Αγκύλη Lie – Ασκήσεις

Ασκ 1 ΝΔο $\forall\, \xi, \eta \in \mathcal{X}(M)$, $\forall\, f, g \in C^\infty(M, \mathbb{R})$:

$$[f\xi, g\eta] = fg[\xi, \eta] + f\xi(g)\cdot\eta - g\eta(f)\cdot\xi$$

Απόδ.

Τα θεωρούμε παραγωγίσεις. Αρκεί νδο $\forall\, h \in C^\infty(M, \mathbb{R})$:

$$[\qquad](h) = \{\qquad\qquad\qquad\}(h).$$

Πράγματι:

$$[f\xi, g\eta](h) =$$

$$= f\xi(g\eta(h)) - g\eta(f\xi(h)) =$$

$$= f\cdot[\xi(g)\cdot\eta(h) + g\cdot\xi(\eta(h))] - g\cdot[\eta(f)\cdot\xi(h) + f\eta(\xi(h))]$$

$$= f\xi(g)\eta(h) + \underline{fg}\,\xi(\eta(h)) - g\cdot\eta(f)\xi(h) - \underline{gf}\,\eta(\xi(h)) =$$

$$= \underline{fg}\cdot\big(\xi(\eta(h)) - \eta(\xi(h))\big) + f\xi(g)\cdot\eta(h) - g\eta(f)\xi(h) =$$

$$= fg\cdot[\xi, \eta](h) + f\xi(g)\cdot\eta(h) - g\eta(f)\cdot\xi(h) =$$

$$= \{fg[\xi, \eta] + f\xi(g)\eta - g\eta(f)\cdot\xi\}(h).$$

Ασκ 2

ΝΔο $\forall (U,\varphi)$: $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$.

Απόδ

(α) Υπολογίζουμε συντεταγμένες:

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]_k = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right](x_k) = \frac{\partial}{\partial x_i}\left(\frac{\partial x_k}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial x_k}{\partial x_i}\right)$$

$$= \frac{\partial}{\partial x_i}(\delta_{kj}) - \frac{\partial}{\partial x_j}(\delta_{ki}) = 0 - 0.$$

(β) Το δείχνουμε σαν παραγώγιση: $\forall h \in C^\infty(M,\mathbb{R})$, $\forall x \in U$:

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right](h) = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}(h)\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}(h)\right) =$$

$$= \left[\frac{\partial}{\partial u_i}\left(\frac{\partial}{\partial x_j}(h)\circ\varphi^{-1}\right)\right]\circ\varphi - \left[\frac{\partial}{\partial u_j}\left(\frac{\partial}{\partial x_i}(h)\circ\varphi^{-1}\right)\right]\circ\varphi =$$

$$= \left[\frac{\partial}{\partial u_i}\left(\frac{\partial}{\partial u_j}(h\circ\varphi^{-1})\circ\varphi\circ\varphi^{-1}\right)\right]\circ\varphi - \left[\frac{\partial}{\partial u_j}\left(\frac{\partial}{\partial u_i}(h\circ\varphi^{-1})\circ\varphi\circ\varphi^{-1}\right)\right]\circ\varphi =$$

$$= \frac{\partial(h\circ\varphi^{-1})}{\partial u_i \partial u_j}\circ\varphi - \frac{\partial(h\circ\varphi^{-1})}{\partial u_j \partial u_i}\circ\varphi \overset{\text{Schwarz}}{=}$$

$$= 0.$$

Ασκ 3

Αν (x,y) οι συντεταγμένες του $(\mathbb{R}^2, id)$ να υπολογίσετε τις αγκύλες Lie :

$$X=\left[\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right]$$

$$T=\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}, e^{-y}\frac{\partial}{\partial y}\right]$$

με 2 τρόπους:

Απαντ.

(1) Υπολογίζουμε συνιστώσες:

$$X_x=[\qquad](x)=$$

$$=\frac{\partial}{\partial x}\left(x\cdot\frac{\partial x}{\partial y}-y\frac{\partial x}{\partial x}\right)-\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\left(\frac{\partial x}{\partial x}\right)=$$

$$=\frac{\partial}{\partial x}(x\cdot 0-y\cdot 1)-\underbrace{\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)(1)}_{=0}=\frac{\partial y}{\partial x}=0.$$

$$X_y=[\qquad](y)=$$

$$=\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial y}-y\frac{\partial y}{\partial x}\right)-\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial y}{\partial y}\right)}_{=0.}$$

$$=\frac{\partial}{\partial x}(x-0)-0=\frac{\partial x}{\partial x}=1.$$

Άρα $X=0\cdot\frac{\partial}{\partial x}+1\cdot\frac{\partial}{\partial y}=\frac{\partial}{\partial y}$

$Y_x = [\quad\quad](x) =$

$= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(e^{-y}\frac{\partial x}{\partial y}\right) - e^{-y}\frac{\partial}{\partial y}\left(x\frac{\partial x}{\partial x} + y\frac{\partial x}{\partial y}\right) =$

$= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(0) - e^{-y}\frac{\partial}{\partial y}(x\cdot 1 + y\cdot 0) =$

$= 0 - e^{-y}\cdot\left(\frac{\partial x}{\partial y} + 0\right) = 0 - e^{-y}0 = 0.$

$Y_y = [\quad\quad](y) =$

$= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(e^{-y}\frac{\partial y}{\partial y}\right) - e^{-y}\frac{\partial}{\partial y}\left(x\frac{\partial y}{\partial x} + y\frac{\partial y}{\partial y}\right) =$

$= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(e^{-y}\right) - e^{-y}\cdot\frac{\partial}{\partial y}(x\cdot 0 + y\cdot 1) =$

$= x\cdot\frac{\partial e^{-y}}{\partial x} + y\frac{\partial e^{-y}}{\partial y} - e^{-y}\cdot 1$

$= x\cdot 0 + y\cdot e^{-y}\cdot(-1) - e^{-y} = -e^{-y}(y+1).$

Άρα $Y = -e^{-y}(1+y)\cdot\frac{\partial}{\partial y}$

(β) Υπολογίζονται μέσω του τύπου της Ασκ. 1:

$$X = \left[\frac{\partial}{\partial x}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right] =$$

$$= \left[\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}\right] - \left[\frac{\partial}{\partial x}, -y\frac{\partial}{\partial x}\right] =$$

$$= x\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] + \frac{\partial x}{\partial x}\cdot\frac{\partial}{\partial y} - 0 -$$

$$-\left\{-y\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right] + \frac{\partial(-y)}{\partial x}\cdot\frac{\partial}{\partial x} - 0\right\} =$$

$$= x\cdot 0 + 1\cdot\frac{\partial}{\partial y} + y\cdot 0 + 0\cdot\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$$

$$Y = \left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, e^{-y}\frac{\partial}{\partial y}\right] =$$

$$= \left[x\frac{\partial}{\partial x}, e^{-y}\frac{\partial}{\partial y}\right] + \left[y\frac{\partial}{\partial y} + e^{-y}\frac{\partial}{\partial y}\right] =$$

$$= xe^{-y}\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] + x\frac{\partial}{\partial x}(e^{-y})\cdot\frac{\partial}{\partial y} - e^{-y}\frac{\partial}{\partial y}(x)\cdot\frac{\partial}{\partial x}$$

$$+ ye^{-y}\left[\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right] + y\frac{\partial}{\partial y}(e^{-y})\cdot\frac{\partial}{\partial y} - e^{-y}\frac{\partial y}{\partial y}\cdot\frac{\partial}{\partial y} =$$

$$= xe^{-y}\cdot 0 + x\cdot 0\cdot\frac{\partial}{\partial y} - e^{-y}\cdot 0\cdot\frac{\partial}{\partial x} +$$

$$+ ye^{-y}\cdot 0 + y(e^{-y}\cdot(-1))\cdot\frac{\partial}{\partial y} - e^{-y}\cdot 1\cdot\frac{\partial}{\partial y} =$$

$$= -ye^{-y}\frac{\partial}{\partial y} - e^{-y}\frac{\partial}{\partial y} =$$

$$= -(1+y)e^{-y}\cdot\frac{\partial}{\partial y}$$

Άσκ 4

$(U,\varphi)\in\mathcal{A}$, $f,h\in C^\infty(M,\mathbb{R})$. NΔο

$$\left[f\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right](h)=-\frac{\partial f}{\partial x_j}\cdot\frac{\partial g}{\partial x_i}$$

Απόδ

$$\left[f\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right](h)=$$

$$=f\frac{\partial}{\partial x_i}\left(\frac{\partial h}{\partial x_j}\right)-\frac{\partial}{\partial x_j}\left(f\frac{\partial}{\partial x_i}(h)\right)=$$

$$=f\cdot\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}(h)\right)-\frac{\partial}{\partial x_j}(f)\cdot\frac{\partial}{\partial x_i}(h)-f\cdot\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}(h)\right)$$

$$=f\cdot\left[\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}(h)\right)-\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}(h)\right)\right]-\frac{\partial f}{\partial x_j}\cdot\frac{\partial h}{\partial x_i}$$

$$=f\cdot\underbrace{\left[\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right]}_{=0}(h)-\frac{\partial f}{\partial x_j}\cdot\frac{\partial h}{\partial x_i}.$$

Ασκ 5

$\xi \in \mathcal{X}(M)$, $\eta \in \mathcal{X}(N)$. Θέτουμε

$$\tilde{\xi} : M \times N \longrightarrow TM \times TN : \tilde{\xi}_{(x,y)} = (\xi_x, 0_y) \in T_x M \times T_y N.$$

$$\hat{\eta} : M \times N \longrightarrow TM \times TN : \hat{\eta}_{(x,y)} = (0_x, \eta_y) \in T_x M \times T_y N.$$

Νδο $\tilde{\xi}, \hat{\eta} \in \mathcal{X}(M \times N)$ με $[\tilde{\xi}, \hat{\eta}] = 0$.

Απάντ.

Υπενθ. ότι $\underset{T_{(x,y)}(M\times N)}{\underset{\cap}{\omega \equiv (u,v)}} \longrightarrow \omega(f) = u(f_y) + v(f_x)$.

$\forall f \in C^\infty(M \times N, \mathbb{R})$:

$$\tilde{\xi}(f)(x,y) = \tilde{\xi}_{(x,y)}(f) \equiv \xi_x(f_y) + 0$$

Αν z_i οι συντεταγμένες ενός $(U \times V, \varphi \times \psi) \Rightarrow$

$\Rightarrow z_i = \varphi_i \circ p_M$ για $i = 1, \ldots, m$ και

$z_i = \psi_i \circ p_N$ για $i = m+1, \ldots, m+n$.

$$\tilde{\xi}_i(x,y) = \tilde{\xi}_{(x,y)}(z_i) = \xi_x(z_{iy}) + 0 \overset{\circledast}{=} \begin{cases} \xi_i(x) & i = 1, \ldots, m \\ 0 & i = m+1, \ldots, m+n. \end{cases}$$

$$\circledast \begin{cases} z_{iy}(x) = z_i(x,y) = \varphi_i(x) & \text{για } i = 1, \ldots, m. \\ z_{iy}(x) = z_i(x,y) = \underbrace{y_i(y)}_{\text{σταθ}} & \text{για } i = m+1, \ldots, m+n \end{cases}$$

Παρόμοια για $\hat{\eta}$.

$$[\tilde{\xi},\tilde{\eta}](f) = \tilde{\xi}(\tilde{\eta}(f)) - \tilde{\eta}(\tilde{\xi}(f))$$

$$[\quad](f)(x_0,y_0) = \tilde{\xi}_{(x_0,y_0)}(\tilde{\eta}(f)) - \tilde{\eta}_{(x_0,y_0)}(\tilde{\xi}(f)) =$$

$$= \xi_{x_0}(\tilde{\eta}(f)_{y_0}) - \eta_{y_0}(\tilde{\xi}(f)_{x_0}) \overset{\circledast}{=}$$

$$\tilde{\eta}(f)_{y_0}(x) = \tilde{\eta}_{(x,y_0)}(f) = \eta_{y_0}(f_x).$$

$$[\tilde{\xi},\tilde{\eta}](z_i)(x,y) = \tilde{\xi}(\tilde{\eta}(z_i))(x,y) - \tilde{\eta}_{(x,y)}(\tilde{\xi}(z_i)) =$$

$$= \xi_x(\tilde{\eta}(z_i)_y) - \eta_y(\tilde{\xi}(z_i)_x) \overset{\circledast}{=}$$

$$\tilde{\eta}(z_i)_y(x) = \tilde{\eta}(z_i)(x,y) = \eta_y(z_i x) \begin{array}{l} \nearrow c \\ \searrow y_i(y) \end{array}$$

(1) $z_i(x,y) = x_i(x) = x_i \circ p_M$

$$[\tilde{\xi},\tilde{\eta}](z_i)(x_0,y_0) = \tilde{\xi}_{(x_0,y_0)}(\tilde{\eta}(x_i\circ p_M)) - \tilde{\eta}_{(x_0,y_0)}(\tilde{\xi}(x_i\circ p_M)) =$$

$$= \xi_{x_0}(\tilde{\eta}(x_i\circ p_M)_{y_0}) - \eta_{y_0}(\tilde{\xi}(x_i\circ p_M)_{x_0}) \overset{\circledast}{=}$$

$$\tilde{\eta}(x_i\circ p_M)_{y_0}(x) = \eta_{y_0}((x_i\circ p_M)_x) \overset{\circledast\circledast}{=} 0.$$

$\circledast\circledast$. $(x_i\circ p_M)_x(y) = (x_i\circ p_M)(x,y) = x_i(x)$ ανεξ του y.

Ασκ 7

$f: M \to N$ διαφ., $(U,\varphi) \in \mathcal{A}$, $(V,\psi) \in \mathcal{B}$ με $f(U) \subseteq V$.

Ζητείται ικανή & αναγκαία συνθήκη: $\frac{\partial}{\partial x_1} - f - \frac{\partial}{\partial y_n}$

Απαντ.

$$\begin{tikzcd} TU \arrow[r, "Tf"] & TV \\ U \arrow[u, "\frac{\partial}{\partial x_1}"] \arrow[r, "f"'] & V \arrow[u, "\frac{\partial}{\partial y_n}"'] \end{tikzcd}$$

$$\frac{\partial}{\partial x_1} - f - \frac{\partial}{\partial y_n} \Longleftrightarrow$$

$$Tf \circ \frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_n} \circ f \Longleftrightarrow$$

$$T_x(f)\left(\frac{\partial}{\partial x_1}\Big|_x\right) = \frac{\partial}{\partial y_n}\Big|_{f(x)} \overset{*}{\Longleftrightarrow}$$

$$\Longleftrightarrow \bar{\psi}\left(T_x f\left(\frac{\partial}{\partial x_1}\Big|_x\right)\right) = \bar{\psi}\left(\frac{\partial}{\partial y_n}\Big|_{f(x)}\right) = e_n$$

$$\Longleftrightarrow DF(\varphi(x))(e_1) = e_n \Longleftrightarrow$$

$$\Longleftrightarrow \frac{\partial F}{\partial u_1}\Big|_{\varphi(x)} = e_n \Longleftrightarrow$$

$$\Longleftrightarrow u_j \circ \frac{\partial F}{\partial u_1}\Big|_{\varphi(x)} = \delta_{jn} \Longleftrightarrow$$

$$\Longleftrightarrow \frac{\partial(y_j \circ f \circ \varphi^{-1})}{\partial u_1}\Big|_{\varphi(x)} = \delta_{jn} \Longleftrightarrow$$

$$\Longleftrightarrow \frac{\partial(y_j \circ f)}{\partial x_1}\Big|_x = \delta_{jn}$$

$$\overset{*}{\Longleftrightarrow} T_x f\left(\frac{\partial}{\partial x_1}\Big|_x\right)(y_j) = \frac{\partial y_j}{\partial y_n}\Big|_{f(x)}$$

$$\Longleftrightarrow \frac{\partial}{\partial x_1}\Big|_x (y_j \circ f) = \delta_{jn}$$

Ασκ 8

$f : \mathbb{R} \to \mathbb{R}^2 : f(t) = (\sin t, \cos t)$.

$\eta \in \mathcal{X}(\mathbb{R}^2)$: $\eta = y \dfrac{\partial}{\partial x} - x \dfrac{\partial}{\partial y}$.

Ν δο $\dfrac{d}{dt}$, η είναι f-συσχ.

$$
\begin{tikzcd}
T\mathbb{R} \arrow[r, "Tf"] & T\mathbb{R}^2 \\
\mathbb{R} \arrow[u, "\frac{d}{dt}"] \arrow[r, "f"'] & \mathbb{R}^2 \arrow[u, "\eta"']
\end{tikzcd}
$$

Αποδ

$$\frac{d}{dt} - f - \eta \iff Tf \circ \frac{d}{dt} = \eta \circ f \iff T_t f\left(\frac{d}{dt}\Big|_t\right) = \eta_{f(t)}, \ \forall t \in \mathbb{R}$$

$$T_t f\left(\frac{d}{dt}\Big|_t\right)(u_1) = \frac{d}{dt}\Big|_t (u_1 \circ f) = (u_1 \circ f)'(t) = (\sin)'(t) = \cos t$$

$$\eta_{f(t)}(u_1) = \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}\right)(u_1 \equiv x)(f(t)) =$$

$$= \left(y \cdot \frac{\partial x}{\partial x} - x \frac{\partial x}{\partial y}\right)(f(t)) = (y \cdot 1 - x \cdot 0)(f(t))$$

$$= (u_2 \equiv y)(f(t)) = u_2(\sin t, \cos t) = \cos t.$$

$$T_t f\left(\frac{d}{dt}\Big|_t\right)(u_2) = (u_2 \circ f)'(t) = (\cos)'(t) = -\sin t.$$

$$\eta_{f(t)}(u_2 \equiv y) = \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}\right)(y)(f(t)) =$$

$$= (y \cdot 0 - x \cdot 1)(f(t)) = -u_1(\sin t, \cos t) =$$

$$= -\sin t.$$

Άρα έχουν τις ίδιες συντεταγμένες.