The spectrum of a Toeplitz operator with continuous symbol

Στο αρχειο (Ind26) δειξαμε οτι ο δεικτης στροφης επεκτεινεται σε ολες τις συνεχεις κλειστες καμπυλες $\gamma : [0,1] \to \mathbb{C}$. Αν $\phi \in C(\mathbb{T})$, γραφουμε $n(\phi;\lambda)$ για τον δεικτη στροφης της $\gamma_\phi(t) := \phi(e^{2\pi i t})$, $t \in [0,1]$.

Θεώρημα 1. *Αν $\phi \in C(\mathbb{T})$, τοτε*

$$\sigma(T_\phi) = \operatorname{ran}(\phi) \ \cup \{\lambda \in \mathbb{C} \setminus \operatorname{ran}(\phi) : n(\phi;\lambda) \neq 0\} .$$

Απόδειξη. Observe that since $\phi \in C(\mathbb{T})$ we have $\operatorname{ran}(\phi) = \operatorname{essran}(\phi)$.

Hence by spectral inclusion,

$$\operatorname{ran}(\phi) = \sigma(M_\phi) \subseteq \sigma(T_\phi) .$$

Thus we have to show that if $\lambda \in \mathbb{C} \setminus \operatorname{ran}(\phi)$, then

$$\lambda \notin \sigma(T_\phi) \iff n(\phi;\lambda) = 0$$
$$\text{Equivalently: } T_{\phi-\lambda} \text{ invertible} \iff n(\phi-\lambda;0) = 0.$$

Replacing ϕ with $\phi - \lambda$ we have to show that if $0 \notin \operatorname{ran}(\phi)$ (i.e. if ϕ vanishes nowhere in $\mathbb{T}$), then

$$T_\phi \text{ invertible} \iff n(\phi;0) = 0.$$

Since $\phi(\mathbb{T})$ is compact and $0 \notin \phi(\mathbb{T})$, the quantity $\delta := \min\{|\phi(w)| : w \in \mathbb{T}\}$ is strictly positive.

Choose a trigonometric polynomial $p \in C(\mathbb{T})$ such that

$$\|\phi - p\| < \frac{\delta}{3}$$

and note that p never vanishes on $\mathbb{T}$ ($|p(w)| > |\phi(w)| - \frac{\delta}{3} \geq \frac{2\delta}{3}$); hence $n(p;0)$ is defined. Also the function

$$\psi := \frac{\phi - p}{p} \quad \text{satisfies} \quad \|\psi\| < \frac{\delta/3}{2\delta/3} = \frac{1}{2} .$$

It follows that $u = \log(1+\psi)$ is a well defined continuous function on $\mathbb{T}$ and

$$1 + \psi = e^u .$$

(Indeed the series $\log(1+\psi) = -\sum_{n=1}^\infty \frac{1}{n}(-\psi)^n$, converges because $\|1-(1+\psi)\| < 1$.) [1]

[1] Λεπτομερειες στο αρχειο (Ind26)

Therefore (from (`Ind26`))

$$n(1+\psi;0)=0\,.$$

Now there exists $n\in\mathbb{Z}$ such that $f_n p := q$ is an analytic polynomial. Extending q to a polynomial on $\mathbb{C}$ and factorising, we may write

$$q(z)=cz^m\prod_{j=1}^{k}(z-z_j)\prod_{j=1}^{\ell}(z-w_j)\,,\qquad 0<|z_j|<1<|w_j|$$

for some nonzero $c\in\mathbb{C}$, where $\{z_j\}$ are the non-zero roots of q in $\mathbb{D}$ and $\{w_j\}$ are the roots of q outside $\overline{\mathbb{D}}$ (observe that q has no roots in $\mathbb{T}$ because p never vanishes on $\mathbb{T}$!). Then we have

$$\begin{aligned} p(e^{it})=e^{-int}q(e^{it}) &= ce^{i(m-n)t}\prod_{j=1}^{k}(e^{it}-z_j)\prod_{j=1}^{\ell}(e^{it}-w_j)\\ \text{so}\quad p &= cf_{(m-n)}\prod_{j=1}^{k}(f_1-z_j)\prod_{j=1}^{\ell}(f_1-w_j)\\ &= cf_{(m-n+k)}\prod_{j=1}^{k}(1-z_j\bar{f}_1)\prod_{j=1}^{\ell}(f_1-w_j)\\ &= c\bar{u}vf_{(m-n+k)} \end{aligned}$$

where

$$u:=\prod_{j=1}^{k}(1-\bar{z}_j f_1)\quad\text{and}\quad v:=\prod_{j=1}^{\ell}(f_1-w_j)\,.$$

Note that v and u are both in $\widetilde{H}^\infty$.

From a calculation in (`Ind26`) we know that

$$n(p;0)=m-n+k\,.$$

Now

$$\phi=p(1+\psi)=c\bar{u}(1+\psi)vf_{(m-n+k)}$$

and so

$$n(\phi;0)=n(p;0)+n(1+\psi;0)=n(p;0)=m-n+k.$$

Thus we have to show that T_ϕ is invertible iff $m-n+k=0$.

Letting $w:=c\bar{u}(1+\psi)v$ we have

$$T_\phi=T_{wf_{(m-n+k)}}=T_{f_{(m-n+k)}w}\,.$$

But we claim that the operator T_w is invertible. Indeed, since $w=(c\bar{u}(1+\psi))v$ and v is in $\widetilde{H}^\infty$ we have $T_w=cT_{\bar{u}(1+\psi)}T_v$; and since u is in $\widetilde{H}^\infty$ we have $T_{\bar{u}(1+\psi)}=T_{\bar{u}}T_{(1+\psi)}$. Thus

$$T_w=T_{\bar{u}}T_{(1+\psi)}T_v=T_u^*T_{(1+\psi)}T_v\,.$$

Since u (extended to $\mathbb{C}$) vanishes nowhere in $\overline{\mathbb{D}}$, the Toeplitz operator T_u is invertible [2] and so $T_{\bar{u}} = T_u^*$ is invertible. By the same argument T_v is invertible. Also $\left\|T_{1+\psi} - I\right\| = \left\|T_\psi\right\| = \|\psi\| < 1$ and so $T_{1+\psi}$ is an invertible operator (as is well known). Thus the product T_w is invertible.

Now, we consider two cases:

• If $m-n+k \geq 0$, then the function $f_{(m-n+k)}$ is in $\widetilde{H}^\infty$ and hence (as we have shown) the Toeplitz operator $T_\phi = T_{wf_{(m-n+k)}}$ is the product $T_w T_{f_{(m-n+k)}}$.

• If $m-n+k \leq 0$, the function $\bar{f}_{(m-n+k)}$ is in $\widetilde{H}^\infty$, hence by the same result $T_\phi = T_{f_{(m-n+k)}w}$ is the product $T_{f_{(m-n+k)}} T_w$.

Thus it follows that in both cases T_ϕ is invertible iff $T_{f_{(m-n+k)}}$ is invertible, which happens iff $f_{(m-n+k)}$ is constant, i.e. iff $m-n+k = 0$, οπως ϑελαμε.

$\square$

[2] u^{-1} is defined and holomorphic near $\overline{\mathbb{D}}$, so $T_u T_{u^{-1}} = T_{uu^{-1}} = I = T_{u^{-1}} T_u$