On the Toeplitz algebra

Εισαγωγη[1] The norm-closed unital subalgebra$\mathcal{A}$ of $\mathcal{B}(\widetilde{H}^2)$ generated by the (unilateral) shift $S = T_1$ is $\{T_f : f \in A(\mathbb{D})\}$ where $A(\mathbb{D}) := \{f \in C(\mathbb{T}) : \hat{f}(-k) = 0\,\forall k > 0\}$ (the disc algebra). The algebra $\mathcal{A}$ is a commutative Banach algebra, isometrically isomorphic to the disc algebra (via the map $A(\mathbb{D}) \to \mathcal{B}(\widetilde{H}^2) : f \mapsto T_f$ which is unital isometric and multiplicative on $A(\mathbb{D})$).

What if we consider the norm-closed algebra $C^*(S)$ generated by the shift S and its adjoint $S^* = T_{-1}$? This certainly contains $\{T_f : f \in C(\mathbb{T})\}$ (see below); but this (closed, unital, selfadjoint) subspace is not an algebra, nor is the unital isometry $C(\mathbb{T}) \to \mathcal{B}(\widetilde{H}^2) : f \mapsto T_f$ multiplicative. The purpose of these notes is to determine the algebra $C^*(S)$.

Ορισμός 1. *For the purpose of these notes, a* C* algebra *is a norm closed *-subalgebra (i.e. closed under sum, product and the map $T \mapsto T^*$) of $\mathcal{B}(H)$ for some Hilbert space H.* [2]

Παραδείγματα 1.
- $\mathcal{B}(L^2(\mathbb{T}))$*: unital, non commutative;*
- *its subalgebra $\{M_f : f \in C(\mathbb{T})\}$: unital, commutative;*
- *the algebra $\mathcal{K}(H)$ of compact operators on a Hilbert space H (for us, this is the norm closure of the *-algebra of finite rank operators on H): non commutative, non unital (iff $\dim H = \infty$).*

Ορισμός 2. *The* Toeplitz algebra *$C^*(S)$ is the C* algebra generated in $\mathcal{B}(\widetilde{H}^2)$ by the shift, i.e. the norm-closed linear span of all products of S and S^*.*

Θεώρημα 2. *The Toeplitz algebra $C^*(S)$ is equal to*

$$\mathcal{T} := \{T_f + K : f \in C(\mathbb{T}),\, K \in \mathcal{K}(\widetilde{H}^2)\} \subseteq \mathcal{B}(\widetilde{H}^2).$$

Απόδειξη. • Recall that

$$T_{f_k} = \begin{cases} S^k, & k \geq 0 \\ (S^*)^k, & k < 0 \end{cases}$$

Thus if $p = \sum_{|k| \leq n} \hat{p}(k) f_k$ is a trigonometric polynomial, then

$$T_p = \sum_{k=-n}^{-1} \hat{p}(k)(S^*)^k + \sum_{m=0}^{n} \hat{p}(m) S^m\,.$$

[1] cstars26, modified 2 Ιουνίου 2026

[2] There is an axiomatic characterization, which will not concern us here.

belongs to $C^*(S)$. Since every $f \in C(\mathbb{T})$ is a norm-limit of trigonometric polynomials and the map $f \to T_f$ is continuous (in fact, isometric) we have that $\{T_f : f \in C(\mathbb{T})\} \subseteq C^*(S)$.

• We show that every rank one operator of the form $f_m f_n^* \, (m, n \in \mathbb{Z}_+)$ is in $C^*(S)$: [3] Indeed recall that $SS^*(f_0) = 0$ and $SS^*(f_n) = f_n$ for $n > 0$; this means that SS^* is the projection onto the closed linear span of $\{f_n : n > 0\}$, and so $I - SS^*$ is the projection onto its orthogonal complement, span f_0, i.e. $I - SS^* = f_0 f_0^*$. (Note that $S^*S = I$ so $f_0 f_0^*$ is in $C^*(S)$.) Hence for $m, n \in \mathbb{Z}_+$ we have

$$S^m (S^*)^n - S^{m+1}(S^*)^{n+1} = S^m (I - SS^*)(S^*)^n = S^m (f_0 f_0^*)(S^*)^n = (S^m f_0)(S^n f_0)^* = f_m f_n^* \,.$$

The left hand side is in $C^*(S)$, hence $f_m f_n^* \in C^*(S)$ as claimed.

• It follows that $\mathcal{K}(\widetilde{H}^2) \subseteq C^*(S)$: indeed, it is a simple exercise (ΑΣΚΗΣΗ!) to show that every rank one operator can be approximated by linear combinations of operators of the form $f_m f_n^*$ and hence the same holds for any operator in the closed linear span of rank one operators, i.e. for any compact operator.

• We have shown that $\mathcal{T} \subseteq C^*(S)$. Note that $\mathcal{T}$ is a unital selfadjoint subspace; we now show that it is an algebra.

Indeed, if $T_f + K_1$ and $T_g + K_2$ are in $\mathcal{T}$ then recalling that $T_f T_g - T_{fg} = K$ is a compact operator (ΑΣΚΗΣΗ!) we have

$$\begin{aligned}(T_f + K_1)(T_g + K_2) &= T_f T_g + (T_f K_2 + K_1 T_g + K_1 K_2) \\ &= T_{fg} + (K + T_f K_2 + K_1 T_g + K_1 K_2) \in \mathcal{T} \end{aligned} \tag{1}$$

since the compact operators form an ideal. Thus $\mathcal{T}$ is an algebra.

• To show that $\mathcal{T}$ is closed in $\mathcal{B}(\widetilde{H}^2)$, we will need the following Lemma:

Λήμμα 3. *If $\phi \in L^\infty(\mathbb{T})$, then for every $K \in \mathcal{K}(\widetilde{H}^2)$ we have*

$$\|T_\phi + K\| \geq \|T_\phi\| \,.$$

Απόδειξη. Στο τελος.

Now let $(T_{f_n} + K_n)_n$ be a sequence in $\mathcal{T}$ which converges to some $X \in \mathcal{B}(\widetilde{H}^2)$. Recalling that $f \mapsto T_f$ is an isometry and using the Lemma, we have

$$\begin{aligned}\|f_n - f_m\|_\infty = \|T_{f_n - f_m}\| &\leq \|T_{f_n - f_m} + (K_n - K_m)\| = \|T_{f_n} - T_{f_m} + (K_n - K_m)\| \\ &= \|(T_{f_n} + K_n) - (T_{f_m} + K_m)\| \end{aligned}$$

and so (f_n) is Cauchy in $C(\mathbb{T})$; hence the limit $f = \lim f_n$ exists in $C(\mathbb{T})$ and thus $T_f = \lim T_{f_n}$ exists in $\mathcal{T}$. Since also $\lim(T_{f_n} + K_n) = X$, it follows that $X - T_f = \lim K_n$ is a compact operator K, [4] and so $X = T_f + K$ is in $\mathcal{T}$ οπως θελαμε.

[3] recall that $(f_m f_n^*)(g) := \langle g, f_n \rangle f_m = \hat{g}(n) f_m$.

[4] here we use that $\mathcal{K}$ is closed - by our definition

• Finally, we have shown that (a) $\mathcal{T}$ is a norm closed selfadjoint subalgebra of $\mathcal{B}(\widetilde{H}^2)$ containing S, so it must contain $C^*(S)$; but also (b) $\mathcal{T}$ is contained in $C^*(S)$; hence $\mathcal{T} = C^*(S)$. □

Παρατήρηση 4. *The decomposition of an element of $\mathcal{T}$ as a sum $T_f + K$ is unique: if $T_f + K = T_g + K'$ where $f, g \in C(\mathbb{T})$ and K, K' are compact, then $f = g$ and $K = K'$.*

Απόδειξη. We have

$$T_{f-g} = T_f - T_g = K' - K \in \mathcal{K}(\widetilde{H}^2)\,.$$

But by Lemma 3 we have $\|T_{f-g}\| \leq \mathrm{dist}(T_{f-g}, \mathcal{K}(\widetilde{H}^2)) = 0$, so $T_{f-g} = 0$ and hence $f - g = 0$. It follows that $K = K'$ as well. □

Πρόταση 5. *The map*

$$\pi : \mathcal{T} \to C(\mathbb{T}) : T_f + K \mapsto f$$

*is a well defined, contractive, *-preserving homomorphism of $\mathcal{T}$ onto $C(\mathbb{T})$, with* $\ker \pi = \mathcal{K}(\widetilde{H}^2)$.

Παρατήρηση 6. *In more algebraic language, the proposition says equivalently that the quotient $\mathcal{T}/\mathcal{K}$, which is a complete *-algebra, is isometrically *-isomorphic to $C(\mathbb{T})$.*

Απόδειξη. The map π is well-defined by the uniqueness of the expression of an element of $\mathcal{T}$ as $T_f + K$ (Remark 4). Its linearity is clear. Also $(T_f + K)^* = T_{\bar{f}} + K^*$ hence $\pi((T_f + K)^*) = \bar{f} = (\pi(T_f + K))^*$.

For multiplicativity, recalling the calculation (1), we have

$$\begin{aligned}
\pi((T_f + K_1)(T_g + K_2)) &= \pi(T_{fg} + K + K_1 T_g + T_f K_2 + K_1 K_2) \\
&= fg \qquad (\text{since } K + K_1 T_g + T_f K_2 + K_1 K_2 \in \mathcal{K}) \\
\text{and } \ \pi(T_f + K_1)\pi(T_g + K_2) &= fg
\end{aligned}$$

οπως θελαμε.

Clearly π maps $\mathcal{T}$ onto $C(\mathbb{T})$; also, $\pi(T_f + K) = 0$ iff $f = 0$, i.e. iff $T_f + K \in \mathcal{K}$. Thus $\ker \pi = \mathcal{K}$.

Finally, contractivity of π follows from the Lemma and the (known) fact that $\|f\|_\infty = \|T_f\|$ for $f \in C(\mathbb{T})$:

$$\|\pi(T_f + K)\| = \|f\|_\infty = \|T_f\| \leq \|T_f + K\|\,.$$

□

Παρατήρηση 7. *In algebraic language, we have defined a so called* exact sequence of C*-algebras and *-homomorphisms

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow{\pi} C(\mathbb{T}) \to 0\,.$$

Exactness means (by definition) that the map $\mathcal{K} \to \mathcal{T}$ is 1-1, its range in $\mathcal{T}$ is exactly the kernel of the map π, and the latter sends $\mathcal{T}$ onto $C(\mathbb{T})$.

We say that $\mathcal{T}$ is the Toeplitz extension of $\mathcal{K}$ by $C(\mathbb{T})$*.*

Let

$$\xi : C(\mathbb{T}) \to \mathcal{T} : f \mapsto T_f$$

be the so-called *symbol map*. As we know, this is a unital, *-preserving isometry, but not an algebra homomorphism (for example, $T_{f_1}T_{f_{-1}} \neq T_{f_1 f_{-1}}$). However, the composition

$$\pi \circ \xi : C(\mathbb{T}) \xrightarrow{\xi} \mathcal{T} \xrightarrow{\pi} C(\mathbb{T}) : f \to T_f \to \pi(T_f) = f$$

is of course a homomorphism, is *-preserving, and an isometry.

Παρατήρηση 8 (Universality). *Let H be any separable Hilbert space; fix an orthonormal basis $\{x_n : n \geq 0\}$ and consider the isometry $X \in \mathcal{B}(H)$ which satisfies $Xx_n = x_{n+1}$, $n \in Z_+$. Then* the C* algebra $C^*(X)$ generated by X *(i.e. the smallest norm-closed selfadjoint subalgebra of $\mathcal{B}(H)$ containing X) is isomorphic as a C* algebra (i.e. via a linear multiplicative *-preserving isometric bijection) with the Toeplitz algebra $\mathcal{T}$.*

*Indeed the map $U : x_n \mapsto f_n$, $n \in Z_+$ extends to a unitary from H onto $\widetilde{H}^2$; and so the map $\alpha : \mathcal{B}(H) \to \mathcal{B}(\widetilde{H}^2) : A \mapsto UAU^*$ is an isometric *-isomorphism sending X to S; hence α sends $C^*(X)$ onto $C^*(S) = \mathcal{T}$.*

*In fact it can be shown [Dav96, Theorem V.2.2] that if X is any isometry on any Hilbert space which is not unitary, then there exists a *-isomorphism $\alpha : C^*(X) \to \mathcal{T}$ such that $\alpha(X) = S$.*

For this reason, the Toeplitz algebra $\mathcal{T}$ is said to be the universal C* algebra generated by a proper (i.e. non-unitary) isometry.

Παρατήρηση 9. *Some denote by $\mathcal{T}(C(\mathbb{T}))$ what we have denoted by $\mathcal{T}$: it is the smallest closed subalgebra of $\widetilde{H}^2$ containing Toeplitz operators with continuous symbol.*

A very interesting object about which much less is known is $\mathcal{T}(L^\infty(\mathbb{T}))$, the smallest closed subalgebra of $\widetilde{H}^2$ containing all Toeplitz operators, namely $\{T_\phi : \phi \in L^\infty(\mathbb{T})\}$.

See for example R. Douglas, [Dou98], Chapter 7.

Αποδειξη του Λημματος 3 For $\phi \in L^\infty(\mathbb{T})$ and every $K \in \mathcal{K}(\widetilde{H}^2)$, noting that $T_{f_{-n}}$ is coanalytic for every $n \in \mathbb{N}$, we have $T_{f_{-n}} T_\phi = T_{f_{-n}\phi}$. Since also $\|T_{f_{-n}}\| = 1$,

$$\|T_\phi - K\| = \|T_{f_{-n}}\| \|T_\phi - K\| \geq \|T_{f_{-n}}(T_\phi - K)\| = \|T_{f_{-n}\phi} - T_{f_{-n}}K\| \geq \|T_{f_{-n}\phi}\| - \|T_{f_{-n}}K\|.$$

Now $\|T_{f_{-n}\phi}\| = \|f_{-n}\phi\|_\infty = \|\phi\|_\infty = \|T_\phi\|$ and so it remains to prove that $\lim_n \|T_{f_{-n}}K\| = 0$.

For this, suppose first that K is a rank one operator $K = fg^*$. Then, since $\|fg^*\| = \|f\|\|g\|$ (easily verified), we have $\lim_n \|T_{f_{-n}}K\| = \lim_n \|T_{f_{-n}}f\|\|g\|$. But if $f = \sum_{k=0}^{\infty} \hat{f}(k) f_k$ is in $\widetilde{H}^2$, then $T_{f_{-n}} f = \sum_{k=n}^{\infty} \hat{f}(k) f_{k-n}$ so $\|T_{f_{-n}}f\|_2^2 = \sum_{k=n}^{\infty} |\hat{f}(k)|^2$ which tends to 0 as $n \to \infty$.

Thus when K is a rank one operator, $\lim_n \|T_{f_{-n}}K\| = 0$; it follows that the same holds if K is a (finite) sum of rank one operators. For an arbitrary compact operator K choose, for any given $\epsilon > 0$, a finite rank operator K' such that $\|K' - K\| < \epsilon$. By the previous paragraph, there is an $n_0 \in \mathbb{N}$ such that $\|T_{f_{-n}}K'\| < \epsilon$ when $n \geq n_0$. Then, for any $n \geq n_0$, since all $T_{f_{-n}}$ have norm 1,

$$\|T_{f_{-n}}K\| \leq \|T_{f_{-n}}K'\| + \|T_{f_{-n}}(K - K')\| \leq \|T_{f_{-n}}K'\| + \|K - K'\| < 2\epsilon$$

οπως θελαμε. $\square$

Αναφορές

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