ΑΣΚΗΣΕΙΣ 11-12

Ολοκληρωτικές καμπύλες - Διαφ. Ροές

Ασκ 1

Βρείτε τις ολοκλ. καμπύλες των επόμενων δ.π $\in \mathcal{X}(\mathbb{R}^2)$:

(α) $\xi = y\dfrac{\partial}{\partial y}$

Απαντ. $\xi_1 = 0$, $\xi_2 = y = pr_2 \circ id = pr_2$

$$\dot{\alpha} = \xi \circ \alpha \Leftrightarrow \begin{cases} \alpha_1'(t) = \tilde{\xi}_1(\alpha_1(t), \alpha_2(t)) \\ \alpha_2'(t) = \tilde{\xi}_2(\alpha_1(t), \alpha_2(t)) \end{cases} \Leftrightarrow \begin{cases} \alpha_1'(t) = 0 \\ \alpha_2'(t) = \alpha_2(t) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_1(t) = c_1 \\ \alpha_2(t) = c_2 e^t \end{cases} \Leftrightarrow \tilde{\alpha}(t) = (c_1, c_2 e^t) \Leftrightarrow$$

$$\Leftrightarrow \alpha(t) = id_{\mathbb{R}^2}^{-1} \circ \tilde{\alpha}(c_1, c_2 e^t) = (c_1, c_2 e^t)$$

Για αρχ. συνθ (x,y) πρέπει $\alpha(0) = (x,y) = (c_1, c_2)$.

$$\boxed{\alpha_{(x,y)}(t) = (x, ye^t)}$$

(β) $\xi = x\dfrac{\partial}{\partial x} + 2y\dfrac{\partial}{\partial y}$

Απαντ. $\xi_1 = x = pr_1 \circ id = pr_1$, $\xi_2 = 2y = 2pr_2$

$$\dot{\alpha} = \xi \circ \alpha \Leftrightarrow \begin{cases} \alpha_1'(t) = \alpha_1(t) \\ \alpha_2'(t) = 2\alpha_2(t) \end{cases} \Leftrightarrow \begin{cases} \alpha_1(t) = c_1 e^t \\ \alpha_2(t) = c_2 e^{2t} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \tilde{\alpha}(t) = \alpha(t) = (c_1 e^t, c_2 e^{2t})$$

Για αρχ. συνθ. $\alpha(0) = (x,y) \Leftrightarrow c_1 = x, \wedge c_2 = y$

$$\boxed{\alpha_{(x,y)}(t) = (xe^t, ye^{2t})}$$

(γ) $\eta = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ | $\eta_1 = y = pr_2$, $\eta_2 = x = pr_1$

$\dot{\alpha} = \eta \circ \alpha \Longleftrightarrow \begin{cases} \alpha_1'(t) = \alpha_2(t) \\ \alpha_2'(t) = \alpha_1(t) \end{cases} \Rightarrow \begin{cases} \alpha_1''(t) = \alpha_1(t) \\ \alpha_2''(t) = \alpha_2(t) \end{cases}$

$\Rightarrow \begin{cases} \alpha_1(t) = c_1 e^{t} + c_2 e^{-t} \\ \alpha_2(t) = c_1 e^{t} - c_2 e^{-t} \end{cases} \Rightarrow$

$\Rightarrow \tilde{\alpha}(t) = \alpha(t) = (c_1 e^{t} + c_2 e^{-t},\ c_1 e^{t} - c_2 e^{t})$.

Για αρχ. συνθ $\alpha(0) = (x, y) \Rightarrow$

$\Rightarrow \begin{cases} c_1 + c_2 = x \\ c_1 - c_2 = y \end{cases} \Rightarrow \begin{cases} c_1 = \frac{x+y}{2} \\ c_2 = \frac{x-y}{2} \end{cases} \Rightarrow$

$\Rightarrow \boxed{\alpha(t) = \left(\frac{x+y}{2}e^{t} + \frac{x-y}{2}e^{-t},\ \frac{x+y}{2}e^{t} - \frac{x-y}{2}e^{t}\right).}$

Παρατ: $(x,y) = (0,0) \Rightarrow \alpha(t) \equiv (0,0) =$ σταθ.

Ασκ 2

Στο $M=\mathbb{R}^2\setminus\{(0,0)\}$ βρείτε τις ολ. καμπ. του

$$\xi=\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$$

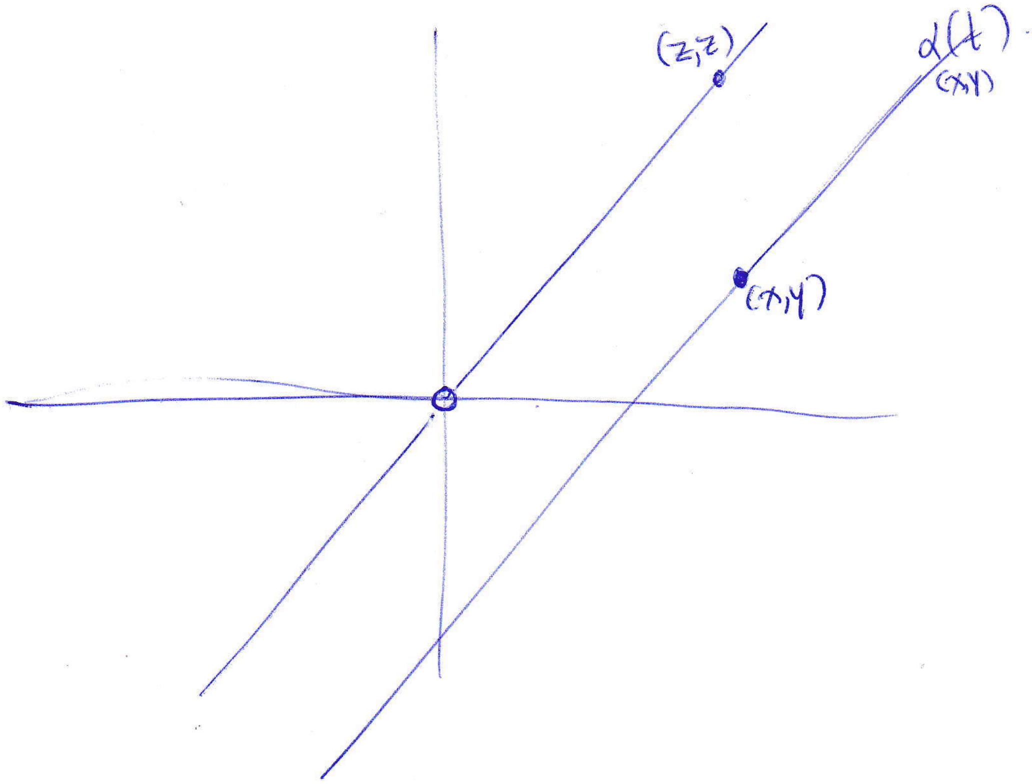
Είναι πλήρεις;

Απάντ. $\xi_1=1=\xi_2$

$$\dot\alpha=\xi\circ\alpha \iff \begin{cases}\alpha_1'(t)=1 \iff \alpha_1(t)=t+c_1\\ \alpha_2'(t)=1 \iff \alpha_2(t)=t+c_2\end{cases} \iff \tilde\alpha(t)=\alpha(t)=(t+c_1,t+c_2)$$

$\alpha(0)=(x,y)\iff(c_1,c_2)=(x,y)$. Άρα

$$\boxed{\alpha_{(x,y)}(t)=(x+t,y+t)}$$



$\forall(x,y)$ με $x\neq y$ $\alpha_{(x,y)}(t)$ ορίζεται $\forall t\in\mathbb{R}$.

για $x=y$ $\alpha_{(x,x)}(t)$ δεν ορίζεται στο 0.

$\boxed{\text{Ασκ } 3}$

$\left.\begin{array}{l} \dot{\alpha} = \xi\circ\alpha . \\ f : M \to \mathbb{R} \end{array}\right\}$ Ν.δ.ο. $\dot{\overbrace{f\circ\alpha}} = \xi(f)\circ\alpha$

Λύση.

$$\dot{\overbrace{f\circ\alpha}}(t) = Tf\circ T\alpha\circ \frac{d}{dt}(t) = Tf\circ\dot{\alpha}(t) = Tf\circ\xi\circ\alpha(t) =$$

$$= \underbrace{Tf(\xi_{\alpha(t)})}_{\in\, T_{f(\alpha(t))}\mathbb{R}} \equiv \overline{id}\circ Tf(\xi_{\alpha(t)}) \overset{(*)}{=} \xi_{\alpha(t)}(f) =$$

$$= \xi(f)(\alpha(t)) =$$

$$= \xi(f)\circ\alpha(t).$$

Υπενθ: $v \in T_x M$, $f : M \to \mathbb{R} \Rightarrow T_x f(v) \equiv v(f)$.

$$T_x f(v) \in T_{f(x)}\mathbb{R} \Rightarrow T_x f(v) \equiv \overline{id}\circ T_x f(v) =$$

$$= D(f\circ\varphi^{-1})(\varphi(x))(\bar{\varphi}(v))$$

$$= D(f\circ\varphi^{-1})(\varphi(x))((\varphi\circ\alpha)'(0))$$

$$= D(f\circ\alpha)(0)(1) = (f\circ\alpha)'(0) =$$

$$= v(f).$$

$$\begin{tikzcd} T_x M \arrow[r, "T_x f"] \arrow[d, "\bar{\varphi}"'] & T_{f(x)}\mathbb{R} \arrow[d, "\overline{id}"] \\ \mathbb{R}^m \arrow[r, "D(f\circ\varphi^{-1})(\varphi(x))"'] & \mathbb{R} \end{tikzcd}$$

Ασκ 5

$\xi \in \mathcal{X}(\mathbb{R})$ $\quad \xi = x^2 \frac{d}{dx}$ ΝΔο ξ όχι πλήρες και να βρείτε την ροή του.

Απάντ.

ξ έχει μια συντελεστένη: $\xi_1 = x^2 = pr_1^2 = id^2$. (δηλ $\xi_1(t) = t^2$).

$\dot{\alpha} = \xi \circ \alpha \Leftrightarrow \alpha_1'(t) \equiv \tilde{\alpha}'(t) = \tilde{\xi}_1(\alpha_1(t)) = \xi_1(\alpha_1(t)) = \alpha_1(t)^2$.

$\xi(t) = -t^2 \frac{d}{dt}\Big|_t$

$\alpha'(t) = -(\alpha_1(t))^2 \Rightarrow \boxed{\alpha(t) = \frac{1}{t+c}} \Rightarrow \alpha'(t) = -\left(\frac{1}{t+c}\right)^2$

$\alpha(0) = s = \frac{1}{c} \Rightarrow c = 1/s$.

$\alpha_s(t) = \frac{1}{t + 1/s} = \frac{s}{1+ts}$ και δεν ορίζεται για $t = -1/s$.

$\theta(t,s) = \frac{s}{1+ts}$ με πο: $\mathcal{D}(\xi) = \{(t,s) \in \mathbb{R}^2 : ts \neq -1\}$

$\boxed{\text{Άσκ. 6}}$

$\theta : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2 : \ \theta(t,(x,y)) = ((2+\sin y)t + x,\ y)$.

Ν.δ.ο θ ροή και να βρείτε τον απειροστικό της γεννήτορα.

<u>Απόδ.</u>

(1) θ διαφορίσιμη.

$$\theta(0,(x,y)) = ((2+\sin y)\cdot 0 + x, y) = (x,y).$$

$$\begin{aligned}
\theta(t,\theta(s,(x,y))) &= \theta(t, (\underbrace{(2+\sin y)s + x}_{x'}, y)) = \\
&= ((2+\sin y)t + x', y) = \\
&= ((2+\sin y)t + (2+\sin y)s + x, y) = \\
&= ((2+\sin y)(t+s) + x, y) = \\
&= \theta(t+s,(x,y)).
\end{aligned}$$

Άρα είναι διαφορική ροή.

(2) Αν ξ ο απειροστικός γεννήτορας, τότε

$$\xi_{(x,y)} = \dot{\theta}_{(x,y)}(0) = T_0\theta_{(x,y)}\left(\frac{d}{dt}\Big|_0\right) \in T_{(x,y)}\mathbb{R}^2$$

Υπολογίζουμε το $\overline{id_{\mathbb{R}^2}}(\xi_{(x,y)})$, χρησιμοποιώντας το μεταθετικό διάγραμμα

$$\begin{tikzcd}
\frac{d}{dt}\big|_0 \in T_0\mathbb{R} \arrow[r, "T_0\theta_{(x,y)}"] \arrow[d, "\overline{id_{\mathbb{R}}}"'] & T_{(x,y)}\mathbb{R}^2 \ni \xi_{(x,y)} \arrow[d, "\overline{id_{\mathbb{R}^2}}"] \\
\mathbb{R} \arrow[r, "D\theta_{(x,y)}(id_{\mathbb{R}}(0))"'] & \mathbb{R}^2
\end{tikzcd}$$

$$\overline{id}_{\mathbb{R}^2}\left(\xi_{(x,y)}\right) = \overline{id}_{\mathbb{R}^2} \circ T_0\, \theta_{(x,y)}\left(\frac{d}{dt}\Big|_0\right) =$$

$$= D\theta_{(x,y)}\left(id_{\mathbb{R}}(0)\right) \circ \overline{id}_{\mathbb{R}}\left(\frac{d}{dt}\Big|_0\right) =$$

$$= D\theta_{(x,y)}(0)(1) = \theta'_{(x,y)}(0) =$$

$$= (2+\sin y, 0) \Longrightarrow$$

$$\Longrightarrow \xi_{(x,y)} = \overline{id}_{\mathbb{R}^2}^{-1}(2+\sin y, 0) =$$

$$= (2+\sin y) \cdot \overline{id}_{\mathbb{R}^2}^{-1}(1,0) =$$

$$= (2+\sin y) \frac{\partial}{\partial x_1}\Big|_{(x,y)}$$