CATEGORICAL RELATIONS AND BIPARTITE ENTANGLEMENT IN TENSOR CONES FOR TOEPLITZ AND FEJÉR-RIESZ OPERATOR SYSTEMS

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ABSTRACT. The present paper aims to understand separability and entanglement in tensor cones, in the sense of Namioka and Phelps [33], that arise from the base cones of operator system tensor products [31]. Of particular interest here are the Toeplitz and Fejér-Riesz operator systems, which are, respectively, operator systems of Toeplitz matrices and Laurent polynomials (that is, trigonometric polynomials), and which are related in the operator system category through duality. Some notable categorical relationships established in this paper are the C*-nuclearity of Toeplitz and Fejér-Riesz operator systems, as well as their unique operator system structures when tensoring with injective operator systems. Among the results of this study are two of independent interest: (i) a matrix criterion, similar to the one involving the Choi matrix [10], for a linear map of the Fejér-Riesz operator system to be completely positive; (ii) a completely positive extension theorem for positive linear maps of $n \times n$ Toeplitz matrices into arbitrary von Neumann algebras, thereby showing that a similar extension theorem of Haagerup [26] for 2×2 Toeplitz matrices holds for Toeplitz matrices of higher dimension.

1. Introduction

1.1. **Tensor cones and the category $\mathfrak{S}_1$.** A cone C in a finite-dimensional topological real vector space V (that is, a subset $C \subseteq V$ such that $\alpha x + \beta y \in C$, for all nonnegative $\alpha, \beta \in \mathbb{R}$ and all $x, y \in C$) is *proper* if it is closed, generating ($C - C = V$), and pointed ($C \cap (-C) = \{0\}$).

The dual cone C^d of a cone C in V is the subset

$$C^d = \left\{\varphi \in V^d \,|\, \varphi(x) \geq 0, \,\forall x \in C\right\}$$

of the dual space V^d of V. If C is a proper cone, then so is C^d. Moreover, in identifying the bidual V^{dd} of V with V, the bipolar theorem yields the identification of C^{dd} with C. A proper cone C of a finite-dimensional topological real vector space V induces a partial order on V in which $x \leq y$ (or $y \geq x$), for $x, y \in V$, is understood to denote $y - x \in C$. Thus, C coincides with those $x \in V$ for which $x \geq 0$; such elements are said to be *positive*.

If C_1 and C_2 are cones in finite-dimensional real topological vector spaces V_1 and V_2, then the (C_1, C_2)-*separability cone* in $V_1 \otimes V_2$, which here will simply be called the *separability cone*, is defined to be the set

$$C_1 \otimes_{\rm sep} C_2 = \left\{ \sum_{j=1}^{k} a_j \otimes b_j \,|\, k \in \mathbb{N},\, a_j \in C_1,\, b_j \in C_2 \right\}.$$

The set $C_1 \otimes_{\text{sep}} C_2$ is a proper cone in $V_1 \otimes V_2$, if each C_i is a proper cone on V_i. The (C_1, C_2)-*dual separable cone* in $V_1 \otimes V_2$, or simply the *dual separable cone*, is the set

$$\begin{aligned} C_1 \otimes_{\text{sep}^*} C_2 &= \left(C_1^d \otimes_{\text{sep}} C_2^d\right)^d \\ &= \left\{\xi \in V_1 \otimes V_2 \,|\, (\varphi_1 \otimes \varphi_2)(\xi) \geq 0, \, \forall \, \varphi_i \in C_i^d\right\}, \end{aligned}$$

and it is a proper cone if C_1 and C_2 are proper cones.

It is straightforward to see that $C_1 \otimes_{\text{sep}} C_2 \subseteq C_1 \otimes_{\text{sep}^*} C_2$. Therefore, a fundamental question is, "Which cones C_1 and C_2 satisfy $C_1 \otimes_{\text{sep}} C_2 = C_1 \otimes_{\text{sep}^*} C_2$?" This long-standing question stemming from [33] has been answered only quite recently by Aubrun, Ludovico, Palazuelos, and Plávala [7], where it is proved that equality holds if and only one of the cones C_i is a simplicial cone. (A proper cone C in a finite-dimensional vector space V is said to be *simplicial* if V has a linear basis B in which C is the cone generated by B.)

Definition 1.1. A cone $C \subset V_1 \otimes V_2$ is a *tensor cone* for C_1 and C_2 if

$$C_1 \otimes_{\text{sep}} C_2 \subseteq C \subseteq C_1 \otimes_{\text{sep}^*} C_2.$$

Elements of $C_1 \otimes_{\text{sep}} C_2$ are said be *separable*; elements of $C \setminus (C_1 \otimes_{\text{sep}} C_2)$ are said to be *entangled*.

Thus, in cases where neither C_1 nor C_2 is classical, one of the main tasks in the analysis of a given tensor cone C for proper cones C_1 and C_2 is to discern which elements of C are separable and which are entangled. Analyses of this type have been carried out, for example, in the recent works [8, 9], providing a mathematical framework for general physical theories other than the purely classical or quantum settings. In the present paper, the tensor cones under study are those that arise as the base positive cone of an operator system tensor product.

The terminology and notation for $C_1 \otimes_{\text{sep}} C_2$ and $C_1 \otimes_{\text{sep}^*} C_2$ above differs from some of the standardly used terminology and notation, namely "min" and "max" and $\otimes_{\min}$ and $\otimes_{\max}$ [7, 8, 9, 23]. The reason for these differences is because the symbols $\otimes_{\min}$ and $\otimes_{\max}$ and terms "min" and "max" will be reserved in the present paper to reference the minimal and maximal operator system tensor products [31].

1.2. **Operator systems.** Formally, an *operator system* is a triple $(\mathcal{R}, \{\mathcal{C}_n\}_{n\in\mathbb{N}}, e_{\mathcal{R}})$ consisting of a complex $*$-vector space $\mathcal{R}$, a family $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ of proper cones in the real vector spaces $\mathcal{M}_n(\mathcal{R})_{\text{sa}}$ satisfying $\alpha^*\mathcal{C}_n\alpha \subseteq \mathcal{C}_m$, for all $n, m \in \mathbb{N}$ and $n \times m$ complex matrices α, and a distinguished element $e_{\mathcal{R}} \in \mathcal{C}_1$ that serves as an Archimedean order unit for the family $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ [11, 34]. Here, the notation $\mathcal{V}_{\text{sa}}$, for a complex $*$-vector space $\mathcal{V}$, means the real vector space $\mathcal{V}_{\text{sa}} = \{v \in \mathcal{V} \,|\, v^* = v\}$ of self adjoint elements. (With matrices over $\mathcal{R}$, the induced adjoint operation is $([x_{ij}]_{i,j=1}^n)^* = [x_{ji}^*]_{i,j=1}^n$.) It is common to dispense with the triple notation, and simply refer to $\mathcal{R}$ as an operator system and denote each proper cone $\mathcal{C}_n$ by $\mathcal{M}_n(\mathcal{R})_+$. The cone $\mathcal{C}_1$ is called the *base (positive) cone* of $\mathcal{R}$, and is denoted by $\mathcal{R}_+$, while the *matrix cones* $\mathcal{C}_n$ are denoted by $\mathcal{M}_n(\mathcal{R})_+$, for each n.

Linear transformations $\phi : \mathcal{R} \to \mathcal{T}$ of operator systems are said to be completely positive if their ampliations $\phi^{(n)} : \mathcal{M}_n(\mathcal{R}) \to \mathcal{M}_n(\mathcal{T})$, which are defined by $\phi^{(n)}([x_{ij}]_{i,j=1}^n) = [\phi(x_ij)]_{i,j=1}^n$, map $\mathcal{M}_n(\mathcal{R})_+$ in $\mathcal{M}_n(\mathcal{T})_+$, for each n, and these maps are unital if $\phi(e_{\mathcal{R}}) = e_{\mathcal{T}}$.

By $\mathfrak{S}_1$ we denote the category whose objects are operator systems and morphisms are unital completely positive (ucp) linear maps. Two operator systems, $\mathcal{R}$ and $\mathcal{T}$, are isomorphic in the category $\mathfrak{S}_1$ if there is linear bijection $\phi : \mathcal{R} \to \mathcal{T}$ such that both ϕ and ϕ^{-1} are ucp maps; such bijections ϕ are called unital complete order isomorphisms and the notation $\mathcal{R} \simeq \mathcal{T}$ is used to denote that $\mathcal{R}$ and $\mathcal{T}$ are isomorphic in $\mathfrak{S}_1$. Unital C*-algebras are among the objects in $\mathfrak{S}_1$, and the theory of operator systems is, in general, concerned with unital $*$-closed subspaces of unital C*-algebras, as the Choi-Effros Embedding Theorem [11] shows that for every operator system $\mathcal{R}$ there are a Hilbert space $\mathcal{H}$ and an operator subsystem $\mathcal{T}$ of the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$ such that $\mathcal{R} \simeq \mathcal{T}$.

The category $\mathfrak{S}_1$ admits a variety of constructions, including quotients and tensor products. With respect to tensor products, there are maximal and minimal operator system tensor product structures, whereby the maximal tensor product has the smallest family of matricial cones, while the minimal tensor product has the largest family of positive cones [31]. Matricial cones are used to norm elements of operator systems; thus, small cones lead to large [e.g., "max"] norms, while large cones lead to small [e.g., "min"] norms.

1.3. **Real versus complex vector spaces.** Operator systems are complex $*$-vector spaces, but the theory of tensor cones is framed in terms of real vector spaces; therefore, it is important to be clear about the definitions of the structures involved.

The dual space $\mathcal{R}^d$ of a finite-dimensional operator system $\mathcal{R}$ is also an operator system [11], and so one can consider operator system tensor products of the form $\mathcal{R}^d \otimes_\sigma \mathcal{T}$ for finite-dimensional operator systems $\mathcal{R}$ and arbitrary operator systems $\mathcal{T}$. At the base level, the base cone $(\mathcal{R}^d)_+$ is given by all linear functionals $\phi : \mathcal{R} \to \mathbb{C}$ such that $\phi(y) \geq 0$, for all $y \in \mathcal{R}_+$.

Consider now both the complex $*$-vector space $\mathcal{R}$ and the real vector space $\mathcal{R}_{\rm sa}$. If $\phi : \mathcal{R} \to \mathbb{C}$ is a linear functional such that $\phi(y) \geq 0$, for all $y \in \mathcal{R}_+$, then necessarily $\phi(x^*) = \overline{\phi(x)}$, for every $x \in \mathcal{R}$; thus, ϕ is also a linear functional $\mathcal{R}_{\rm sa} \to \mathbb{R}$ and is an element of the dual of $\mathcal{R}_+$ when considered as a proper cone in $\mathcal{R}_{\rm sa}$. Conversely, if $\vartheta : \mathcal{R}_{\rm sa} \to \mathbb{R}$ is a linear functional for which $\vartheta(x) \geq 0$ for every $x \in \mathcal{R}_+$, then the linear map $\phi : \mathcal{R} \to \mathbb{C}$ defined by $\phi(x + iy) = \vartheta(x) + i\vartheta(y)$, for $x, y \in \mathcal{R}_{\rm sa}$, defines a complex linear functional for which $\phi(x) \geq 0$ when $x \in \mathcal{R}_+$. Thus, the dual of the cone $\mathcal{R}_+$, when considered as a proper cone in $\mathcal{R}_{\rm sa}$, is canonically identified with the base cone $(\mathcal{R}^d)_+$ of the dual operator system $\mathcal{R}^d$.

Using the notation $\otimes_{\mathbb{R}}$ and $\otimes_{\mathbb{C}}$ to distinguish the tensor product operations in the categories of real and complex vector spaces, respectively, one needs to understand how $\mathcal{V}_{\rm sa} \otimes_{\mathbb{R}} \mathcal{W}_{\rm sa}$ relates to $(\mathcal{V} \otimes_{\mathbb{C}} \mathcal{W})_{\rm sa}$ when $\mathcal{V}$ and $\mathcal{W}$ are complex $*$-vector spaces. In general, $\mathcal{V}_{\rm sa} \otimes_{\mathbb{R}} \mathcal{W}_{\rm sa} \subseteq (\mathcal{V} \otimes_{\mathbb{C}} \mathcal{W})_{\rm sa}$; however, if $\mathcal{W}$ is the algebra $\mathcal{M}_n(\mathbb{C})$ of complex $n \times n$ matrices, then $\mathcal{V}_{\rm sa} \otimes_{\mathbb{R}} \mathcal{W}_{\rm sa} = (\mathcal{V} \otimes_{\mathbb{C}} \mathcal{W})_{\rm sa}$ [35, Lemma 3.7].

Returning to the case of finite-dimensional operator systems, $\mathcal{R}$ and $\mathcal{T}$, there are two possible field-dependent definitions of $\mathcal{R}_+ \otimes_{\rm sep^*} \mathcal{T}_+$, and we distinguish them below by using a superscript to denote the base field:

$$\mathcal{R}_+ \otimes^{\mathbb{R}}_{\rm sep^*} \mathcal{T}_+ = \left\{ x \in \mathcal{R}_{\rm sa} \otimes_{\mathbb{R}} \mathcal{T}_{\rm sa} \,|\, (\phi \otimes \psi)[x] \geq 0, \text{ for all } \phi \in (\mathcal{R}^d_{\rm sa})_+,\, \psi \in (\mathcal{T}^d_{\rm sa})_+ \right\},$$

and

$$\mathcal{R}_+ \otimes^{\mathbb{C}}_{\rm sep^*} \mathcal{T}_+ = \left\{ x \in (\mathcal{R} \otimes_{\mathbb{C}} \mathcal{T})_{\rm sa} \,|\, (\phi \otimes \psi)[x] \geq 0, \text{ for all } \phi \in (\mathcal{R}^d)_+,\, \psi \in (\mathcal{T}^d)_+ \right\}.$$

As explained above, $(\mathcal{R}^d_{\mathrm{sa}})_+$ and $(\mathcal{R}^d)_+$ are canonically identified, and so

$$\mathcal{R}_+ \otimes^{\mathbb{R}}_{\mathrm{sep}^*} \mathcal{T}_+ \subseteq \mathcal{R}_+ \otimes^{\mathbb{C}}_{\mathrm{sep}^*} \mathcal{T}_+$$

in general.

Therefore, to accommodate our interest in complex $*$-vector spaces, the following notation shall be used henceforth.

Notation 1.2. If $\mathcal{R}$ and $\mathcal{T}$ are finite-dimensional operator systems, then

$$\mathcal{R}_+ \otimes_{\mathrm{sep}^*} \mathcal{T}_+ = \left\{ x \in (\mathcal{R} \otimes_{\mathbb{C}} \mathcal{T})_{\mathrm{sa}} \,|\, (\phi \otimes \psi)[x] \geq 0, \text{ for all } \phi \in (\mathcal{R}^d)_+, \, \psi \in (\mathcal{T}^d)_+ \right\}.$$

That is, $\mathcal{R}_+ \otimes_{\mathrm{sep}^*} \mathcal{T}_+$ denotes $\mathcal{R}_+ \otimes^{\mathbb{C}}_{\mathrm{sep}^*} \mathcal{T}_+$.

With the notation above, the main result of [7] still holds.

Theorem 1.3. *If $\mathcal{R}$ and $\mathcal{T}$ are finite-dimensional operator systems, then*

$$\mathcal{R}_+ \otimes_{\mathrm{sep}} \mathcal{T}_+ = \mathcal{R}_+ \otimes_{\mathrm{sep}^*} \mathcal{T}_+$$

if and only if one of the cones $\mathcal{R}_+$ or $\mathcal{T}_+$ is simplicial.

Proof. If $\mathcal{R}_+ \otimes_{\mathrm{sep}} \mathcal{T}_+ = \mathcal{R}_+ \otimes_{\mathrm{sep}^*} \mathcal{T}_+$, then it is also true that $\mathcal{R}_+ \otimes_{\mathrm{sep}} \mathcal{T}_+ = \mathcal{R}_+ \otimes^{\mathbb{R}}_{\mathrm{sep}^*} \mathcal{T}_+$; hence, by the main result of [7], one of $\mathcal{R}_+$ or $\mathcal{T}_+$ is simplicial.

Conversely, assume $\mathcal{R}_+$ is simplicial. Thus, $\mathcal{R}_{\mathrm{sa}}$ has a linear basis $\{e_1, \dots, e_d\}$ consisting of positive e_j such that $\mathcal{R}_+$ is the cone generated by $\{e_1, \dots, e_d\}$. Hence, each $x \in \mathcal{R} \otimes_{\mathbb{C}} \mathcal{T}$ has the form $x = \displaystyle\sum_{j=1}^{d} e_j \otimes t_j$, for some $t_j \in \mathcal{T}$. If, further, $x^* = x$, then $t_j^* = t_j$ by the linear independence of the e_j, which yields $x \in \mathcal{R}_{\mathrm{sa}} \otimes_{\mathbb{R}} \mathcal{T}_{\mathrm{sa}}$. In other words, $\mathcal{R}_{\mathrm{sa}} \otimes_{\mathbb{R}} \mathcal{T}_{\mathrm{sa}} = (\mathcal{R} \otimes_{\mathbb{C}} \mathcal{T})_{\mathrm{sa}}$. Hence, by the main result of [7], the equality

$$\mathcal{R}_+ \otimes_{\mathrm{sep}} \mathcal{T}_+ = \mathcal{R}_+ \otimes^{\mathbb{R}}_{\mathrm{sep}^*} \mathcal{T}_+$$

holds, which yields $\mathcal{R}_+ \otimes_{\mathrm{sep}} \mathcal{T}_+ = \mathcal{R}_+ \otimes_{\mathrm{sep}^*} \mathcal{T}_+$ because $\mathcal{R}_{\mathrm{sa}} \otimes_{\mathbb{R}} \mathcal{T}_{\mathrm{sa}} = (\mathcal{R} \otimes_{\mathbb{C}} \mathcal{T})_{\mathrm{sa}}$. □

With Notation 1.2 in mind, the following example is one of the most important convex cones in quantum theory, consisting of what are called *block-positive matrices*, which are employed as entanglement witnesses.

Example 1.4 (Block-positive matrices)**.** For every $n \in \mathbb{N}$ and finite-dimensional operator system $\mathcal{T}$,

$$\mathcal{T}_+ \otimes_{\mathrm{sep}^*} \mathcal{M}_n(\mathbb{C})_+ = \left\{ [x_{ij}]_{i,j=1}^n \in \mathcal{M}_n(\mathcal{T})_{\mathrm{sa}} \,|\, \sum_{i,j=1}^{n} \alpha_i \overline{\alpha}_j x_{ij} \in \mathcal{T}_+, \text{ for all } \alpha_j \in \mathbb{C} \right\}.$$

Proof. The assertion is Theorem 3.2 of [35]. □

1.4. **Tensor cones with operator systems.** As will be explained in Proposition 2.6, and using Notation 1.2, if both $\mathcal{R}$ and $\mathcal{T}$ are finite-dimensional operator systems, then

$$\mathcal{R}_+ \otimes_{\mathrm{sep}} \mathcal{T}_+ \subseteq (\mathcal{R} \otimes_{\max} \mathcal{T})_+ \subseteq (\mathcal{R} \otimes_{\sigma} \mathcal{T})_+ \subseteq (\mathcal{R} \otimes_{\min} \mathcal{T})_+ \subseteq \mathcal{R}_+ \otimes_{\mathrm{sep}^*} \mathcal{T}_+. \tag{1.1}$$

In other words, the base cone $(\mathcal{R} \otimes_{\sigma} \mathcal{T})_+$ is a tensor cone for $\mathcal{R}_+$ and $\mathcal{T}_+$.

In light of the main result of [7] (in the formulation of Theorem 1.3), equality across (1.1) will occur only if one of $\mathcal{R}_+$ or $\mathcal{T}_+$ is a simplicial cone. However, it is interesting to have a better understanding of the other inclusions in (1.1), and the first purpose of the present paper is to address this when $\mathcal{R}$ and $\mathcal{T}$ are drawn from Toeplitz and Fejér-Riesz operator systems.

1.5. **Toeplitz and Fejér-Riesz operator systems.** A *Toeplitz operator system* is an operator system of $n \times n$ complex matrices, for some $n \geq 2$, in which the matrices are of Toeplitz form [38]:

$$\begin{bmatrix} \alpha_0 & \alpha_{-1} & \alpha_{-2} & \dots & \alpha_{-n+1} \\ \alpha_1 & \alpha_0 & \alpha_{-1} & \ddots & \vdots \\ \alpha_2 & \alpha_1 & \alpha_0 & \ddots & \alpha_{-2} \\ \vdots & \ddots & \ddots & \ddots & \alpha_{-1} \\ \alpha_{n-1} & \dots & \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix}, \tag{1.2}$$

for some $\alpha_\ell \in \mathbb{C}$. The linear space $C(S^1)^{(n)}$ of all such $n\times n$ complex matrices forms an operator subsystem, denoted by $C(S^1)^{(n)}$, of the unital C*-algebra $\mathcal{M}_n(\mathbb{C})$ of $n \times n$ complex matrices, where the $n \times n$. Thus, a Toeplitz matrix x is positive if it is positive as linear operator on the Hilbert space $\mathbb{C}^n$, and the identity matrix 1_n serves as the canonical Archimedean order unit for $\mathcal{M}_n(\mathbb{C})$ and, hence, $C(S^1)^{(n)}$. At the matrix level, $x \in \mathcal{M}_p(C(S^1)^{(n)})$ is positive if x is positive as an operator on the Hilbert space $\mathbb{C}^n \otimes \mathbb{C}^p \cong \bigoplus_1^p \mathbb{C}^n$.

A *Fejér-Riesz operator system* is an operator subsystem of the unital abelian C*-algebra $C(S^1)$ of continuous functions $f : S^1 \to \mathbb{C}$ on the unit circle $S^1 \subset \mathbb{C}$. If $f \in C(S^1)$, then the Fourier coefficients $\hat{f}(k)$ of f are given by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ik\theta}\, d\theta,$$

for every $k \in \mathbb{Z}$. If $n \geq 2$ is fixed, then Fejér-Riesz operator system $C(S^1)_{(n)}$ is defined to be the set of those $f \in C(S^1)$ for which $\hat{f}(k) = 0$, for all $k \in \mathbb{Z}$ with $|k| \geq n$. Thus, each $f \in C(S^1)_{(n)}$ has the form

$$f(z) = \sum_{\ell=-n+1}^{n-1} \alpha_\ell z^\ell,$$

for some $\alpha_\ell \in \mathbb{C}$. The positive elements of $C(S^1)$ are those continuous functions for which $f(z) \geq 0$, for every $z \in S^1$. At the matrix level, an element $F \in \mathcal{M}_p\left(C(S^1)_{(n)}\right)$ is positive if, as a matrix-valued function $F : S^1 \to \mathcal{M}_p(\mathbb{C})$, $F(z) \in \mathcal{M}_p(\mathbb{C})_+$, for every $z \in S^1$. The constant function χ_0, given by $\chi_0(z) = 1$ for all $z \in S^1$, serves as the canonical Archimedean order unit for both $C(S^1)$ and $C(S^1)_{(n)}$. The term "Fejér-Riesz operator system" is used because of the importance of the Fejér-Riesz factorisation theorem [21] in the study and application of trigonometric polynomials with nonnegative values.

Even though Toeplitz and Fejér-Riesz operator systems have been studied intensely since their introduction in classical works devoted quadratic forms and Fourier series, the following categorical relationship between these operator systems is rather recent [12, 16].

Theorem 1.5 (Duality)**.** *The linear map $\delta : C(S^1)^{(n)} \to \left(C(S^1)_{(n)}\right)^d$ that sends a Toeplitz matrix $T = [\tau_{k-\ell}]_{k,\ell=0}^{n-1} \in C(S^1)^{(n)}$ to the linear functional $\varphi_T : C(S^1)_{(n)} \to \mathbb{C}$ defined by*

$$\varphi_T(f) = \sum_{k=-n+1}^{n-1} \tau_{-k}\hat{f}(k), \tag{1.3}$$

for $f \in C(S^1)_{(n)}$, is a unital complete order isomorphism. That is,

$$C(S^1)^{(n)} \simeq \left(C(S^1)_{(n)}\right)^d \quad \text{and} \quad \left(C(S^1)^{(n)}\right)^d \simeq C(S^1)_{(n)}.$$

1.6. **Statement of the main results.**

1.6.1. *Base cones of operator system tensor products are tensor cones.*

Theorem 1.6. *If $\otimes_\sigma$ is an operator system tensor product structure on finite-dimensional operator systems $\mathcal{R}$ and $\mathcal{T}$, then the base positive cone $(\mathcal{R} \otimes_\sigma \mathcal{T})_+$ of the operator system $\mathcal{R} \otimes_\sigma \mathcal{T}$ satisfies the inclusions*

$$\mathcal{R}_+ \otimes_{\rm sep} \mathcal{T}_+ \subseteq (\mathcal{R} \otimes_\sigma \mathcal{T})_+ \subseteq \mathcal{R}_+ \otimes_{\rm sep^*} \mathcal{T}_+.$$

In other words, $(\mathcal{R} \otimes_\sigma \mathcal{T})_+$ is a tensor cone for $\mathcal{R}_+$ and $\mathcal{T}_+$.

1.6.2. *Separability.* In the operator system category $\mathfrak{S}_1$, if $\mathcal{R}$ and $\mathcal{T}$ are Toeplitz or Fejér-Riesz operator systems, then, by the results in [16],

$$\mathcal{R} \otimes_{\rm min} \mathcal{T} \neq \mathcal{R} \otimes_{\rm max} \mathcal{T}.$$

What this means is that the sequence of positive matrix cones for these tensor product operator systems do not coincide somewhere along the sequence. Nevertheless, equality is sometimes possible at the base level, $(\mathcal{R} \otimes_\sigma \mathcal{T})_+$, in an operator system tensor product $\mathcal{R} \otimes_\sigma \mathcal{T}$, as demonstrated by the theorem below.

Theorem 1.7. *Assume that $n, m \in \mathbb{N}$ satisfy $n, m \geq 2$.*

1. *The positive cone of $C(S^1)^{(n)} \otimes_{\rm max} C(S^1)^{(m)}$ coincides with separability cone $(C(S^1)^{(n)})_+ \otimes_{\rm sep} (C(S^1)^{(m)})_+$.*
2. *The positive cone of $C(S^1)^{(2)} \otimes_{\rm min} C(S^1)^{(m)}$ coincides with separability cone $(C(S^1)^{(2)})_+ \otimes_{\rm sep} (C(S^1)^{(m)})_+$.*
3. *If x_0, x_1 are $m \times m$ Toeplitz matrices for which $\begin{bmatrix} x_0 & x_1^* \\ x_1 & x_0 \end{bmatrix}$ is a positive operator, then for every $n \geq 3$ there exist Toeplitz matrices $x_2, \dots, x_{n-1}$ such that*
$$\begin{bmatrix} x_0 & x_1^* & \dots & x_{n-1}^* \\ x_1 & x_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_1^* \\ x_{n-1} & \dots & x_1 & x_0 \end{bmatrix}$$
is a positive operator.

The first two assertions above sharpen Proposition 3.4 and Corollary 3.5, respectively, in [18], as well as some results of Ando in [2, 3], while the third assertion may be viewed as an extension of the Ando moment theorem [1] from the operator algebra category to the operator system category.

In addition, a purely algebraic proof establishes the following theorem, which can also be proved using the main result of [7].

Theorem 1.8. *If* $\mathrm{C}^*(\mathbb{Z}_m)$ *denotes the operator system of* $m \times m$ *complex circulant matrices, then, for every* $n \geq 2$,

$$(C(S^1)^{(n)})_+ \otimes_{\mathrm{sep}} \mathrm{C}^*(\mathbb{Z}_m)_+ = (C(S^1)^{(n)})_+ \otimes_{\mathrm{sep}^*} \mathrm{C}^*(\mathbb{Z}_m)_+.$$

1.6.3. *Entanglement.* The first result is in contrast to Theorem 1.8 above.

Theorem 1.9. *If* $\mathcal{R}_+$ *and* $\mathcal{T}_+$ *are Toeplitz or Fejér-Riesz cones, then*

$$\mathcal{R}_+ \otimes_{\mathrm{sep}} \mathcal{T}_+ \neq \mathcal{R}_+ \otimes_{\mathrm{sep}^*} \mathcal{T}_+.$$

An important matrix in the study of Toeplitz and Fejér-Riesz operator systems is the universal positive $n \times n$ Toeplitz matrix T_n, the element of $C(S^1)^{(n)} \otimes C(S^1)_{(n)}$ defined by

$$T_n(z) = \begin{bmatrix} 1 & z^{-1} & \dots & z^{-n+1} \\ z & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & z^{-1} \\ z^{n-1} & \dots & z & 1 \end{bmatrix},$$

for $z \in S^1$.

Theorem 1.10. *For every* $n \geq 2$, T_n *generates an extremal ray of the positive cone of* $C(S^1)^{(n)} \otimes_{\max} C(S^1)_{(n)}$ *and is entangled.*

Interestingly, the matrix T_n has the same role for determining the complete positivity of linear maps on $C(S^1)_{(n)}$ that the Choi matrix [10] has for determining the complete positivity of linear maps on $\mathcal{M}_n(\mathbb{C})$, as noted below.

Theorem 1.11. *A linear map* $\phi : C(S^1)_{(n)} \to \mathcal{T}$*, for an operator system* $\mathcal{T}$*, is completely positive if and only if* $\phi^{(n)}(T_n)$ *is positive in* $\mathcal{M}_n(\mathcal{T})$*, where* $\phi^{(n)}$ *is the ampliation of* ϕ *to a linear map* $\mathcal{M}_n(C(S^1)_{(n)}) \to \mathcal{M}_n(\mathcal{T})$.

Elements of $C(S^1)_{(n)} \otimes C(S^1)_{(m)}$ may viewed naturally as continuous complex-valued functions on the 2-torus $S^1 \times S^1$. In this regard, we have:

Theorem 1.12. *For every* $n, m \geq 2$,

$$\left(C(S^1)_{(n)} \otimes_{\min} C(S^1)_{(m)}\right)_+ = (C(S^1)_{(n)})_+ \otimes_{\mathrm{sep}^*} (C(S^1)_{(m)})_+$$

and

$$\left(C(S^1)_{(2)} \otimes_{\max} C(S^1)_{(m)}\right)_+ = (C(S^1)_{(2)})_+ \otimes_{\mathrm{sep}^*} (C(S^1)_{(m)})_+.$$

1.6.4. *Categorical relations.*

Theorem 1.13. *If* $\mathcal{R}$ *is a Toeplitz or Fejér-Riesz operator system, then:*

(1) *there are no unital C*-algebras* $\mathcal{A}$ *for which* $\mathcal{R} \simeq \mathcal{A}$*;*
(2) $\mathcal{R}$ *does not have the weak expectation property;*
(3) $\mathcal{R} \otimes_{\min} \mathcal{I} = \mathcal{R} \otimes_{\max} \mathcal{I}$*, for every injective operator system* $\mathcal{I}$*;*
(4) $\mathcal{R} \otimes_{\min} \mathcal{A} = \mathcal{R} \otimes_{\max} \mathcal{A}$*, for every unital C*-algebra* $\mathcal{A}$.

Statement (4) of Theorem 1.13 asserts that Toeplitz and Fejér-Riesz operator systems are *C*-nuclear*, which is to say they have unique operator system tensor product structures when tensoring with a unital C*-algebra.

The truncation of $m \times m$ Toeplitz matrices to $n \times n$ Toeplitz matrices, when $m \geq n$, is a very natural map sending large Toeplitz matrices to smaller ones; it is in fact a complete quotient map in the sense of [19].

Theorem 1.14. *$C(S^1)^{(n)}$ is an operator system quotient of $C(S^1)^{(m)}$, if $m \geq n$.*

Finally, the following extension theorem is established by taking advantage of certain categorical relations involving Toeplitz and Fejér-Riesz operator systems.

Theorem 1.15. *If $\mathcal{N}$ is a von Neumann algebra, then every unital positive linear map $\phi : C(S^1)^{(n)} \to \mathcal{N}$ has an extension to a unital completely positive linear map $\tilde{\phi} : \mathcal{M}_n(\mathbb{C}) \to \mathcal{N}$.*

Theorem 1.15 was established by Haagerup [26] for the case $n = 2$ using more direct operator-theoretic methods. An intriguing feature of the theorem, noted in [26], is that there is no requirement for the von Neumann algebra $\mathcal{N}$ to be injective; nor is there a requirement, *a priori*, for the positive linear map $\phi : C(S^1)^{(n)} \to \mathcal{N}$ to be completely positive.

2. Operator System Duality and Tensor Products

As mentioned earlier, an operator system is a triple $(\mathcal{R}, \{\mathcal{C}_n\}_{n\in\mathbb{N}}, e_{\mathcal{R}})$ consisting of a complex $*$-vector space $\mathcal{R}$, a family $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ of proper cones in the real vector spaces $\mathcal{M}_n(\mathcal{R})_{\rm sa}$ of selfadjoint matrices over $\mathcal{R}$ in which the cones satisfy $\alpha^*\mathcal{C}_n\alpha \subseteq \mathcal{C}_m$, for all $n, m \in \mathbb{N}$ and $n \times m$ complex matrices α, and a distinguished element $e_{\mathcal{R}} \in \mathcal{C}_1$ that serves as an Archimedean order unit for the family $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ [11, 34]. Most often the cones $\mathcal{C}_n$ are denoted by $\mathcal{M}_n(\mathcal{R})_+$, with $\mathcal{R}_+$ denoting $\mathcal{C}_1$, and $\mathcal{R}$ is used to designate the triple $(\mathcal{R}, \{\mathcal{C}_n\}_{n\in\mathbb{N}}, e_{\mathcal{R}})$.

Definition 2.1. An operator subsystem $\mathcal{R}$ of an operator system $\mathcal{S}$ is a unital linear subspace $\mathcal{R} \subseteq \mathcal{S}$, closed under the involution of $\mathcal{S}$, such that $\mathcal{M}_n(\mathcal{R})_+$ is defined to be $\mathcal{M}_n(\mathcal{R})_+ = \mathcal{M}_n(\mathcal{R}) \cap \mathcal{M}_n(\mathcal{S})_+$, for every $n \in \mathbb{N}$, and $e_{\mathcal{R}} = e_{\mathcal{S}}$.

A *unital linear complete order embedding* of an operator system $\mathcal{R}$ into an operator system $\mathcal{T}$ is a linear injection $\phi : \mathcal{R} \to \mathcal{T}$ such that ϕ and the inverse map $\phi(\mathcal{R}) \to \mathcal{R}$ of operator systems are completely positive. If such a map ϕ is also surjective, then ϕ is a *unital linear complete order isomorphism* and we denote this relationship between $\mathcal{R}$ and $\mathcal{T}$ by $\mathcal{R} \simeq \mathcal{T}$. We this notation, if $\mathcal{R}$ and $\mathcal{T}$ are operator systems and if $\phi : \mathcal{R} \to \mathcal{T}$ is a unital linear complete order embedding, then $\mathcal{R} \simeq \phi(\mathcal{R})$.

If $\mathcal{R}$ is a finite-dimensional operator system with dual space $\mathcal{R}^d$, then define an $n \times n$ matrix $\Phi = [\varphi_{ij}]_{i,j=1}^n$ of linear functionals $\varphi_{ij} \in \mathcal{R}^d$ on $\mathcal{R}$ to be positive if the linear map $\hat{\Phi} : \mathcal{R} \to \mathcal{M}_n(\mathbb{C})$, where $\hat{\Phi}(x) = [\varphi_{ij}(x)]_{i,j=1}^n$, is completely positive. The collection $\{\mathcal{M}_n(\mathcal{R}^d)_+\}_{n\in\mathbb{N}}$ satisfies the compatibility requirements for the matrix cones of an operator system. In selecting any faithful positive linear functional δ on $\mathcal{R}$–by which is meant a positive linear functional δ for which $\delta(y) = 0$, for $y \in \mathcal{R}_+$, occurs only with $y = 0$–an Archimedean order unit for the matrix ordering $\{\mathcal{M}_n(\mathcal{R}^d)_+\}_{n\in\mathbb{N}}$ on $\mathcal{R}^d$ is obtained, providing $\mathcal{R}^d$ with the structure of an operator system [11].

Turning to tensor products [31, 32], if $(\mathcal{R}, \{\mathcal{P}_n\}_{n\in\mathbb{N}}, e_{\mathcal{R}})$ and $(\mathcal{T}, \{\mathcal{Q}_n\}_{n\in\mathbb{N}}, e_{\mathcal{T}})$ are operator systems, then an operator system tensor product structure on the algebraic tensor product $\mathcal{R} \otimes \mathcal{T}$ is a family $\sigma = \{\mathcal{C}_n\}_{n\in\mathbb{N}}$ of cones $\mathcal{C}_n \subseteq \mathcal{M}_n(\mathcal{R} \otimes \mathcal{T})$ such that:

(i) $(\mathcal{R} \otimes \mathcal{T}, \{\mathcal{C}_n\}_{n\in\mathbb{N}}, e_{\mathcal{R}} \otimes e_{\mathcal{T}})$ is an operator system, denoted by $\mathcal{R} \otimes_\sigma \mathcal{T}$;
(ii) $\mathcal{P}_{n_1} \otimes_{\rm sep} \mathcal{Q}_{n_2} \subseteq \mathcal{C}_{n_1 n_2}$, for all $n_1, n_2 \in \mathbb{N}$; and

(iii) for all $n_1, n_2 \in \mathbb{N}$, the linear map $\phi \otimes \psi : \mathcal{R} \otimes_\sigma \mathcal{T} \to \mathcal{M}_{n_1}(\mathbb{C}) \otimes \mathcal{M}_{n_2}(\mathbb{C})$ is a ucp map, whenever $\phi : \mathcal{R} \to \mathcal{M}_{n_1}(\mathbb{C})$ and $\psi : \mathcal{T} \to \mathcal{M}_{n_2}(\mathbb{C})$ are ucp maps.

In item (iii) above, $\mathcal{M}_{n_1}(\mathbb{C}) \otimes \mathcal{M}_{n_2}(\mathbb{C})$ is the unique C*-algebra tensor product structure on the tensor product of the unital C*-algebras $\mathcal{M}_{n_1}(\mathbb{C})$ and $\mathcal{M}_{n_2}(\mathbb{C})$.

If $\otimes_\sigma$ and $\otimes_\tau$ are operator system tensor product structures on $\mathcal{R} \otimes \mathcal{T}$, then the notation

$$\mathcal{R} \otimes_\sigma \mathcal{T} \subseteq_+ \mathcal{R} \otimes_\tau \mathcal{T}$$

is used to indicate that the linear identity map $\iota(x) = x$ is a ucp map when considered as a linear map $\iota : \mathcal{R} \otimes_\sigma \mathcal{T} \to \mathcal{R} \otimes_\tau \mathcal{T}$ of operator systems. If, in addition, the ucp map ι is unital complete order isomorphism (that is, if the inverse of ι is completely positive), then we write

$$\mathcal{R} \otimes_\sigma \mathcal{T} = \mathcal{R} \otimes_\tau \mathcal{T}.$$

If $\mathcal{R}_1$ and $\mathcal{T}_1$ are operator subsystems of operator systems $\mathcal{R}_2$ and $\mathcal{T}_2$, and if $\otimes_\sigma$ and $\otimes_\tau$ are operator system tensor product structures on $\mathcal{R}_1 \otimes \mathcal{T}_1$ and $\mathcal{R}_2 \otimes \mathcal{T}_2$, respectively, then the notation

$$\mathcal{R}_1 \otimes_\sigma \mathcal{T}_1 \subseteq_+ \mathcal{R}_2 \otimes_\tau \mathcal{T}_2$$

is used to indicate that the canonical embedding $x \mapsto x$ is a ucp map when considered as a linear map $\mathcal{R}_1 \otimes_\sigma \mathcal{T}_1 \to \mathcal{R}_2 \otimes_\tau \mathcal{T}_2$ of operator systems. If this ucp embedding is also a complete order embedding, by which is meant that a matrix $x \in \mathcal{M}_p(\mathcal{R}_1 \otimes \mathcal{T}_1)$ belongs to $\mathcal{M}_p(\mathcal{R}_1 \otimes_\sigma \mathcal{T}_1)_+$ if and only if x is an element of $\mathcal{M}_p(\mathcal{R}_2 \otimes_\tau \mathcal{T}_2)_+$, for every $p \in \mathbb{N}$, then this situation is denoted by

$$\mathcal{R}_1 \otimes_\sigma \mathcal{T}_1 \subseteq_{\mathrm{coi}} \mathcal{R}_2 \otimes_\tau \mathcal{T}_2.$$

The following definitions were introduced in [31].

Definition 2.2. The *minimal operator system tensor product*, $\otimes_{\min}$, of operator systems $\mathcal{R}$ and $\mathcal{T}$ is the operator system tensor product structure on the algebraic tensor product $\mathcal{R} \otimes \mathcal{T}$ that is obtained by declaring a matrix $x \in \mathcal{M}_p(\mathcal{R} \otimes \mathcal{T})$ to be positive if $(\phi \otimes \psi)^{(p)}[x]$ is a positive element of $\mathcal{M}_p\left(\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{M}_q(\mathbb{C})\right)$, for every $n, q \in \mathbb{N}$ and unital completely positive linear maps $\phi : \mathcal{R} \to \mathcal{M}_n(\mathbb{C})$ and $\psi : \mathcal{T} \to \mathcal{M}_q(\mathbb{C})$.

Definition 2.3. The *maximal operator system tensor product*, $\otimes_{\max}$, of operator systems $\mathcal{R}$ and $\mathcal{T}$ is the operator system tensor product structure on the algebraic tensor product $\mathcal{R} \otimes \mathcal{T}$ that is obtained by declaring a matrix $x \in \mathcal{M}_p(\mathcal{R} \otimes \mathcal{T})$ to be positive if, for each $\varepsilon > 0$, there are $n, q \in \mathbb{N}$, $a \in \mathcal{M}_n(\mathcal{R})_+$, $b \in \mathcal{M}_q(\mathcal{T})_+$, and a linear map $\delta : \mathbb{C}^p \to \mathbb{C}^n \otimes \mathbb{C}^q$ such that

$$\varepsilon(e_{\mathcal{R}} \otimes e_{\mathcal{T}}) + x = \delta^*(a \otimes b)\delta.$$

With respect to the notation established above, we have the following relationships, for all operator systems $\mathcal{R}$ and $\mathcal{T}$, and all operator system tensor product structures $\otimes_\sigma$ on $\mathcal{R} \otimes \mathcal{T}$:

$$\mathcal{R} \otimes_{\max} \mathcal{T} \subseteq_+ \mathcal{R} \otimes_\sigma \mathcal{T} \subseteq_+ \mathcal{R} \otimes_{\min} \mathcal{T}. \tag{2.1}$$

Furthermore, as noted in [31], if unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$ are considered as operator systems, then the operator system tensor products $\mathcal{A} \otimes_{\min} \mathcal{B}$ and $\mathcal{A} \otimes_{\max} \mathcal{B}$ are unitally completely order isomorphic to the image of $\mathcal{A} \otimes \mathcal{B}$ inside the minimal and maximal C*-algebraic tensor products of $\mathcal{A}$ and $\mathcal{B}$, respectively.

An alternative to the defining condition for membership in $(\mathcal{R}\otimes_{\max}\mathcal{T})_+$ is given by the next result.

Proposition 2.4. *If $\mathcal{R}$ and $\mathcal{T}$ are operator systems, then the following statements are equivalent for $x\in\mathcal{R}\otimes\mathcal{T}$:*

1. *$x\in(\mathcal{R}\otimes_{\max}\mathcal{T})_+$;*
2. *for every $\varepsilon>0$, there exist $N\in\mathbb{N}$, $G=[g_{ij}]_{i,j=1}^N\in\mathcal{M}_N(\mathcal{R})_+$, and $H=[h_{ij}]_{i,j=1}^N\in\mathcal{M}_N(\mathcal{T})_+$ such that*
$$x+\varepsilon(e_{\mathcal{R}}\otimes e_{\mathcal{T}})=\sum_{i=1}^N\sum_{j=1}^N g_{ij}\otimes h_{ij}.$$

Proof. Select $x\in(\mathcal{R}\otimes_{\max}\mathcal{T})_+$, and let $\varepsilon>0$ and set $\delta=\varepsilon/2$. The elements $y=x+\varepsilon(e_{\mathcal{R}}\otimes e_{\mathcal{T}})$ and $y-\delta(e_{\mathcal{R}}\otimes e_{\mathcal{T}})$ also belong to $(\mathcal{R}\otimes_{\max}\mathcal{T})_+$, as $(\mathcal{R}\otimes_{\max}\mathcal{T})_+$ is a cone. The condition $y-\delta(e_{\mathcal{R}}\otimes e_{\mathcal{T}})\in(\mathcal{R}\otimes_{\max}\mathcal{T})_+$ implies y is a strictly positive element of $(\mathcal{R}\otimes_{\max}\mathcal{T})_+$. Thus, by [17, Lemma 2.7], there exist $N\in\mathbb{N}$ and matrices $[g_{ij}]_{i,j=1}^N\in\mathcal{M}_N(\mathcal{R})_+$ and $[h_{ij}]_{i,j=1}^N\in\mathcal{M}_N(\mathcal{T})_+$ such that $y=\sum_{i=1}^N\sum_{j=1}^N g_{ij}\otimes h_{ij}$.

Conversely, let $\varepsilon>0$. By assumption, there exist $N\in\mathbb{N}$ and matrices $G=[g_{ij}]_{i,j=1}^N\in\mathcal{M}_N(\mathcal{R})_+$ and $H=[h_{ij}]_{i,j=1}^N\in\mathcal{M}_N(\mathcal{T})_+$ such that $x+\varepsilon(e_{\mathcal{R}}\otimes e_{\mathcal{T}})=\sum_{i=1}^N\sum_{j=1}^N g_{ij}\otimes h_{ij}$. Consider the vector $\xi\in\mathbb{C}^N\otimes\mathbb{C}^N$ given by $\xi=\sum_{i=1}^N\sum_{j=1}^N f_i\otimes f_j$, where $\{f_1,\dots,f_N\}$ are the canonical orthonormal basis vectors for $\mathbb{C}^N$. Hence, for the linear map $\alpha:\mathbb{C}\to\mathbb{C}^N\otimes\mathbb{C}^N$ given by $\alpha(\zeta)=\zeta\xi$, for $\zeta\in\mathbb{C}$, we have
$$\alpha^*(G\otimes H)\alpha=\sum_{i=1}^N\sum_{j=1}^N g_{ij}\otimes h_{ij}=x+\varepsilon(e_{\mathcal{R}}\otimes e_{\mathcal{T}}).$$
As $\varepsilon>0$ is arbitrary, we deduce $x\in(\mathcal{R}\otimes_{\max}\mathcal{T})_+$. □

The following theorem is established in [19].

Theorem 2.5 (Tensor Duality). *If $\mathcal{R}$ and $\mathcal{T}$ are finite-dimensional operator systems, then*
$$(\mathcal{R}\otimes_{\min}\mathcal{T})^d=\mathcal{R}^d\otimes_{\max}\mathcal{T}^d.$$

Turning now to tensor cones, ignoring the matrix cones $\mathcal{M}_n(\mathcal{R}\otimes_\sigma\mathcal{T})_+$ for all $n\geq 2$, the next result creates the setting for the study herein.

Proposition 2.6. *For any finite-dimensional operator systems $\mathcal{R}$ and $\mathcal{T}$, and any operator system tensor product structure $\otimes_\sigma$ on $\mathcal{R}\otimes\mathcal{T}$, the cone $(\mathcal{R}\otimes_\sigma\mathcal{T})_+$ is a tensor cone for $\mathcal{R}_+$ and $\mathcal{T}_+$.*

Proof. Suppose that $x\in\mathcal{R}_+\otimes_{\rm sep}\mathcal{T}_+$; thus, there exist $k\in\mathbb{N}$, $a_j\in\mathcal{R}_+$, and $b_j\in\mathcal{T}_+$ such that $x=\sum_{j=1}^k a_j\otimes b_j$. Let $\varepsilon>0$ and consider $\varepsilon(e_{\mathcal{R}}\otimes e_{\mathcal{T}})+x$. Set $a_0=\varepsilon e_{\mathcal{R}}$ and $b_0=e_{\mathcal{T}}$ so that
$$\varepsilon(e_{\mathcal{R}}\otimes e_{\mathcal{T}})+x=\sum_{j=0}^k a_j\otimes b_j.$$

Let $P = \sum_{p=0}^{k} a_p \otimes e_{pp}$ and $Q = \sum_{q=0}^{k} a_q \otimes e_{qq}$, which are positive matrices in $\mathcal{M}_{k+1}(\mathcal{R})$ and $\mathcal{M}_{k+1}(\mathcal{T})$, respectively, and let $\xi \in \mathbb{C}^{k+1} \otimes \mathbb{C}^{k+1}$ be the vector $\xi = \sum_{j=0}^{k} f_j \otimes f_j$, where $\{f_0, \dots, f_k\}$ denotes the canonical orthonormal basis for $\mathbb{C}^{k+1}$. Hence, if $\alpha : \mathbb{C} \to \mathbb{C}^{k+1} \otimes \mathbb{C}^{k+1}$ is the linear map $\alpha(\zeta) = \zeta\xi$, for $\zeta \in \mathbb{C}$, then

$$\alpha^*(P \otimes Q)\alpha = \sum_{j=0}^{k} a_j \otimes b_j = \varepsilon(e_{\mathcal{R}} \otimes e_{\mathcal{T}}) + x,$$

which proves that $x \in (\mathcal{R} \otimes_{\max} \mathcal{T})_+$. Thus,

$$\mathcal{R}_+ \otimes_{\mathrm{sep}} \mathcal{T}_+ \subseteq (\mathcal{R} \otimes_{\max} \mathcal{T})_+ \subseteq (\mathcal{R} \otimes_{\sigma} \mathcal{T})_+,$$

where the second inclusion is by virtue of the inclusion sequence (2.1).

Consider now an element $x \in (\mathcal{R} \otimes_{\min} \mathcal{T})_+$, and suppose that $\varphi : \mathcal{R} \to \mathbb{C}$ and $\vartheta : \mathcal{T} \to \mathbb{C}$ are positive linear functionals. Without loss of generality, we may assume that have been normalised to be unital. As unital positive linear functionals are ucp maps, $(\varphi \otimes \vartheta)[x]$ is positive in $\mathbb{C} \otimes \mathbb{C} = \mathbb{C}$, by definition of the minimal operator system tensor product. In other words, $x \in \mathcal{R}_+ \otimes_{\mathrm{sep}^*} \mathcal{T}_+$. Thus,

$$(\mathcal{R} \otimes_{\sigma} \mathcal{T})_+ \subseteq (\mathcal{R} \otimes_{\min} \mathcal{T})_+ \subseteq \mathcal{R}_+ \otimes_{\mathrm{sep}^*} \mathcal{T}_+,$$

where the first inclusion is, again, by virtue of the inclusion sequence (2.1).

Hence, $\mathcal{R}_+ \otimes_{\mathrm{sep}} \mathcal{T}_+ \subseteq (\mathcal{R} \otimes_{\sigma} \mathcal{T})_+ \subseteq \mathcal{R}_+ \otimes_{\mathrm{sep}^*} \mathcal{T}_+$, which proves that $(\mathcal{R} \otimes_{\sigma} \mathcal{T})_+$ is a tensor cone for $\mathcal{R}_+$ and $\mathcal{T}_+$. $\square$

3. Toeplitz and Fejér-Riesz Operator Systems

3.1. **Canonical linear bases.** The operator systems under study in this paper are the Toeplitz and Fejér-Riesz operator systems $C(S^1)^{(n)}$ and $C(S^1)_{(n)}$, for $n \geq 2$. The operator system structure on the Toeplitz matrices arises from considering $C(S^1)^{(n)}$ as an operator subsystem of the unital C*-algebra $\mathcal{M}_n(\mathbb{C})$, while the operator system structure on the trigonometric polynomials arises from considering $C(S^1)_{(n)}$ as an operator subsystem of the unital abelian C*-algebra $C(S^1)$ of continuous complex-valued functions on the unit circle S^1. Thus, the Archimedean order unit for $C(S^1)^{(n)}$ is the identity, and for $C(S^1)_{(n)}$ it is the constant function $z \mapsto 1$.

The functions $\chi_\ell : S^1 \to \mathbb{C}$ given by $\chi_\ell(z) = z^\ell$, for $\ell = -n+1, \dots, n-1$, form a linear basis for $C(S^1)_{(n)}$, with the canonical Archimedean order unit for the operator system $C(S^1)_{(n)}$ being the constant function χ_0. It is sometimes useful to note that the operator system $C(S^1)_{(n)}$ is an operator subsystem of $C(S^1)_{(m)}$, whenever $m \geq n$. Finally, if elements t_ℓ are selected from an operator system $\mathcal{T}$, then the element $x = \sum_{\ell=-n+1}^{n-1} \chi_\ell \otimes t_\ell$ in the vector space $C(S^1)_{(n)} \otimes \mathcal{T}$ has a natural representation as a function $S^1 \to \mathcal{T}$ of the form $z \mapsto \sum_{\ell=-n+1}^{n-1} z^\ell t_\ell$.

Turning to the Toeplitz operator system, the canonical linear basis for $C(S^1)^{(n)}$ is $\{r_\ell\}_{\ell=-n+1}^{n-1}$, where

$$r_\ell \;=\; \left\{ \begin{array}{lcl} s^\ell & : & \text{if } \ell \geq 0 \\ (s^*)^\ell & : & \text{if } \ell < 0 \end{array} \right\}, \text{ for } \ell = 0, 1, \dots, n-1,$$

and $s \in \mathcal{M}_n(\mathbb{C})$ is the shift matrix

$$s = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}.$$

Thus, if elements t_ℓ are selected from an operator system $\mathcal{T}$, then the element $x = \displaystyle\sum_{\ell=-n+1}^{n-1} s_\ell \otimes t_\ell$ in the vector space $C(S^1)^{(n)} \otimes \mathcal{T}$ has a natural representation as a matrix with entries from $\mathcal{T}$: namely,

$$x = \begin{bmatrix} t_0 & t_{-1} & \dots & t_{-n+1} \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & \dots & t_1 & t_0 \end{bmatrix} \tag{3.1}$$

The canonical Archimedean order unit for the operator system $C(S^1)^{(n)}$ is the identity matrix, r_0.

It is common in the matrix theory literature to refer to matrices such as those in (3.1) as *block Toeplitz matrices* when the elements t_ℓ are not complex numbers.

3.2. **Generalised circulants.** While the Toeplitz operator system is far from being closed under multiplication, it has numerous operator subsystems that are abelian C*-algebras. For $\theta \in \mathbb{R}$, let u_θ be the unitary Toeplitz matrix

$$u_\theta = r_1 + e^{i\theta} r_{-n+1} = \begin{bmatrix} 0 & \dots & 0 & e^{i\theta} \\ 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix}.$$

The minimal annihilating polynomial for u_θ is $z^n - e^{i\theta}$, which has n distinct roots in S^1, and, for any complex polynomial $g(z) = \displaystyle\sum_{k=0}^{n-1} \alpha_k z^k$, the matrix

$$g(u_\theta) = \begin{bmatrix} \alpha_0 & \alpha_{n-1} e^{i\theta} & \dots & \alpha_1 e^{i\theta} \\ \alpha_1 & \alpha_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha_{n-1} e^{i\theta} \\ \alpha_{n-1} & \dots & \alpha_1 & \alpha_0 \end{bmatrix}$$

is normal and Toeplitz. Moreover, the operator systems

$$C^{n,\theta} = \{ g(u_\theta) \,|\, g \in \mathbb{C}[z] \},$$

for $\theta \in \mathbb{R}$, are unital abelian C*-subalgebras of $C(S^1)^{(n)}$, yielding the algebra of circulant matrices, when $\theta = 0$, and the algebra of skew-circulant matrices, when $\theta = \pi$. Elements of $C^{n,\theta}$ are called *generalised circulants*. Note that, by the Spectral Theorem, the positive cone of $C^{n,\theta}$ is affinely homeomorphic to the nonnegative orthant of $\mathbb{R}^n$; hence, $\left(C^{n,\theta}\right)_+$ is a simplicial cone. By diagonalising matrices in $C^{n,\theta}$ via a single unitary matrix U_θ, for each $\theta \in \mathbb{R}$, we have the following isomorphisms in the category $\mathfrak{S}_1$:

$$C^{n,\theta} \simeq \mathrm{C}^*(\mathbb{Z}_n), \tag{3.2}$$

for all $\theta \in \mathbb{R}$, where $\mathrm{C}^*(\mathbb{Z}_n)$ is the group C*-algebra of the finite abelian group $\mathbb{Z}_n$.

As noted in [12, §5], the operator systems $C^{n,\theta}$ are self-dual and the Toeplitz operator system $C(S^1)^{(n)}$ is the (completely positive) truncation of the circulant operator system $C^{2n-1,\theta}$, for $\theta = 0$, to the $n \times n$ upper-left corner. Unfortunately, this completely positive map does not send the positive cone of $C^{2n-1,\theta}$, for $\theta = 0$, onto the positive cone of $C(S^1)^{(n)}$. For example, the positive Toeplitz matrix $\begin{bmatrix} 1 & \zeta^{-1} \\ \zeta & 1 \end{bmatrix}$, for $z \in S^1$, is the upper 2×2 corner of a 3×3 positive circulant matrix if and only if ζ is a cube root of unity.

3.3. The matrices R_n and T_n.

Definition 3.1. $R_n \in C(S^1)^{(n)} \otimes C(S^1)^{(n)}$ and $T_n \in C(S^1)^{(n)} \otimes C(S^1)_{(n)}$ are the matrices given by

$$R_n = \sum_{\ell=-n+1}^{n-1} r_\ell \otimes r_\ell \text{ and } T_n = \sum_{\ell=-n+1}^{n-1} r_\ell \otimes \chi_\ell.$$

The matrix T_n is called the *universal positive $n \times n$ Toeplitz matrix*.

It is evident T_n is positive. To show R_n is positive, one need only note that R_n coincides with the matrix designated by (*) on page 36 of [34], and it is shown in [34] that the matrix (*) is positive.

In addition to being positive, the matrix R_n separable (see Proposition 4.10), while T_n entangled [16, Corollary 7.7].

An important result of Choi [10] shows that the complete positivity of a linear map ϕ of the matrix algebra $\mathcal{M}_n(\mathbb{C})$ to an operator system $\mathcal{T}$ can be confirmed by determining whether a single matrix, namely the *Choi matrix* $C_\phi = \sum_{i,j} e_{ij} \otimes \phi(e_{ij})$, is positive in $\mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathcal{T}$. The following theorem shows how the matrix T_n functions in a similar way for linear maps on $C(S^1)_{(n)}$.

Proposition 3.2. *Let $\mathcal{T}$ be an operator system.*

1. *Every positive linear map $\phi : C(S^1)^{(n)} \to \mathcal{T}$ is completely positive.*
2. *The following statements are equivalent for a linear map $\phi : C(S^1)_{(n)} \to \mathcal{T}$:*
   - (a) *the matrix* $\displaystyle\sum_{\ell=-n+1}^{n-1} r_\ell \otimes \phi(\chi_\ell)$ *is positive in* $C(S^1)^{(n)} \otimes_{\min} C(S^1)_{(n)}$*;*
   - (b) *ϕ is completely positive.*

Proof. The assertion that positive linear maps $\phi : C(S^1)^{(n)} \to \mathcal{T}$ are completely positive is established in [16, Lemma 2.5].

For the second assertion, because

$$C(S^1)^{(n)} \otimes_{\min} C(S^1)_{(n)} \subseteq_{\text{coi}} \mathcal{M}_n(\mathbb{C}) \otimes_{\min} C(S^1)_{(n)},$$

it is clear, by definition of complete positivity, that $\sum_{\ell=-n+1}^{n-1} r_\ell \otimes \phi(\chi_\ell)$ is positive in $C(S^1)^{(n)} \otimes_{\min} C(S^1)_{(n)}$, if ϕ is completely positive.

Conversely, suppose $\phi : C(S^1)^{(n)} \to \mathcal{T}$ is a linear map for which the matrix $\sum_{\ell=-n+1}^{n-1} r_\ell \otimes \phi(\chi_\ell)$ is positive in $C(S^1)^{(n)} \otimes_{\min} \mathcal{T}$. Again, using the fact that

$$C(S^1)^{(n)} \otimes_{\min} \mathcal{T} \subseteq_{\text{coi}} \mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(\mathcal{T}),$$

we may consider ϕ to be a linear map $\phi : C(S^1)^{(n)} \to \mathrm{C}^*_{\mathrm{e}}(\mathcal{T})$ for which the matrix $\sum_{\ell=-n+1}^{n-1} r_\ell \otimes \phi(\chi_\ell)$ is positive in $C(S^1)^{(n)} \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(\mathcal{T})$, Thus, by the universal property of T_n [16, Theorem 7.7], there is a completely positive linear map $\psi : C(S^1)_{(n)} \to \mathrm{C}^*_{\mathrm{e}}(\mathcal{T})$ such that $\psi(\chi_\ell) = \phi(\chi_\ell)$, for every ℓ. Hence, because ψ and ϕ agree on a linear basis for $C(S^1)_{(n)}$, they must be equal, which implies that ϕ is completely positive. $\square$

Corollary 3.3. *If $\phi : C(S^1)_{(n)} \to \mathcal{T}$ is a positive, but not completely positive, linear map, then the matrix $\sum_{\ell=-n+1}^{n-1} r_\ell \otimes \phi(\chi_\ell)$ is nonpositive in $C(S^1)^{(n)} \otimes_{\min} C(S^1)_{(n)}$.*

Proposition 3.2 may be used to show certain positive linear maps on $C(S^1)_{(n)}$ are not completely positive. The following example is inspired by [5, Appendix A2] (see also [34, Example 2.2]).

Example 3.4. The linear map $\phi : C(S^1)_{(2)} \to C(S^1)^{(2)}$ given by

$$\phi(\alpha_{-1} z^{-1} + \alpha_0 + \alpha_1 z^1) = \begin{bmatrix} \alpha_0 & 2\alpha_{-1} \\ 2\alpha_1 & \alpha_0 \end{bmatrix}$$

is positive but not completely positive.

Proof. Because $\overline{\alpha}_1 z^{-1} + \alpha_0 + \alpha_1 z^1$ is positive if and only if $\alpha_0 + 2\Re(\alpha_1 z) \geq 0$ for all $z \in S^1$, α_0 and α_1 must necessarily satisfy $\alpha_0 \geq 2|\alpha_1|$, which in turn implies that the matrix $\begin{bmatrix} \alpha_0 & 2\overline{\alpha}_1 \\ 2\alpha_1 & \alpha_0 \end{bmatrix}$ is positive. Hence, the linear map ϕ is positive.

Consider the matrix

$$x = \sum_{\ell=-1}^{1} r_\ell \otimes \phi(\chi_\ell) = \begin{bmatrix} r_0 & 2r_{-1} \\ 2r_1 & r_0 \end{bmatrix}.$$

If $\xi \in \mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ is $\xi = e_1 - e_4$, then $\langle x\xi, \xi \rangle = -2 < 0$, which shows that x is not positive; hence, ϕ is not completely positive, by Proposition 3.2. $\square$

On the other hand:

Example 3.5. The linear map $\phi : C(S^1)_{(n)} \to C(S^1)^{(n)}$ defined on the basis elements by $\phi(\chi_\ell) = r_\ell$ is completely positive.

Proof. The matrix $\sum_{\ell=-n+1}^{n-1} r_\ell \otimes \phi(\chi_\ell)$ is R_n, which is positive. Therefore, by Proposition 3.2, ϕ is completely positive. □

3.4. **From positive matrices to completely positive linear maps.** For any operator system $\mathcal{T}$ and Toeplitz matrix $x = \sum_\ell r_\ell \otimes t_\ell \in C(S^1)^{(n)} \otimes \mathcal{T}$, a linear map $\hat{x} : C(S^1)_{(n)} \to \mathcal{T}$ is induced in which the action of $\hat{x}$ is given by

$$\hat{x}(f) = \sum_{\ell=-n+1}^{n-1} \hat{f}(-\ell) t_\ell, \text{ for all } f \in C(S^1)_{(n)}. \tag{3.3}$$

Moreover, the matrix x is positive in $C(S^1)^{(n)} \otimes_{\min} \mathcal{T}$ if and only if the linear map $\hat{x} : C(S^1)_{(n)} \to \mathcal{T}$ is completely positive.

In the case where the matrix in question is R_n, we obtain the map

$$\hat{R}_n(f) = \sum_{\ell=-n+1}^{n-1} f(-\ell) r_\ell = \begin{bmatrix} \hat{f}(0) & \hat{f}(-1) & \dots & \hat{f}(-n+1) \\ \hat{f}(1) & \hat{f}(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \hat{f}(-1) \\ \hat{f}(n-1) & \dots & \hat{f}(1) & \hat{f}(0) \end{bmatrix},$$

which is unitarily equivalent to the Fourier matrix for $f \in C(S^1)_{(n)}$.

In the case where $x = T_n(\lambda) \otimes T_n(\mu)$ for some $\lambda, \mu \in S^1$, expressing x as

$$x = \sum_{\ell=-n+1}^{n-1} r_\ell \otimes \lambda^\ell T_n(\mu)$$

leads to

$$\hat{x}(f) = \left(\sum_{\ell=-n+1}^{n-1} \hat{f}(-\ell) \lambda^\ell \right) T_n(\mu),$$

in which case the positivity of f implies the positivity of the scalar that multiplies the positive rank-1 matrix $T_n(\mu)$. Hence, the following proposition is proved.

Proposition 3.6. *For each $\lambda \in S^1$, the linear functional $\varphi_\lambda : C(S^1)_{(n)} \to \mathbb{C}$, defined by*

$$\varphi_\lambda(f) = \sum_{\ell=-n+1}^{n-1} \hat{f}(-\ell) \lambda^\ell,$$

is positive.

3.5. **Linear $*$-preserving idempotents.** The following observation situates the Toeplitz operator system within the larger matrix algebra.

Proposition 3.7. *There exists a linear map $\mathcal{E}_n : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$, for every $n \geq 2$, such that*

(1) *$\mathcal{E}_n(x^*) = \mathcal{E}_n(x)^*$, for all $x \in \mathcal{M}_n(\mathbb{C})$,*
(2) *$\mathcal{E}_n$ is idempotent, and*
(3) *the range of $\mathcal{E}_n$ is $C(S^1)^{(n)}$.*

Furthermore, $\mathcal{E}_2$ is positive, but not completely positive.

Proof. Let $\mathcal{E}_n$ be the map that averages the entries along each of the super diagonals of a matrix x and replaces each entry along the super diagonal with that average. Thus, $\mathcal{E}_n$ is linear, satisfies $\mathcal{E}_n(x^*) = \mathcal{E}_n(x)^*$, for all $x \in \mathcal{M}_n(\mathbb{C})$, and maps into $C(S^1)^{(n)}$. If x is a Toeplitz matrix, then the super diagonals are constant, leaving an average of the entries unchanged. Hence, $\mathcal{E}_n$ maps onto $C(S^1)^{(n)}$ and $\mathcal{E}_n^2 = \mathcal{E}_n$.

To show that $\mathcal{E}_2$ is positive, let $x = \begin{bmatrix} \alpha & \overline{\gamma} \\ \gamma & \beta \end{bmatrix}$ be a positive 2×2 matrix. Thus, $\alpha \geq 0$, $\beta \geq 0$, and $|\gamma|^2 \leq \alpha\beta$. If

$$y = \mathcal{E}_2(x) = \begin{bmatrix} \frac{\alpha+\beta}{2} & \overline{\gamma} \\ \gamma & \frac{\alpha+\beta}{2} \end{bmatrix},$$

then the diagonal entries of y are nonnegative and

$$|\gamma| \leq \sqrt{\alpha\beta} \leq \frac{\alpha+\beta}{2},$$

where the second inequality is the arithmetic-geometric mean inequality. Thus, the matrix y is positive, proving $\mathcal{E}_2$ is a positive linear map.

The Choi matrix $[\mathcal{E}_2(e_{ij})]_{i,j=1}^2$ for $\mathcal{E}_2$ is easily seen to be nonpositive; hence, $\mathcal{E}_2$ is not completely positive. $\square$

Corollary 3.8 (Selfadjoint Toeplitz matrices)**.** *For every operator system $\mathcal{T}$ and $n \geq 2$,*

$$C(S^1)^{(n)}_{\rm sa} \otimes_{\mathbb{R}} \mathcal{T}_{\rm sa} = \left(C(S^1)^{(n)} \otimes_{\mathbb{C}} \mathcal{T} \right)_{\rm sa}.$$

Proof. If $x = \displaystyle\sum_{r=-n+1}^{n-1} r_\ell \otimes t_\ell \in \left(C(S^1)^{(n)} \otimes_{\mathbb{C}} \mathcal{T} \right)_{\rm sa}$, then $x \in (\mathcal{M}_n(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{T})_{\rm sa}$ as well. Therefore, by [35, Lemma 3.7], there are selfadjoint $h_k \in \mathcal{M}_n(\mathbb{C})$ and $s_k \in \mathcal{T}_{\rm sa}$ such that

$$x = \sum_{r=-n+1}^{n-1} r_\ell \otimes t_\ell = \sum_{j=1}^{m} h_k \otimes s_k.$$

Apply the ampliation $\mathcal{E}_n \otimes \mathrm{Id}_{\mathcal{T}}$ to obtain

$$(\mathcal{E}_n \otimes \mathrm{Id}_{\mathcal{T}})[x] = \sum_{j=1}^{m} \mathcal{E}_n(h_k) \otimes s_k = \sum_{r=-n+1}^{n-1} \mathcal{E}_n(r_\ell) \otimes t_\ell = \sum_{r=-n+1}^{n-1} r_\ell \otimes t_\ell = x.$$

As each $\mathcal{E}_n(h_k)$ is a selfadjoint Toeplitz matrix, the proof is complete. $\square$

As a consequence of Corollary 6.15, if $n \geq 2$ and $\mathcal{G}_n : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ is an idempotent linear map with range $C(S^1)^{(n)}$, then $\mathcal{G}_n$ can not be completely positive. The situation, however, is rather different for the circulant operator systems $C^{n,\theta}$.

Proposition 3.9. *For every $n \geq 2$ and $\theta \in \mathbb{R}$, there exists a linear map $\mathcal{F}_{n,\theta} : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ such that*

1. *$\mathcal{F}_{n,\theta}$ is completely positive,*
2. *$\mathcal{F}_{n,\theta}$ is idempotent, and*
3. *the range of $\mathcal{F}_{n,\theta}$ is $C^{n,\theta}$.*

Proof. Fix $n \geq 2$ and $\theta \in \mathbb{R}$, and let $(\lambda_1, \ldots, \lambda_n)$ be a fixed ordering of the (necessarily distinct) eigenvalues of the unitary matrix u_θ. By the Spectral Theorem, there is a unitary v_θ such that $v_\theta^* f(u_\theta) v_\theta$ is a diagonal matrix, for every polynomial

$f \in \mathbb{C}[z]$. Thus, the automorphism $\vartheta_n(x) = v_\theta^* x v_\theta$ of $\mathcal{M}_n(\mathbb{C})$ maps $C^{n,\theta}$ into the algebra of diagonal matrices. In fact this map is onto, because, for any n-tuple $(w_1, \ldots, w_n)$ of complex numbers, there is an interpolating polynomial $g \in \mathbb{C}[z]$ that sends each λ_j to w_j; that is, the diagonal matrix determined by $(w_1, \ldots, w_n)$ is $v_\theta^* g(u_\theta) v_\theta$. The linear map E_n that sends every $n \times n$ matrix to its diagonal is a completely positive idempotent that maps $\mathcal{M}_n(\mathbb{C})$ onto the subalgebra of diagonal matrices. Thus, define $\mathcal{F}_{n,\theta}$ to be $\vartheta_n^{-1} \circ E_n$. □

3.6. **Geometry of the positive cones.** This section concludes with observations regarding the geometric nature of the positive cones of the operator systems discussed herein.

Evaluating a universal Toeplitz matrix T_n at a point $\lambda \in S^1$ yields a positive Toeplitz matrix $T_n(\lambda)$ in which $n^{-1}T_n(\lambda)$ is a rank-1 projection. Such matrices comprise the extremal rays of the cone $\left(C(S^1)^{(n)}\right)_+$.

Proposition 3.10 (Extremal Rays)**.** [12, Propositions 4.5 and 4.8] *The extremal rays of the cone* $\left(C(S^1)^{(n)}\right)_+$ *are positive scalar multiples of matrices of the form* $T_n(\lambda)$*, for* $\lambda \in S^1$*. The extremal rays of the cone* $\left(C(S^1)_{(n)}\right)_+$ *are positive scalar multiples of those elements* $f \in \left(C(S^1)_{(n)}\right)_+$ *with the property that* $f(\omega) = 0$*, for some* $\omega \in \mathbb{C} \setminus \{0\}$*, only if* $\omega \in S^1$*.*

Proposition 3.11. *If* $n \geq 2$*, the cone* $\left(C^{n,\theta}\right)_+$ *is simplicial, for every* $\theta \in \mathbb{R}$*, whereas the cones* $\left(C(S^1)^{(n)}\right)_+$ *and* $\left(C(S^1)_{(n)}\right)_+$ *are not simplicial.*

Proof. The discussion leading up to the statement of the proposition establishes that $\left(C^{n,\theta}\right)_+$ is a simplicial cone.

By Proposition 3.10, the extremal rays of $\left(C(S^1)^{(n)}\right)_+$ are generated by $T_n(\lambda)$ for $\lambda \in S^1$, and so there cannot be a linear isomorphism that maps $\left(C(S^1)^{(n)}\right)_+$ onto any simplicial cone, as a simplicial cone has only finitely many extremal rays.

Likewise, the extremal rays of $\left(C(S^1)_{(n)}\right)_+$ are generated by Laurent polynomials that are nonnegative on S^1 and whose zeros are contained in S^1. As

$$f_\lambda(z) = \lambda z^{-1} + 2 + \lambda^{-1} z$$

is one such Laurent polynomial, for each $\lambda \in S^1$, the cone $\left(C(S^1)^{(n)}\right)_+$ has infinitely many extremal rays. □

Corollary 3.12. *If* $\mathcal{R}_+$ *and* $\mathcal{T}_+$ *are Toeplitz or Fejér-Riesz cones, then*

$$\mathcal{R}_+ \otimes_{\rm sep} \mathcal{T}_+ \neq \mathcal{R}_+ \otimes_{\rm sep^*} \mathcal{T}_+.$$

Proof. If equality held, then the main result of [7] implies at least one of the Toeplitz cones $\mathcal{R}_+$ or $\mathcal{T}_+$ would be simplicial; however, such an implication is impossible by Proposition 3.11. □

Proposition 3.10 provides a method via convexity to generate all $n \times n$ positive Toeplitz matrices, foreshadowing a similar result (Proposition 4.5) for positive block Toeplitz matrices.

Definition 3.13. For $n \geq 2$, let $\mathbb{T}^n$ denote the n-torus $S^1 \times \cdots \times S^1$ and let $\mathfrak{P}_n \subset \mathbb{T}^n$ be the set of all n-tuples of the form $(\lambda, \lambda^2, \ldots, \lambda^n)$, for $\lambda \in S^1$.

Proposition 3.14. *$\xi = (\xi_1, \dots, \xi_{n-1}) \in \operatorname{Conv}\mathfrak{P}_{n-1}$ if and only if the Toeplitz matrix*

$$x(\xi) = \begin{bmatrix} 1 & \overline{\xi}_1 & \dots & \overline{\xi}_{n-1} \\ \xi_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \overline{\xi}_1 \\ \xi_{n-1} & \dots & \xi_1 & 1 \end{bmatrix}$$

is positive.

Proof. If $x(\xi)$ is positive, then it is a convex combination $\sum_{j=1}^{k} \alpha_j T_n(\lambda_j)$ of matrices of the form $T_n(\lambda_j)$, where $\lambda_j \in S^1$. Thus, $\xi_\ell = \sum_{j=1}^{k} \alpha_j \lambda_j^\ell$, for each ℓ.

Conversely, if $\xi \in \operatorname{Conv}\mathfrak{P}_{n-1}$, then there exist convex coefficients α_j and elements $\lambda_j \in S^1$ such that each $\xi_\ell = \sum_{j=1}^{k} \alpha_j \lambda_j^\ell$. Because $\sum_{j=1}^{k} \alpha_j T_n(\lambda_j) = x(\xi)$, the matrix $x(\xi)$ is positive. □

4. Separability

4.1. **The Gurvits Separation Theorem.** An elegant theorem of Gurvits, explained in a joint paper with Burnam [25], states

$$\left(C(S^1)^{(n)} \otimes_{\min} \mathcal{M}_m(\mathbb{C})\right)_+ = (C(S^1)^{(n)})_+ \otimes_{\text{sep}} (\mathcal{M}_m(\mathbb{C}))_+ .$$

Given the non-simplicial nature of the cones $(C(S^1)^{(n)})_+$ and $(\mathcal{M}_m(\mathbb{C}))_+$, Gurvits' result is truly astonishing, and a great deal of effort has gone into understanding this separability result. For this reason a number of expositions of Gurvits' original proof, such as those in [36, 39], or the development of alternative proofs, such as those in [2, 3, 18], have appeared in the literature.

Although all the steps in Gurvits' argument are indicated in the joint paper with Burnam [25], the proof of one of the key steps, which is stated below as Lemma 4.1, is missing in every exposition I am aware of. Therefore, the purpose of this expository subsection is to fill this gap in the published literature, drawing inspiration from the well-known majorisation theorem of Douglas [13] to prove the aforementioned "missing" lemma of [25]. For completeness, the entirety of Gurvits' argument is also presented.

Lemma 4.1. *The following statements are equivalent for $q \times r$ complex matrices x and y:*

1. *$yy^* = xx^*$;*
2. *there is a $r \times r$ unitary matrix w such that $y = xw$.*

Proof. It is clear that (2) implies (1). To prove that (1) implies (2), let $\operatorname{ran} a$ and $\ker a$ denote, respectively, the range and null-space of a linear transformation a, and view the matrices x and y as linear transformations of $\mathbb{C}^q$ into $\mathbb{C}^r$, which we consider as Hilbert spaces with respect to their standard inner products. The equation $xx^* = yy^*$ implies that the function $y^*\xi \mapsto x^*\xi$ is a well-defined isometric linear transformation $\tilde{w}_0$ of $\operatorname{ran} y^*$ into $\operatorname{ran} x^*$. Because $\tilde{w}_0$ has a linear inverse given by $x^*\xi \mapsto y^*\xi$, the linear transformation $\tilde{w}_0$ is an isomorphism of the vector spaces $\operatorname{ran} y^*$ and $\operatorname{ran} x^*$. Extend the domain of $\tilde{w}_0$ to all of $\mathbb{C}^q$ to obtain a linear transformation $w_0 : \mathbb{C}^q \to \mathbb{C}^r$ whereby $w_0\gamma = \tilde{w}_0\gamma$, for $\gamma \in \operatorname{ran} y^*$, and $w_0\eta = 0$, for $\eta \in (\operatorname{ran} y^*)^\perp = \ker y$.

An arbitrary vector $\omega \in \mathbb{C}^q$ has the form $\omega = \gamma + \eta$, for some $\gamma \in \operatorname{ran} y^*$ and $\eta \in \ker y$. Thus, if $\gamma = y^*\xi \in \operatorname{ran} y^*$, for some $\xi \in \mathbb{C}^r$, then

$$y\omega = y\left(y^*\xi + \eta\right) = yy^*\xi = xx^*\xi = x\left(w_0(y^*\xi) + w_0\eta\right) = xw_0(y^*\xi + \eta) = xw_0\omega.$$

Hence, $y = xw_0\omega$, for all vectors $\omega \in \mathbb{C}^q$, proving that $y = xw_0$.

The isomorphism of the subspaces $\operatorname{ran} y^*$ and $\operatorname{ran} x^*$ of $\mathbb{C}^q$ implies an isomorphism of their orthogonal complements, $\ker y$ and $\ker x$, respectively. Choose any orthonormal bases of $\ker y$ and $\ker x$, respectively, and let $\tilde{w}_1$ be the linear isomorphism that maps the orthonormal basis of $\ker y$ onto the orthonormal basis of $\ker x$, and extend the definition of $\tilde{w}_1$ to a linear transformation w_1 on all of $\mathbb{C}^q$ by setting $w_1\delta = 0$ for all $\delta \in (\ker y)^\perp = \operatorname{ran} y^*$.

The linear transformations w_0 and w_1 have orthogonal ranges, and $w_0^*w_0$ is the projection with range $\operatorname{ran} y^*$ and $w_1^*w_1$ is the (complementary) projection with range $\operatorname{ran} x^*$. Hence, if $w = w_0 + w_1$, then w^*w is the identity map on $\mathbb{C}^r$, which implies $w^{-1} = w^*$, proving w is unitary. Finally, the equality $\operatorname{ran} w_1 = \ker x$ implies $xw = xw_0 + xw_1 = xw_0 = y$. □

Before proving Gurvits' Theorem, note that, if u is a unitary element of a unital C^*-algebra $\mathcal{A}$, then by functional calculus, we may evaluate the universal Toeplitz matrix T_n at u to obtain

$$T_n(u) = \begin{bmatrix} 1 & u^{-1} & \dots & u^{-n+1} \\ u & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & u^{-1} \\ u^{n-1} & \dots & u & 1 \end{bmatrix},$$

which is a positive element of $\mathcal{M}_n(\mathcal{A})$.

Theorem 4.2 (Gurvits)**.** *For all $n, m \geq 2$,*

$$(C(S^1)^{(n)})_+ \otimes_{\text{sep}} (\mathcal{M}_m(\mathbb{C}))_+ = \Big(C(S^1)^{(n)} \otimes_{\min} \mathcal{M}_m(\mathbb{C})\Big)_+ .$$

Proof. Suppose $x = \displaystyle\sum_{\ell=-n+1}^{n-1} r_\ell \otimes a_\ell \in \Big(C(S^1)^{(n)} \otimes_{\min} \mathcal{M}_m(\mathbb{C})\Big)_+$. Set $q = nm$ and $r = \operatorname{rank} x$. By the Spectral Theorem, x has the form $x = udu^*$ for some unitary matrix u and diagonal matrix d whose diagonal entries are eigenvalues α_j of x. Without loss of generality, assume the first r diagonal entries of d are the nonzero eigenvalues of x and let g be the $r \times r$ diagonal matrix with diagonal entries $\alpha_j^{\frac{1}{2}}$ and z be the $q \times r$ rectangular matrix $z = \begin{bmatrix} g \\ 0 \end{bmatrix}$, where "0" in the matrix above is a $(q - r) \times q$ matrix of zeroes if $r < q$, or is absent if $r = q$. Because $zz^* = d$, if we let $y = uz$, which is a $q \times r$ matrix, we then obtain $yy^* = uzz^*u^* = udu^* = x$.

As below, partition the matrix y as a column of n matrices y_k, each of dimension $m \times r$, and then define y_U and y_L:

$$y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}, \quad y_U = \begin{bmatrix} y_0 \\ \vdots \\ y_{n-2} \end{bmatrix}, \quad y_L = \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}.$$

The matrix $y_U y_U^* = y_L y_L^*$ coincides with the upper-left $(n-1) \times (n-1)$ block of the matrix x.

Apply Lemma 4.1 to obtain $y_L = y_U w$ for some $r \times r$ unitary matrix w. Thus, $y_k = y_{k-1} w$, for each $k = 1, \dots n-1$. The Toeplitz structure of x and the equality $y_U y_U^* = y_L y_L^*$ yield:

$$a_0 = y_0 y_0^*, \quad a_1 = y_0 w y_0^*, \quad a_2 = y_0 w^2 y_0^*, \quad \dots \quad a_{n-1} = y_0 w^{n-1} y_0^*.$$

Because $a_{-\ell} = a_\ell^*$ and $w^{-\ell} = (w^\ell)^*$ for every ℓ, we deduce $a_\ell = y_0 w^\ell y_0^*$ for all $\ell = -n+1, ..., n-1$.

Set $z = y_0$ and express x as

$$x = (1_n \otimes z) \left(\sum_{\ell=-n+1}^{n-1} r_\ell \otimes w^\ell \right) (1_n \otimes z)^* = \sum_{\ell=-n+1}^{n-1} r_\ell \otimes (z w^\ell z^*).$$

Let $\lambda_1, \dots, \lambda_k$ denote the distinct eigenvalues of W. By the Spectral Theorem, there are pairwise-orthogonal projections $p_1, \dots, p_k$ such that $w^\ell = \sum_{j=1}^{k} \lambda_j^\ell p_j$, for all $\ell \in \mathbb{Z}$. Therefore,

$$T_n(w) = \sum_{\ell=-n+1}^{n-1} r_\ell \otimes (\sum_{j=1}^{k} \lambda_j^\ell p_j) = \sum_{j=1}^{k} \sum_{\ell=-n+1}^{n-1} \lambda_j^\ell r_\ell \otimes p_j = \sum_{j=1}^{k} T_n(\lambda_j) \otimes p_j.$$

Hence,

$$x = (1_n \otimes z) T_n(w) (1_n \otimes z)^* = \sum_{j=1}^{k} T_n(\lambda_j) \otimes (z p_j z^*) = \sum_{j=1}^{k} T_n(\lambda_j) \otimes b_j,$$

where $b_j = z p_j z^*$ for each j. Thus, $x \in (C(S^1)^{(n)})_+ \otimes_{\mathrm{sep}} (\mathcal{M}_m(\mathbb{C}))_+$. □

It is sometimes convenient to recast Theorem 4.2 as follows.

Corollary 4.3. *If* $x = \sum_{\ell=-n+1}^{n-1} r_\ell \otimes a_\ell \in \left(C(S^1)^{(n)} \otimes_{\min} \mathcal{M}_m(\mathbb{C}) \right)_+$, *then there exist* $k \in \mathbb{N}$, $\lambda_j \in S^1$, *and* $b_j \in \mathcal{M}_m(\mathbb{C})_+$ *such that*

$$a_\ell = \sum_{j=1}^{k} \lambda_j^\ell b_j, \tag{4.1}$$

for each $\ell = -n+1, \dots, n-1$.

4.2. **The matrix convex hull of $\mathfrak{P}_n$.** Recall that $\mathfrak{P}_n$ is the subset of the n-torus consisting of n-tuples of the form $(\lambda, \lambda^2, \dots, \lambda^n)$, for $\lambda \in S^1$.

One can use Gurvits' Theorem to reframe Proposition 3.14 in the setting of matrix convexity [15].

Definition 4.4. The *matrix convex hull* of $\mathfrak{P}_n$ is the sequence m-Conv$\mathfrak{P}_n = (\mathcal{L}_k)_{k\in\mathbb{N}}$ of subsets $\mathcal{L}_k$ of $\mathcal{M}_k(\mathbb{C})^n$ defined by

$$\mathcal{L}_k = \left\{ \left(\sum_{j=1}^{g} \lambda_j q_j, \dots, \sum_{j=1}^{g} \lambda_j^n q_j \right) \mid g \in \mathbb{N},\, \lambda_j \in S^1,\, q_j \in \mathcal{M}_k(\mathbb{C})_+,\, \sum_{j=1}^{g} q_j = 1_k \right\}.$$

Similar to Proposition 3.14, we have the following result.

Proposition 4.5. *A tuple $a = (a_1, \dots, a_{n-1})$ of $k \times k$ matrices is in the matrix convex hull of $\mathfrak{P}_{n-1}$ if and only if the block Toeplitz matrix*

$$x(a) = \begin{bmatrix} 1_k & a_1^* & \dots & a_{n-1}^* \\ a_1 & 1_k & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_1^* \\ a_{n-1} & \dots & a_1 & 1_k \end{bmatrix}$$

is positive.

Proof. The argument is the same as that of Proposition 3.14, except one uses the matrix convex combinations of equation (4.1). □

4.3. **Positive 2×2 block Toeplitz matrices with Toeplitz blocks.**

Proposition 4.6. *For every $n \geq 2$,*

$$C(S^1)^{(2)}{}_+ \otimes_{\text{sep}} C(S^1)_+^{(n)} = \left(C(S^1)^{(2)} \otimes_{\min} C(S^1)^{(n)} \right)_+ .$$

Proof. By the Gurvits Separation Theorem, there exist $c_j \in \mathcal{M}_2(\mathbb{C})_+$ and $s_j \in \mathcal{M}_n(\mathbb{C})_+$ such that

$$x = \sum_{j=1}^{m} c_j \otimes s_j.$$

The idempotent $\mathcal{E}_2 : \mathcal{M}_2(\mathbb{C}) \to C(S^1)^{(2)}$ (Proposition 3.7)) is a positive linear map, and so

$$x = \mathcal{E}_2^{(n)}(x) = \sum_{j=1}^{m} \mathcal{E}_2(c_j) \otimes s_j,$$

which implies the Toeplitz matrices $b_j = \mathcal{E}_2(c_j)$ are positive. □

It was noted in [18] that, if

$$x = \begin{bmatrix} a & c^* \\ c & a \end{bmatrix}$$

is a positive invertible block-Toeplitz matrix, and if both a and c are 2×2 Toeplitz matrices, then x is separable in the operator system $C(S^1)^{(2)} \otimes_{\min} C(S^1)^{(2)}$. The methods of the present paper yield the following improved result, in which invertibility and 2×2 a and c are not required, as a consequence of Proposition 4.6 and the structure of the positive cone of Toeplitz matrices.

Corollary 4.7. *If $a, c \in C(S^1)^{(n)}$ are such that $x = \begin{bmatrix} a & c^* \\ c & a \end{bmatrix}$ is positive, then x is separable. Thus, there exist $k \in \mathbb{N}$, $\alpha_j \in \mathbb{R}_+$, and $\lambda_j, \mu_j \in S^1$ such that*

$$\begin{bmatrix} a & c^* \\ c & a \end{bmatrix} = \sum_{j=1}^{k} \alpha_j \begin{bmatrix} 1_n & \lambda_j^{-1} T_n(\mu) \\ \lambda_j T_n(\mu) & 1_n \end{bmatrix}. \tag{4.2}$$

Proof. Proposition 4.6 asserts $x = \sum_{i=1}^{q} a_i \otimes b_i$, for some positive $a_i \in C(S^1)^{(2)}$ and $b_i \in C(S^1)^{(n)}$. By Proposition 3.10, each a_i and b_i is a linear combination of matrices of the form $T_2(\lambda)$ and $T_n(\mu)$ using nonnegative scalar coefficients. Hence, x can be expressed in the form (4.2) above. □

Another consequence of 2×2 separability is in reference to Toeplitz-matrix moments, which is the problem of extending positive 2×2 block Toeplitz matrices with Toeplitz blocks matrices to positive $n \times n$ block Toeplitz matrices with Toeplitz blocks.

Corollary 4.8. *If x_0, x_1 are $m \times m$ Toeplitz matrices for which $\begin{bmatrix} x_0 & x_1^* \\ x_1 & x_0 \end{bmatrix}$ is positive, then, for every $n \geq 3$, there exist $m \times m$ Toeplitz matrices $x_2, \ldots, x_{n-1}$ such that*

$$\begin{bmatrix} x_0 & x_1^* & \ldots & x_{n-1}^* \\ x_1 & x_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_1^* \\ x_{n-1} & \ldots & x_1 & x_0 \end{bmatrix}$$

is positive.

Proof. Corollary 4.7 shows that

$$\begin{bmatrix} x_0 & x_1^* \\ x_1 & x_0 \end{bmatrix} = \sum_{j=1}^{k} \alpha_j T_2(\lambda_j) \otimes T_m(\mu_j),$$

for some $\lambda_j, \mu_j \in S^1$ and $\alpha_j \in \mathbb{R}_+$. Fix $n \geq 3$ and consider the matrix

$$X = \sum_{j=1}^{k} \alpha_j T_n(\lambda_j) \otimes T_m(\mu_j).$$

Then, X is positive and has the form

$$X = \begin{bmatrix} x_0 & x_1^* & \ldots & x_{n-1}^* \\ x_1 & x_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_1^* \\ x_{n-1} & \ldots & x_1 & x_0 \end{bmatrix},$$

for some $x_2, \ldots, x_{n-1} \in C(S^1)^{(m)}$. □

4.4. **Positive block-Toeplitz matrices with circulant blocks.**

Proposition 4.9. *For any $n, m \geq 2$ and $\theta \in \mathbb{R}$,*

$$\begin{array}{rcl} (C(S^1)^{(n)})_+ \otimes_{\mathrm{sep}} \left(C^{m,\theta}\right)_+ & = & \left(C(S^1)^{(n)} \otimes_{\min} C^{m,\theta}\right)_+ \\ & = & (C(S^1)^{(n)})_+ \otimes_{\mathrm{sep}^*} \left(C^{m,\theta}\right)_+ \end{array}.$$

Proof. If $x = [a_{k-j}]_{k,j=1}^n \in C(S^1)^{(n)} \otimes_{\min} C^{m,\theta}$ is positive, then $x \in \mathcal{M}_m(C(S^1)^{(n)})_+$. Therefore, by Gurvits' Theorem,

$$a_\ell = \sum_{j=1}^k \lambda_j^\ell b_j, \text{ for every } \ell,$$

for some $\lambda_j \in S^1$ and $b_j \in \mathcal{M}_m(\mathbb{C})_+$. Apply the completely positive idempotent map $\mathcal{F}_{m,\theta}$ of Proposition 3.9 to each a_ℓ to obtain

$$a_\ell = \sum_{j=1}^k \lambda_j^\ell \mathcal{F}_{m,\theta}(b_j), \text{ for every } \ell. \tag{4.3}$$

Because the matrices b_j are positive, the generalised circulant matrices $c_j = \mathcal{F}_{m,\theta}(b_j)$ are also positive. Thus,

$$a_\ell = \sum_{j=1}^k \lambda_j^\ell c_j, \text{ for every } \ell,$$

which is equivalent to

$$x = \sum_{j=1}^k T_n(\lambda_j) \otimes c_j.$$

Hence, x is an element of $(C(S^1)^{(n)})_+ \otimes_{\mathrm{sep}} \left(C^{m,\theta}\right)_+$.

Now suppose x is an element of $(C(S^1)^{(n)})_+ \otimes_{\mathrm{sep}^*} \left(C^{m,\theta}\right)_+$. Because of the operator system equivalences $C^{m,\theta} \simeq \mathrm{C}^*(\mathbb{Z}_m) \simeq \mathbb{C}^m$, the cone $\left(C^{m,\theta}\right)_+$ is affinely isomorphic to $(\mathbb{R}^m)_+$ and elements of $C(S^1)^{(n)} \otimes C^{m,\theta}$ arise as elements of $C(S^1)^{(n)} \otimes \mathbb{C}^m$. However, from $C(S^1)^{(n)} \otimes_{\min} \mathbb{C}^m \simeq \bigoplus_1^m C(S^1)^{(n)}$, if φ and ψ are positive linear functionals on $C(S^1)^{(n)}$ and $\mathbb{C}^m$ respectively, then $\varphi \otimes \psi$ applied to $x \in C(S^1)^{(n)} \otimes_{\min} \mathbb{C}^m$ yields

$$\varphi \otimes \psi(x) = \sum_{j=1}^m \alpha_j \varphi(x_j),$$

where $\alpha_j = \psi(e_j)$ and x_j is the j-th direct summand of x. By choosing j and then choosing ψ so that $\alpha_k = 0$ for all $k \neq j$, one gets $\varphi(x_j) \geq 0$ for all positive linear functionals φ on $C(S^1)^{(n)}$, whence $x_j \in \left(C(S^1)^{(n)}\right)_+$. Hence, $\bigoplus_1^m x_j$ is a positive element of

$$\bigoplus_1^m C(S^1)^{(n)} \simeq C(S^1)^{(n)} \otimes_{\min} \mathbb{C}^m,$$

which proves $(C(S^1)^{(n)})_+ \otimes_{\mathrm{sep}^*} \left(C^{m,\theta}\right)_+ \subseteq \left(C(S^1)^{(n)} \otimes_{\min} C^{m,\theta}\right)_+$. $\square$

4.5. **The matrix R_n is separable.**

Proposition 4.10. $R_n \in C(S^1)_+^{(n)} \otimes_{\rm sep} C(S^1)_+^{(n)}$.

Proof. Let q be a prime number such that $q > 2n$, and let $\lambda \in S^1$ be a primitive q-th root of unity. If $\ell \in \{-n+1, \dots, n-1\}$ is such that $\ell \neq 0$, then λ^ℓ is also a primitive q-th root of unity. Hence $\displaystyle\sum_{j=1}^{q} (\lambda^\ell)^j$ is a sum of all the q-th roots of unity, and so $\displaystyle\sum_{j=1}^{q} (\lambda^\ell)^j = 0$. Furthermore, assuming $\ell \in \{-n+1, \dots, n-1\}$ and $k \in \{1, \dots, n-1\}$, the condition $q > 2n$ also implies that $\displaystyle\sum_{j=1}^{q} (\lambda^{k+\ell})^j = 0$, if $\ell \neq -k$, and that $\displaystyle\sum_{j=1}^{q} (\lambda^{k-\ell})^j = 0$, if $\ell \neq k$,

Summing over the primitive roots of unity leads to

$$\sum_{j=1}^{q} T_n(\lambda^j) = \sum_{j=1}^{q} \sum_{\ell=-n+1}^{n-1} (\lambda^j)^\ell r_\ell = \sum_{\ell=-n+1}^{n-1} \left(\sum_{j=1}^{q} (\lambda^\ell)^j \right) r_\ell = q r_0.$$

Next, fix $j \in \{1, \dots, q\}$ and consider $T_n(\lambda^j) \otimes T_n(\lambda^{-j})$, which is the following Toeplitz matrix of Toeplitz matrices:

$$T_n(\lambda^j) \otimes T_n(\lambda^{-j}) = \begin{bmatrix} T_n(\lambda^{-j}) & \lambda^{-j} T_n(\lambda^{-j}) & \dots & \lambda^{-nj+j} T_n(\lambda^{-j}) \\ \lambda^j T_n(\lambda^{-j}) & T_n(\lambda^{-j}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \lambda^{-j} T_n(\lambda^{-j}) \\ \lambda^{nj-j} T_n(\lambda^{-j}) & \dots & \lambda^j T_n(\lambda^{-j}) & T_n(\lambda^{-j}) \end{bmatrix}.$$

Note that, for $k = 1, \dots, n-1$,

$$\lambda^{kj} T_n(\lambda^{-j}) = \sum_{\ell=-n+1}^{n-1} \lambda^{kj} (\lambda^{-j})^\ell r_\ell = \sum_{\ell=-n+1}^{n-1} (\lambda^{k-\ell})^j r_\ell,$$

and so

$$\sum_{j=1}^{q} \left(\lambda^{kj} T_n(\lambda^{-j}) \right) = \sum_{\ell=-n+1}^{n-1} \left(\sum_{j=1}^{q} (\lambda^{k-\ell})^j \right) r_\ell = q r_k.$$

Similarly, for $k = 1, \dots, n-1$,

$$\sum_{j=1}^{q} \left(\lambda^{-kj} T_n(\lambda^{-j}) \right) = \sum_{\ell=-n+1}^{n-1} \left(\sum_{j=1}^{q} (\lambda^{k+\ell})^j \right) r_\ell = q r_{-k}.$$

Hence,

$$\sum_{j=1}^{q} \frac{1}{q} \left(T_n(\lambda^j) \otimes T_n(\lambda^{-j}) \right) = \begin{bmatrix} r_0 & r_{-1} & \dots & r_{-n+1} \\ r_1 & r_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_{-1} \\ r_{n-1} & \dots & r_1 & r_0 \end{bmatrix} = R_n,$$

which proves that R_n is separable. $\square$

The choice of prime $q > 2n$ in the proof of Proposition 4.10, while sufficient for the proof of separability, is not claimed to be optimal. For example, with $n = 2$, one can calculate

$$R_2 = \frac{1}{2}\left(T_2(i) \otimes T_2(-i) \,+\, T_2(-i) \otimes T_2(i)\right).$$

4.6. **The tensor cone** $\left(C(S^1)^{(n)} \otimes_{\max} C(S^1)^{(m)}\right)_+$**.**

Theorem 4.11. *For every $n, m \geq 2$,*

$$C(S^1)_+^{(n)} \otimes_{\text{sep}} C(S^1)_+^{(n)} = \left(C(S^1)^{(n)} \otimes_{\max} C(S^1)^{(m)}\right)_+.$$

Proof. We first show that $C(S^1)_+^{(n)} \otimes_{\text{sep}} C(S^1)_+^{(n)}$ is topologically closed. Consider the compact set $\mathcal{E} = \left\{T_n(\lambda) \otimes T_m(\mu)) \,|\, \lambda, \mu \in S^1\right\}$. Because the extremal rays of $C(S^1)_+^{(n)} \otimes_{\text{sep}} C(S^1)_+^{(n)}$ have the form

$$\left\{\alpha(T_n(\lambda) \otimes T_m(\mu)) \,|\, \alpha \geq 0,\, \lambda, \mu \in S^1\right\},$$

the set $\mathcal{E}$ constitutes the set of extreme points of the convex hull $\mathcal{C}$ of $\mathcal{E}$. By Carathéodory's Theorem, the convex hull of a compact set in a finite-dimensional vector space is compact; thus, $\mathcal{C}$ is compact. But $\mathcal{C}$ is also the base for the cone $C(S^1)_+^{(n)} \otimes_{\text{sep}} C(S^1)_+^{(n)}$, and so $C(S^1)_+^{(n)} \otimes_{\text{sep}} C(S^1)_+^{(n)}$ is closed.

Select $x \in \left(C(S^1)^{(n)} \otimes_{\max} C(S^1)^{(m)}\right)_+$. By definition, for each $\varepsilon > 0$, the matrix $x + \varepsilon(1_n \otimes 1_m)$ has the form $\gamma^*(p \otimes q)\gamma$, for some positive matrices p and q with entries from $C(S^1)^{(n)}$ and $C(S^1)^{(m)}$, respectively, and some rectangular complex matrix γ. Because $x + \varepsilon(1_n \otimes 1_m)$ is strictly positive in $C(S^1)^{(n)} \otimes_{\max} C(S^1)^{(m)}$, there are, by [18, Proposition 3.4], $k_\varepsilon \in \mathbb{N}$, $a_j^{(\varepsilon)} \in C(S^1)_+^{(n)}$, and $b_j^{(\varepsilon)} \in C(S^1)_+^{(m)}$ such that and so, by [18, Proposition 3.4] it may be expressed as

$$x + \varepsilon(1_n \otimes 1_m) = \sum_{j=1}^{k_\varepsilon} a_j^{(\varepsilon)} \otimes b_j^{(\varepsilon)}.$$

Thus, $x + \varepsilon(1_n \otimes 1_m)$ is separable, for every $\varepsilon > 0$. Because the separability cone is closed, we deduce that x itself is separable. $\square$

5. Entanglement

If $\mathcal{R}$ and $\mathcal{T}$ are finite-dimensional operator systems, then

$$\mathcal{R}_+ \otimes_{\text{sep}^*} \mathcal{T}_+ = \left\{x \in (\mathcal{R} \otimes \mathcal{T})_{\text{sa}} \,|\, (\varphi \otimes \psi)[x] \geq 0, \text{ for all } \varphi \in \mathcal{R}_+^d,\, \psi \in \mathcal{T}_+^d\right\}.$$

By the convexity of $\mathcal{T}_+^d$ and $\mathcal{R}_+^d$, one may restrict themselves to those φ and ψ that generate the extremal rays of these cones. Fortunately, these extremal rays have been determined by Connes and van Suijlekom for both Toeplitz and Fejér-Riesz operator systems [12, Propositions 4.5 and 4.8].

Proposition 5.1 (Extremal Rays of the Dual)**.** *For every $n \geq 2$:*

(1) *the extremal positive linear functionals on $C(S^1)^{(n)}$ are those of the form*

$$x \mapsto \langle x\xi, \xi\rangle, \text{ where } \xi = \begin{bmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{n-1} \end{bmatrix} \in \mathbb{C}^n \text{ is a vector of coefficients in which}$$

the polynomial

$$f_\xi(z) = \sum_{\ell=0}^{n-1} \xi_\ell z^{n-1-\ell}$$

has all of its roots on the unit circle S^1*;*

(2) *the extremal positive linear functionals on* $C(S^1)_{(n)}$ *are positive scalar multiples of the point-evaluations* $f \mapsto f(\lambda)$*, for* $\lambda \in S^1$*.*

5.1. **Entangled block Toeplitz matrices.** If $\xi \in \mathbb{C}^2$ is such that $f_\xi(\lambda) = 0$ only for $\lambda \in S^1$, then scaling ξ by an appropriate positive scalar multiple allows us to assume ξ is an element of $S^1 \times S^1 \subset \mathbb{C}^2$. Thus, these are the vector states that generate the extremal rays of the cone of positive linear functionals on $C(S^1)^{(2)}$.

Lemma 5.2. *If* $a = \begin{bmatrix} \alpha & \overline{\beta} \\ \beta & \alpha \end{bmatrix} \in C(S^1)^{(2)}$ *satisfies* $\langle a\xi, \xi\rangle \geq 0$ *for every* $\xi \in S^1 \times S^1$*, then* $\langle a\gamma, \gamma\rangle \geq 0$ *for every* $\gamma \in \mathbb{C}^2$*.*

Proof. In writing $\xi = \lambda e_1 + \mu e_2$, for $\lambda, \mu \in S^1$, we obtain

$$0 \leq \langle a\xi, \xi\rangle = 2\alpha + 2\Re(\lambda\overline{\mu}\beta).$$

Thus, $\langle a\xi, \xi\rangle \geq 0$ for every $\xi \in S^1 \times S^1$ if and only if $\alpha \geq -\Re(e^{i\theta}\beta)$, for every $\theta \in \mathbb{R}$. In selecting θ so that $-\Re(e^{i\theta}\beta) = |\beta|$, one obtains $\alpha \geq |\beta|$, which implies a is a positive operator on $\mathbb{C}^2$. □

Proposition 5.3. *The following statements are equivalent for* $x = \begin{bmatrix} a & b^* \\ b & a \end{bmatrix}$ *in* $C(S^1)^{(2)} \otimes C(S^1)^{(2)}$*:*

(1) $x \in (C(S^1)^{(2)})_+ \otimes_{\mathrm{sep}^*} C(S^1)^{(2)})_+$*;*
(2) $\frac{1}{2}(e^{i\theta}b + e^{-i\theta}b^*) \leq a$*, for all* $\theta \in \mathbb{R}$*.*

Proof. In fixing $\eta \in S^1 \times S^1$ and taking $\xi = \lambda e_1 + \mu e_2$, for $\lambda, \mu \in S^1$, we obtain

$$\langle x(\xi \otimes \eta), (\xi \otimes \eta)\rangle = 2\left(\langle a\eta, \eta\rangle + \Re(\lambda\overline{\mu}\langle b\eta, \eta\rangle)\right).$$

Hence, $x \in (C(S^1)^{(2)})_+ \otimes_{\mathrm{sep}^*} C(S^1)^{(2)})_+$ if and only if, for each $\theta \in \mathbb{R}$,

$$\left\langle \left(a + \Re(e^{i\theta}b)\right)\eta, \eta\right\rangle \geq 0, \text{ for all } \eta \in S^1 \times S^1.$$

However, by Lemma 5.2, the condition above holds if and only if $a + \Re(e^{i\theta}b)$ is a positive operator on $\mathbb{C}^2$, for every $\theta \in \mathbb{R}$. □

Note $(C(S^1)^{(2)})_+ \otimes_{\mathrm{sep}^*} C(S^1)^{(2)})_+$ properly contains $\left(C(S^1)^{(2)} \otimes_{\min} C(S^1)^{(2)}\right)_+$ because, if they were equal, then Proposition 4.6 would imply every element of the cone $(C(S^1)^{(2)})_+ \otimes_{\mathrm{sep}^*} C(S^1)^{(2)})_+$ is separable, contrary to the conclusion of Corollary 3.12

5.2. **Entanglement in tensor products of Fejér-Riesz cones.**

Proposition 5.4. *For all* $n, m \geq 2$*,*

$$\left(C(S^1)_{(n)} \otimes_{\min} C(S^1)_{(m)}\right)_+ = \left(C(S^1)_{(n)}\right)_+ \otimes_{\mathrm{sep}^*} \left(C(S^1)_{(m)}\right)_+$$

and

$$\left(C(S^1)_{(2)} \otimes_{\max} C(S^1)_{(m)}\right)_+ = \left(C(S^1)_{(2)}\right)_+ \otimes_{\mathrm{sep}^*} \left(C(S^1)_{(m)}\right)_+.$$

Proof. If $\lambda, \mu \in S^1$ and if ϱ_λ and ϱ_μ are the corresponding point evaluations, then, for every $f \in \left(C(S^1)_{(n)}\right)_+ \otimes_{\text{sep}^*} \left(C(S^1)_{(m)}\right)_+$,

$$0 \leq (\varrho_\lambda \otimes \varrho_\mu)[f] = f(\lambda, \mu);$$

hence, $f \in \left(C(S^1)_{(n)} \otimes_{\min} C(S^1)_{(m)}\right)_+$.

Proposition 4.6 show that the identity map

$$\iota : C(S^1)^{(2)} \otimes_{\min} C(S^1)^{(m)} \to C(S^1)^{(2)} \otimes_{\max} C(S^1)^{(m)}$$

is a positive linear map of operator systems. Hence, the dual map ι^d is also positive, where

$$\iota^d : \left(C(S^1)^{(2)} \otimes_{\max} C(S^1)^{(m)}\right)^d \to \left(C(S^1)^{(2)} \otimes_{\min} C(S^1)^{(m)}\right)^d.$$

However, in identifying $(\mathcal{R}\otimes_{\min}\mathcal{T})^d$ with $\mathcal{R}^d\otimes_{\max}\mathcal{T}^d$, by Theorem 2.5, then Toeplitz duality yields the positivity of the map

$$\iota^d : C(S^1)_{(2)} \otimes_{\min} C(S^1)_{(m)} \to C(S^1)_{(2)} \otimes_{\max} C(S^1)_{(m)},$$

which proves $\left(C(S^1)_{(2)} \otimes_{\max} C(S^1)_{(m)}\right)_+ = \left(C(S^1)_{(2)}\right)_+ \otimes_{\text{sep}^*} \left(C(S^1)_{(m)}\right)_+$. □

The separability cones for Fejér-Riesz operator systems are really quite small, since, by definition, $f \in \left(C(S^1)_{(n)}\right)_+ \otimes_{\text{sep}} \left(C(S^1)_{(m)}\right)_+$ if and only if there are $g_j \in \left(C(S^1)_{(n)}\right)_+$ and $h_j \in \left(C(S^1)_{(m)}\right)_+$ such that

$$f(z_1, z_2) = \sum_{j=1}^{k} g_j(z_1) h_j(z_2).$$

More natural, and in the spirit of the single-variable Fejér-Riesz factorisation of positive trigonometric polynomials $f(z)$ as $f(z) = |h(z)|^2$ for a suitable polynomial $f \in \mathbb{C}[z]$, are representations of $f \in \left(C(S^1)_{(n)} \otimes_{\min} C(S^1)_{(m)}\right)_+$ as sums of squares, by which is meant

$$f(z_1, z_2) = \sum_{j=1}^{k} \overline{h_j(z_1, z_2)} h_j(z_1, z_2) = \sum_{s=1}^{k} |h_j(z_1, z_2)|^2,$$

for some polynomials $h_j \in \mathbb{C}[z_1, z_2]$.

5.3. **The matrix T_n is entangled.**

Proposition 5.5. *T_n is entangled in $\left(C(S^1)^{(n)} \otimes_{\max} C(S^1)_{(n)}\right)_+$.*

Proof. It is necessary to first demonstrate T_n is positive in $C(S^1)^{(n)} \otimes_{\max} C(S^1)_{(n)}$. To this end, select $\varepsilon > 0$ and consider $x = \varepsilon(1_n \otimes \chi_0) + T_n$, which can be expressed as

$$x = r_0 \otimes (1+\varepsilon)\chi_0 \; + \sum_{\ell \neq 0} r_\ell \otimes \chi_\ell.$$

Let $k = 1$ and take $P \in \mathcal{M}_k(C(S^1)^{(n)})_+$ to be $P = 1_n$. Let $m = n$ and take $Q \in \mathcal{M}_m(C(S^1)_{(n)})_+$ to be

$$Q = \begin{bmatrix} 1+\varepsilon & z^{-1} & \dots & z^{-n+1} \\ z & 1+\varepsilon & \ddots & \vdots \\ \vdots & \ddots & \ddots & z^{-1} \\ z^{n-1} & \dots & z & 1+\varepsilon \end{bmatrix}.$$

Thus, $P \otimes Q$ is a block-diagonal matrix with each diagonal block equal to Q. Use β to compress $P \otimes Q$ to the first of these diagonal blocks to obtain $\beta^*(P \otimes Q)\beta = x$. Thus, as $\varepsilon > 0$ is arbitrary, $T_n \in \left(C(S^1)^{(n)} \otimes_{\max} C(S^1)_{(n)}\right)_+$.

By [18, Theorem 4.4], the matrix T_n is entangled in $\left(\mathcal{M}_n(\mathbb{C}) \otimes_{\min} C(S^1)_{(n)}\right)_+$; thus, T_n is entangled in the smaller cone $\left(C(S^1)^{(n)} \otimes_{\min} C(S^1)_{(n)}\right)_+$, which contains the cone $\left(C(S^1)^{(n)} \otimes_{\max} C(S^1)_{(n)}\right)_+$, which contains T_n. Hence, T_n is entangled in $\left(C(S^1)^{(n)} \otimes_{\max} C(S^1)_{(n)}\right)_+$. □

Corollary 5.6. $(C(S^1)^{(n)})_+ \otimes_{\rm sep} (C(S^1)_{(n)})_+ \neq \left(C(S^1)^{(n)} \otimes_{\max} C(S^1)_{(n)}\right)_+$.

6. Categorical Relations

6.1. **C*- and injective envelopes.** The operator system category contains all unital C*-algebras as objects. For each operator system $\mathcal{R}$, there is a unital C*-algebra $\mathrm{C}^*_{\rm e}(\mathcal{R})$, called the *C*-envelope of* $\mathcal{R}$, and unital linear complete order embedding $\iota_{\rm e} : \mathcal{R} \to \mathrm{C}^*_{\rm e}(\mathcal{R})$ in which the operator subsystem $\iota_{\rm e}(\mathcal{R})$ of $\mathrm{C}^*_{\rm e}(\mathcal{R})$ generates $\mathrm{C}^*_{\rm e}(\mathcal{R})$, such that the pair $(\mathrm{C}^*_{\rm e}(\mathcal{R}), \iota_{\rm e})$ has the following universal property: for every unital C*-algebra $\mathcal{A}$ and every unital linear complete order embedding $\phi : \mathcal{R} \to \mathcal{A}$ for which $\phi(\mathcal{R})$ generates $\mathcal{A}$, there exists a unital surjective $*$-homomorphism $\pi : \mathcal{A} \to \mathrm{C}^*_{\rm e}(\mathcal{R})$ such that $\iota_e = \pi \circ \phi$. The existence of the injective envelope was established by Hamana [27], and it serves as a type of noncommutative Šilov boundary for operator systems. One of the main tasks in the theory of operator systems is to identify $\mathrm{C}^*_{\rm e}(\mathcal{R})$ for various operator systems $\mathcal{R}$.

Definition 6.1. An operator system $\mathcal{I}$ is *injective* if, for each operator system $\mathcal{R}$, ucp map $\alpha : \mathcal{R} \to \mathcal{I}$, and unital linear complete order embedding $\beta : \mathcal{R} \to \mathcal{T}$ of $\mathcal{R}$ into an operator system $\mathcal{T}$, there exists a ucp map $\phi : \mathcal{T} \to \mathcal{I}$ such that $\alpha = \phi \circ \beta$.

The Arveson Extension Theorem [5, 34] establishes the injectivity of the type I factor $\mathcal{B}(\mathcal{H})$, for any Hilbert space $\mathcal{H}$, which is the most most basic of injective operator systems. Choi and Effros [11] developed an abstract approach to injectivity, which Hamana used to establish the existence of the injective envelope of operator systems [27]. For each operator system $\mathcal{R}$, there is an injective operator system $\mathcal{I}(\mathcal{R})$, called the *injective envelope of* $\mathcal{R}$, and unital linear complete order embedding $\iota_{\rm ie} : \mathcal{R} \to \mathcal{I}(\mathcal{R})$ with the property that a sequence of operator subsystems of the form

$$\iota_{\rm ie}(\mathcal{R}) \subseteq \mathcal{I}_0 \subseteq \mathcal{I},$$

with $\mathcal{I}_0$ injective, can occur only if $\iota_0 = \mathcal{I}$.

The relationship between C*-envelopes and injective envelopes is captured by the following theorem [34, Chapter 15].

Theorem 6.2. *If $\mathcal{I}$ is an injective operator system, then there is a unital C*-algebra $\mathcal{B}$ such that $\mathcal{I} \simeq \mathcal{B}$. Furthermore, for every operator system $\mathcal{R}$, there are an injective unital C*-alegbra $\mathcal{B}$, a unital C*-subalgebra $\mathcal{A} \subseteq \mathcal{B}$, and a unital linear complete order embedding $\kappa : \mathcal{R} \to \mathcal{A}$ such that*

$$\mathrm{C}^*\left(\kappa(\mathcal{R})\right) = \mathcal{A} \simeq \mathrm{C}^*_{\rm e}(\mathcal{R}) \tag{6.1}$$

and

$$\mathcal{R} \simeq \kappa(\mathcal{R}) \subseteq \mathcal{A} \subseteq \mathcal{B} \simeq \mathcal{I}(\mathcal{R}), \tag{6.2}$$

where, if $\phi : \mathcal{B} \to \mathcal{I}(\mathcal{R})$ is a unital linear complete order isomorphism implementing the categorical isomorphism $\mathcal{B} \simeq \mathcal{I}(\mathcal{R})$, then $\iota_{\mathrm{ie}} = \phi \circ \kappa$.

Hamana's work on injective envelopes [27] shows that there is an operator subsystem $\mathcal{Q}$ of $\mathcal{I}(\mathcal{R})$ such that $\mathrm{C}^*_{\mathrm{e}}(\mathcal{R}) \simeq \mathcal{Q}$ and $\mathcal{R} \subseteq \mathcal{Q} \subseteq \mathcal{I}(\mathcal{R})$. If $\mathcal{R}$ and $\mathrm{C}^*_{\mathrm{e}}(\mathcal{R})$ have finite dimension, then, because finite-dimensional C*-algebras are injective operator systems, it must be that $\mathcal{Q} = \mathcal{I}(\mathcal{R})$. Hence, $\mathrm{C}^*_{\mathrm{e}}(\mathcal{R}) \simeq \mathcal{I}(\mathcal{R})$, for every operator system $\mathcal{R}$ in which both $\mathcal{R}$ and $\mathrm{C}^*_{\mathrm{e}}(\mathcal{R})$ have finite dimension. This fact makes it relatively easy to compute the injective envelopes of irreducible operator systems of matrices.

Proposition 6.3. *If $\mathcal{R}$ is an operator subsystem of $\mathcal{M}_n(\mathbb{C})$ such that $\mathcal{R} \neq \mathcal{M}_n(\mathbb{C})$ and $\mathcal{R}' \simeq \mathbb{C}$, then*

1. *there is no unital C*-algebra $\mathcal{A}$ for which $\mathcal{R} \simeq \mathcal{A}$,*
2. *$\mathcal{R}$ is not an injective operator system, and*
3. *the C*- and injective envelopes of $\mathcal{R}$ are $\mathcal{M}_n(\mathbb{C})$.*

Proof. If, on the contrary, $\mathcal{R} \simeq \mathcal{A}$ for unital C*-algebra $\mathcal{A}$, then $\mathcal{A}$ has finite dimension and is, therefore, isomorphic to a finite direct sum of type I factors. By Arveson's Extension Theorem [5], type I factors are injective; hence, $\mathcal{A}$ and, thus, $\mathcal{R}$ are injective. With $\mathcal{R}$ being injective, there must exist an idempotent ucp map $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ for which the range of Φ is $\mathcal{R}$. Hence, the fixed point set of Φ contains $\mathcal{R}$. Because $\mathcal{R}' \simeq \mathbb{C}$, the fixed point set of Φ is an irreducible operator subsystem of $\mathcal{M}_n(\mathbb{C})$. By Arveson's Boundary Theorem [6], Φ is necessarily the identity map, and so the range of Φ cannot be a proper subset of $\mathcal{M}_n(\mathbb{C})$. Hence, it cannot be that $\mathcal{R}$ is injective or that $\mathcal{R} \simeq \mathcal{A}$, proving (1) and (2).

To prove (3), recall that the C*-envelope $\mathrm{C}^*_{\mathrm{e}}(\mathcal{R})$ of $\mathcal{R}$ is a quotient of the C^*-algebra generated by $\mathcal{R}$. Because $\mathcal{R}$ is a matrix operator system, $\mathrm{C}^*(\mathcal{R}) = \mathcal{R}'' = \mathcal{M}_n(\mathbb{C})$, which is a simple C*-algebra. Hence, $\mathrm{C}^*_{\mathrm{e}}(\mathcal{R}) = \mathcal{M}_n(\mathbb{C})$ and, by the remarks preceding the statement of the Proposition, $\mathrm{C}^*_{\mathrm{e}}(\mathcal{R}) = \mathcal{I}(\mathcal{R})$. ☐

Corollary 6.4. *The conclusions of Proposition 6.3 apply to Toeplitz operator systems.*

Proof. For each $n \geq 2$, the commutant of $C(S^1)^{(n)}$ in $\mathcal{M}_n(\mathbb{C})$ is contained in the commutant of $\{r_1, r_{-1}\}$, which is simply $\{\alpha 1_n \mid \alpha \in \mathbb{C}\}$. ☐

For Fejér-Riesz operator systems, a result similar to Proposition 6.3 holds.

Proposition 6.5. *There is no unital C*-algebra $\mathcal{A}$ for which $C(S^1)_{(n)} \simeq \mathcal{A}$, if $n \geq 2$.*

Proof. If there were a unital C*-algebra $\mathcal{A}$ for which $C(S^1)_{(n)} \simeq \mathcal{A}$, then the finite-dimensionality of $\mathcal{A}$ would yield the injectivity of $C(S^1)_{(n)}$, implying that the C*-envelope and injective envelope of $C(S^1)_{(n)}$ were also $\mathcal{A}$. However, the C*-envelope of $C(S^1)_{(n)}$ is $C(S^1)$ [12, Proposition 4.3], which has infinite dimension. ☐

As noted in the proof of Proposition 6.5, the C*-envelopes of Fejér-Riesz operator systems have been determined by Connes and van Suijlekom [12, Proposition 4.3], where they proved $\mathrm{C}^*_{\mathrm{e}}\left(C(S^1)_{(n)}\right) = C(S^1)$, for every $n \geq 2$. The injective envelopes of Fejér-Riesz operator systems are also abelian C*-algebras, but with extremely disconnected spectra.

Lemma 6.6. *If $\Delta_{S^1} = \lim\limits_{\leftarrow} \beta X$, the topological inverse limit of the Stone-Čech compactifications βX, partially ordered by reverse inclusion, of dense open subsets $X \subseteq S^1$, then Δ_{S^1} is a compact, Hausdorff, and extremely disconnected topological space, and*

$$\mathcal{I}\left(C(S^1)_{(n)}\right) \simeq C(\Delta_{S^1}),$$

for every $n \geq 2$.

Proof. Fix $n \geq 2$, and consider the canonical embedding of the operator system $C(S^1)_{(n)}$ into its C*-envelope $C(S^1)$. As an operator system and its C*-envelope have the same injective envelope, it is enough to consider the injective envelope of $C(S^1)$.

Let $O_d(S^1) = \{X \subseteq S^1 \,|\, X \text{ is open and dense in } S^1\}$. For each $X \in O_d(S^1)$, the set $C_0(X)$ of continuous complex-valued functions on X that vanish at infinity is an essential ideal of the algebra $C(S^1)$, and $C(\beta\Omega)$ is the multiplier algebra of this essential ideal, where βX denotes the Stone-Čech compactification of X. Further, if $\iota_X : X \to \beta X$ denotes the continuous embedding of X as a dense subset of βX, then, because each $X \in O_d(S^1)$ is open and, hence, locally compact [14, Theorem XI.6.5], the embedding $\iota_X : X \to \beta X$ is an open map [14, Theorem VII.7.3]. Therefore, $\iota_X(X)$ is a dense open subset of βX. If $X, Z \in O_d(S^1)$ satisfy $X \subset Z$, then ι_Z embeds X into βZ as a dense subset. Thus, βZ is a compactification of X and so, by the Stone–Čech Theorem [14, Theorem 8.2], there is a unique continuous function $\Phi_{Z,X} : \beta X \to \beta Z$ for which $\Phi_{Z,X} \circ \iota_X = \iota_Z|_X$. Because $\iota_Z(X)$ is dense in βZ, $\Phi_{Z,X}$ is a surjection. Note that if $X \subset W \subset Z$, for $X, W, Z \in O_d(S^1)$, then $\Phi_{Z,X} = \Phi_{Z,W} \circ \Phi_{W,X}$. Hence, $(\{\beta X : X \in O_d(S^1)\}, \Phi_{Z,X})$ is an inverse spectrum over $O_d(S^1)$ endowed with the order of reversed inclusion. Thus, if $\Delta_{S^1} = \lim\limits_{\leftarrow} \beta X$, then, by [37],

$$C(\Delta_{S^1}) = C\left(\lim_{\leftarrow} \beta X\right) = \lim_{\rightarrow} C(\beta X).$$

Because the direct limit C*-algebra $\lim\limits_{\rightarrow} C(\beta X)$ is the local multiplier algebra [4] of $C(S^1)$, and because the local multiplier algebra of any unital C*-algebra $\mathcal{A}$ is an operator subsystem of the injective envelope of $\mathcal{A}$ [22, Theorem 4.5], we deduce that $C(\Delta_{S^1})$ is an operator subsystem of the injective envelope of $C(S^1)_{(n)}$ and contains $C(S^1)_{(n)}$ as an operator subsystem. However, as $C(\Delta_{S^1})$ is an abelian AW*-algebra [4, Proposition 3.1.5], $C(\Delta_{S^1})$ is injective, and so it must coincide with the injective envelope of $C(S^1)_{(n)}$. Because the maximal ideal space of an abelian AW*-algebra is compact, Hausdorff, and extremely disconnected, the proof is complete. □

6.2. **Tensor relations for Toeplitz and Fejér-Riesz operator systems.** It is known from previous work [16] that the minimal and maximal operator system tensor product structures result in distinct operator systems when the tensor factors are Toeplitz or Fejér-Riesz operator systems. The main results in this direction are stated below.

Theorem 6.7. *If $n, m \geq 2$, then*

$$\begin{aligned} C(S^1)^{(n)} \otimes_{\min} C(S^1)^{(m)} &\neq C(S^1)^{(n)} \otimes_{\max} C(S^1)^{(m)}, \\ C(S^1)_{(n)} \otimes_{\min} C(S^1)_{(m)} &\neq C(S^1)_{(n)} \otimes_{\max} C(S^1)_{(m)}, \text{ and} \\ C(S^1)^{(n)} \otimes_{\min} C(S^1)_{(m)} &\neq C(S^1)^{(n)} \otimes_{\max} C(S^1)_{(m)}. \end{aligned}$$

The remainder of this subsection is devoted to the issue of operator system tensor products when one tensor factor is a Toeplitz or Fejér-Riesz operator system and the other factor has some additional structure (e.g., injectivity) not exhibited by Toeplitz and Fejér-Riesz operator systems. The analysis requires the use of an important intermediate operator system tensor product structure called the *commuting operator system tensor product*.

Recall from basic linear algebra that if $\mathcal{B}$ is an algebra and if $\phi : \mathcal{R} \to \mathcal{B}$ and $\psi : \mathcal{T} \to \mathcal{B}$ are linear maps of vector spaces $\mathcal{R}$ and $\mathcal{T}$ with commuting ranges (i.e., $\phi(x)\psi(y) = \psi(y)\phi(x)$ for all $x \in \mathcal{R}$, $y \in \mathcal{T}$), then there exists a unique linear map $\mathcal{R} \otimes \mathcal{T} \to \mathcal{B}$ in which $x \otimes y \mapsto \phi(x)\psi(y)$. This linear map is denoted here by $\phi \cdot \psi$.

Definition 6.8. The *commuting operator system tensor product*, $\otimes_{\mathrm{c}}$, of operator systems $\mathcal{R}$ and $\mathcal{T}$ is the operator system structure on the algebraic tensor product $\mathcal{R} \otimes \mathcal{T}$ obtained by declaring a matrix $x \in \mathcal{M}_p(\mathcal{R} \otimes \mathcal{T})$ to be positive if $(\phi \cdot \psi)^{(p)}[x]$ is a positive element of $\mathcal{M}_p(\mathcal{B}(\mathcal{H}))$, for every Hilbert space $\mathcal{H}$ and every pair of ucp maps $\phi : \mathcal{R} \to \mathcal{B}(\mathcal{H})$ and $\psi : \mathcal{T} \to \mathcal{B}(\mathcal{H})$ with commuting ranges.

The following definition is motivated by the defining conditions for the maximally entangled state in quantum theory.

Definition 6.9. Suppose that $\{x_1, \dots, x_n\}$ is a linear basis for an operator system $\mathcal{R}$, and that $\{\varphi_1, \dots, \varphi_n\}$ is the associated dual basis for the operator system dual $\mathcal{R}^d$. The element

$$\xi = \sum_{j=1}^{n} x_j \otimes \varphi_j$$

is called the *maximally entangled positive element* of $\mathcal{R} \otimes_{\min} \mathcal{R}^d$.

The terminology above is due to Kavruk [30], who also proves the maximally entangled element ξ is positive and that its definition is independent of the choice of linear basis of $\mathcal{R}$. However, Kavruk does not actually prove ξ is entangled, and so this is explained in Proposition 6.10 below for Toeplitz and Fejér-Riesz operator systems.

In the case of the operator system $C(S^1)^{(n)} \otimes_{\min} C(S^1)_{(n)}$, the maximally entangled positive element is represented with respect to the canonical linear bases by

$$\xi_n = \sum_{\ell=-n+1}^{n-1} r_\ell \otimes \chi_{-\ell}.$$

Proposition 6.10. *ξ_n is entangled in $\left(C(S^1)^{(n)} \otimes_{\min} C(S^1)_{(n)}\right)_+$.*

Proof. The linear map $\phi_n : C(S^1)_{(n)} \to C(S^1)_{(n)}$ defined on the basis elements of $C(S^1)_{(n)}$ by $\phi_n(\chi_\ell) = \chi_{-\ell}$ has the property that

$$\left(\mathrm{Id}_{C(S^1)^{(n)}} \otimes \phi_n\right)[\xi_n] = T_n,$$

the universal positive Toeplitz matrix positive. The positivity of T_n implies the complete positivity of ϕ_n, by Proposition 3.2; therefore, if ξ_n were separable, then T_n would be too, contrary to the conclusions of Proposition 5.5. Hence, ξ_n must be entangled. □

Because the C*-envelopes of Toeplitz and Fejér-Riesz operator systems are nuclear C*-algebras, the following result is a useful tool for determining their categorical properties.

Theorem 6.11. *The following statements are equivalent for a finite-dimensional operator system $\mathcal{R}$ for which $\mathrm{C}^*_{\mathrm{e}}(\mathcal{R})$ is a nuclear C*-algebra:*

1. $\mathcal{R} \otimes_{\min} \mathcal{A} = \mathcal{R} \otimes_{\max} \mathcal{A}$*, for every unital C*-algebra* $\mathcal{A}$*;*
2. $\mathcal{R} \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty) = \mathcal{R} \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)$*;*
3. $\mathcal{R} \otimes_{\min} \mathcal{T} = \mathcal{R} \otimes_{\mathrm{c}} \mathcal{T}$*, for every operator system* $\mathcal{T}$*;*
4. *for every unital complete order embedding* $\kappa : \mathcal{R} \to \mathcal{B}(\mathcal{H})$*, there exists a ucp map* $\phi : \mathcal{B}(\mathcal{H}) \to \kappa(\mathcal{R})''$ *such that* $\phi \circ \kappa = \kappa$*;*
5. *for every unital complete order embedding* $\kappa : \mathcal{R} \to \mathcal{B}(\mathcal{H})$*, there exists a ucp map* $\phi : \mathcal{I}(\mathcal{R}) \to \kappa(\mathcal{R})''$ *such that* $\phi(x) = \kappa(x)$*, for every* $x \in \mathcal{R}$*;*
6. *the maximally entangled element* $\xi \in \mathcal{R} \otimes \mathcal{R}^d$ *is positive in* $\mathcal{R} \otimes_{\mathrm{c}} \mathcal{R}^d$*.*

Proof. Because $\mathrm{C}^*_{\mathrm{e}}(\mathcal{R})$ is a nuclear C*-algebra containing $\mathcal{R}$ as an operator subsystem, the operator system $\mathcal{R}$ is exact in the category $\mathfrak{S}_1$ [29, Proposition 4.10]. Hence, $\mathcal{R} \otimes_{\min} \mathcal{T} = \mathcal{R} \otimes_{\mathrm{el}} \mathcal{T}$, for every operator system $\mathcal{T}$ [32, Theorem 5.7], where $\otimes_{\mathrm{el}}$ is the operator system structure on $\mathcal{R} \otimes \mathcal{T}$ in which a matrix $X \in \mathcal{M}_p(\mathcal{R} \otimes \mathcal{T})$ is positive if X is positive in $\mathcal{M}_p(\mathcal{I}(\mathcal{R}) \otimes_{\max} \mathcal{T})$. By [32, Theorem 7.3], statements (4) and (5) are equivalent, and each of these is equivalent to the statement that $\mathcal{R} \otimes_{\mathrm{el}} \mathcal{T} = \mathcal{R} \otimes_{\mathrm{c}} \mathcal{T}$, for all operator systems $\mathcal{T}$. Hence, for operator systems in which $\mathrm{C}^*_{\mathrm{e}}(\mathcal{R})$ is nuclear, we have (4) or (5) if and only if $\mathcal{R} \otimes_{\min} \mathcal{T} = \mathcal{R} \otimes_{\mathrm{el}} \mathcal{T} = \mathcal{R} \otimes_{\mathrm{c}} \mathcal{T}$ for all operator systems $\mathcal{T}$, thereby showing the equivalence of (4), (5), and (3) for operator systems $\mathcal{R}$ in which $\mathrm{C}^*_{\mathrm{e}}(\mathcal{R})$ is nuclear. The equivalence of statements (3) and (1) is given by [29, Proposition 4.11], and the equivalence of (4) and (2) is given by [32, Theorem 7.6]. Finally, using for the first time the finite-dimensionality of $\mathcal{R}$, the equivalence of (1) and (6) is given by [30, Theorem A.1]. □

It is perhaps worth noting explicitly that, as shown by the proof of Theorem 6.11, statements (1) through (5) are equivalent whenever $\mathrm{C}^*_{\mathrm{e}}(\mathcal{R})$ is nuclear, regardless of whether $\mathcal{R}$ has finite dimension or not.

Proposition 6.12. *If $\mathcal{R}$ is a Toeplitz or Fejér-Riesz operator system, then $\mathcal{R}$ satisfies equivalent condition (6) of Theorem 6.11.*

Proof. Proposition 5.5 shows $T_n \in \left(C(S^1)^{(n)} \otimes_{\max} C(S^1)_{(n)}\right)_+$, while Proposition 6.10 shows ξ_n is the image of T_n under a completely positive linear map of the form $\mathrm{Id}_{C(S^1)^{(n)}} \otimes \phi_n$, for some ucp map $\phi_n : C(S^1)_{(n)} \to C(S^1)_{(n)}$. Therefore, by Proposition 2.4, ξ_n is also an element of $\left(C(S^1)^{(n)} \otimes_{\max} C(S^1)_{(n)}\right)_+$ and, hence, $\left(C(S^1)^{(n)} \otimes_{\mathrm{c}} C(S^1)_{(n)}\right)_+$. The proof is complete by the duality relationship between $C(S^1)^{(n)}$ and $C(S^1)_{(n)}$. □

A number of conclusions follow from Theorem 6.11 and Proposition 6.12, two of which are presented below.

Corollary 6.13. *Toeplitz and Fejér-Riesz operator systems are C*-nuclear.*

The next assertion might also be thought of as a nuclearity-type result (e.g., "injectively nuclear").

Corollary 6.14. *If $\mathcal{R}$ is a Toeplitz or Fejér-Riesz operator system, then*

$$\mathcal{R} \otimes_{\max} \mathcal{I} = \mathcal{R} \otimes_{\min} \mathcal{I},$$

for every injective operator system $\mathcal{I}$.

Proof. An operator system $\mathcal{R}$ for which $\mathcal{R} \otimes_{\min} \mathcal{B}(\mathcal{H}) = \mathcal{R} \otimes_{\max} \mathcal{B}(\mathcal{H})$, for every Hilbert space $\mathcal{H}$, has the property, for every unital C*-algebra $\mathcal{B}$, that a matrix $x \in \mathcal{M}_p(\mathcal{R} \otimes_{\min} \mathcal{B})$ is positive if and only if x is positive in $\mathcal{M}_p(\mathcal{R} \otimes_{\max} \mathcal{I}(\mathcal{B}))$ [32, Theorem 8.1], where $\mathcal{I}(\mathcal{B})$ is the injective envelope of $\mathcal{B}$. Thus, if $\mathcal{I}$ is an injective operator system, then it coincides with its injective envelope and a matrix $X \in \mathcal{M}_p(\mathcal{R} \otimes_{\min} \mathcal{I})$ is positive if and only if x is a positive matrix in $\mathcal{M}_p(\mathcal{R} \otimes_{\max} \mathcal{I})$, which is to say that $\mathcal{R} \otimes_{\min} \mathcal{T} = \mathcal{R} \otimes_{\max} \mathcal{I}$. Because $C(S^1)^{(n)}$ and $C(S^1)_{(n)}$ satisfy the tensorial property with $\mathcal{B}(\mathcal{H})$, we conclude that $C(S^1)^{(n)}$ and $C(S^1)_{(n)}$ satisfy the same property when tensored with any injective operator system. □

6.3. **The weak expectation property.** An operator system $\mathcal{R}$ has the weak expectation property if the canonical embedding of $\mathcal{R}$ into its bidual $\mathcal{R}^{dd}$ has a ucp extension $\mathcal{I}(\mathcal{R}) \to \mathcal{R}^{dd}$. The weak expectation property is motivated by operator algebra theory, where the property has an important role in a number of C*-algebraic problems. However, in the operator system category $\mathfrak{S}_1$, it seems rather difficult for an operator system that is not already a C*-algebra to possess this property. The results in this subsection confirm neither Toeplitz or Fejér-Riesz operator systems possess this property.

Proposition 6.15. *If $\mathcal{R}$ is an operator subsystem of $\mathcal{M}_n(\mathbb{C})$ such that $\mathcal{R} \neq \mathcal{M}_n(\mathbb{C})$ and $\mathcal{R}' \simeq \mathbb{C}$, then $\mathcal{R}$ does not have the weak expectation property.*

Proof. Under the stated hypothesis, $\mathcal{R}^{dd} = \mathcal{R}$ and $\mathcal{I}(\mathcal{R}) = \mathcal{M}_n(\mathbb{C})$, with the canonical embedding of $\mathcal{R}$ into $\mathcal{R}^{dd}$ being the identity map on $\mathcal{R}$. Assume a ucp extension $\phi : \mathcal{I}(\mathcal{R}) \to \mathcal{R}^{dd}$ of the identity on $\mathcal{R}$ exists; then, by considering $\mathcal{R}^{dd}$ as an operator subsystem of $\mathcal{M}_n(\mathbb{C})$, ϕ is a ucp $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ that fixes $\mathcal{R}$. Thus, by Arveson's Boundary Theorem, ϕ is the identity map. However, if ϕ is the identity map, then its range is $\mathcal{M}_n(\mathbb{C})$ rather than $\mathcal{R}$, which contradicts the assumption that ϕ maps onto $\mathcal{R}^{dd} = \mathcal{R}$. □

Corollary 6.16. *For every $n \geq 2$, the operator system $C(S^1)^{(n)}$ does not the weak expectation property.*

Corollary 6.17. *If $n \geq 2$ and $\mathcal{E} : \mathcal{M}_n(\mathbb{C}) \to C(S^1)^{(n)}$ is an idempotent linear transformation, then $\mathcal{E}$ is not completely positive.*

Proof. If $\mathcal{E}$ were completely positive, then $\mathcal{E}$ would be a completely positive extension of the canonical embedding (namely, the identity) of $C(S^1)^{(n)}$ into its double dual, $C(S^1)^{(n)}$, to the injective envelope, $\mathcal{M}_n(\mathbb{C})$, of $C(S^1)^{(n)}$. However, this would imply $C(S^1)^{(n)}$ has the weak expectation property, contrary to Proposition 6.15. □

To show $C(S^1)_{(n)}$ does not have the weak expectation property, the following lemma is required.

Lemma 6.18. *If a finite-dimensional operator system $\mathcal{R}$ has the weak expectation property, then $\mathcal{R} \otimes_{\max} \mathcal{T} \subseteq_{\text{coi}} \mathcal{I}(\mathcal{R}) \otimes_{\max} \mathcal{T}$, for every operator system $\mathcal{T}$.*

Proof. The finite-dimensionality of $\mathcal{R}$ implies that $\mathcal{R}^{dd} \simeq \mathcal{R}$, and so the weak expectation property of $\mathcal{R}$ asserts, assuming the canonical inclusion of $\mathcal{R}$ as an operator subsystem of $\mathcal{I}(\mathcal{R})$, that there is a ucp map $\phi : \mathcal{I}(\mathcal{R}) \to \mathcal{R}$ such that $\phi(x) = x$, for every $x \in \mathcal{R}$.

Select $p \in \mathbb{N}$ and a $X \in \mathcal{M}_p(\mathcal{R} \otimes \mathcal{T})$ such that $X \in \mathcal{M}_p(\mathcal{I}(\mathcal{R}) \otimes_{\max} \mathcal{T})_+$. Thus,

$$\left(\phi^{(p)} \otimes \mathrm{id}_{\mathcal{T}}\right)[X] \in \left(\mathcal{M}_p(\mathcal{R}) \otimes_{\max} \mathcal{T}\right)_+ = \mathcal{M}_p(\mathcal{R} \otimes_{\max} \mathcal{T})_+.$$

Hence, X is a positive matrix of $\mathcal{M}_p(\mathcal{R} \otimes_{\max} \mathcal{T})$. □

Proposition 6.19. *The operator system $C(S^1)_{(n)}$ does not have the weak expectation property, for every $n \geq 2$.*

Proof. Suppose, on the contrary, that $C(S^1)_{(n)}$ has the weak expectation property, for some $n \geq 2$. In Lemma 6.18, let $\mathcal{R} = C(S^1)_{(n)}$ and $\mathcal{T} = C(S^1)^{(n)}$. Thus, using Lemmas 6.6 and 6.18,

$$C(S^1)_{(n)} \otimes_{\max} C(S^1)^{(n)} \subseteq_{\mathrm{coi}} C(\Delta_{S^1}) \otimes_{\max} C(S^1)^{(n)} = C(\Delta_{S^1}) \otimes_{\min} C(S^1)^{(n)}.$$

Therefore, if $p \in \mathbb{N}$ and $X \in \mathcal{M}_p(C(S^1)_{(n)} \otimes C(S^1)^{(n)})$ is such that

$$X \in \mathcal{M}_p(C(\Delta_{S^1}) \otimes_{\min} C(S^1)^{(n)})_+,$$

then $X \in \mathcal{M}_p(C(S^1)_{(n)} \otimes_{\min} C(S^1)^{(n)})$, as $\mathcal{R}_1 \otimes_{\min} \mathcal{Q} \subseteq_{\mathrm{coi}} \mathcal{R}_2 \otimes_{\min} \mathcal{Q}$ for all operator systems $\mathcal{R}_2$ and $\mathcal{Q}$, and all operator subsystems $\mathcal{R}_1 \subseteq \mathcal{R}_2$. Hence,

$$C(S^1)_{(n)} \otimes_{\min} C(S^1)^{(n)} = C(S^1)_{(n)} \otimes_{\max} C(S^1)^{(n)},$$

in contradiction to Theorem 6.7. □

6.4. **Quotient operator systems and purity of truncation.** Drawing on the work in [32], if $\phi : \mathcal{R} \to \mathcal{T}$ is a completely positive linear map of operator systems, and if $\mathcal{J}$ denotes $\ker \phi$, then the quotient vector space $\mathcal{R}/\mathcal{J}$ is an operator system in which $\mathcal{M}_n(\mathcal{R}/\mathcal{J})_+$ is defined to be the set of all (cosets) $\dot{h} \in \mathcal{M}_n(\mathcal{R}/\mathcal{J})$ with the property that, for each $\varepsilon > 0$, there is a selfadjoint matrix $k_\varepsilon \in \mathcal{M}_n(\mathcal{J})$ such that $\varepsilon e_{\mathcal{R}} + h + k_\varepsilon \in \mathcal{M}_n(\mathcal{R})_+$. (This definition makes use of the canonical identification of $\mathcal{M}_n(\mathcal{R}/\mathcal{J})$ with $\mathcal{M}_n(\mathcal{R})/\mathcal{M}_n(\mathcal{J})$.) Furthermore, the First Isomorphism Theorem asserts there is a completely positive linear map $\dot{\phi} : \mathcal{R}/\mathcal{J} \to \mathcal{T}$ such that $\phi = \dot{\phi} \circ \pi_{\mathcal{J}}$, where $\pi_{\mathcal{J}} : \mathcal{R} \to \mathcal{T}/\mathcal{J}$ is the canonical projection onto the quotient vector space. The completely positive linear map ϕ is said to be a *complete quotient map* if $\dot{\phi}$ is a complete order isomorphism.

Proposition 6.20. *If $n, m \in \mathbb{N}$ satisfy $n \leq m$, then the canonical inclusion map*

$$\iota_{n,m} : C(S^1)_{(n)} \to C(S^1)_{(m)}$$

is a unital complete order embedding.

Proof. It is sufficient to proof that $\iota_{n,m}$ is a complete isometry, as a ucp map is a complete isometry if and only if it is a complete order embedding. To this end, select $p \in \mathbb{N}$ and consider

$$\iota_{n,m}^{(p)} : \mathcal{M}_p\left(C(S^1)_{(n)}\right) \to \mathcal{M}_p\left(C(S^1)_{(m)}\right).$$

Regardless of whether $X = [F_{ij}]_{i,j=1}^n$ is considered as an element of $\mathcal{M}_p\left(C(S^1)_{(n)}\right)$ or of $\mathcal{M}_p\left(C(S^1)_{(m)}\right)$, the norm of X in both operator systems is given by

$$\max_{z \in S^1} \left\| [F_{ij}(z)]_{i,j=1}^n \right\|,$$

which implies $\|\iota_{n,m}^{(p)}(X)\| = \|X\|$. □

Corollary 6.21. *If $n, m \in \mathbb{N}$ satisfy $n \leq m$, and if*

$$\amalg_{m,n} : C(S^1)^{(m)} \to C(S^1)^{(n)}$$

is the canonical projection of each matrix in $C(S^1)^{(m)}$ onto its leading $n \times n$ principal submatrix, then $\amalg_{m,n} = \iota_{n,m}^d$, $\amalg_{m,n}$ is a complete quotient map, and

$$C(S^1)^{(n)} \simeq C(S^1)^{(m)} / \ker \amalg_{m,n}$$

in the operator system category $\mathfrak{S}_1$.

Proof. By Proposition 6.20, $\iota_{n,m} : C(S^1)_{(n)} \to C(S^1)_{(m)}$ is a unital complete order embedding; hence, the dual map $\iota_{n,m}^d : (C(S^1)_{(m)})^d \to (C(S^1)_{(n)})^d$ is a complete quotient map [19, Proposition 1.15]. Thus, in the category $\mathfrak{S}_1$,

$$(C(S^1)_{(n)})^d \simeq (C(S^1)_{(m)})^d / \ker \iota_{n,m}^d.$$

The Duality Theorem (Theorem 1.5) identifies each $(C(S^1)_q)^d$ with $C(S^1)^q$; hence, we need only show the identification of $\amalg_{m,n}$ and $\iota_{n,m}^d$. To this end, first note that if φ is a linear functional on $C(S^1)_{(m)}$, then it is also a linear functional on $C(S^1)_{(n)}$ and the adjoint $\iota_{n,m}^d$ of the inclusion map $\iota_{n,m}$ simply sends φ to the functional on $C(S^1)_{(n)}$ whose action on $f \in C(S^1)_{(n)}$ is given by $\varphi(f)$. The Duality Theorem (Theorem 1.5) identifies a linear functional φ on $C(S^1)_{(m)}$ with the Toeplitz matrix $t_\varphi = [\tau_{k-j}]$, where $\tau_\ell = \varphi(\chi_{-\ell})$, for $\ell = -m+1, \dots, m-1$, and identifies the linear functional $\iota_{n,m}^d(\varphi)$ on $C(S^1)_{(n)}$ with the Toeplitz matrix $t_{\iota_{n,m}^d(\varphi)} = [\tau_{k-j}]$, where $\tau_\ell = \varphi(\chi_{-\ell})$, for $\ell = -n+1, \dots, n-1$. Hence, after the identification of $(C(S^1)_q)^d$ with $C(S^1)^q$ via the Duality Theorem, the adjoint map $\iota_{n,m}^d$ is given by $\amalg_{m,n}$. □

Corollary 6.22. *The ucp map*

$$\amalg_{m,n} \otimes \amalg_{m,n} : C(S^1)^{(m)} \otimes_{\max} C(S^1)^{(m)} \to C(S^1)^{(n)} \otimes_{\max} C(S^1)^{(n)}$$

is a complete quotient map.

Proof. If a ucp map $\phi : \mathcal{R} \to \mathcal{T}$ is a complete quotient map, then so is the ucp map $\phi \otimes \phi : \mathcal{R} \otimes_{\max} \mathcal{R} \to \mathcal{T} \otimes_{\max} \mathcal{T}$ [19, Corollary 1.13]. □

Observe that Corollary 6.22 above is a stronger assertion than Theorem 4.11 in that the latter result is in reference to the base cone, while the former is in reference to all matrix cones.

Lastly, we note the embeddings of Fejér-Riesz operator systems and truncations of Toeplitz operator systems are extremal completely positive linear maps. Recall that a completely positive linear map $\phi : \mathcal{R} \to \mathcal{T}$ of operator systems is *pure*, if the only completely positive linear maps $\vartheta, \psi : \mathcal{R} \to \mathcal{T}$ that satisfy $\vartheta + \psi = \phi$ are those in which ϑ and ψ are of the form $\vartheta = s\phi$ and $\psi = (1-s)\phi$ for some real number $s \in [0, 1]$.

Proposition 6.23. *The ucp maps $\iota_{n,m}$ and $\amalg_{m,n}$ are pure, for all $2 \leq n \leq m$.*

Proof. Because $\amalg_{m,n} = \iota_{n,m}^d$, it is enough to prove that $\amalg_{m,n}$ is pure [20, Proposition 2.6]. To this end, suppose $\vartheta, \psi : C(S^1)^{(m)} \to C(S^1)^{(n)}$ are completely positive linear maps such that $\vartheta + \psi = \amalg_{m,n}$. For each $\lambda \in S^1$, $T_n(\lambda) = \amalg_{m,n}(T_m(\lambda))$ and, thus, $T_n(\lambda) = \vartheta(T_m(\lambda)) + \psi(T_m(\lambda))$. Because $T_n(\lambda)$ is a pure element of $C(S^1)^{(n)}$ for every $\lambda \in S^1$ [12, Proposition 4.8], there are scalars $s_\lambda \in [0, 1]$ such that $\vartheta(T_m(\lambda)) = s_\lambda T_n(\lambda)$ and $\psi(T_m(\lambda)) = (1-s_\lambda)T_n(\lambda)$. We first show that the value of s_λ is independent of λ.

Let $a = \vartheta(2\cdot 1_m)$ and let α denote the diagonal entry of a and β denote the subdiagonal entry of a. For each $\lambda \in S^1$, the matrix $T_m(\lambda)+T_m(-\lambda)$ is twice the identity, and so

$$a = \vartheta(T_m(\lambda)+T_m(-\lambda)) = \vartheta(T_m(\lambda)) + \vartheta(T_m(\lambda)) = s_\lambda T_n(\lambda) + s_{-\lambda}T_n(-\lambda).$$

Equating the (1,1) and (2,1) matrix entries of a and $s_\lambda T_n(\lambda)+s_{-\lambda}T_n(-\lambda)$ leads to the linear system of equations given by

$$\begin{array}{rcl} \alpha & = & s_\lambda + s_{-\lambda} \\ \frac{\beta}{\lambda} & = & s_\lambda - s_{-\lambda}. \end{array}$$

Thus, $\alpha + \frac{\beta}{\lambda} = 2s_\lambda$, which implies that $\frac{\beta}{\lambda} \in \mathbb{R}$ for every $\lambda \in S^1$. Hence, β must be zero, which implies $s_\lambda = \frac{\alpha}{2}$, for every $\lambda \in S^1$. Thus, if $s = \frac{\alpha}{2}$, then $\vartheta(T_m(\lambda)) = sT_n(\lambda)$ for every $\lambda \in S^1$.

Now let x be an arbitrary element of the positive cone of $C(S^1)^{(m)}$. As x is a sum of pure elements, there exist $k \in \mathbb{N}$, scalars $\alpha_j \in \mathbb{R}_+$, and $\lambda_j \in S^1$ such that $x = \sum_{j=1}^{k} \alpha_j T_m(\lambda_j)$. Thus,

$$\amalg_{m,n}(x) = \sum_{j=1}^{k} \alpha_j T_n(\lambda_j) \text{ and } \vartheta(x) = \sum_{j=1}^{k} \alpha_j s T_n(\lambda_j) = s\amalg_{m,n}(x).$$

Because ϑ and $s\amalg_{m,n}$ agree on the positive cone of $C(S^1)^{(m)}$, which spans $C(S^1)^{(m)}$, $\vartheta = s\amalg_{m,n}$ and $\psi = (1-s)\amalg_{m,n}$, which proves that $\amalg_{m,n}$ is pure. $\square$

6.5. **Completely positive linear maps with commuting ranges.** Another interesting consequence of Theorem 6.11 and Proposition 6.12 is the following proposition.

Proposition 6.24. *If $\phi : C(S^1)^{(n)} \to \mathcal{B}(\mathcal{H})$ and $\psi : C(S^1)_{(n)} \to \mathcal{B}(\mathcal{H})$ are unital completely positive linear maps with commuting ranges, then*

$$\sum_{\ell=-n+1}^{n-1} \phi(r_{-\ell})\psi(\chi_\ell) \;\; \textit{and} \;\; \sum_{\ell=-n+1}^{n-1} \phi(r_\ell)\psi(\chi_\ell)$$

are positive operators.

Proof. The two operators in question have the form $(\phi\cdot\psi)[\xi_n]$ and $(\phi\cdot\psi)[T_n]$. Because both ξ_n and T_n are elements of the positive cone of $C(S^1)^{(n)} \otimes_{\rm c} C(S^1)_{(n)}$, the operators $(\phi\cdot\psi)[\xi_n]$ and $(\phi\cdot\psi)][T_n]$ are, by definition, positive. $\square$

It would be interesting to have a direct proof of Proposition 6.24 above, because, in general, determining whether an element $x \in \mathcal{R}\otimes\mathcal{T}$ belongs to $(\mathcal{R}\otimes_{\rm c}\mathcal{T})_+$ can be surprisingly deep, as the following example shows.

Consider the operator system

$$\mathfrak{X}_2 = \left\{ \begin{bmatrix} \alpha & \beta_1 \\ \gamma_1 & \alpha \end{bmatrix} \oplus \begin{bmatrix} \alpha & \beta_2 \\ \gamma_2 & \alpha \end{bmatrix} \mid \alpha, \beta_1, \gamma_1, \beta_2, \gamma_2 \in \mathbb{C} \right\},$$

and let $\mathcal{S}_2 \subset \mathrm{C}^*(\mathbb{F}_2)$ be operator system generated by generators u_1 and u_2 of the free group $\mathbb{F}_2$. Consider

$$\xi_2 = \begin{bmatrix} 1 & u_1 \\ u_1^* & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & u_2 \\ u_2^* & 1 \end{bmatrix} \in \mathfrak{X}_2 \otimes \mathcal{S}_2.$$

Proposition 6.25. *The element ξ_2 is positive in $\mathfrak{X}_2 \otimes_{\min} \mathcal{S}_2$. However, there exist a Hilbert space $\mathcal{H}$ and ucp maps $\phi : \mathfrak{X}_2 \to \mathcal{B}(\mathcal{H})$ and $\psi : \mathcal{S}_2 \to \mathcal{B}(\mathcal{H})$ with commuting ranges such that the operator $\phi \cdot \psi(\xi_2) \in \mathcal{B}(\mathcal{H})$ is not positive. In other words, $\xi_2 \notin (\mathfrak{X}_2 \otimes_{\mathrm{c}} \mathcal{S}_2)_+$.*

Proof. The element ξ_2 is positive in $\mathfrak{X}_2 \otimes_{\min} \mathcal{S}_2$ because $\begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix}$ is positive for every element x in which $\|x\| \leq 1$. Assume, to the contrary, that $\phi \cdot \psi(\xi_2) \in \mathcal{B}(\mathcal{H})$ is positive, for all Hilbert spaces $\mathcal{H}$ and ucp maps $\phi : \mathfrak{X}_2 \to \mathcal{B}(\mathcal{H})$ and $\psi : \mathcal{S}_2 \to \mathcal{B}(\mathcal{H})$ with commuting ranges. Because $\mathfrak{X}_2 \simeq \mathcal{S}_2^d$ [19], we have $\mathfrak{X}_2 \otimes_\sigma \mathcal{S}_2 \simeq \mathcal{S}_2^d \otimes_\sigma \mathcal{S}_2$, for every operator system tensor product $\otimes_\sigma$. As shown in [20, Proposition 5.4], ξ_2 can be identified with the maximally entangled element of $\mathcal{S}_2^d \otimes_{\min} \mathcal{S}_2$. Therefore, our hypothesis leads to $\xi_2 \in (\mathcal{S}_2^d \otimes_{\mathrm{c}} \mathcal{S}_2)_+$, which in turn implies that $\mathcal{S}_2 \otimes_{\min} \mathcal{A} = \mathcal{S}_2 \otimes_{\max} \mathcal{A}$ for every unital C*-algebra $\mathcal{A}$ [30, Appendix A]. Hence, by [29, Theorem 4.11],

$$\mathcal{S}_2 \otimes_{\min} \mathcal{S}_2 = \mathcal{S}_2 \otimes_{\mathrm{c}} \mathcal{S}_2.$$

However, the equality above is true if and only if the Connes Embedding Problem has an affirmative solution [29, Theorem 0.2]. As the Connes Embedding Problem fails to have an affirmative solution [24, 28], our original assumption cannot be true, which proves the assertion. □

6.6. **A Hahn-Banach completely positive extension theorem for Toeplitz matrices.** The following theorem was established by Haagerup [26, Remark 3.7] for the case $n = 2$ using direct, operator-theoretic arguments. Taking advantage of the categorical relationships established in Theorem 6.11, one can show Haagerup's result holds for every $n \geq 2$.

Theorem 6.26. *If $\mathcal{N}$ is a von Neumann algebra and $\phi : C(S^1)^{(n)} \to \mathcal{N}$ is a unital positive linear map, then there is a completely positive linear map $\tilde{\phi} : \mathcal{M}_n(\mathbb{C}) \to \mathcal{N}$ such that $\tilde{\phi}(x) = \phi(x)$, for every $x \in C(S^1)^{(n)}$.*

Proof. Let $\mathcal{H}$ denote the Hilbert space on which $\mathcal{N}$ acts, and let $\mathcal{K} = \mathbb{C}^n \oplus \mathcal{H}$. Because positive linear maps of Toeplitz matrices are completely positive, the map ϕ is ucp and completely contractive; thus, the linear map $\kappa : C(S^1)^{(n)} \to \mathcal{B}(\mathcal{K})$ defined by $\kappa(x) = x \oplus \phi(x)$ is a unital complete isometry. Hence, κ is a unital complete order embedding of $C(S^1)^{(n)}$ into $\mathcal{B}(\mathcal{K})$.

Note that the double commutant of the range of κ in $\mathcal{B}(\mathcal{K})$ is contained within the von Neumann algebra $\tilde{\mathcal{N}} = \mathcal{M}_n(\mathbb{C}) \oplus \mathcal{N}$. Using the fact that $\mathcal{M}_n(\mathbb{C})$ is the injective envelope of $C(S^1)^{(n)}$, equivalent condition (5) of Theorem 6.11 asserts there is a ucp map $\tilde{\kappa} : \mathcal{M}_n(\mathbb{C}) \to \tilde{M}$ such that $\tilde{\kappa}(x) = \kappa(x)$, for every $x \in C(S^1)^{(n)}$. If $\mathcal{P} : \tilde{\mathcal{N}} \to \mathcal{N}$ is the projection onto the second direct summand of $\tilde{\mathcal{N}}$, then the linear map $\tilde{\phi} = \mathcal{P} \circ \tilde{\kappa}$ is a ucp extension of ϕ from $C(S^1)^{(n)}$ to $\mathcal{M}_n(\mathbb{C})$. □

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