v) = \operatorname{const} v^{-2(n+1)}$, which was to be proved.

EXAMPLE 14.32. Let X satisfy the Faber-Krahn inequality with the function $\Lambda_1(u) = c_1 u^{-2/n}$; for example, X may be any Cartan-Hadamard manifold of dimension n. Let Y be any compact manifold of dimension m. By Exercise 15.2, Y satisfies the Faber-Krahn inequality with the function

$$\Lambda_2(w) = \begin{cases} c_2 w^{-2/m}, & w \le w_0, \\ 0, & w > w_0, \end{cases}$$

which belongs to $\mathbf{L}_\delta$ by Exercise 14.9. By (14.63), we have

$$\begin{aligned} \Lambda(v) &= \min\left(\inf_{\substack{uw=v \\ w>w_0}} c_1 u^{-2/n}, \inf_{\substack{uw=v \\ w\le w_0}} \left(c_1 u^{-2/n} + c_2 w^{-2/m}\right)\right) \\ &\le c \begin{cases} v^{-2/(n+m)}, & v \le v_0, \\ v^{-2/n}, & v > v_0, \end{cases} \end{aligned} \tag{14.64}$$

with some $c, v_0 > 0$. Hence, we conclude by Theorem 14.29 than $X \times Y$ satisfies the Faber-Krahn inequality with the function (14.64).

Notes

The proof of the classical Faber-Krahn theorem in $\mathbb{R}^n$ as well as its extensions to $\mathbb{S}^n$ and $\mathbb{H}^n$ can be found in the book by I.Chavel [**51**] that is a good general reference for the properties of the eigenvalues of the Laplace operator on manifolds.

The fact that the ultracontractivity of the heat semigroup is equivalent to a heat kernel on-diagonal upper bound is widely known. Moreover, the argument that is used to prove the upper bound, can be turned into the proof of the existence of the heat kernel in a rather general setting – see [**49**], [**96**], [**163**], [**184**].

The method of obtaining the heat kernel upper bound from the Nash inequality, which was used in of Theorem 14.19, goes back to a seminar paper of J. Nash [**292**]. The equivalence of the Sobolev inequality and the heat kernel upper bound

$$p_t(x,x) \le C t^{-n/2} \tag{14.65}$$

was first proved by N.Varopoulos [**353**], [**355**]. The equivalence of the classical Nash inequality and (14.65) was proved by Carlen, Kusuoka and Stroock in [**49**].

The equivalence of the Faber-Krahn inequalities and the heat kernel upper bounds in full generality (Theorems 14.19 and 14.20) was proved in [**146**]. A particular case that (14.65) is equivalent to

$$\lambda_1(\Omega) \ge c\mu(\Omega)^{-2/n} \tag{14.66}$$

was obtained independently by G.Carron [**50**].

The equivalence of the generalized Nash inequality and the heat kernel upper bounds in full generality was proved by T.Coulhon [**77**]. In particular, Lemma 14.21 is taken from [**77**]. A direct derivation of various types of Nash and Sobolev inequalities each from others can be found in [**17**].

The discreteness of the spectrum in the setting of Theorem 14.25 was proved by L. Gross [**188**]. The fact that the lower estimates for $\lambda_1(\Omega)$ implies non-trivial lower estimates for $\lambda_k(\Omega)$ (Corollary 14.28) was proved in [**146**]. Further results in this direction can be found in [**67**] and [**184**].

We do not use here a geometric tool for obtaining Faber-Krahn inequalities: the isoperimetric inequalities of the form

$$\sigma(\partial\Omega) \ge F(\mu(\Omega)), \tag{14.67}$$

where Ω is any relatively compact open subset of M with smooth boundary $\partial\Omega$ and σ is the induced measure on $\partial\Omega$, and F is a non-negative function on $[0, +\infty)$. For example, in $\mathbb{R}^n$ the isoperimetric inequality (14.67) holds with the function

$$F(v) = cv^{\frac{n-1}{n}} \tag{14.68}$$

(see, for example, [**53**], [**196**]). By the Cheeger inequality, if (14.67) holds with function F such that $F(v)/v$ is monotone decreasing then the Faber-Krahn inequality holds with function

$$\Lambda(v) = \frac{1}{4}\left(\frac{F(v)}{v}\right)^2$$

(cf. [**56**], [**146**]).

For example, the isoperimetric inequality with the function (14.68) implies the Faber-Krahn inequality (14.66). This can be used to prove the Faber-Krahn inequality (14.66) on Cartan-Hadamard manifolds and minimal submanifolds of $\mathbb{R}^N$ as was mentioned in Remark 14.6, because the corresponding isoperimetric inequalities on such manifolds are known - see [**203**], [**325**] for Cartan-Hadamard manifolds, [**44**] for minimal submanifolds, and [**67**], [**154**] for the both classes.

A far reaching extension of Cheeger's inequality – Maz'ya's inequality, and its applications can be found in [**153**], [**268**], [**269**], [**270**]. Further relations of isoperimetric inequalities and heat kernels can be found in [**54**], [**55**], [**76**], [**153**]. Faber-Krahn inequalities on direct products were proved in [**83**] without using heat kernels. Isoperimetric inequalities on direct products were proved in [**135**], [**278**].

A powerful isoperimetric inequality on groups and covering manifolds was proved by T.Coulhon and L.Saloff-Coste [**86**], which provides plenty of examples of manifolds with explicit functions F and Λ in the isoperimetric and Faber-Krahn inequalities, respectively.

CHAPTER 15

Pointwise Gaussian estimates I

In this Chapter we obtain the pointwise *Gaussian upper bounds* of the heat kernel, that is, the estimates containing the factor $\exp\left(-\frac{d^2(x,y)}{ct}\right)$. The key ingredient is a mean value inequality that is deduced from the Faber-Krahn inequality. The mean value inequality enables one to obtain upper bounds for a certain weighted L^2-norm of the heat kernel, which then implies the pointwise estimates. In contrast to Chapter 14, the Faber-Krahn inequality is assumed to hold in some balls rather than on the entire manifold.

In the core part of this chapter, we use from the previous chapters only the properties of Lipschitz functions (Section 11.2) and the integrated maximum principle (Section 12.1).

15.1. L^2-mean value inequality

Consider a weighted manifold $N = \mathbb{R} \times M$ with the product measure $d\nu = dtd\mu$. Let I be an interval in $\mathbb{R}$ and Ω be an open set in M so that the cylinder $I \times \Omega$ can be considered as a subset of N. A function $u : I \times \Omega \to \mathbb{R}$ is called a *subsolution* to the heat equation if $u \in C^2(I \times \Omega)$ and

$$\frac{\partial u}{\partial t} \leq \Delta_\mu u. \tag{15.1}$$

THEOREM 15.1. *Let $B(x,R)$ be a relatively compact ball in M and assume that, for some $a, n > 0$, the Faber-Krahn inequality*

$$\lambda_1(U) \geq a\mu(U)^{-2/n}, \tag{15.2}$$

holds for any open set $U \subset B(x,R)$. Then, for any $T > 0$ and for any subsolution $u(t,y)$ of the heat equation in the cylinder $C = (0,T] \times B(x,R)$, we have

$$u_+^2(T,x) \leq \frac{Ca^{-n/2}}{\min\left(\sqrt{T}, R\right)^{n+2}} \int_C u_+^2 d\nu, \tag{15.3}$$

where $C = C(n)$.

Although n does not have to be the dimension of M, in most applications of Theorem 15.1 one has $n = \dim M$. We prove first two lemmas.

LEMMA 15.2. *Let Ω be an open subset of M and $T_0 < T$. Let $\eta(t,x)$ be a Lipschitz function in the cylinder $\mathcal{C} = [T_0, T] \times \Omega$ such that $\eta(t,\cdot)$ is supported in some compact set $K \subset \Omega$ for any t. Let u be a subsolution to the heat equation in $\mathcal{C}$ and set $v = (u-\theta)_+$ with some $\theta \geq 0$. Then the following inequality holds:*

$$\frac{1}{2}\left[\int_\Omega v^2\eta^2(t,\cdot)\,d\mu\right]_{t=T_0}^T + \int_{\mathcal{C}} |\nabla(v\eta)|^2\,d\nu \leq \int_{\mathcal{C}} v^2\left(|\nabla\eta|^2 + \left|\eta\frac{\partial\eta}{\partial t}\right|\right)d\nu. \tag{15.4}$$

In particular, if $\eta(T_0,\cdot) = 0$ then

$$\int_\Omega v^2\eta^2(t,\cdot)\,d\mu \leq 2\int_{\mathcal{C}} v^2\left(|\nabla\eta|^2 + \left|\eta\frac{\partial\eta}{\partial t}\right|\right)d\nu \tag{15.5}$$

for any $t \in [T_0, T]$, and

$$\int_{\mathcal{C}} |\nabla(v\eta)|^2\,d\nu \leq \int_{\mathcal{C}} v^2\left(|\nabla\eta|^2 + \left|\eta\frac{\partial\eta}{\partial t}\right|\right)d\nu. \tag{15.6}$$

PROOF. Since $u(t,\cdot) \in W^1_{loc}(\Omega)$, by Exercise 5.8 we have $v(t,\cdot) \in W^1_{loc}(\Omega)$ and

$$\nabla v = 1_{\{u>\theta\}}\nabla u = 1_{\{v\neq 0\}}\nabla u,$$

which implies

$$\langle\nabla v, \nabla u\rangle = |\nabla v|^2 \quad\text{and}\quad v\nabla u = v\nabla v.$$

Since $\eta(t,\cdot) \in Lip_0(\Omega)$, by Exercise 11.13 we have $v\eta^2 \in W^1_0(\Omega)$ for any fixed time t and

$$\nabla\left(v\eta^2\right) = v\nabla\eta^2 + \eta^2\nabla v = 2v\eta\nabla\eta + \eta^2\nabla v,$$

whence

$$\langle\nabla u, \nabla\left(v\eta^2\right)\rangle = 2v\eta\langle\nabla v, \nabla\eta\rangle + \eta^2|\nabla v|^2.$$

Multiplying the inequality (15.1) by $v\eta^2$ and integrating over $\mathcal{C}$, we obtain

$$\begin{aligned}\int_{\mathcal{C}} \frac{\partial u}{\partial t}v\eta^2 d\nu &\leq \int_{T_0}^T\int_\Omega (\Delta_\mu u)\,v\eta^2 d\mu dt\\ &= -\int_{T_0}^T\int_\Omega \langle\nabla u, \nabla\left(v\eta^2\right)\rangle d\mu dt\\ &= -\int_{\mathcal{C}}\left(2v\eta\langle\nabla v, \nabla v\rangle + \eta^2|\nabla v|^2\right)d\nu,\end{aligned}$$

where we have used the Green formula of Lemma 4.4. Since

$$2v\eta\langle\nabla v, \nabla v\rangle + \eta^2|\nabla v|^2 = |\nabla(v\eta)|^2 - v^2|\nabla\eta|^2,$$

we obtain

$$\int_{\mathcal{C}} \frac{\partial u}{\partial t}v\eta^2 d\nu \leq -\int_{\mathcal{C}} |\nabla(v\eta)|^2\,d\nu + \int_{\mathcal{C}} v^2|\nabla\eta|^2\,d\nu. \tag{15.7}$$

For any fixed x, all functions u, v, η are Lipschitz in $t \in [T_0, T]$. Therefore, using the integration by parts formula for Lipschitz functions (see Exercise 2.25) we obtain, for any fixed $x \in \Omega$,

$$\begin{aligned}
\int_{T_0}^{T} \frac{\partial u}{\partial t} v\eta^2 dt &= \frac{1}{2}\int_{T_0}^{T} \frac{\partial\left(v^2\right)}{\partial t}\eta^2 dt \\
&= \frac{1}{2}\left[v^2\eta^2\right]_{T_0}^{T} - \frac{1}{2}\int_{T_0}^{T} v^2 \frac{\partial}{\partial t}\left(\eta^2\right) dt \\
&= \frac{1}{2}\left[v^2\eta^2\right]_{T_0}^{T} - \int_{T_0}^{T} v^2\eta \frac{\partial \eta}{\partial t} dt.
\end{aligned}$$

Integrating this identity over Ω and combining with (15.7), we obtain (15.4).

The estimate (15.5) follows from (15.4) if one replaces T by t, and (15.6) is an obvious consequence of (15.4). □

LEMMA 15.3. *Under the hypotheses of* Theorem 15.1, *consider the cylinders*

$$C_k = [T_k, T] \times B(x, R_k),\ k = 0, 1,$$

where $0 < R_1 < R_0 \le R$ *and* $0 \le T_0 < T_1 < T$ *(see* Fig. 15.1*). Choose* $\theta_1 > \theta_0 \ge 0$ *and set*

$$J_k = \int_{C_k} (u - \theta_k)_+^2\, d\nu.$$

Then the following inequality holds

$$J_1 \le \frac{C J_0^{1+2/n}}{a\delta^{1+2/n}\left(\theta_1 - \theta_0\right)^{4/n}}, \tag{15.8}$$

where $C = C(n)$ *and*

$$\delta = \min\left(T_1 - T_0, (R_0 - R_1)^2\right).$$

PROOF. Replacing function u by $u - \theta_0$ we can assume that $\theta_0 = 0$ and rename θ_1 to θ. Consider function $\eta(t, y) = \varphi(t)\psi(y)$ where

$$\varphi(t) = \frac{(t - T_0)_+}{T_1 - T_0} \wedge 1 = \begin{cases} 1, & t \ge T_1, \\ \frac{t - T_0}{T_1 - T_0}, & T_0 \le t \le T_1, \\ 0, & t \le T_0 \end{cases} \tag{15.9}$$

and

$$\psi(y) = \frac{\left(R_{1/4} - d(x, y)\right)_+}{R_{1/4} - R_{1/2}} \wedge 1,$$

where $R_\lambda = \lambda R_1 + (1 - \lambda) R_0$. Obviously, $\operatorname{supp}\psi = \overline{B\left(x, R_{1/4}\right)}$ is a compact subset of $B(x, R_0)$ because the ball $B(x, R_0)$ is relatively compact by hypothesis.

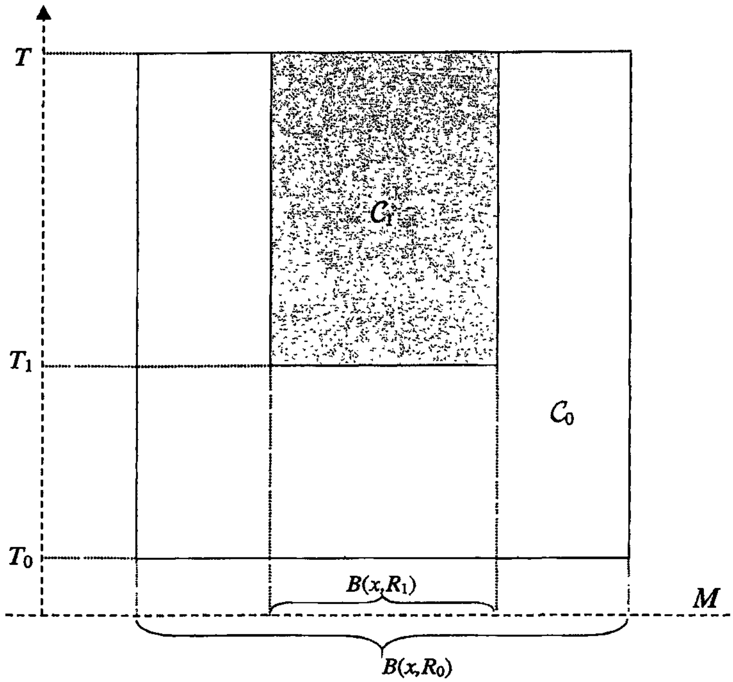

FIGURE 15.1. Cylinders $\mathcal{C}_0$ and $\mathcal{C}_1$

Applying the estimate (15.6) of Lemma 15.2 in the cylinder $\mathcal{C}_0$ for function $v = u_+$ with $t \in [T_1, T]$ and noticing that $\varphi(t) = 1$ in this range and $\psi = 1$ in $B(x, R_{1/2})$, we obtain

$$\int_{B(x,R_{1/2})} u_+^2(t,\cdot)\, d\mu \le 2\int_{\mathcal{C}_0} u_+^2 \left(|\nabla\eta|^2 + \left|\eta\frac{\partial\eta}{\partial t}\right|\right) d\nu \le \frac{34}{\delta}J_0, \tag{15.10}$$

where we have also used that

$$|\nabla\eta|^2 \le \frac{1}{\left(R_{1/4} - R_{1/2}\right)^2} = \frac{16}{\left(R_0 - R_1\right)^2} \le \frac{16}{\delta}$$

and

$$\left|\eta\frac{\partial\eta}{\partial t}\right| \le \frac{1}{T_1 - T_0} \le \frac{1}{\delta}.$$

For any $t \in [T_1, T]$, consider the set

$$U_t = \left\{y \in B\left(x, R_{3/4}\right) : u(t,y) > \theta\right\}$$

so that by (15.10)

$$\mu\left(\overline{U_t}\right) \le \frac{1}{\theta^2}\int_{\overline{B(x,R_{3/4})}} u_+^2(t,\cdot)\, d\mu \le \frac{1}{\theta^2}\int_{B(x,R_{1/2})} u_+^2(t,\cdot)\, d\mu \le \frac{34J_0}{\theta^2\delta}. \tag{15.11}$$

Consider now a different function ψ:

$$\psi(y) = \frac{\left(R_{3/4} - d(x,y)\right)_+}{R_{3/4} - R_1} \wedge 1. \tag{15.12}$$

Applying (15.6) for function $v = (u-\theta)_+$ with $\eta(t,x) = \varphi(t)\psi(y)$ where φ is given by (15.9) and ψ is given by (15.12), we obtain

$$\int_{\mathcal{C}_0} |\nabla(v\eta)|^2 d\nu \le \int_{\mathcal{C}_0} v^2 \left(|\nabla\eta|^2 + \left|\eta\frac{\partial\eta}{\partial t}\right|\right) d\nu \le \frac{17}{\delta}\int_{\mathcal{C}_0} v^2 d\nu \le \frac{17}{\delta} J_0. \tag{15.13}$$

For a fixed t, the function $v\eta(t,y)$ can take a non-zero value only if $y \in B(x, R_{3/4})$ and $u(t,y) > \theta$. It follows that

$$\operatorname{supp}(v\eta(t,\cdot)) \subset \overline{U_t},$$

whence $v\eta(t,\cdot) \in Lip_0(\Omega)$ for any open set Ω containing $\overline{U_t}$. Choose such an Ω with additional conditions $\Omega \subset B(x,R_0)$ and

$$\mu(\Omega) \le 2\mu\left(\overline{U_t}\right) \le \frac{68}{\theta^2\delta} J_0,$$

where we have used (15.11) (still assuming that $t \in [T_1, T]$). Then we obtain by the variational principle and (15.2)

$$\begin{aligned}
\int_{B(x,R_0)} |\nabla(v\eta)|^2(t,\cdot)\,d\mu &\ge \lambda_{\min}(\Omega)\int_{B(x,R_0)} (v\eta)^2(t,\cdot)\,d\mu \\
&\ge a\mu(\Omega)^{-2/n}\int_{B(x,R_0)} (v\eta)^2(t,\cdot)\,d\mu \\
&\ge a\left(\frac{\theta^2\delta}{68}\right)^{2/n} J_0^{-2/n}\int_{B(x,R_1)} v^2(t,\cdot)\,d\mu,
\end{aligned}$$

where we have used that $\eta(t,y) = 1$ for $t \in [T_1, T]$ and $y \in B(x,R_1)$. Integrating this inequality from T_1 to T and using (15.13), we obtain

$$\begin{aligned}
\frac{17}{\delta} J_0 &\ge \int_{T_1}^{T}\int_{B(x,R_0)} |\nabla(v\eta)|^2 d\nu \\
&\ge a\left(\frac{\theta^2\delta}{68}\right)^{2/n} J_0^{-2/n}\int_{T_1}^{T}\int_{B(x,R_1)} v^2(t,\cdot)\,d\mu dt \\
&= a\left(\frac{\theta^2\delta}{68}\right)^{2/n} J_0^{-2/n} J_1,
\end{aligned}$$

whence (15.8) follows. □

PROOF OF THEOREM 15.1. Consider a sequence of cylinders

$$\mathcal{C}_k = [T_k, T] \times B(x, R_k),$$

where $\{T_k\}_{k=0}^{\infty}$ is a strictly increasing sequence such that $T_0 = 0$ and $T_k \leq T/2$ for all k, $\{R_k\}_{k=0}^{\infty}$ is a strictly decreasing sequence such that $R_0 = R$ and $R_k \geq R/2$ for all k. Assume also that

$$(R_k - R_{k+1})^2 = T_{k+1} - T_k =: \delta_k. \tag{15.14}$$

In particular, the sequence of cylinders $\{\mathcal{C}_k\}_{k=0}^{\infty}$ is nested, $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_k$ contains $[T/2, T] \times B(x, R/2)$ for all k.

Fix some $\theta > 0$ and set $\theta_k = \left(1 - 2^{-(k+1)}\right)\theta$ and

$$J_k = \int_{\mathcal{C}_k} (u - \theta_k)^2 \, d\nu.$$

Clearly, the sequence $\{J_k\}_{k=0}^{\infty}$ is decreasing. We will find θ such that $J_k \to 0$ as $k \to \infty$, which will implies that

$$\int_{T/2}^{T} \int_{B(x,R/2)} (u - \theta)_+^2 \, d\nu = 0.$$

In particular, it follows that $u(T, x) \leq \theta$ and, hence, $u_+(T, x) \leq \theta^2$. For an appropriate choice of θ, this will lead us to (15.3).

Applying Lemma 15.3 for two consecutive cylinders $\mathcal{C}_k \supset \mathcal{C}_{k+1}$, we obtain

$$J_{k+1} \leq \frac{C J_k^{1+2/n}}{a\delta_k^{1+2/n} (\theta_{k+1} - \theta_k)^{4/n}} = \frac{C' 16^{k/n} J_k^{1+2/n}}{a\delta_k^{1+2/n}\theta^{4/n}} \tag{15.15}$$

where $C' = 16^{2/n} C$. Assume that δ_k is chosen so that for any k

$$\frac{C' 16^{-k/n} J_0^{2/n}}{a\delta_k^{1+2/n}\theta^{4/n}} = \frac{1}{16}. \tag{15.16}$$

We claim that then

$$J_k \leq 16^{-k} J_0, \tag{15.17}$$

which in particular yields $J_k \to 0$. Indeed, for $k = 0$ (15.17) is trivial. If (15.17) is true for some k then (15.15) and (15.16) imply

$$J_{k+1} \leq \frac{C' 16^{k/n} \left(16^{-k} J_0\right)^{2/n}}{a\delta_k^{1+2/n}\theta^{4/n}} J_k = \frac{C' 16^{-k/n} J_0^{2/n}}{a\delta_k^{1+2/n}\theta^{4/n}} J_k = \frac{1}{16} J_k$$

whence $J_{k+1} \leq 16^{-(k+1)} J_0$.

The equation (15.16) can be used to define δ_k, that is,

$$\delta_k = \left(\frac{C' 16^{1-k/n} J_0^{2/n}}{a\theta^{4/n}}\right)^{\frac{n}{n+2}} = \frac{C'' 16^{-\frac{k}{n+2}} J_0^{\frac{2}{n+2}}}{a^{\frac{n}{n+2}} \theta^{\frac{4}{n+2}}}, \tag{15.18}$$

where $C'' = (16C')^{\frac{n}{n+2}}$, but we must make sure that this choice of δ_k does not violate the conditions $T_k \leq T/2$ and $R_k \geq R/2$. Since by (15.14)

$$T_k = \sum_{i=0}^{k-1} \delta_i \quad \text{and} \quad R_k = R - \sum_{i=0}^{k-1} \sqrt{\delta_k},$$

the sequence $\{\delta_k\}$ must satisfy the inequalities

$$\sum_{k=0}^{\infty} \delta_k \leq T/2 \text{ and } \sum_{k=0}^{\infty} \sqrt{\delta_k} \leq R/2.$$

Substituting δ_k from (15.18) and observing that $\{\delta_k\}$ is a decreasing geometric sequence, we obtain that the following inequalities must be satisfied:

$$\frac{J_0^{\frac{2}{n+2}}}{a^{\frac{n}{n+2}} \theta^{\frac{4}{n+2}}} \leq c^2 T \text{ and } \left(\frac{J_0^{\frac{2}{n+2}}}{a^{\frac{n}{n+2}} \theta^{\frac{4}{n+2}}}\right)^{1/2} \leq cR,$$

for some $c = c(n) > 0$. There conditions can be satisfied by choosing θ as follows:

$$\theta^2 \geq \frac{a^{-n/2} J_0}{\left(c\sqrt{T}\right)^{n+2}} \text{ and } \theta^2 \geq \frac{a^{-n/2} J_0}{(cR)^{n+2}}.$$

Taking

$$\theta^2 = \frac{a^{-n/2} J_0}{c^{n+2} \min\left(\sqrt{T}, R\right)^{n+2}}$$

and recalling that $u_+^2(T,x) \leq \theta^2$, we finish the proof. $\square$

Exercises.

15.1. Fix $x_0 \in M$, $R > r > 0$ and let the ball $B(x_0, R)$ be relatively compact. Assume also that, for some $a, n > 0$, the Faber-Krahn inequality

$$\lambda_{\min}(U) \geq a\mu(U)^{-2/n}, \tag{15.19}$$

holds for any open set $U \subset B(x_0, r)$. Let $u(t,x)$ be a non-negative bounded C^2-function $(0,T) \times B(x_0, R)$, where $T > 0$, such that

(*i*) $\frac{\partial u}{\partial t} - \Delta_\mu u \leq 0$,

(*ii*) $u(t,\cdot) \to 0$ as $t \to 0$ in $L^2(B(x_0,R))$.

Prove that, for all $x \in B(x_0, r/2)$ and $t \in (0,T)$,

$$u(t,x) \leq C\|u\|_{L^\infty} \frac{\mu(B(x_0,R))^{\frac{1}{2}}}{(at)^{n/4}} \max\left(1, \frac{\sqrt{t}}{r}\right)^{\frac{n}{2}+1} \max\left(1, \frac{\delta}{\sqrt{t}}\right) e^{-\frac{\delta^2}{4t}} \tag{15.20}$$

where $\delta = R - r$ and $C = C(n)$.

15.2. Faber-Krahn inequality in balls

Here we show how the local geometry of a manifold and the mean value inequality of Theorem 15.1 can be used to give an alternative proof (of the improved version) of the key estimate (7.18) of Theorem 7.6.

THEOREM 15.4. *On any weighted manifold (M, g, μ) of dimension n there is a continuous function $r(x) > 0$ and a constant $a = a(n) > 0$ such that any ball $B(x, r(x))$ is relatively compact and, for any open set $U \subset B(x, r(x))$,*

$$\lambda_{\min}(U) \geq a\mu(U)^{-2/n}. \tag{15.21}$$

PROOF. For any point $x \in M$, one can always choose $\rho = \rho(x)$ so small that the ball $B(x, \rho(x))$ is relatively compact and is contained in a chart. Furthermore, reducing ρ further, one can achieve that the Riemannian metric $\mathbf{g}$ and the Euclidean metric $\mathbf{e}$ in this ball are in a fixed finite ratio, say

$$\frac{1}{2}\mathbf{e} \le \mathbf{g} \le 2\mathbf{e}$$

(cf. Lemma 3.24), and the density function Υ of measure μ with respect to the Lebesgue measure λ in the chart is almost constant, say

$$\sup \Upsilon \le 2 \inf \Upsilon.$$

Then the Faber-Krahn inequality (14.5) in $\mathbb{R}^n$ implies the Faber-Krahn inequality (15.21) in $B(x, \rho(x))$ with $n = \dim M$ and with a fixed constant $a = a(n)$.

To make a continuous function from $\rho(x)$, denote by $\rho_0(x)$ the supremum of all possible values of $\rho(x)$ such that the above conditions are satisfied, capped by 1 (the latter is to ensure the finiteness of $\rho_0(x)$). Let us show that the function $\rho_0(x)$ is continuous. Indeed, if $y \in B(x, \rho_0(x))$ then the ball $B(y, \rho(y))$ satisfies the above conditions with $\rho(y) = \rho_0(x) - d(x, y)$, which implies

$$\rho_0(y) \ge \rho_0(x) - d(x, y).$$

Swapping x and y, we obtain

$$|\rho_0(x) - \rho_0(y)| \le d(x, y),$$

which proves the continuity of $\rho_0(x)$.

Finally, setting $r(x) = \frac{1}{2}\rho_0(x)$, we obtain the required function $r(x)$. $\square$

REMARK 15.5. If M has bounded geometry (see Example 11.12) then the function $r(x)$ is uniformly bounded below by some $\varepsilon > 0$.

COROLLARY 15.6. *Under the hypotheses of* Theorem 15.1, *for any* $f \in L^2(M)$ *and for all* $t > 0$,

$$\sup_{B(x,R/2)} |P_t f| \le C a^{-n/4} \left(R^{-n/2} + t^{-n/4}\right) \|f\|_{L^2}, \tag{15.22}$$

where $C = C(n)$.

PROOF. The function $u(t, \cdot) = P_t f$ satisfies the hypotheses of Theorem 15.1. Since $\|u(t, \cdot)\|_{L^2} \le \|f\|_{L^2}$, we obtain

$$\int_0^t \int_{B(x,R)} u_+^2 d\nu \le t \|f\|_{L^2}^2$$

whence

$$u_+^2(t, x) \le \frac{C a^{-n/2} t}{\min\left(\sqrt{t}, R\right)^{n+2}} \|f\|_{L^2}^2 \le C a^{-n/2} \left(R^{-n} + t^{-n/2}\right) \|f\|_{L^2}^2.$$

Applying the same argument to $u = -P_t f$, we obtain a similar estimate for $|P_t f(x)|^2$. Finally, replacing x by any point $x' \in B(x, R/2)$ and applying the above estimates in the ball $B(x', R/2)$ instead of $B(x, R)$, we obtain (15.22). □

Now we can improve the inequality (7.18) of Theorem 7.6 as follows.

COROLLARY 15.7. *For any weighted manifold M of dimension n and for any set $K \Subset M$, there exists a constant C such that, for any $f \in L^2(M)$ and all $t > 0$,*

$$\sup_K |P_t f| \le C\left(1 + t^{-n/4}\right) \|f\|_{L^2(M)}. \tag{15.23}$$

PROOF. Let $r(x)$ be the function from Theorem 15.4. Then any ball $B(x, r(x))$ satisfies the hypotheses of Theorem 15.1, and we obtain by Corollary 15.6 that, for all $x \in M$,

$$|P_t f(x)| \le C\left(r(x)^{-n/2} + t^{-n/4}\right) \|f\|_{L^2},$$

where $C = C(n)$. Replacing $r(x)$ by $\inf_K r(x)$, which is positive by the continuity of $r(x)$, we obtain (15.23). □

15.3. The weighted L^2-norm of heat kernel

For any $D > 0$, define the function $E_D(t, x)$ on $\mathbb{R}_+ \times M$ by

$$E_D(t, x) = \int_M p_t^2(x, y) \exp\left(\frac{d^2(x, y)}{Dt}\right) d\mu(y). \tag{15.24}$$

This quantity may take value ∞. For example, in $\mathbb{R}^n$ we have $E_D(t, x) = \infty$ for any $D \le 2$.

If $D \ge 2$ then the function

$$\xi(t, y) = \frac{d^2(x, y)}{Dt}$$

satisfies (12.1). It follows from Theorem 12.1 that $E_D(t, x)$ is a non-increasing function of $t > 0$, because we can represent it in the form

$$E_D(t, x) = \int_M (P_{t-s} f)^2 e^{\xi(t,\cdot)} d\mu$$

where $0 < s < t$ and $f = p_s(x, \cdot) \in L^2$. Furthermore, by (12.3)

$$E_D(t, x) \le E_D(t_0, x) e^{-2\lambda_{\min}(M)(t - t_0)}, \tag{15.25}$$

for all $t \ge t_0 > 0$.

One can naturally extend the definition (15.24) to $D = \infty$ by setting

$$E_\infty(t, x) = \int_M p_t^2(x, y) d\mu(y) = p_{2t}(x, x).$$

Then (15.25) remains true also for $D = \infty$ (cf. (10.84) and Exercise 10.29). Observe also that $E_D(t, x)$ is non-increasing in $D \in (0, +\infty]$.

THEOREM 15.8. *Let $B(x,r)$ be a relatively compact ball on a weighted manifold M. Assume that the following Faber-Krahn inequality holds: for any open set $U \subset B(x,r)$,*

$$\lambda_1(U) \geq a\mu(U)^{-2/n}, \tag{15.26}$$

where a, n are positive constants. Then, for any $t > 0$ and $D \in (2, +\infty]$

$$E_D(t,x) \leq \frac{C(a\delta)^{-n/2}}{\min(t,r^2)^{n/2}}, \tag{15.27}$$

where $C = C(n)$ and $\delta = \min(D-2, 1)$.

By Theorem 15.4, for any $x \in M$ there exists $r > 0$ that satisfies the hypotheses of Theorem 15.8 with $n = \dim M$. In particular, this implies the following statement:

COROLLARY 15.9. *On any weighted manifold M and for all $t > 0$, $x \in M$, and $D > 2$,*

$$E_D(t,x) < \infty.$$

The main part of the proof of Theorem 15.8 is contained in the following lemma.

LEMMA 15.10. *Under the conditions of* Theorem 15.8, *set*

$$\rho(y) = (d(x,y) - r)_+.$$

Then, for all $t > 0$,

$$\int_M p_t^2(x,y)\exp\left(\frac{\rho^2(y)}{2t}\right)d\mu(y) \leq \frac{Ca^{-n/2}}{\min(t,r^2)^{n/2}}, \tag{15.28}$$

where $C = C(n)$.

PROOF. By Theorem 12.1, the left hand side of (15.28) is a non-increasing function of $t > 0$. Hence, it suffices to prove (15.28) for $t \leq r^2$. Fix a function $f \in L^2(M)$ and set $u = P_t f$. Applying the mean value inequality of Theorem 15.1 in cylinder $[t/2, t] \times B(x,r)$, we obtain

$$u^2(t,x) \leq \frac{Ca^{-n/2}}{t^{1+n/2}} \int_0^t \int_{B(x,r)} u^2(s,y)\,d\mu(y)\,ds. \tag{15.29}$$

Consider the function

$$\xi(s,y) = \frac{\rho^2(y)}{2(s-t)},$$

defined for $0 \leq s < t$ and $y \in M$. Since function ξ vanishes in $B(x,r)$, we can rewrite (15.29) as follows:

$$u^2(t,x) \leq \frac{Ca^{-n/2}}{t^{1+n/2}} \int_0^t \int_{B(x,r)} u^2 e^{\xi} d\mu ds. \tag{15.30}$$

Function ξ obviously satisfies the condition (12.1) of Theorem 12.1. Hence, the function

$$J(s) := \int_M u^2(s,\cdot)\, e^{\xi(s,\cdot)} d\mu$$

is non-increasing in $s \in [0,t)$, in particular, $J(s) \le J(0)$ for all $s \in [0,t)$. It follows from (15.30) that

$$u^2(t,x) \le C\,(at)^{-n/2}\, J(0).$$

Since

$$J(0) = \int_M f^2 \exp\left(-\frac{\rho^2}{2t}\right) d\mu,$$

we obtain

$$u^2(t,x) \le C\,(at)^{-n/2} \int_M f^2 \exp\left(-\frac{\rho^2}{2t}\right) d\mu. \tag{15.31}$$

Now choose function f as follows

$$f(y) = p_t(x,y) \exp\left(\frac{\rho^2(y)}{2t}\right) \varphi(y),$$

where φ is any cutoff function. Applying (15.31) with this function f, we obtain

$$\left(\int_M p_t^2(x,\cdot) \exp\left(\frac{\rho^2}{2t}\right) \varphi d\mu\right)^2 \le C\,(at)^{-n/2} \int_M p_t^2(x,\cdot) \exp\left(\frac{\rho^2}{2t}\right) \varphi^2 d\mu.$$

Using the inequality $\varphi^2 \le \varphi$ and cancelling by the integral in the right hand side, we obtain

$$\int_M p_t^2(x,\cdot) \exp\left(\frac{\rho^2}{2t}\right) \varphi d\mu \le C\,(at)^{-n/2},$$

whence (15.28) follows. $\square$

PROOF OF THEOREM 15.8. Since $E_D(t,x)$ is decreasing in D, it suffices to prove (15.27) for $D \le 3$. Since $E_D(t,x)$ is decreasing in t, it suffices to prove (15.27) for $t \le r^2$.

Set $\delta = D - 2$ and observe that $\sqrt{\delta t} \le r$, so that the Faber-Krahn inequality (15.26) holds in $B\left(x, \sqrt{\delta t}\right)$. Applying Lemma 15.10 with $\sqrt{\delta t}$ in place of r, we obtain

$$\int_M p_t^2(x,y) \exp\left(\frac{\left(d(x,y) - \sqrt{\delta t}\right)_+^2}{2t}\right) d\mu(y) \le \frac{C a^{-n/2}}{(\delta t)^{n/2}}. \tag{15.32}$$

Using the elementary inequality

$$\frac{a^2}{t} + \frac{b^2}{s} \ge \frac{(a+b)^2}{t+s},$$

which is true for real a, b and positive t, s, we obtain

$$\frac{\left(d(x,y)-\sqrt{\delta t}\right)_+^2}{2t}+\frac{\left(\sqrt{\delta t}\right)^2}{\delta t}\geq\frac{d^2(x,y)}{(2+\delta)\,t},$$

whence

$$\frac{\left(d(x,y)-\sqrt{\delta t}\right)_+^2}{2t}\geq\frac{d^2(x,y)}{Dt}-1.$$

Substituting into (15.32), we obtain (15.27). $\square$

15.4. Faber-Krahn inequality in unions of balls

In this Section we demonstrate the heat kernel techniques for merging the Faber-Krahn inequalities, that is, obtaining the Faber-Krahn inequality in a union Ω of a family of balls assuming that it holds in each ball separately. At the same token, we obtain non-trivial lower estimate for higher eigenvalues, similarly to Corollary 14.26 but in a different setting.

THEOREM 15.11. *Let a, n, r be positive numbers and $\{x_i\}_{i\in I}$ be a family of points on M where I is any index set. Assume that, for any $i \in I$ the ball $B(x_i, r)$ is relatively compact and, for any open set $U \subset B(x_i, r)$,*

$$\lambda_1(U)\geq a\mu(U)^{-2/n}. \tag{15.33}$$

Let Ω be the union of all the balls $B(x_i, \frac{1}{2}r)$, $i \in I$. Then, for any open set $U \subset \Omega$ with finite measure (see Fig. 15.2), the spectrum of $\mathcal{L}^U$ is discrete and

$$\lambda_k(U)\geq ca\left(\frac{k}{\mu(U)}\right)^{2/n}$$

for any k such that

$$k\geq Ca^{-n/2}r^{-n}\mu(U),$$

where c, C are positive constants depending only on n. In particular,

$$\mu(U)\leq C^{-1}a^{n/2}r^n \implies \lambda_1(U)\geq ca\mu(U)^{-2/n}$$

PROOF. If $x \in \Omega$ then $x \in B(x_i, \frac{1}{2}r)$ for some $i \in I$. Therefore, $B(x, \frac{1}{2}r) \subset B(x_i, r)$ which implies that the Faber-Krahn inequality (15.33) holds for any open set $U \subset B(x, \frac{1}{2}r)$. By Theorem 15.8 with $D = \infty$, we obtain

$$p_t(x,x)\leq\frac{Ca^{-n/2}}{\min(t,r^2)^{n/2}}. \tag{15.34}$$

Fix now an open set $U \subset \Omega$ with finite measure. It follows from (15.34) that

$$\sup_{x\in U}p_t^U(x,x)\leq Ca^{-n/2}\left(t^{-n/2}+r^{-n}\right)=\frac{1}{\gamma(t)}$$

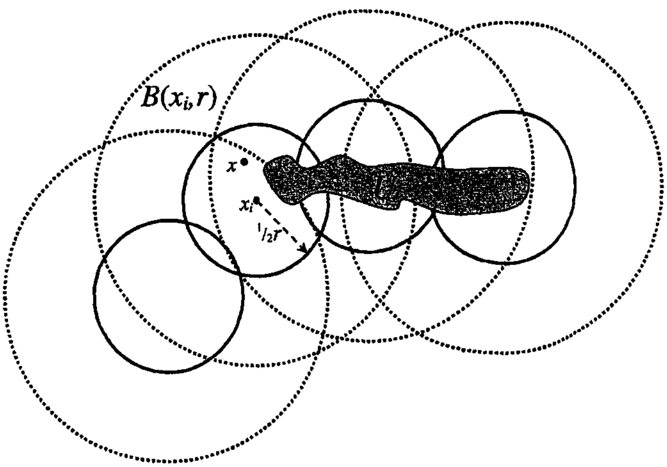

FIGURE 15.2. Set U

where

$$\gamma(t) = \frac{C^{-1}a^{n/2}}{t^{-n/2} + r^{-n}}.$$

By Corollary 14.26, the spectrum of $\mathcal{L}^U$ is discrete and

$$\lambda_k(U) \geq c\Lambda\left(\frac{\mu(U)}{k}\right), \tag{15.35}$$

where $c = c(n) > 0$ and Λ is the L-transform of γ.

If $v = \gamma(t)$ for some $t \leq r^2$ then

$$v \geq \frac{(at)^{n/2}}{2C},$$

and by (14.27)

$$\Lambda(v) = \frac{\gamma'(t)}{\gamma(t)} = \frac{n}{2}\frac{t^{-n/2-1}}{t^{-n/2} + r^{-n}} \geq \frac{n}{4t} \geq cav^{-2/n},$$

where $c = c(n) > 0$. The condition $t \leq r^2$ is equivalent to $\gamma(t) \leq (2C)^{-1} a^{n/2} r^n$. Hence, we conclude that

$$v \leq (2C)^{-1} a^{n/2} r^n \implies \Lambda(v) \geq cav^{-2/n},$$

which together with (15.35) finishes the proof. □

COROLLARY 15.12. *On any connected weighted manifold M and for any relatively compact open set $\Omega \subset M$, such that $M \setminus \overline{\Omega}$ is non-empty, there exists $a > 0$ such for any open subset $U \subset \Omega$ and any $k \in \mathbb{N}$,*

$$\lambda_k(U) \geq a\left(\frac{k}{\mu(U)}\right)^{2/n}, \tag{15.36}$$

where $n = \dim M$. In particular, the Faber-Krahn inequality holds in Ω with function $\Lambda(v) = av^{-2/n}$.

PROOF. Since the function $r(x)$ from Theorem 15.15 is continuous, we have

$$R := \inf_{x \in \Omega} r(x) > 0.$$

By Theorem 15.15, the family of ball $\{B(x, R)\}_{x \in \Omega}$ satisfies the hypotheses of Theorem 15.11 with $n = \dim M$. By Theorem 15.11, we obtain

$$\lambda_k(U) \geq c_0 \left(\frac{k}{\mu(U)} \right)^{2/n},$$

where $c_0 = c_0(n) > 0$, provided k satisfies $k \geq C\mu(U)$, where $C = C(n, \Omega)$.

Since $M \setminus \overline{\Omega}$ is non-empty, by Theorem 10.22 we have $\lambda_1(\Omega) > 0$. Therefore, if $k \leq C\mu(U)$ then

$$\lambda_k(U) \geq \lambda_1(\Omega) \geq \lambda_1(\Omega) \left(C^{-1} \frac{k}{\mu(U)} \right)^{2/n} = c_1 \left(\frac{k}{\mu(U)} \right)^{2/n}$$

where $c_1 = \lambda_1(\Omega) C^{-2/n}$. Hence, (15.36) holds with $a = \min(c_0, c_1)$ for all $k \geq 1$. □

Exercises.

15.2. Prove that the Faber-Krahn inequality holds on a weighted n-dimensional manifold M with function

$$\Lambda(v) = \begin{cases} cv^{-2/n}, & v < v_0, \\ 0, & v \geq v_0, \end{cases}$$

where c, v_0 are some positive constants, provided M belongs to one of the following classes:

(a) M is compact;

(b) M has bounded geometry (see Example 11.12).

REMARK. If M is non-compact and has bounded geometry then the Faber-Krahn function Λ can be improved by setting $\Lambda(v) = cv^{-2}$ for $v \geq v_0$ – see [**148**].

15.5. Off-diagonal upper bounds

Our main result in this section is Theorem 15.14 that provides Gaussian upper bounds of the heat kernel assuming the validity of the Faber-Krahn inequalities in some balls. It is preceded by a lemma showing how the weighted norm $E_D(t, x)$, defined by (15.24), can be used to obtain pointwise upper bounds of the heat kernel.

LEMMA 15.13. *For any weighted manifold M, for any $D > 0$ and all $x, y \in M$, $t \geq t_0 > 0$, the following inequality is true:*

$$p_t(x,y) \leq \sqrt{E_D(\tfrac{1}{2}t_0, x) E_D(\tfrac{1}{2}t_0, y)} \exp\left(-\frac{\rho^2}{2Dt} - \lambda(t - t_0) \right), \qquad (15.37)$$

where $\rho = d(x,y)$ and $\lambda = \lambda_{\min}(M)$.

In particular, setting $t = t_0$ we obtain

$$p_t(x,y) \leq \sqrt{E_D(\tfrac{1}{2}t, x) E_D(\tfrac{1}{2}t, y)} \exp\left(-\frac{d^2(x,y)}{2Dt} \right). \qquad (15.38)$$

PROOF. For any points $x, y, z \in M$, let us denote $\alpha = d(y,z)$, $\beta = d(x,z)$ and $\gamma = d(x,y)$ (see Fig. 15.3). By the triangle inequality, we have $\alpha^2 + \beta^2 \geq \frac{1}{2}\gamma^2$.

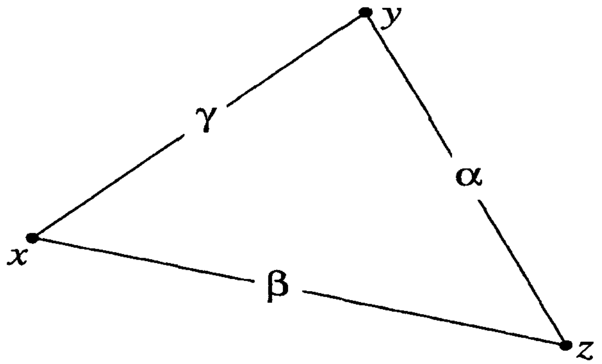

FIGURE 15.3. Distances α, β, γ

Applying the semigroup identity (7.51), we obtain

$$\begin{aligned}
p_t(x,y) &= \int_M p_{t/2}(x,z)p_{t/2}(y,z)d\mu(z) \\
&\leq \int_M p_{t/2}(x,z)e^{\frac{\beta^2}{Dt}}p_{t/2}(y,z)e^{\frac{\alpha^2}{Dt}}e^{-\frac{\gamma^2}{2Dt}}d\mu(z) \\
&\leq \left(\int_M p_{t/2}^2(x,z)e^{\frac{2\beta^2}{Dt}}d\mu(z)\right)^{\frac{1}{2}}\left(\int_M p_{t/2}^2(y,z)e^{\frac{2\alpha^2}{Dt}}d\mu(z)\right)^{\frac{1}{2}}e^{-\frac{\gamma^2}{2Dt}} \\
&= \sqrt{E_D(\tfrac{1}{2}t,x)E_D(\tfrac{1}{2}t,y)}\exp\left(-\frac{d^2(x,y)}{2Dt}\right),
\end{aligned}$$

which proves (15.38). Combining (15.38) with (15.25), we obtain (15.37). □

THEOREM 15.14. *Let M be a weighted manifold and let $\{B(x_i, r_i)\}_{i\in I}$ be a family of relatively compact balls in M, where I is an arbitrary index set. Assume that, for any $i \in I$, the Faber-Krahn inequality holds*

$$\lambda_{\min}(U) \geq a_i\mu(U)^{-2/n} \tag{15.39}$$

for any open set $U \subset B(x_i, r_i)$, where $a_i > 0$. Let Ω be the union of all the balls $B(x_i, \frac{1}{2}r_i)$, $i \in I$. Then, for all $x, y \in \Omega$ and $t \geq t_0 > 0$,

$$p_t(x,y) \leq \frac{C(n)\left(1+\frac{\rho^2}{t}\right)^{n/2}\exp\left(-\frac{\rho^2}{4t}-\lambda(t-t_0)\right)}{\left(a_ia_j\min(t_0,r_i^2)\min(t_0,r_j^2)\right)^{n/4}}, \tag{15.40}$$

where $\rho = d(x,y)$, i *and* j *are the indices such that* $x \in B(x_i, \frac{1}{2}r_i)$ *and* $y \in B(x_j, \frac{1}{2}r_j)$, *and* $\lambda = \lambda_{\min}(M)$.

PROOF. If $x \in B(x_i, \frac{1}{2}r_i)$ then $B(x, \frac{1}{2}r_i) \subset B(x_i, r_i)$ so that the Faber-Krahn inequality (15.39) holds for any open set $U \subset B(x, \frac{1}{2}r_i)$. Applying Theorem 15.8, we obtain, for all $t > 0$ and $D > 2$,

$$E_D(t,x) \leq \frac{C(a_i\delta)^{-n/2}}{\min(t, r_i)^{n/2}}.$$

Using a similar inequality for $E_D(t,y)$, we obtain by (15.37)

$$p_t(x,y) \leq \frac{C\delta^{-n/2}\exp\left(-\frac{\rho^2}{2Dt} - \lambda_{\min}(M)(t-t_0)\right)}{\left(a_i a_j \min(t_0, r_i^2)\min(t_0, r_j^2)\right)^{n/4}}. \tag{15.41}$$

Setting here

$$\delta = \left(1 + \frac{\rho^2}{t}\right)^{-1} \tag{15.42}$$

and, consequently, $D = 2 + \delta$, we obtain that

$$\frac{\rho^2}{4t} - \frac{\rho^2}{2Dt} = \frac{\delta}{4D}\frac{\rho^2}{t} < 1,$$

so that the term $\frac{\rho^2}{2Dt}$ in (15.41) can be replaced by $\frac{\rho^2}{4t}$. Substituting (15.42) into (15.41), we obtain (15.40). □

COROLLARY 15.15. *On any weighted manifold* M *there is a continuous function* $r(x) > 0$ *such that, for all* $x, y \in M$ *and* $t \geq t_0 > 0$,

$$p_t(x,y) \leq \frac{C\left(1 + \frac{\rho^2}{t}\right)^{n/2}\exp\left(-\frac{\rho^2}{4t} - \lambda(t-t_0)\right)}{\min(t_0, r(x)^2)^{n/4}\min(t_0, r(y)^2)^{n/4}}, \tag{15.43}$$

where $n = \dim M$, $\rho = d(x,y)$, $\lambda = \lambda_{\min}(M)$, *and* $C = C(n)$.

PROOF. Let $r(x)$ be the function from Theorem 15.4. Then the family $\{B(x, r(x))\}_{x\in M}$ of balls satisfies the hypotheses of Theorem 15.14, and (15.43) follows from (15.40). □

COROLLARY 15.16. *For any weighted manifold, and for all* $x, y \in M$,

$$\limsup_{t\to 0+} 4t \log p_t(x,y) \leq -d^2(x,y). \tag{15.44}$$

PROOF. Indeed, setting in (15.43) $t = t_0 < \min(r(x), r(y))$, we obtain

$$t\log p_t(x,y) \leq t\log\left(Ct^{-n/2}\right) + t\log\left(1 + \frac{\rho^2}{t}\right)^{n/2} - \frac{\rho^2}{4}.$$

Letting $t \to 0$, we obtain (15.44). □

In fact, the matching lower bound in (15.44) is also true. More precisely, on any weighted manifold the following asymptotic takes place:

$$\lim_{t\to 0+} 4t \log p_t(x,y) = -d^2(x,y). \tag{15.45}$$

Hence, the term $\frac{\rho^2}{4t}$ in (15.43) is sharp. Note also that if $\lambda_{\min}(M) > 0$ then the term $\lambda_{\min}(M)\,t$ in (15.43) gives a sharp exponential rate of decay of $p_t(x,y)$ as $t \to \infty$ – see Theorem 10.24. The term $\left(1+\frac{\rho^2}{t}\right)^{n/2}$ in (15.43) is almost sharp: as it was shown in [**277**], on n-dimensional sphere the following asymptotic is true

$$p_t(x,y) \sim \frac{c}{t^{n/2}}\left(\frac{\rho^2}{t}\right)^{n/2-1} \exp\left(-\frac{\rho^2}{4t}\right),$$

provided x and y are the conjugate points and $t \to 0$.

COROLLARY 15.17. *Let M be a complete weighted manifold of dimension n. Then the following conditions are equivalent:*

(a) *The Faber-Krahn inequality holds on M with function $\Lambda(v) = cv^{-2/n}$ for some positive constant c.*

(b) *The heat kernel on M satisfies for all $x,y \in M$ and $t \geq t_0 > 0$ the estimate*

$$p_t(x,y) \leq \frac{C}{t_0^{n/2}}\left(1+\frac{\rho^2}{t}\right)^{n/2} \exp\left(-\frac{\rho^2}{4t} - \lambda(t-t_0)\right), \tag{15.46}$$

where $\rho = d(x,y)$, $\lambda = \lambda_{\min}(M)$, and C is a positive constant.

(c) *The heat kernel on M satisfies the estimate*

$$p_t(x,x) \leq Ct^{-n/2} \tag{15.47}$$

for all $x \in M$, $t > 0$, and for some positive constant C.

PROOF. The implication $(a) \Rightarrow (b)$ follows from Theorem 15.14 by taking $r_i = r_j = \sqrt{t}$ (by the completeness of M, all balls $B(x,\sqrt{t})$ are relatively compact). The implication $(b) \Rightarrow (c)$ is trivial, and $(c) \Rightarrow (a)$ is true by Theorem 14.20 (or Corollary 14.26). □

Sometimes it is convenient to use (15.46) in the following form:

$$p_t(x,y) \leq \frac{C}{\min(t,T)^{n/2}}\left(1+\frac{\rho^2}{t}\right)^{n/2} \exp\left(-\frac{\rho^2}{4t} - \lambda(t-T)_+\right), \tag{15.48}$$

for all $t, T > 0$, which is obtained from (15.46) by setting $t_0 = \min(t,T)$.

The statement of Corollary 15.17 remains true without assuming the completeness of M. Indeed, the only place where the completeness was used is $(a) \Rightarrow (b)$, and this can be proved by a different method without completeness – see Exercise 16.4.

Examples of manifolds satisfying (a) were mentioned in Remark 14.24. Hence, on such manifolds, the heat kernel satisfies the estimates (15.46) and (15.47).

Observe that, under the conditions of Corollary 15.17, the heat kernel satisfies also the following estimate, for any $\varepsilon > 0$ and all $x, y \in M$, $t > 0$:

$$p_t(x,y) \le Ct^{-n/2} \exp\left(-\frac{\rho^2}{(4+\varepsilon)\,t}\right), \tag{15.49}$$

where the constant C depends on ε. Indeed, this follows from (15.46) by setting $t_0 = t$ and then using the inequality

$$(1+\xi)^{n/2} \exp\left(-\frac{\xi}{4}\right) \le C_{\varepsilon,n} \exp\left(-\frac{\xi}{4+\varepsilon}\right), \tag{15.50}$$

which is true for all $\xi \ge 0$.

COROLLARY 15.18. *Let (M, g, μ) be a complete weighted manifold whose heat kernel satisfies any of the equivalent upper bounds* (15.46), (15.47), (15.49). *If a weighted manifold $(\widetilde{M}, \widetilde{\mathrm{g}}, \widetilde{\mu})$ is quasi-isometric to (M, g, μ) then the heat kernel on $(\widetilde{M}, \widetilde{\mathrm{g}}, \widetilde{\mu})$ satisfies all the estimates* (15.46), (15.47), (15.49).

PROOF. Thanks to Corollary 15.17, it suffices to prove that the Faber-Krahn inequality with function $\Lambda(v) = cv^{-2/n}$ is stable under quasi-isometry. For simplicity of notation, let us identify M and $\widetilde{M}$ as smooth manifolds. It follows from Exercise 10.7 or 3.44 that, for any open set $U \subset M$,

$$\lambda_{\min}(U) \simeq \widetilde{\lambda}_{\min}(U),$$

where $\lambda_{\min}(U)$ and $\widetilde{\lambda}_{\min}(U)$ are the bottoms of the spectrum of the Dirichlet Laplacians in U on the manifolds (M, g, μ) and $(\widetilde{M}, \widetilde{\mathrm{g}}, \widetilde{\mu})$ respectively. By Exercise 3.44, we have also

$$\mu(U) \simeq \widetilde{\mu}(U).$$

Hence, the Faber-Krahn inequality

$$\lambda_{\min}(U) \ge c\mu(U)^{-2/n}$$

implies

$$\widetilde{\lambda}_{\min}(U) \ge \widetilde{c}\widetilde{\mu}(U)^{-2/n},$$

which was to be proved. □

Exercises.

In the following exercises, we use the notation $\rho = d(x,y)$.

15.3. Prove that, on any weighted manifold M there is a positive continuous function $F(x,s)$ on $M \times \mathbb{R}_+$, which is monotone increasing in s and such that the heat kernel on M satisfies the following estimate

$$p_t(x,y) \le \frac{C\left(1+\frac{\rho^2}{t}\right)^{n/2}}{F\left(x,\sqrt{t}\right)^{1/2} F\left(y,\sqrt{t}\right)^{1/2}} \exp\left(-\frac{\rho^2}{4t}\right), \tag{15.51}$$

for all $x, y \in M$ and $t > 0$,where $n = \dim M$ and $C = C(n)$ (cf. Exercise 16.3).

15.4. Prove that if M has bounded geometry then, for some constant C,

$$p_t(x,y) \leq \frac{C\left(1+\frac{\rho^2}{t}\right)^{n/2}}{\min(1,t)^{n/2}} \exp\left(-\frac{\rho^2}{4t}\right), \tag{15.52}$$

for all $x, y \in M$ and $t > 0$.

15.5. Under the hypotheses of Corollary 15.17, assume in addition that $n > 2$ and

$$\mu(B(x,r)) \leq Cr^n$$

for all $r > 0$. Prove that each of the conditions $(a) - (c)$ is equivalent to the following estimate of the Green function:

$$g(x,y) \leq Cd(x,y)^{2-n},$$

for all distinct $x, y \in M$.

REMARK. Note for comparison that the Faber-Krahn inequality of Corollary 15.17 implies $\mu(B(x,r)) \geq \operatorname{const} r^n$ – see Exercise 14.5.

15.6. Under conditions of Corollary 15.17, let $n \geq 2$ and $\lambda := \lambda_{\min}(M) > 0$. Prove that, for any $\varepsilon \in (0,1)$, the Green function of M satisfies the estimate

$$g(x,y) \leq Ce^{-(1-\varepsilon)\sqrt{\lambda}\rho} \begin{cases} \rho^{2-n}, & n > 2, \\ \left(1+\log_+ \frac{1}{\rho}\right), & n = 2, \end{cases} \tag{15.53}$$

for all $x \neq y$, where $C = C(n, \varepsilon, \lambda, c)$.

15.7. Let M be an arbitrary weighted manifold of dimension $n \geq 2$. Prove that if the Green function of M is finite then, for any $x \in M$ and for all y close enough to x,

$$g(x,y) \leq C \begin{cases} \rho^{2-n}, & n > 2, \\ \log \frac{1}{\rho}, & n = 2, \end{cases} \tag{15.54}$$

where $C = C(n)$.

15.6. Relative Faber-Krahn inequality and Li-Yau upper bounds

DEFINITION 15.19. We say that a weighted manifold M admits *the relative Faber-Krahn inequality* if there exist positive constants b and ν such that, for any ball $B(x,r) \subset M$ and for any relatively compact open set $U \subset B(x,r)$,

$$\lambda_1(U) \geq \frac{b}{r^2}\left(\frac{\mu(B(x,r))}{\mu(U)}\right)^{2/\nu}. \tag{15.55}$$

In $\mathbb{R}^n$ (15.55) holds with $\nu = n$, because it amounts to (14.5). It is possible to prove that the relative Faber-Krahn inequality holds on any complete non-compact manifold of non-negative Ricci curvature – see the Notes at the end of this Chapter for bibliographic references.

DEFINITION 15.20. We say that the measure μ on M is *doubling* if the *volume function*

$$V(x,r) := \mu(B(x,r))$$

satisfies the inequality

$$V(x,2r) \leq CV(x,r), \tag{15.56}$$

for some constant C and for all $x \in M$ and $r > 0$.

If (15.56) holds then one also says that the manifold M satisfies the *doubling volume property*.

Now we can state and prove the main theorem of this section.

THEOREM 15.21. *Let $(M, \mathbf{g}, \mu)$ be a connected, complete, non-compact manifold. Then the following conditions are equivalent:*

(a) *M admits the relative Faber-Krahn inequality* (15.55).

(b) *The measure μ is doubling and the heat kernel satisfies the upper bound*

$$p_t(x,y) \le \frac{C\left(1+\frac{\rho^2}{t}\right)^{\nu/2}}{V\left(x,\sqrt{t}\right)^{1/2} V\left(y,\sqrt{t}\right)^{1/2}} \exp\left(-\frac{\rho^2}{4t}\right), \tag{15.57}$$

for all for all $x, y \in M$, $t > 0$, and for some positive constants C, ν, where $\rho = d(x,y)$.

(c) *The measure μ is doubling and the heat kernel satisfies the inequality*

$$p_t(x,x) \le \frac{C}{V\left(x,\sqrt{t}\right)}, \tag{15.58}$$

for all for all $x \in M$, $t > 0$, and for some constant C.

REMARK 15.22. For the implication $(a) \Rightarrow (b)$, the value of ν in (15.57) is the same as in (15.55). In this case, the estimate (15.57) can be slightly improved by replacing $\nu/2$ by $(\nu-1)/2$ – see Exercise 15.9.

REMARK 15.23. As we will see later (cf. Corollary 16.7), under any of the conditions $(a)-(c)$ of Theorem 15.21 we have also the matching lower bound

$$p_t(x,x) \ge \frac{c}{V\left(x,\sqrt{t}\right)},$$

for all $x \in M$, $t > 0$ and for some constant $c > 0$.

PROOF. $(a) \Longrightarrow (b)$ By Definition 15.19, we have, for any ball $B(x,r) \subset M$ and any relatively compact open set $U \subset B(x,r)$,

$$\lambda_1(U) \ge a(x,r)\,\mu(U)^{-2/\nu}, \tag{15.59}$$

where

$$a(x,r) = \frac{b}{r^2} V(x,r)^{2/\nu}. \tag{15.60}$$

Applying Theorem 15.14 with the family of balls $\{B(x,r)\}_{x\in M}$, we obtain that, for all $x, y \in M$ and $r, t > 0$,

$$p_t(x,y) \le \frac{C\left(1+\frac{\rho^2}{t}\right)^{\nu/2}}{\left(a(x,r)\,a(y,r)\min(t,r^2)\min(t,r^2)\right)^{\nu/4}} \exp\left(-\frac{\rho^2}{4t}\right).$$

Note that r is arbitrary here. Choosing $r = \sqrt{t}$ and substituting a from (15.60) we obtain (15.57). An alternative proof of this part will be given in Section 16.2.

By Exercise 14.5, the Faber-Krahn inequality (15.59) in the ball $B(x,R)$ implies that, for any $r \le R$.

$$\mu(B(x,r)) \ge ca(x,R)^{\nu/2} r^{\nu},$$

where $c = c(\nu) > 0$. Substituting $a(x,R)$ from (15.60), we obtain

$$V(x,r) \ge c' \left(\frac{r}{R}\right)^{\nu} V(x,R), \tag{15.61}$$

where the doubling property follows.

$(b) \Longrightarrow (c)$ Trivial.

$(c) \Longrightarrow (a)$ It follows easily from the volume doubling property (15.56) that there exists $\nu > 0$ such that

$$\frac{V(x,R)}{V(x,r)} \le C\left(\frac{R}{r}\right)^{\nu}, \tag{15.62}$$

for all $x \in M$ and $0 < r \le R$. Fix a ball $\Omega = B(x,r)$ and consider an open set $U \subset \Omega$. Then, by (14.58) and (15.58),

$$e^{-\lambda_1(U)t} \le \int_U p_t^U(y,y)\, d\mu(y) \le C \int_U \frac{d\mu(y)}{V(y,\sqrt{t})}. \tag{15.63}$$

For any $y \in U$ and $t \le r^2$, we have by (15.62)

$$\frac{V(x,r)}{V(y,\sqrt{t})} \le \frac{V(y,2r)}{V(y,\sqrt{t})} \le C\left(\frac{r}{\sqrt{t}}\right)^{\nu}.$$

Therefore,

$$\int_U \frac{d\mu(y)}{V(y,\sqrt{t})} \le \frac{\mu(U)}{V(x,r)} C\left(\frac{r}{\sqrt{t}}\right)^{\nu}. \tag{15.64}$$

Now choose t from the condition

$$\left(\frac{r}{\sqrt{t}}\right)^{\nu} = \varepsilon \frac{V(x,r)}{\mu(U)} \tag{15.65}$$

for some $\varepsilon \in (0,1)$. Since we need to have $t \le r^2$, we have to assume for a while that

$$\mu(U) \le \varepsilon V(x,r). \tag{15.66}$$

If so then we obtain from (15.63), (15.64), and (15.65) that

$$\lambda_1(U) \ge \frac{1}{t} \log \frac{1}{C^2 \varepsilon}.$$

Choosing $\varepsilon = e^{-1}C^{-2}$ and evaluating t from (15.65), we obtain

$$\lambda_1(U) \ge \frac{b}{r^2}\left(\frac{V(x,r)}{\mu(U)}\right)^{2/\nu} \tag{15.67}$$

where $b > 0$ is a positive constant, which was to be proved.

We are left to extend (15.67) to any $U \subset B(x,r)$ without the restriction (15.66). For that, we will use the following fact.

CLAIM. *If M is connected, complete, non-compact and satisfies the doubling volume property then there are positive numbers c, ν' such that*

$$\frac{V(x,R)}{V(x,r)} \geq c\left(\frac{R}{r}\right)^{\nu'} \tag{15.68}$$

for all $x \in M$ and $0 < r \leq R$.

The inequality (15.68) is called the *reverse volume doubling.* Assume first $R = 2r$. The connectedness of M implies that there is a point $y \in M$ such that $d(x,y) = \frac{3}{2}r$. Then $B\left(y, \frac{1}{2}r\right) \leq B(x,2r) \setminus B(x,r)$, which implies

$$V(x,2r) \geq V(x,r) + V(y, \frac{1}{2}r).$$

By (15.56), we have

$$\frac{V(x,r)}{V(y,\frac{1}{2}r)} \leq \frac{V(y,4r)}{V\left(y,\frac{1}{2}r\right)} \leq C^3,$$

whence

$$V(x,2r) \geq \left(1 + C^{-3}\right) V(x,r).$$

Iterating this inequality, we obtain (15.68) with $\nu' = \log_2\left(1 + C^{-3}\right)$.

Returning to the proof of (15.67), find $R > r$ so big that

$$\frac{V(x,R)}{V(x,r)} \geq \frac{1}{\varepsilon},$$

were ε was chosen above. Due to (15.68), we can take R in the form $R = Ar$, where A is a constant, depending on the other constants in question. Then $U \subset B(x,R)$ and

$$\mu(U) \leq \varepsilon V(x,R),$$

which implies by the first part of the proof that

$$\lambda_1(U) \geq \frac{b}{R^2}\left(\frac{V(x,R)}{\mu(U)}\right)^{2/\nu} \geq \frac{b}{(Ar)^2}\left(\frac{V(x,r)}{\mu(U)}\right)^{2/\nu},$$

which was to be proved. □

Using (15.61), we obtain, for $\rho = d(x,y)$,

$$\frac{V\left(x,\sqrt{t}\right)}{V\left(y,\sqrt{t}\right)} \leq \frac{V\left(y,\sqrt{t}+\rho\right)}{V\left(y,\sqrt{t}\right)} \leq C\left(\frac{\sqrt{t}+\rho}{\sqrt{t}}\right)^{\nu} \leq C\left(1+\frac{\rho^2}{t}\right)^{\nu/2}.$$

Replacing $V\left(y,\sqrt{t}\right)$ in (15.57) according to the above estimate, we obtain

$$p_t(x,y) \leq \frac{C}{V\left(x,\sqrt{t}\right)}\left(1+\frac{\rho^2}{t}\right)^{3\nu/4} \exp\left(-\frac{\rho^2}{4t}\right) \tag{15.69}$$

where $C = C(b,\nu)$. Absorbing the term $\left(1+\frac{\rho^2}{t}\right)^{3\nu/4}$ in (15.69) into the exponential by means of the inequality (15.50), we obtain

$$p_t(x,y) \le \frac{C}{V\left(x,\sqrt{t}\right)} \exp\left(-\frac{\rho^2}{(4+\varepsilon)\,t}\right), \tag{15.70}$$

where $\varepsilon > 0$ is arbitrary and $C = C(b,\nu,\varepsilon)$.

It follows from Theorem 15.21 that all the estimates (15.57), (15.58), (15.69), (15.70) are equivalent[1], provided the measure μ is doubling.

DEFINITION 15.24. Each of the equivalent estimates (15.57), (15.58), (15.69), (15.70) are referred to as the *Li-Yau upper estimate* of the heat kernel.

COROLLARY 15.25. *Assume that (M, g, μ) is connected, complete, non-compact weighted manifold and measure μ be doubling. Let $(\widetilde{M}, \widetilde{\mathrm{g}}, \widetilde{\mu})$ be another weighed manifold that is quasi-isometric to M. If the heat kernel on M admits the the Li-Yau upper estimate then so does the heat kernel on $\widetilde{M}$. In other words, the Li-Yau upper estimates are stable under quasi-isometry.*

Consequently, the Li-Yau upper estimate holds on manifolds that are quasi-isometric to complete non-compact manifolds of non-negative Ricci curvature.

PROOF. In the view of Theorem 15.21, it suffices to prove that the relative Faber-Krahn inequality (15.55) is stable under quasi-isometry. For simplicity of notation, let us identify M and $\widetilde{M}$ as smooth manifolds. Let $\widetilde{d}$ be the geodesic distance of the metric $\widetilde{\mathrm{g}}$, $\widetilde{B}(x,r)$ be a metric ball of $\widetilde{d}$. By Exercise 3.44, there is a constant $K > 1$ such that

$$K^{-1} d(x,y) \le \widetilde{d}(x,y) \le K d(x,y).$$

Hence, any open set $U \subset \widetilde{B}(x,r)$ is also contained in $B(x,Kr)$, and we obtain by the relative Faber-Krahn inequality on (M, g, μ) that

$$\begin{aligned} \lambda_{\min}(U) &\ge \frac{b}{(Kr)^2}\left(\frac{\mu\left(B(x,Kr)\right)}{\mu(U)}\right)^{2/\nu} \\ &\ge \frac{b}{K^2 r^2}\left(\frac{\mu(\widetilde{B}(x,r))}{\mu(U)}\right)^{2/\nu} \\ &\simeq \frac{1}{r^2}\left(\frac{\widetilde{\mu}(\widetilde{B}(x,r))}{\widetilde{\mu}(U)}\right)^{2/\nu}. \end{aligned} \tag{15.71}$$

[1] Note that the equivalence of (15.58) and (15.70) can be proved without the hypothesis of completeness – see Exercise 16.5.

Let $\widetilde{\lambda}_{\min}(U)$ be the bottom of the spectrum of the Dirichlet Laplacian in U on the manifold $(\widetilde{M}, \widetilde{\mathbf{g}}, \widetilde{\mu})$. By Exercise 10.7, we have that $\widetilde{\lambda}_{\min}(U) \simeq \lambda_{\min}(U)$, which together with (15.71) implies the relative Faber-Krahn inequality on $(\widetilde{M}, \widetilde{\mathbf{g}}, \widetilde{\mu})$. □

Exercises.

15.8. Let M be a complete manifold satisfying the relative Faber-Krahn inequality. Prove that the Green function $g(x,y)$ is finite if and only if, for all $x \in M$,

$$\int^{\infty} \frac{r dr}{V(x,r)} < \infty.$$

Prove also the estimate for all $x, y \in M$:

$$g(x,y) \leq C \int_{d(x,y)}^{\infty} \frac{r dr}{V(x,r)}.$$

15.9. Under conditions of Theorem 15.21, prove that the relative Faber-Krahn inequality (15.55) implies the following enhanced version of (15.57):

$$p_t(x,y) \leq \frac{C\left(1+\frac{\rho^2}{t}\right)^{\frac{\nu-1}{2}}}{V\left(x,\sqrt{t}\right)^{1/2} V\left(y,\sqrt{t}\right)^{1/2}} \exp\left(-\frac{\rho^2}{4t}\right). \tag{15.72}$$

HINT. *Use the mean-value inequality of Theorem 15.1 and (12.18).*

Notes

The L^2-mean value inequality of Section 15.1 was introduced and proved by J.Moser [**279**], [**280**] for solutions of uniformly elliptic and parabolic equations in $\mathbb{R}^n$. Moser used for the proof his celebrated iteration techniques when one estimates the L^∞-norm of a solution via its L^2-norm in a larger ball my means of a series of iterations through L^p-norms with $p \to \infty$. A possibility to increase the exponent of p comes from the Sobolev inequality.

Here we use a different approach, which goes back to de Giorgi and employs the Faber-Krahn inequality in the level sets of a solution. We follow the account of this method in [**145**], which in turn is based on [**241**] and [**242**]. A more general mean value inequality under the Faber-Krahn inequality with an arbitrary function $\Lambda(v)$ was proved in [**145**] and [**146**]. See also [**105**] for extension of the mean value inequality to non-linear operators.

The use of the weighted L^2-norm of the heat kernel in conjunction with the mean value inequality was introduced by Aronson[**9**], [**10**] in the context of parabolic equations in $\mathbb{R}^n$. A good account of Aronson's estimates of the heat kernel as well as of the Harnack inequality of Moser can be found in [**306**], [**333**]. The relation between the mean value inequality and the heat kernel upper bound was extensively studied in [**255**]. An alternative method of obtaining Gaussian upper bounds can be found [**94**], [**95**].

Our treatment of Gaussian upper bounds follows [**141**] and [**146**], where all the results of Sections 15.3 and 15.5 were obtained. Let us emphasize that Theorem 15.14, which provides the main Gaussian upper bound of the heat kernel, applies even when the Faber-Krahn inequality is known locally, in some balls. The output of this theorem varies depending on how much is known about the Faber-Krahn properties of the manifold in question.

The short time asymptotics (15.45) of $\log p_t(x,y)$ was proved by Varadhan [**351**] (see also [**296**], [**326**]).

The relative Faber-Krahn inequality (15.55) was introduced in [**145**], where it was shown that (15.55) holds on complete non-compact Riemannian manifolds of non-negative Ricci curvature (see [**128**], [**154**], [**325**] for alternative proofs). The equivalence of (15.55) and the Li-Yau upper estimate was proved in [**146**]. The two-sided Li-Yau estimates for the heat kernel on complete manifolds of non-negative Ricci curvature was first proved by P.Li and S.-T. Yau [**258**], using the gradient estimates.

Let us briefly outline an alternative approach to obtaining the Li-Yau upper bound from the relative Faber-Krahn inequality (Theorem 15.21), which is due to [**81**], [**154**] and which avoids using Theorem 15.14.

Step1. Observe that under the relative Faber-Krahn inequality, the L^2-mean value inequality Theorem 15.1 becomes

$$u_+^2(T,x) \leq \frac{CT}{\mu(B(x,\sqrt{T}))} \iint_C u_+^2 \, d\mu dt, \tag{15.73}$$

provided $R = \sqrt{T}$, which follows from (15.3) by substitution

$$a = \frac{c}{R^2} \mu\left(B\left(x, R\right)\right)^{2/\nu}.$$

Step 2. The L^2-mean value inequality (15.73) together with the doubling volume property (which is also a consequence of the relative Faber-Krahn inequality – see the proof of Theorem 15.21) implies its L^1-counterpart – see [**248**], [**255**].

Step 3. Combining the L^1-mean value inequality with the Davies-Gaffney inequality (12.17) yields (15.57).

Denoting the integral in (7.88) by I and using (7.87) and the fundamental theorem of calculus, we obtain, for any $\varphi \in \mathcal{H}$,

$$\begin{aligned}(I,\varphi) &= \int_0^T (h_i(\gamma(t)),\varphi)\dot{\gamma}^i(t)\,dt \\ &= \int_0^T \partial_i (h(\cdot),\varphi)|_{\gamma(t)}\dot{\gamma}^i(t)\,dt \\ &= \int_0^T \frac{d}{dt}(h(\gamma(t)),\varphi)\,dt \\ &= (h(x),\varphi)-(h(x_0),\varphi),\end{aligned}$$

whence (7.88) follows.

Now fix a point $x \in \Omega$ and choose $\varepsilon > 0$ so that the cube $(x-\varepsilon, x+\varepsilon)^n$ lies in Ω. For simplicity of notation, assume that the origin 0 of $\mathbb{R}^n$ is contained in this cube, and consider the polygonal path γ connecting 0 and x inside the cube, whose consecutive vertices are as follows:

$$(0,0,...,0,0),\ (x^1,0,...,0,0),\ ...,\ (x^1,x^2,...,x^{n-1},0),\ (x^1,x^2,...,x^{n-1},x^n).$$

By (7.88), we have

$$h(x) = h(0) + \int_0^T h_i(\gamma(t))\dot{\gamma}^i(t)\,dt. \tag{7.89}$$

The integral in (7.89) splits into the sum of n integral over the legs of γ, and only the last one depends on x^n. Hence, to differentiate (7.89) in x^n, it suffices to differentiate the integral over the last leg of γ. Parametrizing this leg by

$$\gamma(t) = (x^1,x^2,...,x^{n-1},t),\ \ 0 \le t \le x^n,$$

we obtain

$$\partial_n h(x) = \frac{\partial}{\partial x^n}\int_0^{x^n} h_i(\gamma(t))\dot{\gamma}^i(t)\,dt = \frac{\partial}{\partial x^n}\int_0^{x^n} h_n(x^1,...,x^{n-1},t)\,dt = h_n(x),$$

which was to be proved. □

SECOND PROOF OF THEOREM 7.20. Let Ω be a chart on the manifold $\mathbb{R}_+ \times M$, and consider $p_{t,x}$ as a mapping $\Omega \to L^2(M)$. By Theorem 7.10, for any $f \in L^2(M)$, the function $P_t f(x) = (p_{t,x}, f)_{L^2}$ is C^∞-smooth in t, x. Hence, the mapping $p_{t,x}$ is weakly C^∞. By Lemma 7.21, the mapping $p_{t,x}$ is strongly C^∞. Let Ω' be another chart on $\mathbb{R}_+ \times M$ which will be the range of the variables s, y. Since $p_{s,y}$ is also strongly C^∞ as a mapping from $\Omega' \to L^2(M)$, we obtain by (7.56)

$$p_{t+s}(x,y) = (p_{t,x}, p_{s,y})_{L^2} = C^\infty(\Omega \times \Omega'),$$

which implies that $p_t(x,y)$ is C^∞-smooth in t, x, y.

Let D^α be a partial differential operator in variables $(t,x) \in \Omega$. By (7.84), we have, for any $f \in L^2(M)$,

$$D^\alpha(p_{t,x}, f) = (D^\alpha p_{t,x}, f), \tag{7.90}$$

where $D^\alpha p_{t,x}$ is understood as the Gâteaux derivative. Since the left hand sides of (7.82) and (7.90) coincide, so do the right hand sides, whence we obtain by Lemma 3.13

$$D^\alpha p_t(x,\cdot) = D^\alpha p_{t,x} \text{ a.e.}$$

Consequently, $D^\alpha p_t(x,\cdot) \in L^2(M)$ and, for any $f \in L^2(M)$,

$$D^\alpha \int_M p_t(x,y) f(y)\,d\mu = D^\alpha(p_{t,x}, f) = (D^\alpha p_{t,x}, f) = \int_M D^\alpha p_t(x,y) f(y)\,d\mu,$$

which finishes the proof. □

CHAPTER 16

Pointwise Gaussian estimates II

In this Chapter we describe another approach to the off-diagonal upper bounds of the heat kernel. This method allows to deduce the Gaussian estimates for $p_t(x,y)$ directly from the estimates of $p_t(x,x)$ and $p_t(y,y)$ and does not require the completeness of the manifold in question.

16.1. The weighted L^2-norm of $P_t f$

DEFINITION 16.1. We say that a function γ defined on an interval $(0,T)$ is *regular* if γ is an increasing positive function such that, for some $A \geq 1$, $a > 1$ and all $0 < t_1 < t_2 < T/a$,

$$\frac{\gamma(at_1)}{\gamma(t_1)} \leq A\frac{\gamma(at_2)}{\gamma(t_2)}. \tag{16.1}$$

Here are two simple situations when (16.1) holds:

- $\gamma(t)$ satisfies *the doubling condition,* that is, for some $A > 1$, and all $0 < t < T/2$
$$\gamma(2t) \leq A\gamma(t). \tag{16.2}$$
Then (16.1) holds with $a = 2$ because
$$\frac{\gamma(2t_1)}{\gamma(t_1)} \leq A \leq A\frac{\gamma(2t_2)}{\gamma(t_2)}.$$
- $\gamma(t)$ has *at least a polynomial growth* in the sense that, for some $a > 1$, the function $\gamma(at)/\gamma(t)$ is increasing in t. Then (16.1) holds for $A = 1$.

Let $T = +\infty$ and γ be differentiable. Set $l(\tau) := \log\gamma(e^\tau)$ and observe that l is defined on $(-\infty,+\infty)$. We claim that γ is regular provided one of the following two conditions holds:

- *l' is uniformly bounded* (for example, this is the case when $\gamma(t) = t^N$ or $\gamma(t) = \log^N(1+t)$ where $N > 0$);
- *l' is monotone increasing* (for example, $\gamma(t) = \exp(t^N)$).

On the other hand, (16.1) fails if $l' = \exp(-\tau)$ (which is unbounded as $\tau \to -\infty$), that is $\gamma(t) = \exp\left(-t^{-1}\right)$. Also, (16.1) may fail if l' is oscillating.

THEOREM 16.2. *Let (M, g, μ) be a weighted manifold and $S \subset M$ be a a non-empty measurable subset of M. For any function $f \in L^2(M)$ and $t > 0$*

and $D > 0$ set

$$E_D(t,f) = \int_M (P_t f)^2 \exp\left(\frac{d^2(\cdot, S)}{Dt}\right) d\mu. \tag{16.3}$$

Assume that, for some $f \in L^2(S)$ and for all $t > 0$,

$$\|P_t f\|_2^2 \le \frac{1}{\gamma(t)}, \tag{16.4}$$

where $\gamma(t)$ is a regular function on $(0, +\infty)$ in the sense of Definition 16.1. *Then, for all $D > 2$ and $t > 0$,*

$$E_D(t,f) \le \frac{6A}{\gamma(ct)}, \tag{16.5}$$

where $c = c(a, D) > 0$ and A, a are the constants from (16.1).

PROOF. The proof will be split into four steps.

Step 1. For any $t > 0$ and $r > 0$ define the following quantity

$$J_r(t) := \int_{S_r^c} (P_t f)^2 \, d\mu,$$

where S_r is the open r-neighborhood of S. By the inequality (12.11) of Theorem 12.3, we have, for all $0 < r < R$ and $0 < t < T$,

$$\int_{S_R^c} (P_T f)^2 \, d\mu \le \int_{S_r^c} (P_t f)^2 \, d\mu + \exp\left(-\frac{(R-r)^2}{2(T-t)}\right) \int_{S_r} (P_t f)^2 \, d\mu.$$

By (16.4), we have

$$\int_{S_r} (P_t f)^2 \, d\mu \le \frac{1}{\gamma(t)},$$

whence it follows that

$$J_R(T) \le J_r(t) + \frac{1}{\gamma(t)} \exp\left(-\frac{(R-r)^2}{2(T-t)}\right). \tag{16.6}$$

Step 2. Let us prove that

$$J_r(t) \le \frac{3A}{\gamma(t/a)} \exp\left(-\varepsilon \frac{r^2}{t}\right), \tag{16.7}$$

for some $\varepsilon = \varepsilon(a) > 0$. Let $\{r_k\}_{k=0}^{\infty}$ and $\{t_k\}_{k=0}^{\infty}$ be two strictly decreasing sequences of positive reals such that

$$r_0 = r, \quad r_k \downarrow 0, \quad t_0 = t, \ t_k \downarrow 0$$

as $k \to \infty$. By (16.6), we have, for any $k \ge 1$,

$$J_{r_{k-1}}(t_{k-1}) \le J_{r_k}(t_k) + \frac{1}{\gamma(t_k)} \exp\left(-\frac{(r_{k-1} - r_k)^2}{2(t_{k-1} - t_k)}\right). \tag{16.8}$$

When $k \to \infty$ we obtain

$$J_{r_k}(t_k) = \int_{S^c_{r_k}} (P_{t_k} f)^2 \, d\mu \le \int_{S^c} (P_{t_k} f)^2 \, d\mu \to \int_{S^c} f^2 d\mu = 0, \tag{16.9}$$

where we have used the fact that $P_t f \to f$ in $L^2(M)$ as $t \to 0+$ (cf. Theorem 4.9) and the hypothesis that $f \equiv 0$ in S^c.

Adding up the inequalities (16.8) for all k from 1 to ∞ and using (16.9), we obtain

$$J_r(t) \le \sum_{k=1}^{\infty} \frac{1}{\gamma(t_k)} \exp\left(-\frac{(r_{k-1} - r_k)^2}{2(t_{k-1} - t_k)}\right). \tag{16.10}$$

Let us specify the sequences $\{r_k\}$ and $\{t_k\}$ as follows:

$$r_k = \frac{r}{k+1} \quad \text{and} \quad t_k = t/a^k,$$

where a is the constant from (16.1). For all $k \ge 1$ we have

$$r_{k-1} - r_k = \frac{r}{k\,(k+1)} \quad \text{and} \quad t_{k-1} - t_k = \frac{(a-1)t}{a^k},$$

whence

$$\frac{(r_{k-1} - r_k)^2}{2(t_{k-1} - t_k)} = \frac{a^k}{2(a-1)k^2(k+1)^2} \frac{r^2}{t} \ge \varepsilon(k+1)\frac{r^2}{t}$$

where

$$\varepsilon = \varepsilon(a) = \inf_{k \ge 1} \frac{a^k}{2(a-1)k^2(k+1)^3} > 0. \tag{16.11}$$

By the regularity condition (16.1) we have

$$\frac{\gamma(t_{k-1})}{\gamma(t_k)} \le A\frac{\gamma(t_0)}{\gamma(t_1)},$$

which implies

$$\frac{\gamma(t)}{\gamma(t_k)} = \frac{\gamma(t_0)}{\gamma(t_1)}\frac{\gamma(t_1)}{\gamma(t_2)} \cdots \frac{\gamma(t_{k-1})}{\gamma(t_k)} \le \left(A\frac{\gamma(t_0)}{\gamma(t_1)}\right)^k.$$

Substituting into (16.10), we obtain

$$\begin{aligned} J_r(t) &\le \frac{1}{\gamma(t)} \sum_{k=1}^{\infty} \left(A\frac{\gamma(t_0)}{\gamma(t_1)}\right)^k \exp\left(-\varepsilon(k+1)\frac{r^2}{t}\right) \\ &= \frac{\exp\left(-\varepsilon\frac{r^2}{t}\right)}{\gamma(t)} \sum_{k=1}^{\infty} \exp\left(kL - \varepsilon k\frac{r^2}{t}\right), \end{aligned}$$

where

$$L := \log\left(A\frac{\gamma(t_0)}{\gamma(t_1)}\right).$$

Consider the following two cases:

(1) If $\varepsilon\frac{r^2}{t} - L \geq 1$ then

$$J_r(t) \leq \frac{\exp\left(-\varepsilon\frac{r^2}{t}\right)}{\gamma(t)} \sum_{k=1}^{\infty} \exp(-k) \leq \frac{2}{\gamma(t)} \exp\left(-\varepsilon\frac{r^2}{t}\right).$$

(2) If $\varepsilon\frac{r^2}{t} - L < 1$ then we estimate $J_r(t)$ in a trivial way:

$$J_r(t) \leq \int_M (P_t f)^2 \, d\mu \leq \frac{1}{\gamma(t)},$$

whence

$$\begin{aligned} J_r(t) &\leq \frac{1}{\gamma(t)} \exp\left(1 + L - \varepsilon\frac{r^2}{t}\right) = \frac{e}{\gamma(t)} A \frac{\gamma(t_0)}{\gamma(t_1)} \exp\left(-\varepsilon\frac{r^2}{t}\right) \\ &\leq \frac{3A}{\gamma(t/a)} \exp\left(-\varepsilon\frac{r^2}{t}\right). \end{aligned}$$

Hence, in the both cases we obtain (16.7).

Step 3. Let us prove the inequality

$$E_D(t, f) \leq \frac{6A}{\gamma(t/a)} \tag{16.12}$$

under the additional restriction that

$$D \geq 5\varepsilon^{-1}, \tag{16.13}$$

where ε was defined by (16.11) in the previous step.

Set $\rho(x) = d(x, S)$ and split the integral in the definition (16.3) of $E_D(t, f)$ into the series

$$E_D(t, f) = \left(\int_{\{\rho \leq r\}} + \sum_{k=1}^{\infty} \int_{\{2^{k-1}r < \rho \leq 2^k r\}}\right) (P_t f)^2 \exp\left(\frac{\rho^2}{Dt}\right) d\mu, \tag{16.14}$$

where r is a positive number to be chosen below. The integral over the set $\{\rho \leq r\}$ is estimated using (16.4):

$$\begin{aligned} \int_{\{\rho \leq r\}} (P_t f)^2 \exp\left(\frac{\rho^2}{Dt}\right) d\mu &\leq \exp\left(\frac{r^2}{Dt}\right) \int_M (P_t f)^2 \, d\mu \\ &\leq \frac{1}{\gamma(t)} \exp\left(\frac{r^2}{Dt}\right). \end{aligned} \tag{16.15}$$

The k-th term in the sum in (16.14) is estimated by (16.7) as follows

$$\begin{aligned}
&\int_{\{2^{k-1}r<\rho\le 2^k r\}} (P_t f)^2 \exp\left(\frac{\rho^2}{Dt}\right) d\mu \\
\le\ & \exp\left(\frac{4^k r^2}{Dt}\right) \int_{S^c_{2^{k-1}r}} (P_t f)^2\, d\mu \\
=\ & \exp\left(\frac{4^k r^2}{Dt}\right) J_{2^{k-1}r}(t) \\
\le\ & \frac{3A}{\gamma(t/a)} \exp\left(\frac{4^k r^2}{Dt} - \varepsilon \frac{4^{k-1} r^2}{t}\right) \\
\le\ & \frac{3A}{\gamma(t/a)} \exp\left(-\frac{4^{k-1} r^2}{Dt}\right),
\end{aligned} \tag{16.16}$$

where in the last line we have used (16.13).

Let us choose $r = \sqrt{Dt}$. Then we obtain from (16.14), (16.15), and (16.16)

$$E_D(t,f) \le \frac{3}{\gamma(t)} + \sum_{k=1}^{\infty} \frac{3A}{\gamma(t/a)} \exp\left(-4^{k-1}\right) \le \frac{3+3A}{\gamma(t/a)},$$

whence (16.12) follows.

Step 4. We are left to prove (16.5) in the case

$$2 < D < D_0 := 5\varepsilon^{-1}. \tag{16.17}$$

By Theorem 12.1, we have for any $s > 0$ and all $0 < \tau < t$

$$\int_M (P_t f)^2 \exp\left(\frac{\rho^2}{2(t+s)}\right) d\mu \le \int_M (P_\tau f)^2 \exp\left(\frac{\rho^2}{2(\tau+s)}\right) d\mu. \tag{16.18}$$

Given $t > 0$ and D as in (16.17), let us choose the values of s and τ so that the left hand side of (16.17) be equal to $E_D(t,f)$ whereas the right hand side be equal to $E_{D_0}(\tau,f)$. In other words, s and τ must satisfy the simultaneous equations

$$\begin{cases} 2(t+s) = Dt, \\ 2(\tau+s) = D_0\tau, \end{cases}$$

whence we obtain

$$s = \frac{D-2}{2} t \quad \text{and} \quad \tau = \frac{D-2}{D_0-2} t < t.$$

Hence, we can rewrite (16.18) in the form

$$E_D(t,f) \le E_{D_0}(\tau,f).$$

By (16.12), we have

$$E_{D_0}(\tau,f) \le \frac{6A}{\gamma(a^{-1}\tau)},$$

whence we conclude

$$E_D(t,f) \le \frac{6A}{\gamma\left(\frac{D-2}{D_0-2}a^{-1}t\right)},$$

thus finishing the proof of (16.5). □

16.2. Gaussian upper bounds of the heat kernel

We will again use the notation

$$E_D(t,x) := \int_M p_t^2(x,y)\exp\left(\frac{d^2(x,y)}{Dt}\right)d\mu(y).$$

THEOREM 16.3. *If, for some $x \in M$ and all $t > 0$,*

$$p_t(x,x) \le \frac{1}{\gamma(t)}, \tag{16.19}$$

where γ is a regular function on $(0,+\infty)$ then, for all $D > 2$ and $t > 0$,

$$E_D(t,x) \le \frac{C}{\gamma(ct)}, \tag{16.20}$$

where $C = 6A$, $c = c(a,D) > 0$, and a, A are the constants from (16.1).

PROOF. Let U be an open relatively compact neighborhood of the point x, and let φ be a cutoff function of $\{x\}$ in U. For any $s > 0$ define the function φ_s on M by

$$\varphi_s(z) = p_s(x,z)\varphi(z).$$

Clearly, we have $\varphi_s \le p_s(x,\cdot)$ whence

$$P_t\varphi_s \le P_t p_s(x,\cdot) = p_{t+s}(x,\cdot)$$

and

$$\|P_t\varphi_s\|_2^2 \le \|p_{t+s}(x,\cdot)\|_2^2 \le \|p_t(x,\cdot)\|_2^2 = p_{2t}(x,x) \le \frac{1}{\gamma(2t)}.$$

By Theorem 16.2, we conclude that, for any $D > 2$,

$$\int_M (P_t\varphi_s)^2 \exp\left(\frac{d^2(\cdot,U)}{Dt}\right)d\mu \le \frac{C}{\gamma(ct)}. \tag{16.21}$$

Fix $y \in M$ and observe that, by the definition of φ_s,

$$P_t\varphi_s(y) = \int_M p_t(y,z)\,p_s(x,z)\,\varphi(z)\,d\mu(z) = P_s\psi_t(x),$$

where

$$\psi_t(z) := p_t(y,z)\varphi(z)$$

Since function $\psi_t(\cdot)$ is continuous and bounded (cf. Exercise 7.27), we conclude by Theorem 7.16 that

$$P_s\psi_t(x) \to \psi_t(x) \text{ as } s \to 0,$$

that is,

$$P_t\varphi_s(y) \to p_t(x,y) \quad \text{as } s \to 0.$$

Passing to the limit in (16.21) as $s \to 0$, we obtain by Fatou's lemma

$$\int_M p_t^2(x,\cdot)\exp\left(\frac{d^2(\cdot,U)}{Dt}\right)d\mu \le \frac{C}{\gamma(ct)}.$$

Finally, shrinking U to the point x, we obtain (16.20). □

COROLLARY 16.4. *Let γ_1 and γ_2 be two regular functions on $(0,+\infty)$, and assume that, for two points $x,y \in M$ and all $t>0$*

$$p_t(x,x) \le \frac{1}{\gamma_1(t)} \quad \textit{and} \quad p_t(y,y) \le \frac{1}{\gamma_2(t)}. \tag{16.22}$$

Then, for all $D>2$ and $t>0$,

$$p_t(x,y) \le \frac{C}{\sqrt{\gamma_1(ct)\gamma_2(ct)}}\exp\left(-\frac{d^2(x,y)}{2Dt}\right),$$

where C and c depend on the constants from the regularity condition, and c depends in addition on D.

PROOF. By Theorem 16.3, we obtain

$$E_D(t,x) \le \frac{C}{\gamma_1(ct)} \quad \text{and} \quad E_D(t,y) \le \frac{C}{\gamma_2(ct)}.$$

Substituting these inequalities into the estimate (15.38) of Lemma 15.13, we finish the proof. □

In particular, if $\gamma(t)$ is regular and

$$p_t(x,x) \le \frac{1}{\gamma(t)}$$

for all $x \in M$ and $t>0$ then

$$p_t(x,y) \le \frac{C}{\gamma(ct)}\exp\left(-\frac{d^2(x,y)}{2Dt}\right),$$

for all $x,y \in M$ and $t>0$. If the manifold M is complete and the function γ is of the form $\gamma(t) = ct^{n/2}$ then this was proved in Corollary 15.17.

Exercises.

16.1. Let for some $x \in M$ and all $t \in (0,T)$

$$p_t(x,x) \le \frac{1}{\gamma(t)}, \tag{16.23}$$

where $T \in (0,+\infty]$ and γ is a monotone increasing function on $(0,T)$ satisfying the doubling property

$$\gamma(2t) \le A\gamma(t), \tag{16.24}$$

for some $A \ge 1$ and all $t < T/2$. Prove that, for all $D>2$ and $t>0$,

$$E_D(t,x) \le \frac{C}{\gamma(t\wedge T)}, \tag{16.25}$$

where $C = C(A)$.

16.2. Using Exercise 16.1, give an alternative proof of Corollary 15.9: on any weighted manifold M,

$$E_D(t,x) < \infty$$

for all $D > 2$, $x \in M$, $t > 0$.

16.3. Using Lemma 15.13, prove that on any weighted manifold M, for any $D > 2$ there exists a function $\Phi(t,x)$ that is decreasing in t and such that the following inequality holds

$$p_t(x,y) \le \Phi(t,x)\Phi(t,y) \exp\left(-\frac{d^2(x,y)}{2Dt} - \lambda_{\min}(M)\,t\right), \tag{16.26}$$

for all $x, y \in M$ and $t > 0$ (cf. Exercise 15.3).

16.4. Assume that a weighted manifold M admits the Faber-Krahn inequality with a function $\Lambda \in \mathbf{L}$ and let γ be its $\mathbf{L}$-transform. Assume that γ is regular in the sense of Definition 16.1. Prove that, for any $D > 2$ and for all $t > 0$ and $x, y \in M$,

$$p_t(x,y) \le \frac{C}{\gamma(ct)} \exp\left(-\frac{d^2(x,y)}{2Dt}\right),$$

where C depends on D and on the regularity constants of γ.

16.5. Assume that the volume function $V(x,r) = \mu(B(x,r))$ of a weighted manifold M is doubling and that the heat kernel of M admits the estimate

$$p_t(x,x) \le \frac{C}{V(x,\sqrt{t})},$$

for all $x \in M$ and $t \in (0,T)$, where $T \in (0,+\infty]$ and C is a constant. Prove that

$$p_t(x,y) \le \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{2Dt}\right),$$

for all $D > 2$, $x, y \in M$, $t \in (0,T)$ and some constant C.

REMARK. If $T = +\infty$ and the manifold M is complete and non-compact, then this follows from Theorem 15.21.

16.3. On-diagonal lower bounds

Here we demonstrate the use of the quantity $E_D(t,x)$ for the proof of some lower bounds of the heat kernel in two settings. For any $x \in M$ and $r > 0$, set

$$V(x,r) = \mu(B(x,r)).$$

Observe that $V(x,r)$ is positive and finite provided M is complete.

THEOREM 16.5. *Let M be a complete weighted manifold. Assume that, for some $x \in M$ and all $r \ge r_0$,*

$$V(x,r) \le Cr^\nu, \tag{16.27}$$

where C, ν, r_0 are positive constants. Then, for all $t \ge t_0$,

$$p_t(x,x) \ge \frac{1/4}{V(x,\sqrt{Kt\log t})}, \tag{16.28}$$

where $K = K(x, r_0, C, \nu) > 0$ and $t_0 = \max(r_0^2, 3)$.

Furthermore, for any $K > \nu$ there exists large enough t_0 such that (16.28) *holds for all $t \ge t_0$.*

Of course, (16.28) implies

$$p_t(x,x) \geq c\,(t\log t)^{-\nu/2}.$$

In general, one cannot get rid of $\log t$ here – see [**169**].

PROOF. For any $r > 0$, we obtain by the semigroup identity and the Cauchy-Schwarz inequality

$$\begin{aligned} p_{2t}(x,x) &= \int_M p_t^2(x,\cdot)d\mu \geq \int_{B(x,r)} p_t^2(x,\cdot)d\mu \\ &\geq \frac{1}{V(x,r)}\left(\int_{B(x,r)} p_t(x,\cdot)d\mu\right)^2. \end{aligned} \tag{16.29}$$

Since M is complete and the condition (16.27) obviously implies (11.22), we obtain by Theorem 11.8 that M is stochastically complete, that is

$$\int_M p_t(x,\cdot)d\mu = 1.$$

Using also that $p_t(x,x) \geq p_{2t}(x,x)$ (cf. Exercise 7.22) we obtain from (16.29)

$$p_t(x,x) \geq \frac{1}{V(x,r)}\left(1 - \int_{M\setminus B(x,r)} p_t(x,\cdot)d\mu\right)^2. \tag{16.30}$$

Choose $r = r(t)$ so that

$$\int_{M\setminus B(x,r)} p_t(x,\cdot)d\mu \leq \frac{1}{2}. \tag{16.31}$$

Assume for a moment that (16.31) holds. Then (16.30) yields

$$p_t(x,x) \geq \frac{1/4}{V(x,r(t))}.$$

To match (16.28), we need the following estimate of $r(t)$:

$$r(t) \leq \sqrt{Kt\log t}. \tag{16.32}$$

Let us prove (16.31) with $r = r(t)$ satisfying (16.32). Setting $\rho = d(x,\cdot)$ and fixing some $D > 2$, we obtain by the Cauchy-Schwarz inequality

$$\begin{aligned} &\left(\int_{M\setminus B(x,r)} p_t(x,\cdot)d\mu\right)^2 \\ \leq\ & \int_M p_t^2(x,\cdot)\exp\left(\frac{\rho^2}{Dt}\right)d\mu \int_{M\setminus B(x,r)} \exp\left(-\frac{\rho^2}{Dt}\right)d\mu \\ =\ & E_D(t,x)\int_{M\setminus B(x,r)} \exp\left(-\frac{\rho^2}{Dt}\right)d\mu, \end{aligned} \tag{16.33}$$

where $E_D(t,x)$ is defined by (15.24). By Theorem 12.1 and Corollary 15.9, we have, for all $t > t_0$,

$$E_D(t,x) \leq E_D(t_0,x) < \infty. \tag{16.34}$$

Since x is fixed, we can consider $E_D(t_0,x)$ as a constant.

Let us now estimate the integral in (16.33) assuming that

$$r = r(t) \geq r_0. \tag{16.35}$$

By splitting the complement of $B(x,r)$ into the union of the annuli

$$B(x,2^{k+1}r) \setminus B(x,2^k r), \quad k = 0,1,2,...,$$

and using the hypothesis (16.27), we obtain

$$\int_{M\setminus B(x,r)} \exp\left(-\frac{\rho^2}{Dt}\right) d\mu \leq \sum_{k=0}^{\infty} \exp\left(-\frac{4^k r^2}{Dt}\right) V(x,2^{k+1}r) \tag{16.36}$$

$$\leq Cr^\nu \sum_{k=0}^{\infty} 2^{\nu(k+1)} \exp\left(-\frac{4^k r^2}{Dt}\right). \tag{16.37}$$

Assuming

$$\frac{r^2}{Dt} \geq 1, \tag{16.38}$$

the sum in (16.37) is majorized by a geometric series whence

$$\int_{M\setminus B(x,r)} \exp\left(-\frac{\rho^2}{Dt}\right) d\mu \leq C' r^\nu \exp\left(-\frac{r^2}{Dt}\right), \tag{16.39}$$

where C' depends on C and ν. Set

$$r(t) = \sqrt{Kt \log t}, \tag{16.40}$$

where the constant K will be chosen below; in any case, it will be larger than D. If so then assuming that

$$t \geq t_0 = \max\left(r_0^2, 3\right)$$

we obtain that both conditions (16.35) and (16.38) are satisfied.

Substituting (16.40) into (16.39), we obtain

$$\int_{M\setminus B(x,r)} \exp\left(-\frac{\rho^2}{Dt}\right) d\mu \leq C' K^{\nu/2} \left(\frac{\log t}{t^{\frac{2K}{\nu D}-1}}\right)^{\nu/2}. \tag{16.41}$$

If $\alpha > 1$ then the function $\frac{\log t}{t^\alpha}$ is decreasing for $t > 3$. Hence assuming $K > ND$ we obtain from (16.41) and (16.33)

$$\left(\int_{M\setminus B(x,r)} p_t(x,\cdot) d\mu\right)^2 \leq C' K^{\nu/2} \left(\frac{\log t_0}{t_0^{\frac{2K}{\nu D}-1}}\right)^{\nu/2} E_D(t_0,x). \tag{16.42}$$

Choosing K large enough, we can make the right hand side arbitrarily small, which finishes the proof of the first claim.

If $K > \nu$ then choosing D close enough to 2 we can ensure that

$$\frac{2K}{\nu D} - 1 > 0.$$

Therefore, the right hand side in (16.42) can be made arbitrarily small provided t_0 large enough, whence the second claim follows. $\square$

THEOREM 16.6. *Let M be a complete weighted manifold. Assume that, for some point $x \in M$ and all $r > 0$*

$$V(x, 2r) \leq CV(x, r), \tag{16.43}$$

and, for all $t \in (0, T)$,

$$p_t(x, x) \leq \frac{C}{V(x, \sqrt{t})}, \tag{16.44}$$

where $T \in (0, +\infty]$ and $C > 0$. Then, for all $t \in (0, T)$,

$$p_t(x, x) \geq \frac{c}{V(x, \sqrt{t})}, \tag{16.45}$$

where $c > 0$ depends on C.

PROOF. It follows from (16.43) that $V(x, r) \leq Cr^\nu$ for all $r \geq 1$ and some ν. Hence, by Theorem 11.8, M is stochastically complete. Following the argument in the proof of Theorem 16.5, we need to find $r = r(t)$ so that

$$\int_{M \setminus B(x,r)} p_t(x, \cdot) d\mu \leq \frac{1}{2},$$

which implies

$$p_t(x, x) \geq \frac{1/4}{V(x, r(t))}. \tag{16.46}$$

If in addition $r(t) \leq K\sqrt{t}$ for some constant K then (16.45) follows from (16.46) and (16.43).

Let us use the estimate (16.33) from the proof of Theorem 16.5, that is,

$$\left(\int_{M \setminus B(x,r)} p_t(x, \cdot) d\mu \right)^2 \leq E_D(t, x) \int_{M \setminus B(x,r)} \exp\left(-\frac{\rho^2}{Dt} \right) d\mu \tag{16.47}$$

where $\rho = d(x, \cdot)$ and $D > 2$ (for example, set $D = 3$). Next, instead of using the monotonicity of $E_D(t, x)$ as in the proof of Theorem 16.5, we apply Theorem 16.3. Indeed, by Theorem 16.3 and Exercise 16.1, the hypotheses (16.43) and (16.44) yield, for all $t \in (0, T)$,

$$E_D(t, x) \leq \frac{C}{V(x, \sqrt{t})}. \tag{16.48}$$

Applying the doubling property (16.43) we obtain

$$\begin{aligned}\int_{M\setminus B(x,r)} \exp\left(-\frac{\rho^2}{Dt}\right) d\mu &\le \sum_{k=0}^{\infty} \exp\left(-\frac{4^k r^2}{Dt}\right) V(x, 2^{k+1}r)\\ &\le \sum_{k=0}^{\infty} C^{k+1} \exp\left(-\frac{4^k r^2}{Dt}\right) V(x,r)\\ &\le C'V(x,r)\exp\left(-\frac{r^2}{Dt}\right), \qquad (16.49)\end{aligned}$$

provided $r^2 \ge Dt$. It follows from (16.47), (16.48), and (16.49) that, for any $t \in (0,T)$,

$$\left(\int_{M\setminus B(x,r)} p_t(x,\cdot)d\mu\right)^2 \le C'' \exp\left(-\frac{r^2}{Dt}\right).$$

Obviously, the expression $\exp\left(-\frac{r^2}{Dt}\right)$ can be made arbitrarily small by choosing $r = \sqrt{Kt}$ with K large enough, which finishes the proof. ☐

COROLLARY 16.7. *If M is a complete non-compact weighted manifold and M admits the relative Faber-Krahn inequality then*

$$p_t(x,x) \ge \frac{c}{V\left(x, \sqrt{t}\right)}$$

for some $c > 0$ and for all $x \in M$, $t > 0$.

PROOF. Indeed, by Theorem 15.21(a), the relative Faber-Krahn inequality implies both conditions (16.43) and (16.44) with $T = +\infty$, whence the claim follows from Theorem 16.6. ☐

16.4. Epilogue: alternative ways of constructing the heat kernel

Recall that the existence of the heat kernel was proved in Chapter 7 using the key estimate (7.18) of the semigroup P_t, that is,

$$\sup_K |P_t f| \le C\left(1 + t^{-\sigma}\right) \|f\|_{L^2(M)}, \qquad (16.50)$$

for all $t > 0$ and $f \in L^2(M)$, where K is any relatively compact subset of M, $\sigma = \sigma(n) > 0$, and the constant C depends on K. The estimate (16.50) obtained in Theorem 7.6 as a consequence of the Sobolev embedding
$C(K)$ provided σ is large enough.
have methods of construction of the heat kernel, which
le use of the smoothness properties as possible, to be
ing of metric measure spaces. Let us sketch three alter-
constructing the heat kernel, which satisfy this require-
size that in all cases the existence of the heat kernel is
rtain local properties of the underlying space.

1. The estimate (16.50) has received one more proof in Corollary 15.7, which depends on the following two ingredients:

- Theorem 15.4 that provides the Faber-Krahn inequality in small balls;
- Theorem 15.1 that provides a mean-value inequality for solutions of the heat equation assuming the Faber-Krahn inequality.

The properties of manifolds that are used in this approach are the Faber-Krahn inequality in small ball, which is a consequence of a local Euclidean structure, and $|\nabla d| \leq 1$, which reflects a special role of the geodesic distance in contrast to other possible distance functions. Although the proof of the key Theorem 15.1 is relatively long and technical, this approach has a certain advantage since it allows to jump quickly to Gaussian off-diagonal estimates of the heat kernel (cf. Theorem 15.14 and Corollaries 15.17, 15.15).

2. It suffices to construct heat kernels p_t^Ω in all relatively compact open subsets Ω of M. Indeed, then one proves define p_t as the limit of $p_t^{\Omega_k}$ as $k \to \infty$ where $\{\Omega_k\}$ is any compact exhaustion sequence. The fact that the limit exists follows from the monotonicity of p_t^Ω with respect to Ω (cf. Theorem 5.23) and from

$$\left\|p_t^\Omega\right\|_{L^1} \leq 1. \tag{16.51}$$

That p_t is indeed the integral kernel of the heat semigroup P_t, was proved by J. Dodziuk [**108**] (see also [**51**, p.188]).

Let us describe two methods for construction of p_t^Ω.

(*i*) One first shows that the Dirichlet Laplace operator $\mathcal{L}^\Omega$ has a discrete spectrum. This was proved in Theorem 10.13 using the heat kernel and the estimate (16.50). However, to realize the present approach, it is necessary to have a proof of of the discreteness of the spectrum without using the heat kernel – see, for example, the second proof of Corollary 10.21. Then one can define the heat kernel p_t^Ω by the explicit formula (10.33) via the eigenfunctions of $\mathcal{L}^\Omega$. Certain efforts are needed to verify that $p_t^\Omega \geq 0$ and (16.51), for which one employs the maximum principle.

(*ii*) Starting with the Faber-Krahn inequality in small balls (cf. Theorem 15.4), one first obtains the same inequality in any relatively compact open set Ω. This follows from Theorem 15.11, but one should have a different proof for merging Faber-Krahn inequalities without using the heat kernel (a relevant argument can be found in [**179**]). Then, by Theorem 14.19, the heat semigroup P_t^Ω is $L^1 \to L^2$ ultracontractive, and the ultracontractivity implies the existence of the heat kernel (see, for example, [**49**], [**96**], or [**163**]).

Yet another method for construction of the heat kernel under a weaker version of the Faber-Krahn inequality can be found in [**184**].

Notes and further references

The key idea of Theorem 16.2, that the upper bound (16.4) for $\|P_t f\|$ implies a Gaussian estimate (16.7) for the the integral of $(P_t f)^2$ away from the support of f, is due to Ushakov [350] in the context of parabolic equations in $\mathbb{R}^n$ (see also [193]). The present proofs of Theorems 16.2 and 16.3 were taken from [151].

The finiteness of $E_D(t,x)$ on any manifold for $D > 2$ was proved in [141] and [146].

The on-diagonal lower bound of the heat kernel in Theorems 16.5 and 16.6 were proved in [80], [79]. Moreover, Theorem 16.5 was proved in [80] under the hypothesis $V(x,r) \leq F(r)$ with a more general function $F(r)$ than in (16.27).

Further results on the topics related to this book can be found in the following references.

– the Gaussian bounds of heat kernels and Harnack inequalities: [13], [15], [35], [54], [55], [74], [75], [78], [87], [94], [95], [96], [98], [101], [120], [145], [148], [149], [150], [157], [176], [177], [179], [180], [198], [199], [230], [295], [312], [319], [320], [322], [324], [345];

– a short time behavior of the heat kernel: [36], [131], [132], [267], [275], [276], [277], [295], [336], [349].

– heat semigroups and functional inequalities: [20], [94], [95], [96], [188], [190], [313], [314], [315], [369];

– heat kernels and curvature: [18], [61], [258], [267], [307], [361], [365], [366];

– heat kernels on symmetric spaces and groups: [2], [3], [7], [8], [33], [34], [84], [114], [189], [218], [219], [287], [303], [304], [312], [355];

– heat kernels on metric measure spaces and fractals: [14], [22], [23], [24], [27], [124], [125], [156], [158], [162], [163], [164], [165], [166], [167], [172], [195], [202], [224], [225], [238], [331], [332], [337], [339];

– heat kernels and random walks on graphs: [5], [25], [26], [64], [68], [70], [69], [82], [99], [111], [129], [181], [182], [191], [233], [234], [323], [358], [359];

– heat kernels of non-linear operators: [105], [107];

– heat kernels of higher order elliptic operators: [19], [100], [237], [312];

– heat kernels of non-symmetric operators: [117], [236], [300];

– heat kernels of subelliptic operators: [30], [31], [215], [216], [239];

– heat kernels in infinite dimensional spaces: [32], [34], [112], [113];

– heat kernels and stochastic processes: [152], [155], [161], [168], [169], [170], [171], [178], [210], [334];

– heat kernels for Schrödinger operators: [21], [71], [157], [160], [201], [259], [260], [282], [283], [344], [371], [370];

– Liouville theorems and related topics: [4], [6], [28], [62], [72], [90], [91], [109], [138], [139], [140], [143], [144], [159], [173], [204], [205], [220], [247], [248], [250], [251], [252], [254], [261], [262], [264], [284], [285], [288], [301], [338], [341].

– eigenvalues and eigenfunctions on manifolds: [36], [51], [59], [60], [174], [183], [185], [232], [247], [253], [256], [257], [289], [290], [291], [326], [362], [367];

– various aspects of isoperimetric inequalities: [11], [40], [42], [48], [53], [116], [128], [249], [302], [305], [321], [354], [362].

APPENDIX A

Reference material

For convenience of the reader, we briefly review here some background material frequently used in the main body of the book. The detailed accounts can be found in numerous textbooks on Functional Analysis and Measure Theory, see for example, [**73**], [**88**], [**89**], [**194**], [**226**], [**229**], [**235**], [**263**], [**281**], [**310**], [**318**], [**335**], [**346**], [**368**].

A.1. Hilbert spaces

We assume throughout that $\mathcal{H}$ is a real Hilbert space with the inner product (x, y) and the associated norm $\|x\| = (x, x)^{1/2}$.

CAUCHY-SCHWARZ INEQUALITY. *For all $x, y \in \mathcal{H}$,*

$$|(x, y)| \leq \|x\| \|y\|.$$

PROJECTION. *If S is a closed subspace of $\mathcal{H}$ then for any $x \in \mathcal{H}$ there is a unique point $y \in S$ such that $(x - y) \perp S$.*

The point y is called the *projection* of x onto S. The mapping $P : \mathcal{H} \to \mathcal{H}$ defined by $Px = y$ is called the *projector* onto S. In fact, P is a linear bounded self-adjoint operator in $\mathcal{H}$, and $\|P\| \leq 1$ (see Section A.5 below).

Let D be a dense subspace of $\mathcal{H}$ and $l : D \to \mathbb{R}$ be a linear functional. The norm of l is defined by

$$\|l\| = \sup_{x \in D \setminus \{0\}} \frac{l(x)}{\|x\|}.$$

The functional l is said to be *bounded* if $\|l\| < \infty$. The boundedness of l is equivalent to the continuity and to the uniform continuity of l. Hence, a bounded linear functional uniquely extends to a bounded linear functional defined on the whole space $\mathcal{H}$.

For example, any vector $a \in \mathcal{H}$ gives rise to a bounded linear functional l_a as follows: $l_a(x) = (x, a)$. It follows from the Cauchy-Schwarz inequality that $\|l_a\| = \|a\|$. The next theorem implies that the family $\{l_a\}_{a \in \mathcal{H}}$ exhausts all the bounded linear functional.

RIESZ REPRESENTATION THEOREM. *For any bounded linear functional l on $\mathcal{H}$, there exists a unique vector $a \in \mathcal{H}$ such that $l(x) = (x, a)$ for all x from the domain of l. Furthermore, $\|l\| = \|a\|$.*

BESSEL'S INEQUALITY. *Let $\{v_k\}$ be a orthonormal sequence in a Hilbert space $\mathcal{H}$. Fix a vector $x \in \mathcal{H}$ and set*

$$x_k = (x, v_k).$$

Then

$$\sum_k |x_k|^2 \le \|x\|^2.$$

PARSEVAL'S IDENTITY. *Let $\{v_k\}$ be an orthonormal sequence in a Hilbert space. Fix a vector $x \in \mathcal{H}$ and a sequence of reals $\{x_k\}$. Then the identity*

$$x = \sum_k x_k v_k$$

holds if and only if $x_k = (x, v_k)$ for all k, and

$$\sum_k |x_k|^2 = \|x\|^2.$$

ORTHONORMAL BASIS. *In any separable Hilbert space $\mathcal{H}$, there is an at most countable orthonormal basis, that is, a finite or countable sequence $\{v_k\}_{k=1}^N$ such that*

$$(v_k, v_l) = \begin{cases} 0, & k \neq l, \\ 1, & k = l, \end{cases}$$

and that any vector $x \in \mathcal{H}$ can be uniquely represented as the sum

$$x = \sum_{k=1}^N x_k v_k,$$

for some real x_k. In the case $N = \infty$ the series converges in the norm of $\mathcal{H}$.

The series $\sum_k x_k v_k$ is called the *Fourier series* of the vector x in the basis $\{v_k\}$, and the numbers x_k are called the *coordinates* (or the *Fourier coefficients*) of x.

A.2. Weak topology

A sequence $\{x_k\}$ in a Hilbert space $\mathcal{H}$ converges *weakly* to $x \in \mathcal{H}$ if for all $y \in \mathcal{H}$

$$(x_k, y) \to (x, y).$$

In this case one writes $x_k \rightharpoonup x$ or $x = w\text{-}\lim x_k$. Alternatively, the weak convergence is determined by the *weak topology* of $\mathcal{H}$, which is defined by the family of semi-norms

$$N_y(x) = |(x, y)|,$$

where y varies in $\mathcal{H}$.

In contrast to that, the topology of $\mathcal{H}$ that is determined by the norm of $\mathcal{H}$, is called the *strong topology*. Clearly, the strong convergence implies the weak convergence.

PRINCIPLE OF UNIFORM BOUNDEDNESS. *Any weakly bounded (that is, bounded in any semi-norm) subset of a Hilbert space is strongly bounded (that is, bounded in the norm).*

Hence, the boundedness of a subset in the weak sense is equivalent to that in the strong sense.

We say that a set $S \subset \mathcal{H}$ is *weakly compact* if any sequence $\{x_k\} \subset S$ contains a subsequence $\{x_{k_i}\}$ that converges weakly to some $x \in S$.

WEAK COMPACTNESS OF A BALL. *In any Hilbert space $\mathcal{H}$, the ball*

$$B := \{x \in \mathcal{H} : \|x\| \leq 1\}$$

is weakly compact.

For comparison let us mention that the ball B is strongly compact if and only if $\dim \mathcal{H} < \infty$.

It is also worth mentioning that any strongly closed subspace of $\mathcal{H}$ is weakly closed, too – see Exercise A.5. Hence, the closedness of a subspace in the weak sense is equivalent to that in the strong sense.

Exercises.

A.1. Prove that if $\{x_k\}$ and $\{y_k\}$ are two sequences in $\mathcal{H}$ such that $x_k \rightharpoonup x$ and $y_k \to y$ then

$$(x_k, y_k) \to (x, y).$$

A.2. Prove that if $x_k \rightharpoonup x$ then

$$\|x\| \leq \liminf_{k\to\infty} \|x_k\|.$$

A.3. Let $\{x_k\}$ be a sequence of vectors in a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$.

(*a*) Prove that $x_k \to x$ if and only of

$$x_k \rightharpoonup x \text{ and } \|x_k\| \to \|x\|.$$

That is, the strong convergence is equivalent to the weak convergence and the convergence of the norms.

(*b*) Prove that $x_k \rightharpoonup x$ if and only if the numerical sequence $\{\|x_k\|\}$ is bounded and, for a dense subset $\mathcal{D}$ of $\mathcal{H}$,

$$(x_k, y) \to (x, y) \text{ for any } y \in \mathcal{D}.$$

That is, the weak convergence is equivalent to the convergence "in distribution" and the boundedness of the norms.

A.4. Let $\{v_k\}_{k=1}^{\infty}$ be an orthonormal sequence in $\mathcal{H}$.

(*a*) Prove that $v_k \rightharpoonup 0$ as $k \to \infty$.

(*b*) Prove that, for any sequence of reals c_k, the series

$$\sum_{k=1}^{\infty} c_k v_k$$

converges weakly if and only if it converges strongly.

A.5. A subset S of a Hilbert space $\mathcal{H}$ if called weakly closed if it contains all weak limits of all sequences from S. Prove that any closed subspace of $\mathcal{H}$ is also weakly closed.

A.3. Compact operators

Let X, Y be two Banach spaces. A linear operator $A : X \to Y$ is called bounded if

$$\|A\| := \inf_{x \in X \setminus \{0\}} \frac{\|Ax\|_Y}{\|x\|_X} < \infty.$$

A linear operator $A : X \to Y$ is called *compact* if, for any bounded set $S \subset X$, its images $A(S)$ is a relatively compact subset of Y. Equivalently, this means that, for any bounded sequence $\{x_n\}_{n=1}^{\infty} \subset X$, there exists a subsequence $\{x_{n_i}\}$ such that $A(x_{n_i})$ converges in Y.

THEOREM. *Any compact operator is bounded. Composition of a compact operator and a bounded operator is compact.*

Let $\mathcal{H}$ be a Hilbert space. A a bounded operator $A : \mathcal{H} \to \mathcal{H}$ is called *self-adjoint* if

$$(Ax, y) = (x, Ay) \quad \text{for all } x, y \in \mathcal{H}.$$

A vector $x \in \mathcal{H}$ is called an *eigenvector* of A if $x \neq 0$ and $Ax = \lambda x$ for some scalar λ, which is called the *eigenvalue* of x.

THE HILBERT-SCHMIDT THEOREM. *Let A be a compact self-adjoint operator in a Hilbert space $\mathcal{H}$. Then, in the orthogonal complement* $(\ker A)^{\perp}$ *of the kernel of A, there exists an at most countable orthonormal basis $\{v_k\}_{k=1}^{N}$, such that each v_k is an eigenvectors of the operator A. The corresponding eigenvalues λ_k are real, and if $N = \infty$ then the sequence $\{\lambda_k\}$ tends to* 0 *as* $k \to \infty$.

Note that all the eigenvalues λ_k are non-zero because $v_k \notin \ker A$. If the space $\mathcal{H}$ is separable then $\ker A$ admits at most countable orthonormal basis, say $\{w_i\}$. Obviously, w_i is an eigenvector of A with the eigenvalue 0. Merging the bases $\{v_k\}$ and $\{w_i\}$, we obtain an orthonormal basis in $\mathcal{H}$ that consists of the eigenvectors of A.

A.4. Measure theory and integration

A.4.1. Measure and extension. Let M be a set and S be a family of subsets of M containing the empty set $\emptyset$. A *measure* on S is a function $\mu : S \to [0, +\infty]$ such that $\mu(\emptyset) = 0$ and, for any finite or countable sequence $\{E_i\}$ of disjoint sets from S, if the union $E = \sqcup_i E_i$ is in S then

$$\mu(E) = \sum_i \mu(E_i).$$

Measure μ is called *σ-finite* if there exists a countable sequence of sets $\{E_i\}_{i=1}^{\infty}$ from S covering M and such that $\mu(E_i) < \infty$.

A non-empty family S of subsets of M is called a *semi-ring* if the following two conditions holds:

- $E, F \in S \implies E \cap F \in S$;
- $E, F \in S \implies E \setminus F$ is a disjoint union of a finite family of sets from S.

For example, the family of all intervals[1] in $\mathbb{R}$ is a semi-ring, and so is the family of all boxes[2] in $\mathbb{R}^n$.

A non-empty family R of subsets of M is called a *ring* if

$$E, F \in R \implies E \cup F \text{ and } E \setminus F \text{ are also in } R. \tag{A.1}$$

It follows from (A.1) that also $E \cap F \in R$. Hence, a ring is a semi-ring.

A family Σ of subsets of M is called a *σ-ring* if it is a ring and, for any countable sequence $\{E_i\}_{i=1}^{\infty}$ of sets from Σ, also their union $\cup_i E_i$ is in Σ. This implies that the intersection $\cap_i E_i$ is also in Σ. Hence, a σ-ring Σ is closed with respect to the set-theoretic operations $\cap, \cup, \setminus$ on countable sequences of sets. A σ-ring Σ is called a *σ-algebra* if $M \in \Sigma$.

For any semi-ring S, there is the minimal ring $R = R(S)$ containing S, which is obtained as the intersection of all rings containing S. In fact, $R(S)$ consists of sets which are finite disjoint unions of sets from S. If μ is a measure on S then μ uniquely extends to a measure on $R(S)$, also denoted by μ, and this extension is given by

$$\mu(A) = \sum_{i=1}^{n} \mu(A_i) \tag{A.2}$$

whenever $A \in R(S)$ is a finite disjoint union of sets $A_1, ..., A_n \in S$.

For any ring R, there is the minimal σ-ring $\Sigma = \Sigma(R)$ containing R, which is obtained as the intersection of all σ-rings containing R.

Carathéodory Extension Theorem. *Any measure μ defined on a ring R extends to a measure on the σ-ring $\Sigma(R)$; besides, this extension is unique provided measure μ is σ-finite.*

The extended measure, again denoted by μ, is defined for any $A \in \Sigma(R)$ by

$$\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) : A_i \in R, \quad A \subset \bigcup_{i=1}^{\infty} A_i \right\}. \tag{A.3}$$

Let μ be a measure defined on a σ-ring Σ. A subset of M is called a *null set* if it is a subset of a set from Σ with μ-measure 0. Measure μ is called *complete* if all null sets belong to Σ. An arbitrary measure can be extended to a compete measure as follows. Denote by $\overline{\Sigma}$ the family of sets of the form $E \cup N$ where $E \in \Sigma$ and N is a null set. Then $\overline{\Sigma}$ is a σ-ring, and μ can be extended to $\overline{\Sigma}$ by setting $\mu(E \cup N) = \mu(E)$. Measure μ with domain $\overline{\Sigma}$ is complete.

In fact, the formula (A.3) is valid for all $A \in \overline{\Sigma}(R)$. The extension of a measure μ from a ring R (or from a semi-ring S) to a complete measure on $\overline{\Sigma}(R)$ is called the *Carathéodory extension*.

[1] Including open, closed, and semi-open intervals.

[2] A box in $\mathbb{R}^n$ is a set of the form

$$I_1 \times ... \times I_n$$

where $I_k \subset \mathbb{R}$ are intervals.

As an example of application of this procedure, consider construction of the Lebesgue measure in $\mathbb{R}^n$. Let S be the semi-ring of all intervals in $\mathbb{R}$ and $\lambda(I)$ be the length of any interval $I \in S$; in other words, if $a \leq b$ are the ends of I (that is, I is one of the sets (a,b), $[a,b]$, $[a,b)$, $(a,b]$) then set $\lambda(I) = b - a$. It is easy to see that λ is a measure on S and, moreover, this measure is σ-finite. More generally, for any box $E \subset \mathbb{R}^n$, define its n-volume $\lambda_n(E)$ as the product of the lengths of the sides of E. It is not difficult to prove that λ_n is a σ-finite measure on the semi-ring S of all boxes. Hence, λ_n admits a unique extension to the minimal σ-ring (in fact, σ-algebra) containing S. This σ-algebra is denoted by $\mathcal{B}(\mathbb{R}^n)$, and the elements of $\mathcal{B}(\mathbb{R}^n)$ are called *Borel sets.*[3] Null sets in $\mathbb{R}^n$ are not necessarily Borel. Completing the σ-algebra $\mathcal{B}(\mathbb{R}^n)$ by adding the null sets, we obtain the σ-algebra $\Lambda(\mathbb{R}^n)$ of *Lebesgue measurable* sets. The extension of measure λ_n to $\Lambda(\mathbb{R}^n)$ is called the n-dimensional *Lebesgue measure.*

The Lebesgue measure has an additional property, stated in the next theorem, which is called *regularity* and which links the measure with the topology.

THEOREM. *Let μ be the Lebesgue measure in $\mathbb{R}^n$. For any compact set $K \subset \mathbb{R}^n$, the measure $\mu(K)$ is finite, and, for any Lebesgue measurable set $A \subset \mathbb{R}^n$, $\mu(A)$ satisfies the identities:*

$$\mu(A) = \sup\{\mu(K) : K \subset A, \ K \ \textit{compact}\} \tag{A.4}$$

and

$$\mu(A) = \inf\{\mu(U) : A \subset \Omega, \ \Omega \ \textit{open}\}. \tag{A.5}$$

A.4.2. Measurable functions. Let μ be a measure defined on a σ-algebra in a set M. We say that a subset $A \subset M$ is *measurable* (or μ-measurable) if A belongs to the domain of measure μ. We will be considering functions on M, taking values in the extended real line $[-\infty, +\infty]$. A function $f : M \to [-\infty, +\infty]$ is called *measurable* if the set

$$\{x \in M : f(x) < t\}$$

is measurable for any real t.

All algebraic operations on measurable functions result in measurable functions provided they do not contain indeterminacies $\frac{0}{0}$, $\frac{\infty}{\infty}$, and $\infty - \infty$. Moreover, if $f_1, ..., f_n$ are measurable functions on M taking values in $\mathbb{R}$ and $F(x_1, ..., x_n)$ is a continuous function on $\mathbb{R}^n$ then $F(f_1, ..., f_n)$ is a

[3]The class $\mathcal{B}(\mathbb{R}^n)$ of Borel sets is very large. In particular, it contains all open and closed sets in $\mathbb{R}^n$. Denote by $\mathcal{G}$ the family of all open sets and by $\mathcal{F}$ the family of all closed sets. Next, denote by $\mathcal{G}_\delta$ the family of all countable intersections of open sets, and by $\mathcal{F}_\sigma$ the family of all countable unions of closed sets. Similarly, one defines even larger families $\mathcal{G}_{\delta\sigma}$, $\mathcal{F}_{\sigma\delta}$, $\mathcal{G}_{\delta\sigma\delta}$, $\mathcal{F}_{\sigma\delta\sigma}$, etc, which all are called *Baire classes.* Since $\mathcal{B}(\mathbb{R}^n)$ is a σ-algebra, it contains all the Baire classes, even those of transfinite order. It is a deep result from the set theory that, in fact, $\mathcal{B}(\mathbb{R}^n)$ coincides with the union of all the Baire classes.

measurable function on M. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions then $\limsup f_n$ and $\liminf f_n$ are always measurable.

A.4.3. Lebesgue integration. We say that a couple (M, μ) is a *measure space* if μ is a complete σ-finite measure defined on a σ-algebra in the set M.

Let (M, μ) be a measure space and $f : M \to [0, +\infty]$ be a measurable function on M. Define a *partition* of $[0, +\infty]$ as a finite increasing sequence of positive reals. For any partition $\{t_k\}_{k=1}^{n}$, consider the corresponding *Lebesgue integral sum*:

$$\sum_{k=1}^{n} t_k \mu \{x \in M : t_k \leq f(x) < t_{k+1}\},$$

where we set $t_{n+1} := +\infty$. The supremum of all Lebesgue integral sums over all partitions $\{t_k\}$ is called the *Lebesgue integral* of f against measure μ and is denoted by

$$\int_M f d\mu.$$

Note that this integral takes values in $[0, +\infty]$.

Another point of view on this definition is as follows. By definition, a *simple function* is a finite linear combination of the indicator functions of disjoint measurable sets. Any partition $\{t_k\}_{k=1}^{n}$ of $[0, +\infty]$ is associated with the simple function

$$g(x) = \sum_{k=1}^{n} t_k 1_{E_k}, \tag{A.6}$$

where

$$E_k := \mu \{x \in M : t_k \leq f(x) < t_{k+1}\}.$$

It is natural to define the integral of $g(x)$ by

$$\int_M g d\mu := \sum_{k=1}^{n} t_k \mu(E_k).$$

Obviously, we have $f \geq g$. On the other hand, considering a sequence of nested partitions like $\{2^k\}_{k=-n}^{n}$, we obtain an increasing sequence $\{g_n\}$ of simple functions that converges to f pointwise as $n \to \infty$. Hence, it is natural to set

$$\int_M f d\mu = \sup_g \left\{ \int_M g d\mu \right\},$$

where sup is taken over all simple functions g as in (A.6). Clearly, this definition of the integral is equivalent to the one with the Lebesgue integral sums.

A measurable function $f : M \to [-\infty, +\infty]$ is said to be *integrable* against μ if

$$\int_M |f| d\mu < \infty.$$

Equivalently, f is integrable if both function f_+ and f_- are integrable, where $f_\pm := \frac{|f|\pm f}{2}$ are the positive and negative parts of f. For an integrable function, define its integral against μ by

$$\int_M f d\mu := \int_M f_+ d\mu - \int_M f_- d\mu. \tag{A.7}$$

In this case, the value of $\int_M f d\mu$ is a real number. The set of integrable functions is a linear space, and the integral is a linear functional on this space.

If $f \geq 0$ is a measurable function and A is a measurable subset of M then set

$$\int_A f d\mu := \int_M 1_A f d\mu.$$

The identity $\nu(A) := \int_A f d\mu$ defines a measure ν on the same σ-algebra as μ. The function f is called the *density* of measure ν with respect to μ (or the *Radon-Nikodym derivative* of ν) and is denoted by $\frac{d\nu}{d\mu}$. The integral $\int_A f d\mu$ for a signed function f is defined similarly to (A.7).

We say that two measurable functions f, g are equal *almost everywhere* and write

$$f = g \text{ a.e.}$$

if $f(x) = g(x)$ everywhere except for a null set, that is,

$$\mu\{x \in M : f(x) \neq g(x)\} = 0.$$

In the same way, a.e. applies to inequalities and other relations.

It follows easily from the definition of the integral that, for a non-negative measurable function f,

$$\int_M f d\mu < \infty \implies f < \infty \text{ a.e.}$$

and

$$\int_M f d\mu = 0 \iff f = 0 \text{ a.e.}$$

Since the measure μ is complete, changing a measurable function on a null set results in a measurable function. If $f = g$ a.e. then the properties of f and g with respect to integration are identical: they are integrable or not synchronously, and in the former case their integrals are equal. This allows to extend the notion of integral to functions that are defined almost everywhere (that is, outside a set of measure 0).

A.4.4. Convergence theorems. We state below the theorems about passage to the limit under the integral sign, which are most useful in applications.

FATOU'S LEMMA. *If $\{f_k\}$ is a sequence of non-negative measurable functions then*

$$\int_M \liminf_{k\to\infty} f_k \, d\mu \leq \liminf_{k\to\infty} \int_M f_k d\mu.$$

MONOTONE CONVERGENCE THEOREM. (B. Levi's theorem)*If* $\{f_k\}$ *is an increasing sequence of non-negative measurable functions then*

$$\int_M \lim_{k\to\infty} f_k \, d\mu = \lim_{k\to\infty} \int_M f_k d\mu. \tag{A.8}$$

DOMINATED CONVERGENCE THEOREM. (Lebesgue's theorem) *Let* $\{f_k\}$ *be a sequence of measurable functions satisfying the following two conditions:*

- *for some non-negative integrable function* F,

$$|f_k(x)| \le F(x) \text{ a.e.}$$

- *the limit* $\lim_{k\to\infty} f_k(x)$ *exists* a.e..

Then (A.8) holds.

If $\mu(M) < \infty$ then any bounded measurable function is integrable. Hence, we obtain the following particular case of the dominated convergence theorem.

BOUNDED CONVERGENCE THEOREM. *Let* $\mu(M) < \infty$ *and* $\{f_k\}$ *be a sequence of measurable functions satisfying the following two conditions:*

- *for some positive constant* C,

$$|f_k(x)| \le C \text{ a.e.}$$

- *the limit* $\lim_{k\to\infty} f_k(x)$ *exists* a.e..

Then (A.8) holds.

A.4.5. Lebesgue function spaces. Let (M,μ) be a measure space. The relation $f = g$ a.e. is obviously an equivalence relation between measurable functions. For any measurable function f, denote its equivalence class by $[f]$.

For any $1 \le p \le \infty$, define the *p-norm* of a measurable function f on M by

$$\begin{aligned} \|f\|_p &= \left(\int_M |f|^p \, d\mu\right)^{1/p}, \quad 1 \le p < \infty \\ \|f\|_\infty &= \operatorname*{esup}_M |f| := \inf_{g\in[f]} \sup_M |g|. \end{aligned}$$

We say that the parameters $p, q \in [1,+\infty]$ are *Hölder conjugate* if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

HÖLDER INEQUALITY. *If* p, q *are Hölder conjugate then, for all measurable functions* f, g,

$$\int_M |fg| \, d\mu \le \|f\|_p \|g\|_q.$$

Furthermore, the following identity holds

$$\|f\|_p = \sup_{\|g\|_q \le 1} \left(\int_M fg\, d\mu \right)$$

Two important particular cases of the Hölder inequality are:

$$\int_M |fg|\, d\mu \le \|f\|_\infty \|g\|_1$$

and the Cauchy-Schwarz inequality

$$\int_M |fg|\, d\mu \le \|f\|_2 \|g\|_2.$$

The p-norm possesses also the following properties:

- $\|f + g\|_p \le \|f\|_p + \|g\|_p$.
- $\|\lambda f\|_p = |\lambda|\, \|f\|_p$ for any real λ.
- $\|f\|_p = 0$ if and only if $f = 0$ a.e. that is, $[f] = 0$.

In particular, $[f] = [g]$ implies $\|f\|_p = \|g\|_p$, which allows to extend the p-norm to the equivalence classes of measurable functions.

Define the *Lebesgue space* $L^p = L^p(M) = L^p(M, \mu)$ by

$$L^p = \{[f] : f \text{ is a measurable function on } M \text{ and } \|f\|_p < \infty\}.$$

It follows from the above properties of the p-norm, that L^p is a linear space and the p-norm is a norm on L^p. In fact, we will only use the spaces L^1, L^2, L^∞, where the space L^2 is of particular importance.

To simplify the terminology, it is customary to say that a certain function f belongs to L^p and to write $f \in L^p$, while in fact $[f] \in L^p$. The p-norm of a function $f \in L^p$ will also be denoted by $\|f\|_{L^p}$ or $\|f\|_{L^p(M)}$ or $\|f\|_{L^p(M,\mu)}$

The following theorem is of paramount importance and justifies alone the measure theory and Lebesgue integration.

THEOREM. *The normed space $L^p(M, \mu)$ is complete for any $1 \le p \le \infty$; that is, $L^p(M, \mu)$ is a Banach space.*

The space $L^2(M, \mu)$ is particularly useful, because its norm is associated with the following inner product[4]

$$(f, g)_{L^2} = \int_M fg d\mu.$$

Hence, $L^2(M, \mu)$ is a Hilbert space.

[4]Here we consider only real valued functions. For complex valued functions, the definition of the inner product in L^2 should be modified as follows:

$$(f, g)_{L^2} = \int_M f\bar{g} d\mu,$$

where $\bar{g}$ is the complex conjugate of g.

We say that a sequence $\{f_k\}$ of functions converges to a function f almost everywhere and write $f_k \overset{\text{a.e.}}{\to} f$ (or $f_k \to f$ a.e.) if

$$\mu\{x \in M : f_k(x) \not\to f(x)\} = 0.$$

Also, we write $f_k \xrightarrow{L^p} f$ if f_k converges to f in the norm $L^p(M,\mu)$, that is, $\|f_k - f\|_p \to 0$.

The following result is a version of the dominated convergence theorem.

DOMINATED CONVERGENCE THEOREM IN L^p. *Let $p \in [1,+\infty)$ and $\{f_k\}$ be a sequence of functions from $L^p(M,\mu)$ satisfying the following two conditions:*

- *for some non-negative function $F \in L^p(M,\mu)$,*

$$|f_k(x)| \le F(x) \text{ a.e.}$$

- *$f_k \overset{\text{a.e.}}{\to} f$ for some function f.*

Then $f \in L^p(M,\mu)$ and $f_k \xrightarrow{L^p} f$.

PROOF. Function f is measurable as the limit of measurable functions. Since $|f_k| \le F$ a.e. and, hence, also $|f| \le F$ a.e., function f belongs to $L^p(M,\mu)$. Since $|f - f_k|^p \to 0$ a.e.,

$$|f - f_k|^p \le 2^p F^p,$$

and the function F^p is integrable, the dominated convergence theorem yields

$$\|f - f_k\|_p^p = \int_M |f - f_k|^p \, d\mu \to 0.$$

□

The following partial converse is true.

THEOREM. *If $\{f_k\}$ a sequence of functions $L^p(M,\mu)$ and $f_k \xrightarrow{L^p} f$ then there exists a subsequence $\{f_{k_i}\}$ such that $f_{k_i} \overset{\text{a.e.}}{\to} f$.*

Recall that a simple function is a function of the form

$$f = \sum_{k=1}^{n} c_k 1_{E_k},$$

where $n \in \mathbb{N}$, $c_k \in \mathbb{R} \setminus \{0\}$ and E_k are disjoint measurable sets. Assuming that all $c_k \neq 0$, it is easy to see that a simple function f is integrable if and only if all sets E_k have finite measures, and in this case $f \in L^p(M,\mu)$ for any $p \in [1,\infty]$. The following useful property of simple functions follows directly from the definition of the integral.

THEOREM. *The set of integrable simple functions is dense in $L^p(M,\mu)$ for any $p \in [1,\infty)$.*

Exercises.

A.6. Let $\{f_k\}$ be a sequence of functions from $L^2(M,\mu)$ such that $f_k \xrightarrow{L^2} f$. Prove that

$$\operatorname{esup} f \le \liminf_{k\to\infty} (\operatorname{esup} f_k) \tag{A.9}$$

and

$$\operatorname{einf} f \ge \limsup_{k\to\infty} (\operatorname{einf} f_k). \tag{A.10}$$

A.7. Prove that if $f_k \xrightarrow{L^2} f$ then $f_k^2 \xrightarrow{L^1} f^2$. Hence or otherwise show that, for any function $g \in L^\infty$,

$$\int_M f_k^2 g \, d\mu \to \int_M f^2 g \, d\mu.$$

A.4.6. Product measures. Let X_1, X_2 be two non-empty sets and S_1, S_2 be semi-rings of subsets of X_1 and X_2, respectively. Consider the following family of subsets of the set $X_1 \times X_2$:

$$S_1 \times S_2 := \{A \times B : A \in S_1, B \in S_2\},$$

which is also a semi-ring.

THEOREM. *Assume that μ_1 and μ_2 are σ-finite measures on semi-rings S_1 and S_2 on the sets X_1 and X_2 respectively. Then the functional*

$$\mu(A \times B) := \mu_1(A) \times \mu_2(B),$$

is a σ-finite measure on $S_1 \times S_2$.

By Carathéodory's extension theorem, μ can be then uniquely extended to the σ-algebra of measurable sets on $X_1 \times X_2$. The extended measure is denoted by $\mu_1 \times \mu_2$ or $\mu_1 \otimes \mu_2$ and is called the *product measure* of μ_1 and μ_2.

Observe that the product measure is σ-finite (and complete). Hence, one can define by induction the product $\mu_1 \times ... \times \mu_n$ of a finite sequence of σ-finite measures $\mu_1, ..., \mu_n$. This product satisfies the associative law.

The following property of the product measures is frequently useful.

THEOREM. *The set of all finite linear combinations of functions of the form $1_{A\times B}$ is dense in $L^p(X \times Y, \mu \times \nu)$, $1 \le p < \infty$, where A and B run over all measurable subsets with finite measures of X and Y, respectively.*

A.4.7. Fubini's theorem. Given a function $f(x,y)$ on the product $X \times Y$ of two sets X, Y, we refer to the function $y \mapsto f(x,y)$ for a fixed $x \in X$ as the *x-section* of f. Similarly, the function $x \mapsto f(x,y)$ for a fixed $y \in Y$ is called the y-section of f.

FUBINI'S THEOREM. *Let μ and ν be complete σ-finite measure on sets X and Y, respectively. Consider the product space $X \times Y$ with measure $\mu \times \nu$.*

(i) *If a function $f : X \times Y \to [0, +\infty]$ is measurable on $X \times Y$ then μ-almost all x- and ν-almost all y-sections of f are measurable, the functions*

$$x \mapsto \int_Y f(x,y)\, d\nu(y) \quad \textit{and} \quad y \mapsto \int_X f(x,y)\, d\mu(x) \tag{A.11}$$

are measurable on X and Y, respectively, and

$$\begin{aligned} \int_{X\times Y} f d(\mu \times \nu) &= \int_X \left[\int_Y f(x,y)\, d\nu(y)\right] d\mu(x) \\ &= \int_Y \left[\int_X f(x,y)\, d\mu(x)\right] d\nu(y). \end{aligned} \tag{A.12}$$

(ii) *If $f : X \times Y \to \mathbb{R}$ is integrable on $X \times Y$ then μ-almost all x- and ν-almost all y-sections of f are integrable, the functions* (A.11) *are integrable on X and Y respectively, and* (A.12) *holds.*

The part (i) is known also as *Tonelli's theorem.*
The following corollary is frequently useful.

COROLLARY. *If a function $f : X \times Y \to \mathbb{R}$ is measurable and its almost all x-sections are integrable on Y then the function*

$$x \mapsto \int_Y f(x,y)\, d\nu(y) \tag{A.13}$$

is measurable.

PROOF. The integrability of the x-sections of f means that the x-sections of f_+ and f_- are integrable. Hence, the functions

$$x \mapsto \int_Y f_+(x,y)\, d\nu(y) \quad \text{and} \quad x \mapsto \int_Y f_-(x,y)\, d\nu(y)$$

are finite almost everywhere. By part (i) of Fubini's theorem, these two functions are measurable. Therefore, the function (A.13) is measurable as the difference of two finite measurable functions. $\square$

EXAMPLE A.1. A classical application of Fubini's theorem is the evaluation of the following integral

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx.$$

Indeed, using Fubini's theorem, we obtain

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) e^{-x^2} dx \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)} dy\right) dx \\ &= \int_{\mathbb{R}^2} e^{-\left(x^2+y^2\right)} d\lambda, \end{aligned}$$

where λ is the Lebesgue measure in $\mathbb{R}^2$. Switching to the polar coordinates (cf. Example 3.23), we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} e^{-\left(x^2+y^2\right)} d\lambda &= \int_0^{\infty}\left(\int_{\mathbb{S}^1} e^{-r^2} d\theta\right) r dr \\ &= 2\pi \int_0^{\infty} e^{-r^2} r dr = \pi \int_0^{\infty} e^{-r^2} dr^2 = \pi. \end{aligned}$$

Hence, $I^2 = \pi$ and $I = \sqrt{\pi}$, that is,

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

A.5. Self-adjoint operators

A.5.1. Operators in a Hilbert space. An *operator* A in a Hilbert space $\mathcal{H}$ is a linear mapping $A : D \to \mathcal{H}$ where D is a linear subspace of $\mathcal{H}$ called the *domain* of A and denoted by $\operatorname{dom} A$. We say that A is *densely defined* if $\operatorname{dom} A$ is dense in $\mathcal{H}$. Define the *norm* of the operator A by

$$\|A\| = \sup_{x \in \operatorname{dom} A \setminus \{0\}} \frac{\|Ax\|}{\|x\|}. \tag{A.14}$$

The finiteness of the norm $\|A\|$ is equivalent to the continuity of A and to the uniform continuity of A. Hence, if A is densely defined and $\|A\| < \infty$ then the operator A uniquely extends to an operator defined on $\mathcal{H}$ and having the same norm. The operator A is said to be *bounded*[5] if $\operatorname{dom} A = \mathcal{H}$ and $\|A\| < \infty$.

For any two operators A and B, their sum $A + B$ and product AB are defined as follows:

$$(A + B)\, x = Ax + Bx \;\text{ for } x \in \operatorname{dom}(A + B) := \operatorname{dom} A \cap \operatorname{dom} B$$

and

$$(AB)(x) = A(B(x)) \;\text{ for } x \in \operatorname{dom}(AB) := \{x \in \operatorname{dom} B : Bx \in \operatorname{dom} A\}.$$

[5]Unlike the notion of a bounded linear functional, the definition of a bounded operator includes the requirement that the domain is the whole space $\mathcal{H}$, which is motivated by certain results of the spectral theory.

Clearly, if A and B are bounded then also $A+B$ and AB are bounded and $\|A+B\| \leq \|A\|+\|B\|$ and $\|AB\| \leq \|A\|\|B\|$. Therefore, the set $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ is a *normed algebra.* Furthermore, it is complete and hence is a *Banach algebra.* Alongside the topology associated with the operator norm, $\mathcal{B}(\mathcal{H})$ features the *strong operator topology*, which induces the *strong convergence* as follows: $A_k \to A$ strongly if $A_k x \to Ax$ for all $x \in \mathcal{H}$. The *weak operator topology* corresponds to the *weak convergence* defined as follows: $A_k \to A$ weakly if $(A_k x, y) \to (Ax, y)$ for all $x, y \in \mathcal{H}$.

Let A be a densely defined linear operator in $\mathcal{H}$ with the domain D. Consider the subspace

$$D^* = \{y \in \mathcal{H} : x \mapsto (Ax, y) \text{ is a bounded functional in } x \in D\}. \tag{A.15}$$

Using the Riesz representation theorem, (A.15) can be rewritten as follows:

$$D^* = \{y \in \mathcal{H} : \exists! a \in \mathcal{H} \quad \text{such that } (Ax, y) = (x, a) \text{ for all } x \in D\}. \tag{A.16}$$

The *adjoint* operator A^* is defined as follows: set $\operatorname{dom} A^* = D^*$ and, for any $y \in D^*$, set $A^* y = a$. As a consequence of this definition, we obtain the identity

$$(Ax, y) = (x, A^* y) \quad \text{for all } x \in \operatorname{dom} A \text{ and } y \in \operatorname{dom} A^*.$$

Let us emphasize that the adjoint operator is defined only if A is densely defined, but in general A^* is not necessarily densely defined.

An operator B is called an *extension* of an operator A if

$$\operatorname{dom} A \subset \operatorname{dom} B \text{ and } B|_{\operatorname{dom} A} = A.$$

In this case we write $A \subset B$. Obviously, if $A \subset B$ then $B^* \subset A^*$.

An operator A is called *symmetric* if

$$(Ax, y) = (x, Ay) \quad \text{for all } x, y \in \operatorname{dom} A. \tag{A.17}$$

If A is a symmetric (and densely defined) operator then obviously $A \subset A^*$. In particular, for any densely defined symmetric operator, its adjoint is also densely defined.

We say that the operator A is *self-adjoint* if $A^* = A$. Any self-adjoint operator is symmetric but the converse is in general not true. However, if A is a bounded operator then it is self-adjoint if and only if it is symmetric. If A is a symmetric operator and B is a self-adjoint extension of A then we have

$$A \subset B = B^* \subset A^*.$$

The operator A is called *non-negative definite* if

$$(Ax, x) \geq 0 \text{ for all } x \in \operatorname{dom} A.$$

Exercises.

A.8. If an operator A in $\mathcal{H}$ is injective and surjective then one defines the inverse operator A^{-1} such that, for any $x \in \mathcal{H}$, $A^{-1}x$ is the unique vector $y \in \operatorname{dom} A$ such that $Ay = x$.

(a) Prove that if A^{-1} exists then $AA^{-1} = \mathrm{id}$ and $A^{-1}A \subset \mathrm{id}$.

(b) Prove that if A and B are two operators such that

$$AB = \mathrm{id} \text{ and } BA \subset \mathrm{id}$$

then A^{-1} exists and $A^{-1} = B$.

A.9. Prove that, for any operator A in a Hilbert space,

$$\|A\| = \sup_{x \in \operatorname{dom} A, \|x\| \le 1, \|y\| \le 1} (Ax, y). \tag{A.18}$$

A.10. Prove that, for any bounded operator A, the adjoint operator A^* is also bounded and

$$\|A\| = \|A^*\| \quad \text{and} \quad \|A^*A\| = \|A\|^2.$$

A.11. Let A be a densely defined symmetric non-negative definite operator.

(a) Prove that, for all $x, y \in \operatorname{dom} A$,

$$(Ax, y)^2 \le (Ax, x)\,(Ay, y). \tag{A.19}$$

(b) Prove that

$$\|A\| = \sup_{x \in \operatorname{dom} A, \|x\| \le 1} (Ax, x).$$

A.12. Let A be a densely defined self-adjoint operator.

(a) Prove that $(\operatorname{ran} A)^{\perp} = \ker A$ and $(\ker A)^{\perp} = \overline{\operatorname{ran} A}$.

(b) Prove that A is invertible and the inverse A^{-1} is bounded if and only if there exists $c > 0$ such that

$$\|Ax\| \ge c\|x\| \text{ for all } x \in \operatorname{dom} A. \tag{A.20}$$

A.13. A densely defined operator A in a Hilbert space $\mathcal{H}$ is called closed if, for any sequence $\{x_k\} \subset \operatorname{dom} A$, the conditions $x_k \to x$ and $Ax_k \to y$ imply $x \in \operatorname{dom} A$ and $Ax = y$.

(a) Prove that any self-adjoint operator is closed.

(b) Prove that if A is a non-negative definite self-adjoint operator then $\operatorname{dom} A$ is a Hilbert space with respect to the following inner product:

$$(x, y) + (Ax, Ay).$$

A.5.2. Lebesgue-Stieltjes integration. Let $F(\lambda)$ be a function on $\mathbb{R}$ satisfying the following conditions:

$$F \text{ is monotone increasing, left-continuous, } F(-\infty) = 0,\ F(-\infty) < \infty. \tag{A.21}$$

Function F can be used to define a new Borel measure F_U on $\mathbb{R}$. Indeed, first define F_U when U is a semi-open interval $[a, b)$:

$$F_{[a,b)} = F(b) - F(a).$$

This includes also $F_{(-\infty,b)} = F(b)$. It is obvious that all semi-open intervals form a semi-ring. It is possible to prove that F_U is a σ-additive function on this semi-ring and hence, by the Carathéodory extension theorem, F_U can be extended to all Borel sets U. The measure F_U on Borel sets is called the *Lebesgue-Stieltjes* measure. Note that the measure F_U is finite because

$F_{(-\infty,+\infty)} = F(+\infty) < \infty$, which implies, in particular, that the extension F_U is unique.

Hence, we can integrate any Borel function[6] $\varphi(\lambda)$ on $\mathbb{R}$ against the measure F_U. Such an integral is called the *Lebesgue-Stieltjes* integral of φ against F and it is denoted by

$$\int_{-\infty}^{+\infty} \varphi(\lambda)\, dF(\lambda), \tag{A.22}$$

when it exists. The function φ is called integrable against F if

$$\int_{-\infty}^{+\infty} |\varphi(\lambda)|\, dF(\lambda) < \infty.$$

In particular, any bounded Borel function is integrable against F. For example, for $\varphi \equiv 1$ we have

$$\int_{-\infty}^{+\infty} dF(\lambda) = F(+\infty) < \infty, \tag{A.23}$$

and, for all $a < b$,

$$\int_{-\infty}^{+\infty} 1_{[a,b)} dF(\lambda) = F_{[a,b)} = F(b) - F(a).$$

Let us now extend the definition of the integral (A.22) to a larger class of functions F. Let F be any function on $\mathbb{R}$. We say that a Borel function φ is integrable against F if there are two functions F_1 and F_2 satisfying (A.21) such that $F = F_1 - F_2$ and φ is integrable against F_1 and F_2. In this case, set

$$\int_{-\infty}^{+\infty} \varphi(\lambda)\, dF(\lambda) := \int_{-\infty}^{+\infty} \varphi(\lambda)\, dF_1(\lambda) - \int_{-\infty}^{+\infty} \varphi(\lambda)\, dF_2(\lambda). \tag{A.24}$$

In fact, the value of the right hand side in (A.24) does not depend on the choice of F_1 and F_2 (see Exercise A.18).

Exercises.

A.14. Let F be a function satisfying (A.21), and let F_U be the associated Lebesgue-Stieltjes measure on $\mathbb{R}$. Set $F(a+) := \lim_{\lambda \to a+} F(\lambda)$ and prove that, for all $a < b$,

$$\begin{aligned} F_{(a,b)} &= F(b) - F(a+), \\ F_{[a,b]} &= F(b+) - F(a), \\ F_{\{a\}} &= F(a+) - F(a), \\ F_{(a,b]} &= F(b+) - F(a+). \end{aligned}$$

A.15. Let $\{s_k\}_{k=-\infty}^{\infty}$ be a double sequence of reals and let $\{t_k\}_{k=-\infty}^{\infty}$ be a double sequence of positive reals such that $\sum_k t_k < \infty$. Define function F by

$$F(\lambda) = \sum_{\{k.s_k < \lambda\}} t_k. \tag{A.25}$$

(a) Prove that F satisfies the conditions (A.21).

[6] A function φ is called Borel if the set $\{\lambda : \varphi(\lambda) < c\}$ is Borel for any real c. This condition enables one to define Lebesgue integral sums of φ against any Borel measure.

(b) Prove that, for any Borel set U,

$$F_U = \sum_{\{k:s_k \in U\}} t_k. \tag{A.26}$$

(c) Prove that, for any non-negative Borel function φ on $\mathbb{R}$,

$$\int_{-\infty}^{+\infty} \varphi(\lambda)\, dF(\lambda) = \sum_{k\in\mathbb{Z}} t_k \varphi(s_k). \tag{A.27}$$

(d) Prove that a Borel function φ on $\mathbb{R}$ is integrable against F if and only if

$$\sum_{k\in\mathbb{Z}} t_k |\varphi(s_k)| < \infty,$$

and its integral against F is given by (A.27).

A.16. Prove that if function F satisfies (A.21) and F is continuously differentiable then

$$\int_{-\infty}^{+\infty} \varphi(\lambda)\, dF(\lambda) = \int_{-\infty}^{+\infty} \varphi(\lambda) F'(\lambda)\, d\lambda, \tag{A.28}$$

for any non-negative Borel function φ.

A.17. For any function F on $\mathbb{R}$, defined its total variation on $\mathbb{R}$ by

$$\operatorname{var} F := \sup_{\{\lambda_k\}} \sum_{k\in\mathbb{Z}} |F(\lambda_{k+1}) - F(\lambda_k)| \tag{A.29}$$

where the supremum is taken over all increasing double sequences $\{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\lambda_k \to -\infty$ as $k \to -\infty$ and $\lambda_k \to +\infty$ as $k \to +\infty$.

(a) Show that F is the difference of two bounded monotone increasing functions if and only if $\operatorname{var} F < \infty$.
(b) Show that F is the difference of two functions satisfying (A.21) if and only if F is left-continuous and $\operatorname{var} F < \infty$.
(c) Let F be a left-continuous function on $\mathbb{R}$ such that $\operatorname{var} F < \infty$. Prove that

$$\operatorname{var} F = \sup_{|\varphi|\le 1} \int_{-\infty}^{+\infty} \varphi(\lambda)\, dF(\lambda),$$

where the supremum is taken over all continuous functions φ on $\mathbb{R}$ such that $|\varphi(\lambda)| \le 1$ for all λ.
(d) Show that if $F \in C^1(\mathbb{R})$ then

$$\operatorname{var} F = \int_{-\infty}^{+\infty} |F'(\lambda)|\, d\lambda.$$

A.18. Let F be any function on $\mathbb{R}$. We say that a Borel function φ is integrable against F if there are two functions $F^{(1)}$ and $F^{(2)}$ satisfying (A.21) such that $F = F^{(1)} - F^{(2)}$ and φ is integrable against $F^{(1)}$ and $F^{(2)}$. In this case, set

$$\int_{-\infty}^{+\infty} \varphi(\lambda)\, dF(\lambda) := \int_{-\infty}^{+\infty} \varphi(\lambda)\, dF^{(1)}(\lambda) - \int_{-\infty}^{+\infty} \varphi(\lambda)\, dF^{(2)}(\lambda). \tag{A.30}$$

Prove that the value of the right hand side of (A.30) does not depend on the choice of $F^{(1)}$ and $F^{(2)}$.

A.5.3. Spectral resolution. With any closed subspace $S \subset \mathcal{H}$ and any $x \in \mathcal{H}$, there is a unique vector $y \in S$ such that $x - y \perp S$. The vector y is called the (orthogonal) *projection* of x onto S. The *projector* P is the operator with the domain $\mathcal{H}$ such that Px is the projection of x onto S. For example, the identity operator id is the projector onto $\mathcal{H}$, and zero operator is the projector onto the trivial subspace $\{0\}$. It turns out that P is a bounded linear operator and $\|P\| \leq 1$. It is obvious that $P^2 = P$. Furthermore, P is non-negative definite and self-adjoint.

DEFINITION A.2. A family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ of operators in $\mathcal{H}$ is called a *spectral resolution* (or resolution of identity) if the following conditions are satisfied:

- E_λ is a projector for any $\lambda \in \mathbb{R}$.
- The mapping $\lambda \mapsto E_\lambda$ is monotone in the sense that $\lambda < \lambda'$ implies $\operatorname{ran} E_\lambda \subset \operatorname{ran} E_{\lambda'}$.
- The mapping $\lambda \mapsto E_\lambda$ is strongly left continuous, and
$$\lim_{\lambda \to -\infty} E_\lambda = 0, \qquad \lim_{\lambda \to +\infty} E_\lambda = \mathrm{id},$$
where the limits are understood also in the strong sense.

It follows from this definition that, for any $x \in \mathcal{H}$, the function $F(\lambda) := \|E_\lambda x\|^2$ satisfies the condition (A.21); in addition, we have $F(+\infty) = \|x\|^2$.

EXAMPLE A.3. It follows from (A.23) that
$$\int_{-\infty}^{+\infty} d\|E_\lambda x\|^2 = \|x\|^2. \tag{A.31}$$
For any two vectors $x, y \in \mathcal{H}$, we have
$$(E_\lambda x, y) = \left(E_\lambda^2 x, y\right) = (E_\lambda x, E_\lambda y) = \frac{1}{4}\left(\|E_\lambda (x+y)\|^2 - \|E_\lambda (x-y)\|^2\right).$$
We see that the function $F(\lambda) = (E_\lambda x, y)$ is the difference of two functions satisfying (A.21) and hence can be integrated against. In particular, we obtain from the above two lines
$$\begin{aligned}
\int_{-\infty}^{+\infty} d(E_\lambda x, y) &= \frac{1}{4}\int_{-\infty}^{+\infty} d\|E_\lambda (x+y)\|^2 - \frac{1}{4}\int_{-\infty}^{+\infty} d\|E_\lambda (x-y)\|^2 \\
&= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) \\
&= (x, y).
\end{aligned} \tag{A.32}$$

LEMMA. *Let $\varphi(\lambda)$ be a Borel function on $\mathbb{R}$. If*
$$\int_{-\infty}^{+\infty} |\varphi(\lambda)|^2 \, d\|E_\lambda x\|^2 < \infty$$
then the integral
$$\int_{-\infty}^{+\infty} \varphi(\lambda)\, d(E_\lambda x, y)$$
exists for any $y \in \mathcal{H}$ and determines a bounded linear functional of y.

Hence, by the Riesz representation theorem, there exists a unique vector, denoted by $J_\varphi x$, which satisfies the identity

$$(J_\varphi x, y) = \int_{-\infty}^{+\infty} \varphi(\lambda)\, d(E_\lambda x, y). \tag{A.33}$$

THEOREM. *Mapping $x \mapsto J_\varphi x$ is a linear operator in $\mathcal{H}$ with domain*

$$\operatorname{dom} J_\varphi := \left\{ x \in \mathcal{H} : \int_{-\infty}^{+\infty} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 < \infty \right\}, \tag{A.34}$$

which is a dense linear subspace of $\mathcal{H}$. Moreover, J_φ is a self-adjoint operator. Also, for any $x \in \operatorname{dom} J_\varphi$,

$$\|J_\varphi x\|^2 = \int_{-\infty}^{+\infty} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2. \tag{A.35}$$

Now, we can define the integral $\int_{-\infty}^{+\infty} \varphi(\lambda) dE_\lambda$ by

$$\int_{-\infty}^{+\infty} \varphi(\lambda) dE_\lambda := J_\varphi. \tag{A.36}$$

We will also use the notation

$$\int_{-\infty}^{+\infty} \varphi(\lambda) dE_\lambda x := J_\varphi x, \tag{A.37}$$

so that the defining identity (A.33) becomes

$$\left(\int_{-\infty}^{+\infty} \varphi(\lambda) dE_\lambda x, y\right) = \int_{-\infty}^{+\infty} \varphi(\lambda)\, d(E_\lambda x, y). \tag{A.38}$$

If S is a Borel subset of $\mathbb{R}$ and φ is a Borel function on S then set

$$\int_S \varphi(\lambda) dE_\lambda := \int_{-\infty}^{+\infty} \widetilde{\varphi}(\lambda)\, dE_\lambda$$

where $\widetilde{\varphi}$ is the extension of φ to $\mathbb{R}$, which vanishes outside S.

EXAMPLE A.4. If $\varphi \equiv 0$ then (A.33) implies that $J_\varphi = 0$. If $\varphi \equiv 1$ then it follows from (A.32) and (A.33) that

$$(J_\varphi x, y) = \int_{-\infty}^{+\infty} d(E_\lambda x, y) = (x, y)$$

whence $J_\varphi = \mathrm{id}$. In other words,

$$\int_{-\infty}^{+\infty} dE_\lambda = \mathrm{id}.$$

EXAMPLE A.5. Let φ be a bounded Borel function. It follows from (A.31) and (A.34) that $\operatorname{dom} J_\varphi = \mathcal{H}$. Moreover, (A.35) implies that

$$\|J_\varphi x\|^2 \le \sup |\varphi|^2 \|x\|^2.$$

Hence, in this case J_φ is a bounded operator and

$$\|J_\varphi\| \leq \sup|\varphi| .$$

Exercises.

A.19. Prove the following properties of projectors in a Hilbert space.

(a) Any projector P is a linear bounded self-adjoint operator and $P^2 = P$.
(b) For any bounded self-adjoint operator A such that $A^2 = A$, its range $\operatorname{ran} A$ is a closed subspace and A is the projector onto $\operatorname{ran} A$.
(c) Any projector P is non-negative definite, and $\|P\| = 1$ unless $P = 0$.

A.20. Let P be a projector and let $\{v_k\}$ be an orthonormal basis in $\mathcal{H}$. Prove that

$$\sum_k \|Pv_k\|^2 = \dim \operatorname{ran} P.$$

A.21. Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be a spectral resolution in a Hilbert space $\mathcal{H}$.

(a) Prove that if $a \leq b$ then

$$E_a E_b = E_b E_a = E_a. \tag{A.39}$$

(b) Prove that $E_b - E_a$ is a projector for all $a \leq b$. Hence or otherwise prove that the function $\lambda \mapsto \|E_\lambda x\|$ is monotone increasing, for any $x \in \mathcal{H}$.
(c) For a Borel set $U \subset \mathbb{R}$, define the operator

$$E_U := \int_U dE_\lambda.$$

Prove that, for all $-\infty < a < b < +\infty$,

$$E_{[a,b)} = E_b - E_a. \tag{A.40}$$

(d) Prove that if the intervals $[a_1, b_1)$ and $[a_2, b_2)$ are disjoint then the subspaces $\operatorname{ran} E_{[a_1,b_1)}$ and $\operatorname{ran} E_{[a_2,b_2)}$ are orthogonal.

A.22. Let $P_1, ..., P_k$ be projectors in $\mathcal{H}$ such that $\operatorname{ran} P_i \perp \operatorname{ran} P_j$ for $i \neq j$. Consider the operator

$$A = \sum_{i=1}^k \lambda_i P_i,$$

where λ_i are reals. Let $\varphi(\lambda) = \alpha_0 + \alpha_1\lambda + ...\alpha_n\lambda^n$ be a polynomial with real coefficients, and define the operator

$$\varphi(A) := \alpha_0 \operatorname{id} + \alpha_1 A + ... + \alpha_n A^n.$$

Prove that

$$\varphi(A) = \sum_{i=1}^k \varphi(\lambda_i) P_i \tag{A.41}$$

and, for any $x \in \mathcal{H}$,

$$\|\varphi(A)x\|^2 = \sum_{i=1}^k \varphi(\lambda_i)^2 \|P_i x\|^2.$$

Prove also that if φ and ψ are two polynomials then

$$\varphi(A) + \psi(A) = (\varphi + \psi)(A) \tag{A.42}$$

and

$$\varphi(A)\psi(A) = (\varphi\psi)(A). \tag{A.43}$$

A.5.4. Spectrum and Spectral Theorem. For any linear operator A in a *complex* Hilbert space $\mathcal{H}$, we say that a complex number λ is a *regular value* of A if the operator $A - \lambda\,\mathrm{id}$ has a bounded inverse; that is, if

$$\ker(A - \lambda\,\mathrm{id}) = \{0\}, \ \operatorname{ran}(A - \lambda\,\mathrm{id}) = \mathcal{H}, \text{ and } \|(A - \lambda\,\mathrm{id})^{-1}\| < \infty.$$

Any $\lambda \in \mathbb{C}$ which is not a regular value is called a *singular value.* The set of all regular values is called the *resolvent* of A, and the set of all singular values is called the *spectrum* of A and is denoted by $\operatorname{spec} A$. It is known that $\operatorname{spec} A$ is a closed subset of $\mathbb{C}$.

For example, if λ is an *eigenvalue* of A, that A if $Ax = \lambda x$ for some $x \neq 0$ then λ belongs to the spectrum of A. In this case, $\dim\ker(A - \lambda\,\mathrm{id})$ is positive and is called the *multiplicity* of λ. If the multiplicity of λ is 1 then λ is called a *simple eigenvalue.*

If A is an operator in a real Hilbert space $\mathcal{H}$ then define first the complexified space $\mathcal{H}^{\mathbb{C}} = \mathcal{H} \oplus i\mathcal{H}$, extend A by linearity to an operator $A^{\mathbb{C}}$ in $\mathcal{H}^{\mathbb{C}}$, and then set $\operatorname{spec} A = \operatorname{spec} A^{\mathbb{C}}$. We will apply the notion of spectrum only to self-adjoint operators.

THEOREM. *For any self-adjoint operator A, the spectrum* $\operatorname{spec} A$ *is a non-empty closed subset of* $\mathbb{R}$. *Also, we have*

$$\|A\| = \sup_{\lambda \in \operatorname{spec} A} |\lambda|. \tag{A.44}$$

Hence, *a posteriori*, we do not need to consider a complexified operator $A^{\mathbb{C}}$ when we deal with the spectrum of a self-adjoint operator. It follows from (A.44) that A is bounded if and only if its spectrum is bounded.

The full strength of the notion of spectrum is determined by the following theorem.

SPECTRAL THEOREM. *Let A be a self-adjoint operator in a real Hilbert space $\mathcal{H}$. Then there exists a unique spectral resolution $\{E_\lambda\}$ in $\mathcal{H}$ such that the following spectral decomposition takes place:*

$$A = \int_{-\infty}^{+\infty} \lambda dE_\lambda. \tag{A.45}$$

Furthermore, for any Borel function φ vanishing on $\operatorname{spec} A$,

$$\int_{-\infty}^{+\infty} \varphi(\lambda)\, dE_\lambda = 0.$$

In particular, the second statement implies that also

$$A = \int_{\operatorname{spec} A} \lambda dE_\lambda. \tag{A.46}$$

The integrals (A.45) and (A.46) are understood in the sense (A.36). In other words, the operator A coincides with the operator J_φ defined by (A.36) with the function $\varphi(\lambda) \equiv \lambda$ or, more generally, with any function $\varphi(\lambda)$ that is equal to λ on $\operatorname{spec} A$.

For any Borel function φ defined on $\operatorname{spec} A$, define the operator $\varphi(A)$ by setting $\varphi(A) = J_\varphi$; that is,

$$\varphi(A) := \int_{\operatorname{spec} A} \varphi(\lambda)\, dE_\lambda, \tag{A.47}$$

and

$$\operatorname{dom} \varphi(A) := \left\{ x \in \mathcal{H}\colon \int_{\operatorname{spec} A} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 < \infty \right\}. \tag{A.48}$$

Hence, $\varphi(A)$ is a self-adjoint operator. It follows from (A.38), for all $x \in \operatorname{dom} \varphi(A)$ and $y \in \mathcal{H}$,

$$(\varphi(A)\, x, y) = \int_{\operatorname{spec} A} \varphi(\lambda)\, d(E_\lambda x, y). \tag{A.49}$$

Also, (A.35) implies, for all $x \in \operatorname{dom} \varphi(A)$,

$$\|\varphi(A)x\|^2 = \int_{\operatorname{spec} A} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2. \tag{A.50}$$

In particular, if the function φ is bounded on $\operatorname{spec} A$ then $\varphi(A)$ is a bounded operator and

$$\|\varphi(A)\| \le \sup_{\operatorname{spec} A} |\varphi|. \tag{A.51}$$

EXAMPLE A.6. If $\varphi(\lambda) = \lambda$ then $\varphi(A) = A$. If $\varphi \equiv 1$ then $\varphi(A) = \mathrm{id}$. More generally, if $\varphi(\lambda) = 1_{[a,b)}$ then $\varphi(A) = E_b - E_a$.

SPECTRAL MAPPING THEOREM. *For any Borel function φ on* $\operatorname{spec} A$*,*

$$\operatorname{spec} \varphi(A) \subset \overline{\varphi(\operatorname{spec} A)}, \tag{A.52}$$

where the bar means taking closure in $\mathbb{R}$. If φ is continuous then, in fact,

$$\operatorname{spec} \varphi(A) = \overline{\varphi(\operatorname{spec} A)}.$$

It follows that, for a continuous function φ,

$$\|\varphi(A)\| = \sup_{s \in \operatorname{spec} \varphi(A)} |s| = \sup_{\lambda \in \operatorname{spec} A} |\varphi(\lambda)|. \tag{A.53}$$

In the case when $\operatorname{spec} A$ is compact, this implies that the mapping

$$\varphi \mapsto \varphi(A) \tag{A.54}$$

is a norm-preserving mapping between the Banach spaces $C(\operatorname{spec} A)$ and $\mathcal{B}(\mathcal{H})$.

FUNCTIONAL CALCULUS OF OPERATORS. *For any self-adjoint operator A and for all Borel functions φ, ψ on* $\operatorname{spec} A$*, we have*

$$\varphi(A) + \psi(A) \subset (\varphi + \psi)(A), \quad \operatorname{dom}(\varphi(A) + \psi(A)) = \operatorname{dom} \varphi(A) \cap \operatorname{dom} \psi(A). \tag{A.55}$$

and

$$\varphi(A)\psi(A) \subset (\varphi\psi)(A), \quad \operatorname{dom}(\varphi(A)\psi(A)) = \operatorname{dom}(\varphi\psi)(A) \cap \operatorname{dom}\psi(A). \tag{A.56}$$

In particular, if both functions φ and ψ are bounded then we have

$$\varphi(A) + \psi(A) = (\varphi + \psi)(A) \text{ and } \varphi(A)\psi(A) = (\varphi\psi)(A). \tag{A.57}$$

Hence, the mapping (A.54) is in fact a homomorphism from the Banach algebra of functions $C(\operatorname{spec} A)$ to the Banach algebra of operators $\mathcal{B}(\mathcal{H})$.

In fact, (A.57) holds if only ψ is assumed bounded.

Exercises.

A.23. Let A be self-adjoint operator, and φ and ψ be Borel functions on $\operatorname{spec} A$.

(*a*) Prove that
$$\varphi(A) + \psi(A) = (\varphi + \psi)(A),$$
provided either both functions φ, ψ are non-negative or one of them is bounded.

(*b*) Prove that
$$\varphi(A)\psi(A) = (\varphi\psi)(A),$$
provided ψ is bounded.

A.24. Let A be a densely defined self-adjoint operator.

(*a*) Prove that if the inverse A^{-1} exists and is a bounded operator then $A^{-1} = \frac{1}{A}$. Here the operator $\frac{1}{A}$ is defined by $\frac{1}{A} := \psi(A)$ where $\psi(\lambda) = \frac{1}{\lambda}$ on $\operatorname{spec} A$.

(*b*) Prove that if $\operatorname{spec} A \subset [0, +\infty)$ then there exists a non-negative definite self-adjoint operator X such that $X^2 = A$.

(*c*) Prove that if $\operatorname{spec} A \subset [0, +\infty)$ then $\operatorname{ran} e^{-A} \subset \operatorname{dom} A$.

A.25. Let A be a compact self-adjoint operator, and let $\{v_k\}$ be an orthonormal basis in $(\ker A)^{\perp}$ of the eigenvectors of A with the eigenvalues $\{\lambda_k\}$, which is guaranteed by the Hilbert-Schmidt theorem. Prove that $\operatorname{spec} A$ consists of the sequence $\{\lambda_k\}$ and, possibly, 0.

A.26. Let A be a densely defined self-adjoint operator.

(*a*) Prove that A is non-negative definite if and only if $\operatorname{spec} A \subset [0, +\infty)$.

(*b*) Set
$$a = \inf_{\substack{x \in \operatorname{dom} A \\ \|x\|=1}} (Ax, x) \quad \text{and} \quad b = \sup_{\substack{x \in \operatorname{dom} A \\ \|x\|=1}} (Ax, x).$$
Prove that
$$\inf \operatorname{spec} A = a \quad \text{and} \quad \sup \operatorname{spec} A = b.$$

A.27. Let $\{E_\lambda\}$ be a spectral resolution of a self-adjoint operator A. For any Borel set $U \subset \mathbb{R}$, define the operator E_U by
$$E_U := 1_U(A) = \int_U dE_\lambda.$$
The mapping $U \mapsto E_U$ is called a spectral measure.

(*a*) Prove that E_U is a projector. Show that if $U = [a, b)$ where $a < b$ then $E_U = E_b - E_a$. In particular, $E_{(-\infty, b)} = E_b$.

(*b*) Prove that if $U_1 \subset U_2$ then $\operatorname{ran} E_{U_1} \subset \operatorname{ran} E_{U_2}$.

(*c*) Prove that if U_1 and U_2 are disjoint then $\operatorname{ran} E_{U_1} \perp \operatorname{ran} E_{U_2}$.

(d) Prove that if $\{U_i\}_{i=1}^{\infty}$ is an increasing sequence of Borel sets in $\mathbb{R}$ and $U = \bigcup_{i=1}^{\infty} U_i$ then $E_{U_i} \to E_U$ in the strong topology. Prove that the same is true if the sequence $\{U_i\}$ is decreasing and $U = \bigcap_{i=1}^{\infty} U_i$.

A.28. Let A be a densely defined self-adjoint operator and $\{E_\lambda\}$ be its spectral resolution.

(a) Prove that if a is an eigenvalue of A then $E_{\{a\}} := 1_{\{a\}}(A)$ is the projector onto the eigenspace of a.

(b) Prove that if a is an eigenvalue of A with an eigenvector x then, for any Borel function φ on $\operatorname{spec} A$, $x \in \operatorname{dom} \varphi(A)$ and

$$\varphi(A)\, x = \varphi(\alpha)\, x.$$

A.29. Let A be a self-adjoint operator whose spectrum consists of a finite sequence $\lambda_1, \dots, \lambda_k$. Let P_i be the projector onto the eigenspace of λ_i, that is, $\operatorname{ran} P_i = \ker(A - \lambda_i \operatorname{id})$. Prove that $A = \sum_{i=1}^{k} \lambda_i P_i$.

A.30. Let A be a densely defined non-negative definite self-adjoint operator in $\mathcal{H}$ and $\{E_\lambda\}$ be its spectral resolution. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of Borel functions on $[0, +\infty)$ such that, for all n and $\lambda \in [0, +\infty)$,

$$|\varphi_n(\lambda)| \leq \Phi(\lambda),$$

where Φ is a non-negative Borel function on $[0, +\infty)$ such that

$$\int_0^\infty \Phi^2(\lambda)\, d\|E_\lambda x\|^2 < \infty, \tag{A.58}$$

for some $x \in \mathcal{H}$. Prove that if $\varphi_n(\lambda) \to \varphi(\lambda)$ for any $\lambda \in [0, +\infty)$ then $x \in \operatorname{dom} \varphi(A) \cap \operatorname{dom} \varphi_n(A)$ and

$$\varphi_n(A)\, x \to \varphi(A)\, x.$$

A.6. Gamma function

The gamma function is defined by the identity

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \tag{A.59}$$

for all $z > 0$. Here are some useful properties of $\Gamma(z)$.

(1) $\Gamma(z+1) = z\Gamma(z)$ for all $z > 0$, which follows from (A.59) by integration by parts using $zt^{z-1}dt = d(t^z)$.

(2) $\Gamma(1) = 1$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (the latter follows from (1.13)).

(3) If n is a positive integer then $\Gamma(n) = (n-1)!$ and

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n} 1 \cdot 3 \cdot \ldots \cdot (2n-1).$$

(4) For all $\alpha > 2$ and $r > 0$, the following identity holds

$$\int_0^\infty s^{-\alpha/2} \exp\left(-\frac{r^2}{4s}\right) ds = \Gamma(\alpha/2 - 1)\, 4^{\alpha/2-1} r^{2-\alpha}, \tag{A.60}$$

which is proved by the change $t = \frac{r^2}{4s}$ in the integral.

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