# Solutions Manual

*A First Course in*

# PROBABILITY

Seventh Edition

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# Chapter 1

## Problems

1. (a) By the generalized basic principle of counting there are

$$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 67{,}600{,}000$$

(b) $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 19{,}656{,}000$

2. $6^4 = 1296$

3. An assignment is a sequence $i_1, \ldots, i_{20}$ where $i_j$ is the job to which person $j$ is assigned. Since only one person can be assigned to a job, it follows that the sequence is a permutation of the numbers 1, …, 20 and so there are 20! different possible assignments.

4. There are 4! possible arrangements. By assigning instruments to Jay, Jack, John and Jim, in that order, we see by the generalized basic principle that there are $2 \cdot 1 \cdot 2 \cdot 1 = 4$ possibilities.

5. There were $8 \cdot 2 \cdot 9 = 144$ possible codes. There were $1 \cdot 2 \cdot 9 = 18$ that started with a 4.

6. Each kitten can be identified by a code number i, j, k, l where each of i, j, k, l is any of the numbers from 1 to 7. The number $i$ represents which wife is carrying the kitten, $j$ then represents which of that wife's 7 sacks contain the kitten; $k$ represents which of the 7 cats in sack $j$ of wife $i$ is the mother of the kitten; and $l$ represents the number of the kitten of cat $k$ in sack $j$ of wife $i$. By the generalized principle there are thus $7 \cdot 7 \cdot 7 \cdot 7 = 2401$ kittens

7. (a) $6! = 720$
(b) $2 \cdot 3! \cdot 3! = 72$
(c) $4!3! = 144$
(d) $6 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 72$

8. (a) $5! = 120$
(b) $\dfrac{7!}{2!2!} = 1260$
(c) $\dfrac{11!}{4!4!2!} = 34{,}650$
(d) $\dfrac{7!}{2!2!} = 1260$

9. $\dfrac{(12)!}{6!4!} = 27{,}720$

10. (a) $8! = 40{,}320$
(b) $2 \cdot 7! = 10{,}080$
(c) $5!4! = 2{,}880$
(d) $4!2^4 = 384$

11. (a) 6!
(b) 3!2!3!
(c) 3!4!

12. (a) $30^5$
(b) $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26$

13. $\binom{20}{2}$

14. $\binom{52}{5}$

15. There are $\binom{10}{5}\binom{12}{5}$ possible choices of the 5 men and 5 women. They can then be paired up in 5! ways, since if we arbitrarily order the men then the first man can be paired with any of the 5 women, the next with any of the remaining 4, and so on. Hence, there are $5!\binom{10}{5}\binom{12}{5}$ possible results.

16. (a) $\binom{6}{2}+\binom{7}{2}+\binom{4}{2}$ = 42 possibilities.
(b) There are $6 \cdot 7$ choices of a math and a science book, $6 \cdot 4$ choices of a math and an economics book, and $7 \cdot 4$ choices of a science and an economics book. Hence, there are 94 possible choices.

17. The first gift can go to any of the 10 children, the second to any of the remaining 9 children, and so on. Hence, there are $10 \cdot 9 \cdot 8 \cdots 5 \cdot 4 = 604{,}800$ possibilities.

18. $\binom{5}{2}\binom{6}{2}\binom{4}{3} = 600$

19. (a) There are $\binom{8}{3}\binom{4}{3}+\binom{8}{3}\binom{2}{1}\binom{4}{2}$ = 896 possible committees.

There are $\binom{8}{3}\binom{4}{3}$ that do not contain either of the 2 men, and there are $\binom{8}{3}\binom{2}{1}\binom{4}{2}$ that contain exactly 1 of them.

(b) There are $\binom{6}{3}\binom{6}{3}+\binom{2}{1}\binom{6}{2}\binom{6}{3}$ = 1000 possible committees.

(c) There are $\binom{7}{3}\binom{5}{3}+\binom{7}{2}\binom{5}{3}+\binom{7}{3}\binom{5}{2}$ = 910 possible committees. There are $\binom{7}{3}\binom{5}{3}$ in which neither feuding party serves; $\binom{7}{2}\binom{5}{3}$ in which the feuding women serves; and $\binom{7}{3}\binom{5}{2}$ in which the feuding man serves.

20. $\binom{6}{5}+\binom{2}{1}\binom{6}{4}, \binom{6}{5}+\binom{6}{3}$

21. $\dfrac{7!}{3!4!} = 35$. Each path is a linear arrangement of 4 $r$'s and 3 $u$'s ($r$ for right and $u$ for up). For instance the arrangement $r, r, u, u, r, r, u$ specifies the path whose first 2 steps are to the right, next 2 steps are up, next 2 are to the right, and final step is up.

22. There are $\dfrac{4!}{2!2!}$ paths from A to the circled point; and $\dfrac{3!}{2!1!}$ paths from the circled point to B. Thus, by the basic principle, there are 18 different paths from A to B that go through the circled piont.

23. $3!2^3$

25. $\binom{52}{13, 13, 13, 13}$

27. $\binom{12}{3, 4, 5} = \dfrac{12!}{3!4!5!}$

28. Assuming teachers are distinct.
(a) $4^8$
(b) $\binom{8}{2,2,2,2} = \dfrac{8!}{(2)^4} = 2520.$

29. (a) (10)!/3!4!2!

(b) $3\binom{3}{2}\dfrac{7!}{4!2!}$

30. $2 \cdot 9! - 2^2 8!$ since $2 \cdot 9!$ is the number in which the French and English are next to each other and $2^2 8!$ the number in which the French and English are next to each other and the U.S. and Russian are next to each other.

31. (a) number of nonnegative integer solutions of $x_1 + x_2 + x_3 + x_4 = 8$.

Hence, answer is $\binom{11}{3} = 165$

(b) here it is the number of positive solutions—hence answer is $\binom{7}{3} = 35$

32. (a) number of nonnegative solutions of $x_1 + \ldots + x_6 = 8$

answer = $\binom{13}{5}$

(b) (number of solutions of $x_1 + \ldots + x_6 = 5$) × (number of solutions of $x_1 + \ldots + x_6 = 3$) =
$\binom{10}{5}\binom{8}{5}$

33. (a) $x_1 + x_2 + x_3 + x_4 = 20$, $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$

Let $y_1 = x_1 - 1$, $y_2 = x_2 - 1$, $y_3 = x_3 - 2$, $y_4 = x_4 - 3$

$$y_1 + y_2 + y_3 + y_4 = 13, y_i > 0$$

Hence, there are $\binom{12}{3} = 220$ possible strategies.

(b) there are $\binom{15}{2}$ investments only in 1, 2, 3

there are $\binom{14}{2}$ investments only in 1, 2, 4

there are $\binom{13}{2}$ investments only in 1, 3, 4

there are $\binom{13}{2}$ investments only in 2, 3, 4

$\binom{15}{2} + \binom{14}{2} + 2\binom{13}{2} + \binom{12}{3} = 552$ possibilities

## Theoretical Exercises

2. $\sum_{i=1}^{m} n_i$

3. $n(n-1)\cdots(n-r+1) = n!/(n-r)!$

4. Each arrangement is determined by the choice of the $r$ positions where the black balls are situated.

5. There are $\binom{n}{j}$ different $0-1$ vectors whose sum is $j$, since any such vector can be characterized by a selection of $j$ of the $n$ indices whose values are then set equal to 1. Hence there are $\sum_{j=k}^{n}\binom{n}{j}$ vectors that meet the criterion.

6. $\binom{n}{k}$

7.
$$\binom{n-1}{r}+\binom{n-1}{r-1} = \frac{(n-1)!}{r!(n-1-r)!}+\frac{(n-1)!}{(n-r)!(r-1)!}$$
$$= \frac{n!}{r!(n-r)!}\left[\frac{n-r}{n}+\frac{r}{n}\right]=\binom{n}{r}$$

8. There are $\binom{n+m}{r}$ gropus of size $r$. As there are $\binom{n}{i}\binom{m}{r-i}$ groups of size $r$ that consist of $i$ men and $r-i$ women, we see that

$$\binom{n+m}{r}=\sum_{i=0}^{r}\binom{n}{i}\binom{m}{r-i}.$$

9.
$$\binom{2n}{n}=\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n}\binom{n}{i}^2$$

10. Parts (a), (b), (c), and (d) are immediate. For part (e), we have the following:

$$k\binom{n}{k} = \frac{k!n!}{(n-k)!k!}=\frac{n!}{(n-k)!(k-1)!}$$
$$(n-k+1)\binom{n}{k-1} = \frac{(n-k+1)n!}{(n-k+1)!(k-1)!}=\frac{n!}{(n-k)!(k-1)!}$$
$$n\binom{n-1}{k-1} = \frac{n(n-1)!}{(n-k)!(k-1)!}=\frac{n!}{(n-k)!(k-1)!}$$

11. The number of subsets of size $k$ that have $i$ as their highest numbered member is equal to $\binom{i-1}{k-1}$, the number of ways of choosing $k - 1$ of the numbers $1, \ldots, i - 1$. Summing over $i$ yields the number of subsets of size $k$.

12. Number of possible selections of a committee of size $k$ and a chairperson is $k\binom{n}{k}$ and so $\sum_{k=1}^{n} k\binom{n}{k}$ represents the desired number. On the other hand, the chairperson can be anyone of the $n$ persons and then each of the other $n - 1$ can either be on or off the committee. Hence, $n2^{n-1}$ also represents the desired quantity.

(i) $\binom{n}{k}k^2$

(ii) $n2^{n-1}$ since there are $n$ possible choices for the combined chairperson and secretary and then each of the other $n - 1$ can either be on or off the committee.

(iii) $n(n - 1)2^{n-2}$

(c) From a set of $n$ we want to choose a committee, its chairperson its secretary and its treasurer (possibly the same). The result follows since

(a) there are $n2^{n-1}$ selections in which the chair, secretary and treasurer are the same person.

(b) there are $3n(n - 1)2^{n-2}$ selection in which the chair, secretary and treasurer jobs are held by 2 people.

(c) there are $n(n - 1)(n - 2)2^{n-3}$ selections in which the chair, secretary and treasurer are all different.

(d) there are $\binom{n}{k}k^3$ selections in which the committee is of size $k$.

13. $(1 - 1)^n = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-1}$

14. (a) $\binom{n}{j}\binom{j}{i} = \binom{n}{i}\binom{n-i}{j-i}$

(b) From (a), $\sum_{j=i}^{n}\binom{n}{j}\binom{j}{i} = \binom{n}{i}\sum_{j=i}^{n}\binom{n-i}{j-1} = \binom{n}{i}2^{n-i}$

(c) 
$$\sum_{j=i}^{n}\binom{n}{j}\binom{j}{i}(-1)^{n-j} = \binom{n}{i}\sum_{j=i}^{n}\binom{n-i}{j-1}(-1)^{n-j}$$
$$= \binom{n}{i}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^{n-i-k} = 0$$

15. (a) The number of vectors that have $x_k = j$ is equal to the number of vectors $x_1 \le x_2 \le \ldots \le x_{k-1}$ satisfying $1 \le x_i \le j$. That is, the number of vectors is equal to $H_{k-1}(j)$, and the result follows.

(b)

$$
\begin{aligned}
&H_2(1) = H_1(1) = 1\\
&H_2(2) = H_1(1) + H_1(2) = 3\\
&H_2(3) = H_1(1) + H_1(2) + H_1(3) = 6\\
&H_2(4) = H_1(1) + H_1(2) + H_1(3) + H_1(4) = 10\\
&H_2(5) = H_1(1) + H_1(2) + H_1(3) + H_1(4) + H_1(5) = 15\\
&H_3(5) = H_2(1) + H_2(2) + H_2(3) + H_2(4) + H_2(5) = 35
\end{aligned}
$$

16. (a) $1 < 2 < 3$, $1 < 3 < 2$, $2 < 1 < 3$, $2 < 3 < 1$, $3 < 1 < 2$, $3 < 2 < 1$,
$1 = 2 < 3$, $1 = 3 < 2$, $2 = 3 < 1$, $1 < 2 = 3$, $2 < 1 = 3$, $3 < 1 = 2$, $1 = 2 = 3$

(b) The number of outcomes in which $i$ players tie for last place is equal to $\binom{n}{i}$, the number of ways to choose these $i$ players, multiplied by the number of outcomes of the remaining $n - i$ players, which is clearly equal to $N(n - i)$.

(c)
$$
\begin{aligned}
\sum_{i=1}^{n}\binom{n}{i}N(n-1) &= \sum_{i=1}^{n}\binom{n}{n-i}N(n-i)\\
&= \sum_{j=0}^{n-1}\binom{n}{j}N(j)
\end{aligned}
$$

where the final equality followed by letting $j = n - i$.

(d) $N(3) = 1 + 3N(1) + 3N(2) = 1 + 3 + 9 = 13$
$N(4) = 1 + 4N(1) + 6N(2) + 4N(3) = 75$

17. A choice of $r$ elements from a set of $n$ elements is equivalent to breaking these elements into two subsets, one of size $r$ (equal to the elements selected) and the other of size $n - r$ (equal to the elements not selected).

18. Suppose that $r$ labelled subsets of respective sizes $n_1, n_2, \ldots, n_r$ are to be made up from elements $1, 2, \ldots, n$ where $n = \sum_{i=1}^{r} n_i$. As $\binom{n-1}{n_1,\ldots,n_i-1,\ldots n_r}$ represents the number of possibilities when person $n$ is put in subset $i$, the result follows.

19. By induction:

$$(x_1 + x_2 + \ldots + x_r)^n$$

$$= \sum_{i_1=0}^{n} \binom{n}{i_1} x_1^{i_1} (x_2 + \ldots + x_r)^{n-i_1} \text{ by the Binomial theorem}$$

$$= \sum_{i_1=0}^{n} \binom{n}{i_1} x_1^{i_1} \underset{i_2+\ldots+i_r=n-i_1}{\sum_{i_2,\ldots,i_r} \cdots \sum} \binom{n-i_1}{i_2,\ldots,i_r} x_1^{i_2} \ldots x_r^{i_2}$$

$$= \underset{i_1+i_2+\ldots+i_r=n}{\sum_{i_1,\ldots,i_r} \sum \cdots \sum} \binom{n}{i_1,\ldots,i_r} x_1^{i_1} \ldots x_r^{i_r}$$

where the second equality follows from the induction hypothesis and the last from the identity $\binom{n}{i_1}\binom{n-i_1}{i_2,\ldots,i_n} = \binom{n}{i_1,\ldots,i_r}$.

20. The number of integer solutions of

$$x_1 + \ldots + x_r = n,\ x_i \geq m_i$$

is the same as the number of nonnegative solutions of

$$y_1 + \ldots + y_r = n - \sum_1^r m_i\ ,\ y_i \geq 0.$$

Proposition 6.2 gives the result $\binom{n - \sum_1^r m_i + r - 1}{r-1}$.

21. There are $\binom{r}{k}$ choices of the $k$ of the $x$'s to equal 0. Given this choice the other $r - k$ of the $x$'s must be positive and sum to $n$.

By Proposition 6.1, there are $\binom{n-1}{r-k-1} = \binom{n-1}{n-r+k}$ such solutions.

Hence the result follows.

22. $\binom{n+r-1}{n-1}$ by Proposition 6.2.

23. There are $\binom{j+n-1}{j}$ nonnegative integer solutions of

$$\sum_{i=1}^{n} x_i = j$$

Hence, there are $\sum_{j=0}^{k} \binom{j+n-1}{j}$ such vectors.

# Chapter 2

## Problems

1. (a) $S = \{(r, r), (r, g), (r, b), (g, r), (g, g), (g, b), (b, r), b, g), (b, b)\}$
(b) $S = \{(r, g), (r, b), (g, r), (g, b), (b, r), (b, g)\}$

2. $S = \{(n, x_1, \ldots, x_{n-1}), n \geq 1, x_i \neq 6, i = 1, \ldots, n-1\}$, with the interpretation that the outcome is $(n, x_1, \ldots, x_{n-1})$ if the first 6 appears on roll $n$, and $x_i$ appears on roll, $i$, $i = 1, \ldots, n-1$. The event $(\cup_{n=1}^{\infty} E_n)^c$ is the event that 6 never appears.

3. $EF = \{(1, 2), (1, 4), (1, 6), (2, 1), (4, 1), (6, 1)\}$.
$E \cup F$ occurs if the sum is odd or if at least one of the dice lands on 1. $FG = \{(1, 4), (4, 1)\}$.
$EF^c$ is the event that neither of the dice lands on 1 and the sum is odd. $EFG = FG$.

4. $A = \{1,0001,0000001, \ldots\}$ $B = \{01, 00001, 00000001, \ldots\}$
$(A \cup B)^c = \{00000 \ldots, 001, 000001, \ldots\}$

5. (a) $2^5 = 32$
(b)
$W = \{(1, 1, 1, 1, 1), (1, 1, 1, 1, 0), (1, 1, 1, 0, 1), (1, 1, 0, 1, 1), (1, 1, 1, 0, 0), (1, 1, 0, 1, 0)$
$(1, 1, 0, 0, 1), (1, 1, 0, 0, 0), (1, 0, 1, 1, 1), (0, 1, 1, 1, 1), (1, 0, 1, 1, 0), (0, 1, 1, 1, 0), (0, 0, 1, 1, 1)$
$(0, 0, 1, 1, 0), (1, 0, 1, 0, 1)\}$

(c) 8
(d) $AW = \{(1, 1, 1, 0, 0), (1, 1, 0, 0, 0)\}$

6. (a) $S = \{(1, g), (0, g), (1, f), (0, f), (1, s), (0, s)\}$
(b) $A = \{(1, s), (0, s)\}$
(c) $B = \{(0, g), (0, f), (0, s)\}$
(d) $\{(1, s), (0, s), (1, g), (1, f)\}$

7. (a) $6^{15}$
(b) $6^{15} - 3^{15}$
(c) $4^{15}$

8. (a) .8
(b) .3
(c) 0

9. Choose a customer at random. Let A denote the event that this customer carries an American Express card and V the event that he or she carries a VISA card.

$$P(A \cup V) = P(A) + P(V) - P(AV) = .24 + .61 - .11 = .74.$$

Therefore, 74 percent of the establishment's customers carry at least one of the two types of credit cards that it accepts.

10. Let $R$ and $N$ denote the events, respectively, that the student wears a ring and wears a necklace.

(a) $P(R \cup N) = 1 - .6 = .4$

(b) $.4 = P(R \cup N) = P(R) + P(N) - P(RN) = .2 + .3 - P(RN)$
Thus, $P(RN) = .1$

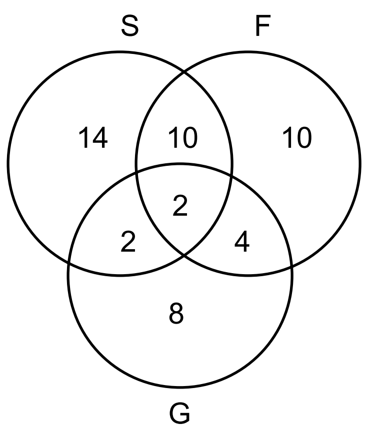
11. Let $A$ be the event that a randomly chosen person is a cigarette smoker and let $B$ be the event that she or he is a cigar smoker.

(a) $1 - P(A \cup B) = 1 - (.07 + .28 - .05) = .7$. Hence, 70 percent smoke neither.

(b) $P(A^cB) = P(B) - P(AB) = .07 - .05 = .02$. Hence, 2 percent smoke cigars but not cigarettes.

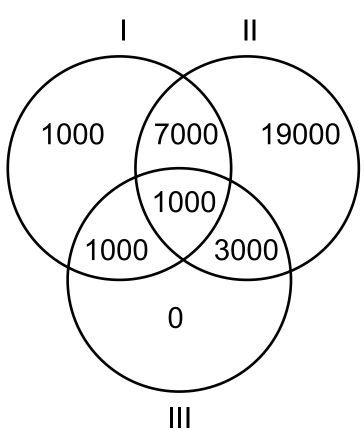
12. (a) $P(S \cup F \cup G) = (28 + 26 + 16 - 12 - 4 - 6 + 2)/100 = 1/2$
The desired probability is $1 - 1/2 = 1/2$.

(b) Use the Venn diagram below to obtain the answer 32/100.




(c) since 50 students are not taking any of the courses, the probability that neither one is taking a course is $\binom{50}{2}\Big/\binom{100}{2} = 49/198$ and so the probability that at least one is taking a course is 149/198.

13.




(a) 20,000
(b) 12,000
(c) 11,000
(d) 68,000
(e) 10,000

14. $P(M) + P(W) + P(G) - P(MW) - P(MG) - P(WG) + P(MWG) = .312 + .470 + .525 - .086 - .042 - .147 + .025 = 1.057$

15. (a) $4\binom{13}{5}\Big/\binom{52}{5}$

(b) $13\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}\Big/\binom{52}{5}$

(c) $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}\Big/\binom{52}{5}$

(d) $13\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}\Big/\binom{52}{5}$

(e) $13\binom{4}{4}\binom{48}{1}\Big/\binom{52}{5}$

16. (a) $\dfrac{6\cdot5\cdot4\cdot3\cdot2}{6^5}$ (b) $\dfrac{6\binom{5}{2}5\cdot4\cdot3}{6^5}$ (c) $\dfrac{\binom{6}{2}4\binom{5}{2}\binom{3}{2}}{6^5}$

(d) $\dfrac{6\cdot5\cdot4\binom{5}{3}}{21}$ (e) $\dfrac{6\cdot5\binom{5}{3}}{6^5}$ (f) $\dfrac{6\cdot5\binom{5}{4}}{6^5}$

(g) $\dfrac{6}{6^5}$

17. $\dfrac{\prod_{i=1}^{8} i^2}{64\cdot63\cdots58}$

18. $\dfrac{2\cdot4\cdot16}{52\cdot51}$

19. $4/36 + 4/36 + 1/36 + 1/36 = 5/18$

20. Let A be the event that you are dealt blackjack and let B be the event that the dealer is dealt blackjack. Then,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(AB) \\ &= \frac{4\cdot4\cdot16}{52\cdot51} + \frac{4\cdot4\cdot16\cdot3\cdot15}{52\cdot51\cdot50\cdot49} \\ &= .0983 \end{aligned}$$

where the preceding used that $P(A) = P(B) = 2 \times \dfrac{4\cdot16}{52\cdot51}$. Hence, the probability that neither is dealt blackjack is .9017.

21. (a) $p_1 = 4/20, p_2 = 8/20, p_3 = 5/20, p_4 = 2/20, p_5 = 1/20$

(b) There are a total of $4 \cdot 1 + 8 \cdot 2 + 5 \cdot 3 + 2 \cdot 4 + 1 \cdot 5 = 48$ children. Hence,

$$q_1 = 4/48, q_2 = 16/48, q_3 = 15/48, q_4 = 8/48, q_5 = 5/48$$

22. The ordering will be unchanged if for some $k$, $0 \le k \le n$, the first $k$ coin tosses land heads and the last $n - k$ land tails. Hence, the desired probability is $(n + 1/2^n$

23. The answer is 5/12, which can be seen as follows:

$$\begin{aligned} 1 &= P\{\text{first higher}\} + P\{\text{second higher}\} + p\{\text{same}\} \\ &= 2P\{\text{second higher}\} + p\{\text{same}\} \\ &= 2P\{\text{second higher}\} + 1/6 \end{aligned}$$

Another way of solving is to list all the outcomes for which the second is higher. There is 1 outcome when the second die lands on two, 2 when it lands on three, 3 when it lands on four, 4 when it lands on five, and 5 when it lands on six. Hence, the probability is $(1 + 2 + 3 + 4 + 5)/36 = 5/12$.

25. $P(E_n) = \left(\dfrac{26}{36}\right)^{n-1} \dfrac{6}{36}, \quad \sum_{n=1}^{\infty} P(E_n) = \dfrac{2}{5}$

27. Imagine that all 10 balls are withdrawn

$$P(A) = \frac{3 \cdot 9! + 7 \cdot 6 \cdot 3 \cdot 7! + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 5! + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 3 \cdot 3!}{10!}$$

28. $P\{\text{same}\} = \dfrac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{19}{3}}$

$$P\{\text{different}\} = \binom{5}{1}\binom{6}{1}\binom{8}{1}\Big/\binom{19}{3}$$

If sampling is with replacement

$$P\{\text{same}\} = \frac{5^3 + 6^3 + 8^3}{(19)^3}$$

$$\begin{aligned} P\{\text{different}\} &= P(RBG) + P\{BRG\} + P(\text{RGB}) + \ldots + P(GBR) \\ &= \frac{6 \cdot 5 \cdot 6 \cdot 8}{(19)^3} \end{aligned}$$

29. (a) $\dfrac{n(n-1)+m(m-1)}{(n+m)(n+m-1)}$

(b) Putting all terms over the common denominator $(n + m)^2(n + m - 1)$ shows that we must prove that

$n^2(n + m - 1) + m^2(n + m - 1) \geq n(n - 1)(n + m) + m(m - 1)(n + m)$

which is immediate upon multiplying through and simplifying.

30. (a) $\dfrac{\binom{7}{3}\binom{8}{3}3!}{\binom{8}{4}\binom{9}{4}4!} = 1/18$

(b) $\dfrac{\binom{7}{3}\binom{8}{3}}{\binom{8}{4}\binom{9}{4}} - 1/18 = 1/6$

(c) $\dfrac{\binom{7}{3}\binom{8}{4}+\binom{7}{4}\binom{8}{3}}{\binom{8}{4}\binom{9}{4}} = 1/2$

31. $P(\{\text{complete}\} = \dfrac{3\cdot 2\cdot 1}{3\cdot 3\cdot 3} = \dfrac{2}{9}$

$P\{\text{same}\} = \dfrac{3}{27} = \dfrac{1}{9}$

32. $\dfrac{g(b+g-1)!}{(b+g)!} = \dfrac{g}{b+g}$

33. $\dfrac{\binom{5}{2}\binom{15}{2}}{\binom{20}{4}} = \dfrac{70}{323}$

34. $\binom{32}{13}\Big/\binom{52}{13}$

35. $1 - \binom{30}{3}\Big/\binom{54}{3} \approx .8363$

36. (a) $\binom{4}{2}\Big/\binom{52}{2} \approx .0045,$

(b) $13\binom{4}{2}\Big/\binom{52}{2} = 1/17 \approx .0588$

37. (a) $\binom{7}{5}\Big/\binom{10}{5} = 1/12 \approx .0833$

(b) $\binom{7}{4}\binom{3}{1}\Big/\binom{10}{5} + 1/12 = 1/2$

38. $1/2 = \binom{3}{2}\Big/\binom{n}{2}$ or $n(n-1) = 12$ or $n = 4$.

39. $\dfrac{5\cdot4\cdot3}{5\cdot5\cdot5} = \dfrac{12}{25}$

40. $P\{1\} = \dfrac{4}{44} = \dfrac{1}{64}$

$$P\{2\} = \binom{4}{2}\left[4+\binom{4}{2}+4\right]\Big/4^4 = \frac{84}{256}$$

$$P\{3\} = \binom{4}{3}\binom{3}{1}\frac{4!}{2!}\Big/4^4 = \frac{36}{64}$$

$$P\{4\} = \frac{4!}{4^4} = \frac{6}{64}$$

41. $1 - \dfrac{5^4}{6^4}$

42. $1 - \left(\dfrac{35}{36}\right)^n$

43. $\dfrac{2(n-1)(n-2)}{n!} = \dfrac{2}{n}$ in a line

$\dfrac{2n(n-2)!}{n!} = \dfrac{2}{n-1}$ if in a circle, $n \geq 2$

44. (a) If $A$ is first, then $A$ can be in any one of 3 places and $B$'s place is determined, and the others can be arranged in any of 3! ways. As a similar result is true, when $B$ is first, we see that the probability in this case is $2 \cdot 3 \cdot 3!/5! = 3/10$

(b) $2 \cdot 2 \cdot 3!/5! = 1/5$

(c) $2 \cdot 3!/5! = 1/10$

45. $1/n$ if discard, $\dfrac{(n-1)^{k-1}}{n^k}$ if do not discard

46. If $n$ in the room,

$$P\{\text{all different}\} = \frac{12 \cdot 11 \cdot \qquad \cdot (13-n)}{12 \cdot 12 \cdot \qquad \cdot 12}$$

When $n = 5$ this falls below 1/2. (Its value when $n = 5$ is .3819)

47. $12!/(12)^{12}$

48. $\binom{12}{4}\binom{8}{4}\dfrac{(20)!}{(3!)^4(2!)^4}\Big/(12)^{20}$

49. $\binom{6}{3}\binom{6}{3}\Big/\binom{12}{6}$

50. $\binom{13}{5}\binom{39}{8}\binom{8}{8}\binom{31}{5}\Big/\binom{52}{13}\binom{39}{13}$

51. $\binom{n}{m}(n-1)^{n-m} / N^n$

52. (a) $\dfrac{20 \cdot 18 \cdot 16 \cdot 14 \cdot 12 \cdot 10 \cdot 8 \cdot 6}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}$

(b) $\dfrac{\binom{10}{1}\binom{9}{6}\frac{8!}{2!}2^6}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}$

53. Let $A_i$ be the event that couple $i$ sit next to each other. Then

$$P(\cup_{i=1}^{4} A_i) = 4\frac{2 \cdot 7!}{8!} - 6\frac{2^2 \cdot 6!}{8!} + 4\frac{2^3 \cdot 5!}{8!} - \frac{2^4 \cdot 4!}{8!}$$

and the desired probability is 1 minus the preceding.

54. $P(S \cup H \cup D \cup C) = P(S) + P(H) + P(D) + P(\mathrm{C}) - P(SH) - \ldots - P(SHDC)$

$$= \frac{4\binom{39}{13}}{\binom{52}{13}} - \frac{6\binom{26}{13}}{\binom{52}{13}} + \frac{4\binom{13}{13}}{\binom{52}{13}}$$

$$= \frac{4\binom{39}{13} - 6\binom{26}{13} + 4}{\binom{52}{13}}$$

55. (a) $P(S \cup H \cup D \cup C) = P(S) + \ldots - P(SHDC)$

$$= \frac{4\binom{2}{2}}{\binom{52}{13}} - \frac{6\binom{2}{2}\binom{2}{2}\binom{48}{9}}{\binom{52}{13}} + \frac{4\binom{2}{2}^3\binom{46}{7}}{\binom{52}{13}} - \frac{\binom{2}{2}^4\binom{44}{5}}{\binom{52}{13}}$$

$$= \frac{4\binom{50}{11} - 6\binom{48}{9} + 4\binom{46}{7} - \binom{44}{5}}{\binom{52}{13}}$$

(b) $P(1 \cup 2 \cup \ldots \cup 13) = \dfrac{13\binom{48}{9}}{\binom{52}{13}} - \dfrac{\binom{13}{2}\binom{44}{5}}{\binom{52}{13}} + \dfrac{\binom{13}{3}\binom{40}{1}}{\binom{52}{13}}$

56. Player B. If Player A chooses spinner (a) then B can choose spinner (c). If A chooses (b) then B chooses (a). If A chooses (c) then B chooses (b). In each case B wins probability 5/9.

## Theoretical Exercises

5. $F_i = E_i \bigcap_{j=1}^{i=1} E_j^c$

6. (a) $EF^cG^c$

   (b) $EF^cG$

   (c) $E \cup F \cup G$

   (d) $EF \cup EG \cup FG$

   (e) $EFG$

   (f) $E^cF^cG^c$

   (g) $E^cF^cG^c \cup EF^cG^c \cup E^cFG^c \cup E^cF^cG$

   (h) $(EFG)^c$

   (i) $EFG^c \cup EF^cG \cup E^cFG$

   (j) $S$

7. (a) $E$

   (b) $EF$

   (c) $EG \cup F$

8. The number of partitions that has $n + 1$ and a fixed set of $i$ of the elements 1, 2, …, $n$ as a subset is $T_{n-i}$. Hence, (where $T_0 = 1$). Hence, as there are $\binom{n}{i}$ such subsets.

$$T_{n+1} = \sum_{i=0}^{n} \binom{n}{i} T_{n-i} = 1 + \sum_{i=0}^{n-1} \binom{n}{i} T_{n-i} = 1 + \sum_{k=1}^{n} \binom{n}{k} T_k \ .$$

11. $1 \geq P(E \cup F) = P(E) + P(F) - P(EF)$

12. $P(EF^c \cup E^cF) = P(EF^c) + P(E^cF)$
$\qquad = P(E) - P(EF) + P(F) - P(EF)$

13. $E = EF \cup EF^c$

15. $\dfrac{\binom{M}{k}\binom{N}{r-k}}{\binom{M+N}{r}}$

16. $P(E_1 \ldots E_n) \geq P(E_1 \ldots E_{n-1}) + P(E_n) - 1$ by Bonferonni's Ineq.

$$\geq \sum_{1}^{n-1} P(E_i) - (n-2) + P(E_n) - 1 \text{ by induction hypothesis}$$

19. $\dfrac{\binom{n}{r-1}\binom{m}{k-r}(n-r+1)}{\binom{n+m}{k-1}(n+m-k+1)}$

21. Let $y_1, y_2, \ldots, y_k$ denote the successive runs of losses and $x_1, \ldots, x_k$ the successive runs of wins. There will be $2k$ runs if the outcome is either of the form $y_1, x_1, \ldots, y_k\, x_k$ or $x_1 y_1, \ldots x_k, y_k$ where all $x_i$, $y_i$ are positive, with $x_1 + \ldots + x_k = n$, $y_1 + \ldots + y_k = m$. By Proposition 6.1 there are $2\binom{n-1}{k-1}\binom{m-1}{k-1}$ number of outcomes and so

$$P\{2k \text{ runs}\} = 2\binom{n-1}{k-1}\binom{m-1}{k-1}\Big/\binom{m+n}{n}.$$

There will be $2k+1$ runs if the outcome is either of the form $x_1, y_1, \ldots, x_k, y_k, x_{k+1}$ or $y_1, x_1, \ldots, y_k, x_k\, y_{k+1}$ where all are positive and $\sum x_i = n$, $\sum y_i = m$. By Proposition 6.1 there are $\binom{n-1}{k}\binom{m-1}{k-1}$ outcomes of the first type and $\binom{n-1}{k-1}\binom{m-1}{k}$ of the second.

# Chapter 3

## Problems

1. $P\{6 \mid \text{different}\} = P\{6, \text{different}\}/P\{\text{different}\}$

$$= \frac{P\{\text{1st} = 6, \text{2nd} \neq 6\} + P\{\text{1s}t \neq 6, \text{2nd} = 6\}}{5/6}$$

$$= \frac{2 \;\; 1/6 \;\; 5/6}{5.6} = 1/3$$

could also have been solved by using reduced sample space—for given that outcomes differ it is the same as asking for the probability that 6 is chosen when 2 of the numbers 1, 2, 3, 4, 5, 6 are randomly chosen.

2. $P\{6 \mid \text{sum of } 7\} = P\{(6, 1)\}/1/6 = 1/6$

$P\{6 \mid \text{sum of } 8\} = P\{(6, 2)\}/5/36 = 1/5$

$P\{6 \mid \text{sum of } 9\} = P\{(6, 3)\}/4/36 = 1/4$

$P\{6 \mid \text{sum of } 10\} = P\{(6, 4)\}/3/36 = 1/3$

$P\{6 \mid \text{sum of } 11\} = P\{(6, 5)\}/2/36 = 1/2$

$P\{6 \mid \text{sum of } 12\} = 1.$

3. $P\{E \text{ has } 3 \mid N - S \text{ has } 8\} = \dfrac{P\{E \text{ has } 3, N - S \text{ has } 8\}}{P\{N - S \text{ has } 8\}}$

$$= \frac{\binom{13}{8}\binom{39}{18}\binom{5}{3}\binom{21}{10} \Big/ \binom{52}{26}\binom{26}{13}}{\binom{13}{8}\binom{39}{18} \Big/ \binom{52}{26}} = .339$$

4. $P\{\text{at least one } 6 \mid \text{sum of } 12\} = 1$. Otherwise twice the probability given in Problem 2.

5. $\dfrac{6}{15}\dfrac{5}{14}\dfrac{9}{13}\dfrac{8}{12}$

6. In both cases the one black ball is equally likely to be in either of the 4 positions. Hence the answer is 1/2.

7. $P1\ g \text{ and } 1\ b \mid \text{at least one } b\} = \dfrac{1/2}{3/4} = 2/3$

8. 1/2

9. $P\{A = w \mid 2w\} = \dfrac{P\{A = w, 2w\}}{P\{2w\}}$

$$= \frac{P\{A = w, B = w, C \neq w\} + P\{A = w, B \neq w, C = w\}}{P\{2w\}}$$

$$= \frac{\frac{1}{3}\frac{2}{3}\frac{3}{4} + \frac{1}{3}\frac{1}{3}\frac{1}{4}}{\frac{1}{2}\frac{2}{3}\frac{3}{4} + \frac{1}{3}\frac{1}{3}\frac{1}{4} + \frac{2}{3}\frac{2}{3}\frac{1}{4}} = \frac{7}{11}$$

10. 11/50

11. (a) $P(B \mid A_s) = \dfrac{P(BA_s)}{P(A_s)} = \dfrac{\frac{1}{52}\frac{3}{21} + \frac{3}{52}\frac{1}{51}}{\frac{2}{52}} = \dfrac{1}{17}$

Which could have been seen by noting that, given the ace of spades is chosen, the other card is equally likely to be any of the remaining 51 cards, of which 3 are aces.

(b) $P(B \mid A) = \dfrac{P(B)}{P(A)} = \dfrac{\frac{4}{52}\frac{3}{51}}{1 - \frac{48}{52}\frac{47}{51}} = \dfrac{1}{33}$

12. (a) $(.9)(.8)(.7) = .504$

(b) Let $F_i$ denote the event that she failed the $i$th exam.

$$P(F_2 \mid (F_1^c F_2^c F_3^c)^c) = \frac{P(F_1^c F_2)}{1 - .504} = \frac{(.9)(.2)}{.496} = .3629$$

13. $P(E_1) = \binom{4}{1}\binom{48}{12} \Big/ \binom{52}{13}$, $\qquad P(E_2 \mid E_1) = \binom{3}{1}\binom{36}{12} \Big/ \binom{39}{13}$

$P(E_3 \mid E_1E_2) = \binom{2}{1}\binom{24}{12} \Big/ \binom{26}{13}$, $\qquad P(E_4 \mid E_1E_2E_3) = 1.$

Hence,

$$p = \binom{4}{1}\binom{48}{12} \Big/ \binom{52}{13} \cdot \binom{3}{1}\binom{36}{12} \Big/ \binom{39}{13} \cdot \binom{2}{1}\binom{24}{12} \Big/ \binom{26}{13}$$

14. $\dfrac{5}{12}\dfrac{7}{14}\dfrac{7}{16}\dfrac{9}{18} - \dfrac{35}{768}$.

15. Let $E$ be the event that a randomly chosen pregnant women has an ectopic pregnancy and $S$ the event that the chosen person is a smoker. Then the problem states that

$$P(E \mid S) = 2P(E \mid S^c),\ P(S) = .32$$

Hence,

$$\begin{aligned}
P(S \mid E) &= P(SE)/P(E) \\
&= \frac{P(E \mid S)P(S)}{P(E \mid S)P(S) + P(E \mid S^c)P(S^C)} \\
&= \frac{2P(S)}{2P(S) + P(S^c)} \\
&= 32/66 \approx .4548
\end{aligned}$$

16. With $S$ being survival and $C$ being $C$ section of a randomly chosen delivery, we have that

$$\begin{aligned}
.98 = P(S) &= P(S \mid C).15 + P(S \mid C^2)\ .85 \\
&= .96(.15) + P(S \mid C^2)\ .85
\end{aligned}$$

Hence

$$P(S \mid C^c) \approx .9835.$$

17. $P(D) = .36$, $P(\mathrm{C}) = .30$, $P(C \mid D) = .22$

(a) $P(DC) = P(D)\ P(C \mid D) = .0792$
(b) $P(D \mid C) = P(DC)/P(C) = .0792/.3 = .264$

18. (a) $P(\text{Ind} \mid \text{voted}) = \dfrac{P(\text{voted} \mid \text{Ind})P(\text{Ind})}{\sum P(\text{voted} \mid type)P(\text{type})}$

$$= \frac{.35(.46)}{.35(.46) + .62(.3) + .58(.24)} \approx 331$$

(b) $P\{\text{Lib} \mid \text{voted}\} = \dfrac{.62(.30)}{.35(.46) + .62(.3) + .58(.24)} \approx .383$

(c) $P\{\text{Con} \mid \text{voted}\} = \dfrac{.58(.24)}{.35(.46) + .62(.3) + .58(.24)} \approx .286$

(d) $P\{\text{voted}\} = .35(.46) + .62(.3) + .58(.24) = .4862$
That is, 48.62 percent of the voters voted.

19. Choose a random member of the class. Let $A$ be the event that this person attends the party and let $W$ be the event that this person is a woman.

(a) $P(W|A) = \dfrac{P(A|W)P(W)}{P(A|W)P(W)+P(A|M)P(M)}$ where $M = W^c$

$$= \frac{.48(.38)}{.48(.38)+.37(.62)} \approx .443$$

Therefore, 44.3 percent of the attendees were women.

(b) $P(A) = .48(.38) + .37(.62) = .4118$

Therefore, 41.18 percent of the class attended.

20. (a) $P(F|C) = \dfrac{P(FC)}{P(C)} = .02/.05 = .40$

(b) $P(C|F) = P(FC)/P(F) = .02/.52 = 1/26 \approx .038$

21. (a) $P\{\text{husband under } 25\} = (212 + 36)/500 = .496$

(b) $P\{\text{wife over} \mid \text{husband over}\} = P\{\text{both over}\}/P\{\text{husband over}\}$

$$= (54/500)/(252/500)$$
$$= 3/14 \approx .214$$

(c) $P\{\text{wife over} \mid \text{husband under}\} = 36/248 \approx .145$

22. a. $\dfrac{6\cdot 5\cdot 4}{6\cdot 6\cdot 6} = \dfrac{5}{9}$

b. $\dfrac{1}{3!} = \dfrac{1}{6}$

c. $\dfrac{5}{9}\dfrac{1}{6} = \dfrac{5}{54}$

23. $P(w \mid w \text{ transferred}\}P\{w \text{ tr.}\} + P(w \mid R \text{ tr.}\}P\{R \text{ tr.}\} = \dfrac{2}{3}\dfrac{1}{3} + \dfrac{1}{3}\dfrac{2}{3} = \dfrac{4}{9}$.

$$P\{w \text{ transferred} \mid w\} = \frac{P\{w|w \text{ tr.}\}P\{w \text{ tr.}\}}{P\{w\}} = \frac{\frac{2}{3}\frac{1}{3}}{\frac{4}{9}} = 1/2.$$

24. (a) $P\{g - g \mid \text{at least one } g\} = \dfrac{1/4}{3/4} = 1/3.$

(b) Since we have no information about the ball in the urn, the answer is 1/2.

26. Let $M$ be the event that the person is male, and let $C$ be the event that he or she is color blind. Also, let $p$ denote the proportion of the population that is male.

$$P(M \mid C) = \frac{P(C|M)P(M)}{P(C|M)P(M) + P(C|M^c)P(M^c)} = \frac{(.05)p}{(.05)p + (.0025)(1-p)}$$

27. Method (b) is correct as it will enable one to estimate the average number of workers per car. Method (a) gives too much weight to cars carrying a lot of workers. For instance, suppose there are 10 cars, 9 transporting a single worker and the other carrying 9 workers. Then 9 of the 18 workers were in a car carrying 9 workers and so if you randomly choose a worker then with probability 1/2 the worker would have been in a car carrying 9 workers and with probability 1/2 the worker would have been in a car carrying 1 worker.

28. Let $A$ denote the event that the next card is the ace of spades and let $B$ be the event that it is the two of clubs.

(a) $P\{A\} = P\{\text{next card is an ace}\}P\{A \mid \text{next card is an ace}\}$

$$= \frac{3}{32}\frac{1}{4} = \frac{3}{128}$$

(b) Let $C$ be the event that the two of clubs appeared among the first 20 cards.

$$P(B) = P(B \mid C)P(C) + P(B \mid C^c)P(C^c)$$
$$= 0\frac{19}{48} + \frac{1}{32}\frac{29}{48} = \frac{29}{1536}$$

29. Let $A$ be the event that none of the final 3 balls were ever used and let $B_i$ denote the event that $i$ of the first 3 balls chosen had previously been used. Then,

$$P(A) = P(A \mid B_0)P(B_0) + P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3)$$
$$= \sum_{i=0}^{3} \frac{\binom{6+i}{3}}{\binom{15}{3}} \frac{\binom{6}{i}\binom{9}{3-i}}{\binom{15}{3}}$$
$$= .083$$

30. Let $B$ and $W$ be the events that the marble is black and white, respectively, and let $B$ be the event that box $i$ is chosen. Then,

$$P(B) = P(B \mid B_1)P(B_1) + P(B \mid B_2)P(B_2) = (1/2)(1/2) = (2/3)(1/2) = 7/12$$
$$P(B_1 \mid W) = \frac{P(W|B_1)P(B_1)}{P(W)} = \frac{(1/2)(1/2)}{5/12} = 3/5$$

31. Let $C$ be the event that the tumor is cancerous, and let $N$ be the event that the doctor does not call. Then

$$\beta = P(C \mid N) = \frac{P(NC)}{P(N)}$$

$$= \frac{P(N|C)P(C)}{P(N|C)P(C) + P(N|C^c)P(C^c)}$$

$$= \frac{\alpha}{\alpha + \frac{1}{2}(1-\alpha)}$$

$$= \frac{2\alpha}{1+\alpha} \geq \alpha$$

with strict inequality unless $\alpha = 1$.

32. Let $E$ be the event the child selected is the eldest, and let $F_j$ be the event that the family has $j$ children. Then,

$$P(F_j \mid E) = \frac{P(EF_j)}{P(E)}$$

$$= \frac{P(F_j)P(E|F_j)}{\sum_j P(F_j)P(E|F_j)}$$

$$= \frac{p_j(1/j)}{.1 + .25(1/2) + .35(1/3) + .3(1/4)} = .24$$

Thus, $P(F_1 \mid E) = .24$, $P(F_4 \mid E) = .18$.

33. Let $V$ be the event that the letter is a vowel. Then

$$P(E \mid V) = \frac{P(V|E)P(E)}{P(V|E)P(E) + P(V|A)P(A)} = \frac{(1/2)(2/5)}{(1/2)(2/5) + (2/5)(3/5)} = 5/11$$

34. $$P(G \mid C) = \frac{P(C|G)P(G)}{P(C|G)P(G) + P(C|G^c)P(G^c)} = 54/62$$

35. $P\{A = \text{superior} \mid A \text{ fair}, B \text{ poor}\}$

$$= \frac{P\{A \text{ fair}, B \text{ poor}|A \text{ superior}|A \text{ superior}\}}{P\{A \text{ fair}, B \text{ poor}\}}$$

$$= \frac{\frac{10}{30}\frac{15}{30}\frac{1}{2}}{\frac{10}{30}\frac{15}{30}\frac{1}{2} + \frac{10}{30}\frac{5}{30}\frac{1}{2}} = \frac{3}{4}.$$

36. $P\{C \mid \text{woman}\} = \dfrac{P\{\text{women} \mid C\}P\{C\}}{P\{\text{women} \mid A\}P\{A\} + P\{\text{women} \mid B\}P\{B\} + P\{\text{women} \mid C\}P\{C\}}$

$$= \frac{.7\frac{100}{225}}{.5\frac{50}{225} + .6\frac{75}{225} + .7\frac{100}{225}} = \frac{1}{2}$$

37. (a) $P\{\text{fair} \mid h\} = \dfrac{\frac{1}{2}\frac{1}{2}}{\frac{1}{2}\frac{1}{2} + \frac{1}{2}} = \dfrac{1}{3}$.

(b) $P\{\text{fair} \mid hh\} = \dfrac{\frac{1}{4}\frac{1}{2}}{\frac{1}{4}\frac{1}{2} + \frac{1}{2}} = \dfrac{1}{5}$.

(c) 1

38. $P\{\text{tails} \mid w\} = \dfrac{\frac{3}{15}\frac{1}{2}}{\frac{3}{15}\frac{1}{2} + \frac{5}{12}\frac{1}{2}} = \dfrac{36}{36+75} = \dfrac{36}{111}$.

39. $P\{\text{acc.} \mid \text{no acc.}\} = \dfrac{P\{\text{no acc., acc.}}{P\{\text{no acc.}\}}$

$$= \frac{\frac{3}{10}(.4)(.6) + \frac{7}{10}(.2)(.8)}{\frac{3}{10}(.6) + \frac{7}{10}(.8)} = \frac{46}{185}.$$

40. (a) $\dfrac{7}{12}\dfrac{8}{13}\dfrac{9}{14}$

(b) $3\dfrac{7 \cdot 8 \cdot 5}{12 \cdot 13 \cdot 14}$

(c) $\dfrac{5 \cdot 6 \cdot 7}{12 \cdot 13 \cdot 14}$

(d) $3\dfrac{5 \cdot 6 \cdot 7}{12 \cdot 13 \cdot 14}$

41. $P\{\text{ace}\} = P\{\text{ace} \mid \text{interchanged selected}\}\ \frac{1}{27}$

$$+P\{\text{ace} \mid \text{interchanged not selected}\}\ \frac{26}{27}$$

$$= 1\frac{1}{27} + \frac{3}{51}\frac{26}{27} = \frac{129}{51 \cdot 27}.$$

42. $P\{A \mid \text{failure}\} = \dfrac{(.02)(.5)}{(.02)(.5) + (.03)(.3) + (.05)(.2)} = \dfrac{10}{29}$

43. $P\{2 \text{ headed} \mid \text{heads}\} = \dfrac{\frac{1}{3}(1)}{\frac{1}{3}(1) + \frac{1}{3}\frac{1}{2} + \frac{1}{3}\frac{3}{4}} = \dfrac{4}{4+2+3} = \dfrac{4}{9}$.

45. $P\{5\text{th} \mid \text{heads}\} = \dfrac{P\{\text{heads} \mid 5^{\text{th}}\}P\{5^{\text{th}}\}}{\sum_i P\{h \mid i^{\text{th}}\}P\{i^{th}\}}$

$$= \frac{\frac{5}{10}\frac{1}{10}}{\sum_{i=1}^{10} \frac{i}{10}\frac{1}{10}} = \frac{1}{11}.$$

46. Let $M$ and $F$ denote, respectively, the events that the policyholder is male and that the policyholder is female. Conditioning on which is the case gives the following.

$$P(A_2 \mid A_1) = \frac{P(A_1A_2)}{P(A_1)}$$

$$= \frac{P(A_1A_2 \mid M)\alpha + P(A_1A_2 \mid F)(1-\alpha)}{P(A_1 \mid M)\alpha + P(A_1 \mid F)(1-\alpha)}$$

$$= \frac{p_m^2\alpha + p_f^2(1-\alpha)}{p_m\alpha + p_f(1-\alpha)}$$

Hence, we need to show that

$$p_m^2\alpha + p_f^2[1-\alpha) > (p_m\alpha + p_f(1-\alpha))^2$$

or equivalently, that

$$p_m^2(\alpha - \alpha^2) + p_f^2[1 - \alpha - (1-a)^2] > 2\alpha(1-\alpha)p_fp_m$$

Factoring out $\alpha(1-\alpha)$ gives the equivalent condition

$$p_m^2 + p_f^2 > 2pf_m$$

or

$$(p_m - p_f)^2 > 0$$

which follows because $p_m \neq p_f$. Intuitively, the inequality follows because given the information that the policyholder had a claim in year 1 makes it more likely that it was a type policyholder having a larger claim probability. That is, the policyholder is more likely to me male if $p_m > p_f$ (or more likely to be female if the inequality is reversed) than without this information, thus raising the probability of a claim in the following year.

47. $P\{\text{all white}\} = \dfrac{1}{6}\left[\dfrac{5}{15} + \dfrac{5}{15}\dfrac{4}{14} + \dfrac{5}{15}\dfrac{4}{14}\dfrac{3}{13} + \dfrac{5}{15}\dfrac{4}{14}\dfrac{3}{13}\dfrac{2}{12} + \dfrac{5}{15}\dfrac{4}{14}\dfrac{3}{13}\dfrac{2}{12}\dfrac{1}{11}\right]$

$$P\{3 \mid \text{all white}\} = \frac{\frac{1}{6}\frac{5}{15}\frac{4}{14}\frac{3}{13}}{P\{\text{all white}\}}$$

48. (a) $P\{\text{silver in other} \mid \text{silver found}\}$

$$= \frac{P\{S \text{ in other}, S \text{ found}\}}{P\{S \text{ found}\}}.$$

To compute these probabilities, condition on the cabinet selected.

$$= \frac{1/2}{P\{S \text{ found} \mid A\}1/2 + P\{S \text{ found} \mid B\}1/2}$$

$$= \frac{1}{1+1/2} = \frac{2}{3}.$$

49. Let $C$ be the event that the patient has cancer, and let $E$ be the event that the test indicates an elevated PSA level. Then, with $p = P(C)$,

$$P(C \mid E) = \frac{P(E \mid C)P(C)}{P(E \mid C)P(C) + P(E \mid C^c)P(C^c)}$$

Similarly,

$$P(C \mid E^c) = \frac{P(E^c \mid C)P(C)}{P(E^c \mid C)P(C) + P(E^c \mid C^c)P(C^c)}$$

$$= \frac{.732p}{.732p + .865(1-p)}$$

50. Choose a person at random

$$
\begin{aligned}
P\{\text{they have accident}\} &= P\{\text{acc.} \mid \text{good}\}P\{g\} + P\{\text{acc.} \mid \text{ave.}\}P\{\text{ave.}\} \\
&\quad + P\{\text{acc.} \mid \text{bad } P(\text{b})\} \\
&= (.05)(.2) + (.15)(.5) + (.30)(.3) = .175
\end{aligned}
$$

$$P\{A \text{ is good} \mid \text{no accident}\} = \frac{.95(2)}{.825}$$

$$P\{A \text{ is average} \mid \text{no accident}\} = \frac{(.85)(.5)}{.825}$$

51. Let $R$ be the event that she receives a job offer.

(a) $P(R) = P(R \mid \text{strong})P(\text{strong}) + P(R \mid \text{moderate})P(\text{moderate}) + P(R \mid \text{weak})P(\text{weak})$
$= (.8)(.7) + (.4)(.2) + (.1)(.1) = .65$

(b)
$$
\begin{aligned}
P(\text{strong} \mid R) &= \frac{P(R \mid \text{strong})P(\text{strong})}{P(R)} \\
&= \frac{(.8)(.7)}{.65} = \frac{56}{65}
\end{aligned}
$$

Similarly,

$$P(\text{moderate} \mid R) = \frac{8}{65},\ P(\text{weak} \mid R) = \frac{1}{65}$$

(c)
$$
\begin{aligned}
P(\text{strong} \mid R^c) &= \frac{P(R^c \mid \text{strong})P(\text{strong})}{P(R^c)} \\
&= \frac{(.2)(.7)}{.35} = \frac{14}{35}
\end{aligned}
$$

Similarly,

$$P(\text{moderate} \mid R^c) = \frac{12}{35},\ P(\text{weak} \mid R^c) = \frac{9}{35}$$

52. Let $M$, $T$, $W$, $Th$, $F$ be the events that the mail is received on that day. Also, let $A$ be the event that she is accepted and $R$ that she is rejected.

(a) $P(M) = P(M \mid A)P(A) + P(M \mid R)P(R) = (.15)(.6) + (.05)(.4) = .11$

(b) $P(T \mid M^c) = \dfrac{P(T)}{P(M^c)}$

$$= \frac{P(T|A)P(A) + P(T|R)P(R)}{1 - P(M)}$$

$$= \frac{(.2)(.6) + (.1)(.4)}{.89} \frac{16}{89}$$

(c) $P(A \mid M^cT^cW^c) = \dfrac{P(M^cT^cW^c|A)P(A)}{P(M^cT^cW^c)}$

$$= \frac{(1 - .15 - .20 - .25)(.6)}{(.4)(.6) + (.75)(.4)} = \frac{12}{27}$$

(d) $P(A \mid Th) = \dfrac{P(Th|A)P(A)}{P(Th)}$

$$= \frac{(.15)(.6)}{(.15)(.6) + (.15)(.4)} = \frac{3}{5}$$

(e) $P(A \mid \text{no mail}) = \dfrac{P(\text{no mail}|A)P(A)}{P(\text{no mail})}$

$$= \frac{(.15)(.6)}{(.15)(.6) + (.4)(.4)} = \frac{9}{25}$$

53. Let $W$ and $F$ be the events that component 1 works and that the system functions.

$$P(W \mid F) = \frac{P(WF)}{P(F)} = \frac{P(W)}{1 - P(F^c)} = \frac{1/2}{1 - (1/2)^{n-1}}$$

55. $P\{\text{Boy}, F\} = \dfrac{4}{16 + x}$ $\quad P\{\text{Boy}\} = \dfrac{10}{16 + x}$ $\quad P\{F\} = \dfrac{10}{16 + x}$

so independence $\Rightarrow 4 = \dfrac{10 \cdot 10}{16 + x} \Rightarrow 4x = 36$ or $x = 9$.

A direct check now shows that 9 sophomore girls (which the above shows is necessary) is also sufficient for independence of sex and class.

56. $P\{\text{new}\} = \sum_i P\{\text{new} \mid \text{type } i\}p_i = \sum_i (1 - p_i)^{n-1} p_i$

57. (a) $2p(1-p)$

(b) $\binom{3}{2}p^2(1-p)$

(c) $P\{\text{up on first} \mid \text{up 1 after 3}\}$
$= P\{\text{up first, up 1 after 3}\}/[3p^2(1-p)]$
$= p2p(1-p)/[3p^2(1-p)] = 2/3.$

58. (a) All we know when the procedure ends is that the two most flips were either $H$, $T$, or $T$, $H$. Thus,

$$P(\text{heads}) = P(H, T \mid H, T \text{ or } T, H) = \frac{P(H,T)}{P(H,T)+P(T,H)} = \frac{p(1-p)}{p(1-p)+(1-p)p} = \frac{1}{2}$$

(b) No, with this new procedure the result will be heads (tails) whenever the first flip is tails (heads). Hence, it will be heads with probability $1-p$.

59. (a) 1/16

(b) 1/16

(c) The only way in which the pattern $H$, $H$, $H$, $H$ can occur first is for the first 4 flips to all be heads, for once a tail appears it follows that a tail will precede the first run of 4 heads (and so $T$, $H$, $H$, $H$ will appear first). Hence, the probability that $T$, $H$, $H$, $H$ occurs first is 15/16.

60. From the information of the problem we can conclude that both of Smith's parents have one blue and one brown eyed gene. Note that at birth, Smith was equally likely to receive either a blue gene or a brown gene from each parent. Let $X$ denote the number of blue genes that Smith received.

(a) $P\{\text{Smith blue gene}\} = P\{X = 1 \mid X \leq 1\} = \dfrac{1/2}{1-1/4} = 2/3$

(b) Condition on whether Smith has a blue-eyed gene.
$P\{\text{child blue}\} = P\{\text{blue} \mid \text{blue gene}\}(2/3) + P\{\text{blue} \mid \text{no blue}\}(1/3)$
$= (1/2)(2/3) = 1/3$

(c) First compute
$$P\{\text{Smith blue} \mid \text{child brown}\} = \frac{P\{\text{child brown} \mid \text{Smith blue}\}2/3}{2/3} = 1/2$$

Now condition on whether Smith has a blue gene given that first child has brown eyes.
$P\{\text{second child brown}\} = P\{\text{brown} \mid \text{Smith blue}\}1/2 + P\{\text{brown} \mid \text{Smith no blue}\}1/2$
$= 1/4 + 1/2 = 3/4$

61. Because the non-albino child has an albino sibling we know that both its parents are carriers. Hence, the probability that the non-albino child is not a carrier is

$$P(A, A \mid A, a \text{ or } a, A \text{ or } A, A) = \frac{1}{3}$$

Where the first gene member in each gene pair is from the mother and the second from the father. Hence, with probability 2/3 the non-albino child is a carrier.

(a) Condition on whether the non-albino child is a carrier. With $C$ denoting this event, and $O_i$ the event that the $i^{\text{th}}$ offspring is albino, we have:

$$\begin{aligned} P(O_1) &= P(O_1 \mid C)P(C) + P(O_1 \mid C^c)P(C^c) \\ &= (1/4)(2/3) + 0(1/3) = 1/6 \end{aligned}$$

(b)
$$\begin{aligned} P(O_2 \mid O_1^c) &= \frac{P(O_1^c O_2)}{P(O_1^c)} \\ &= \frac{P(O_1^c O_2 \mid C)P(C) + P(O_1^c O_2 \mid C^c)P(C^c)}{5/6} \\ &= \frac{(3/4)(1/4)(2/3) + 0(1/3)}{5/6} = \frac{3}{20} \end{aligned}$$

62. (a)
$$P\{\text{both hit} \mid \text{at least one hit}\} = \frac{P\{\text{both hit}\}}{P\{\text{at least one hit}\}}$$

$$= p_1p_2/(1 - q_1q_2)$$

(b) $P\{\text{Barb hit} \mid \text{at least one hit}\} = p_1/(1 - q_1q_2)$
$Q_i = 1 - p_i$, and we have assumed that the outcomes of the shots are independent.

63. Consider the final round of the duel. Let $q_x = 1 - p_x$

(a)
$$\begin{aligned} P\{A \text{ not hit}\} &= P\{A \text{ not hit} \mid \text{at least one is hit}\} \\ &= P\{A \text{ not hit}, B \text{ hit}\}/P\{\text{at least one is hit}\} \\ &= q_Bp_A/(1 - q_Aq_B) \end{aligned}$$

(b)
$$\begin{aligned} P\{\text{both hit}\} &= P\{\text{both hit} \mid \text{at least one is hit}\} \\ &= P\{\text{both hit}\}/P\{\text{at least one hit}\} \\ &= p_Ap_B/(1 - q_Aq_B) \end{aligned}$$

(c) $(q_Aq_B)^{n-1}(1 - q_Aq_B)$

(d) $P\{n \text{ rounds} \mid A \text{ unhit}\} = P\{n \text{ rounds}, A \text{ unhit}\}/P\{A \text{ unhit}\}$

$$= \frac{(q_Aq_B)^{n-1}p_Aq_B}{q_Bp_A/(1 - q_Aq_B)}$$

$$= (q_Aq_B)^{n-1}(1 - q_Aq_B)$$

(e) $P(n \text{ rounds} \mid \text{both hit}) = P\{n \text{ rounds both hit}\}/P\{\text{both hit}\}$

$$= \frac{(q_A q_B)^{n-1} p_A p_B}{p_B p_A/(1-q_A q_B)}$$

$$= (q_A q_B)^{n-1}(1 - q_A q_B)$$

Note that (c), (d), and (e) all have the same answer.

64. If use (a) will win with probability $p$. If use strategy (b) then

$$P\{\text{win}\} = P\{\text{win} \mid \text{both correct}\}p^2 + P\{\text{win} \mid \text{exactly 1 correct}\}2p(1-p) + P\{\text{win} \mid \text{neither correct}\}(1-p)^2$$
$$= p^2 + p(1-p) + 0 = p$$

Thus, both strategies give the same probability of winning.

65. (a) $P\{\text{correct} \mid \text{agree}\} = P\{\text{correct, agree}\}/P\{\text{agree}\}$
$= p^2/[p^2 + (1-p)^2]$
$= 36/52 = 9/13$ when $p = .6$

(b) 1/2

66. (a) $[I - (1 - P_1P_2)(1 - P_3P_4)]P_5 = (P_1P_2 + P_3P_4 - P_1P_2P_3P_4)P_5$

(b) Let $E_1 = \{1 \text{ and } 4 \text{ close}\}$, $E_2 = \{1, 3, 5 \text{ all close}\}$

$E_3 = \{2, 5 \text{ close}\}$, $E_4 = \{2, 3, 4 \text{ close}\}$. The desired probability is

67. $$P(E_1 \cup E_2 \cup E_3 \cup E_4) = P(E_1) + P(E_2) + P(E_3) + P(E_4) - P(E_1E_2) - P(E_1E_3) - P(E_1E_4) - P(E_2E_3) - P(E_2E_4) + P(E_3E_4) + P(E_1E_2E_3) + P(E_1E_2E_4) + P(E_1E_3E_4) + P(E_2E_3E_4) - P(E_1E_2E_3E_4)$$

$$= P_1P_4 + P_1P_3P_5 + P_2P_5 + P_2P_3P_4 - P_1P_3P_4P_5 - P_1P_2P_4P_5 - P_1P_2P_3P_4 - P_1P_2P_3P_5 - P_2P_3P_4P_5 - 2P_1P_2P_3P_4P_5 + 3P_1P_2P_3P_4P_5.$$

(a) $P_1P_2(1 - P_3)(1 - P_4) + P_1(1 - P_2)P_3(1 - P_4) + P_1(1 - P_2(1 - P_3)P_4$
$+ P_2P_3(1 - P_1)(1 - P_4) + (1 - P_1)P_2(1 - P_3)P_4 + (1 - P_1)(1 - P_2)P_3P_4$
$+ P_1P_2P_3(1 - P_4) + P_1P_2(1 - P_3)P_4 + P_1(1 - P_2)P_3P_4 + (1 - P_1)P_2P_3P_4 + P_1P_2P_3P_4.$

(c) $\displaystyle\sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$

68. Let $C_i$ denote the event that relay $i$ is closed, and let $F$ be the event that current flows from $A$ to $B$.

$$
\begin{aligned}
P(C_1C_2 \mid F) &= \frac{P(C_1C_2F)}{P(F)} \\
&= \frac{P(F \mid C_1C_2)P(C_1C_2)}{p_5(p_1p_2 + p_3p_4 - p_1p_2p_3p_4)} \\
&= \frac{p_5p_1p_2}{p_5(p_1p_2 + p_3p_4 - p_1p_2p_3p_4)}
\end{aligned}
$$

69. 1. (a) $\frac{1}{2}\frac{3}{4}\frac{1}{2}\frac{3}{4}\frac{1}{2} = \frac{9}{128}$

(b) $\frac{1}{2}\frac{3}{4}\frac{1}{2}\frac{3}{4}\frac{1}{2} = \frac{9}{128}$

(c) $\frac{18}{128}$

(d) $\frac{110}{128}$

2. (a) $\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{32}$

(b) $\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{32}$

(c) $\frac{1}{16}$

(d) $\frac{15}{16}$

70. (a) $P\{\text{carrier} \mid 3 \text{ without}\}$

$$= \frac{1/8\ 1/2}{1/8\ 1/2 + 1\ 1/2} = 1/9.$$

(b) 1/18

71. $P\{\text{Braves win}\} = P\{B \mid B \text{ wins 3 of 3}\}\ 1/8 + P\{B \mid B \text{ wins 2 of 3}\}\ 3/8$

$+ P\{B \mid B \text{ wins 1 of 3}\}\ 3/8 + P\{B \mid B \text{ wins 0 of 3}\}\ 1/8$

$$= \frac{1}{8} + \frac{3}{8}\left[\frac{1}{4}\frac{1}{2} + \frac{3}{4}\right] + \frac{3}{8} \quad \frac{3}{4}\frac{1}{2} = \frac{38}{64}$$

where $P\{B \mid B \text{ wins } i \text{ of } 3\}$ is obtained by conditioning on the outcome of the other series. For instance

$$
\begin{aligned}
P\{B \mid B \text{ win 2 of 3}\} &= P\{B \mid D \text{ or } G \text{ win 3 of 3, } B \text{ win 2 of 3}\}\ 1/4 \\
&= P\{B \mid D \text{ or } G \text{ win 2 of 3, } B \text{ win 2 of 3}\}\ 3/4 \\
&= \frac{1}{2}\frac{1}{4} + \frac{3}{4}.
\end{aligned}
$$

By symmetry $P\{D \text{ win}\} = P\{G \text{ win}\}$ and as the probabilities must sum to 1 we have.

$$P\{D \text{ win}\} = P\{G \text{ win}\} = \frac{13}{64}.$$

72. Let $f$ denote for and $a$ against a certain place of legislature. The situations in which a given steering committees vote is decisive are as follows:

| given member | other members of S.C. | other council members |
|---|---|---|
| for | both for | 3 or 4 against |
| for | one for, one against | at least 2 for |
| against | one for, one against | at least 2 for |
| against | both for | 3 of 4 against |

$$\begin{aligned}P\{\text{decisive}\} &= p^3 4p(1-p)^3 + p^2 p(1-p)(6p^2(1-p)^2 + 4p^3(1-p) + p^4) \\ &\quad + (1-p)2p(1-p)(6p^2(1-p)^2 + 4p^3(1-p) + p^4) \\ &\quad + (1-p)p^2 4p(1-p)^3.\end{aligned}$$

73. (a) 1/16, (b) 1/32, (c) 10/32, (d) 1/4, (e) 31/32.

74. Let $P_A$ be the probability that $A$ wins when $A$ rolls first, and let $P_B$ be the probability that $B$ wins when $B$ rolls first. Using that the sum of the dice is 9 with probability 1/9, we obtain upon conditioning on whether $A$ rolls a 9 that

$$P_A = \frac{1}{9} + \frac{8}{9}(1 - P_B)$$

Similarly,

$$P_B = \frac{5}{36} + \frac{31}{36}(1 - P_A)$$

Solving these equations gives that $P_A = 9/19$ (and that $P_B = 45/76$.)

75. (a) The probability that a family has 2 sons is 1/4; the probability that a family has exactly 1 son is 1/2. Therefore, on average, every four families will have one family with 2 sons and two families with 1 son. Therefore, three out of every four sons will be eldest sons.

Another argument is to choose a child at random. Letting $E$ be the event that the child is an eldest son, letting $S$ be the event that it is a son, and letting $A$ be the event that the child's family has at least one son,

$$\begin{aligned}P(E \mid S) &= \frac{P(ES)}{P(S)} \\ &= 2P(E) \\ &= 2\left[P(E|A)\frac{3}{4} + P(E|A^c)\frac{1}{4}\right] \\ &= 2\left[\frac{1}{2}\frac{3}{4} + 0\frac{1}{4}\right] = 3/4\end{aligned}$$

(b) Using the preceding notation

$$\begin{aligned}P(E \mid S) &= \frac{P(ES)}{P(S)} \\ &= 2P(E) \\ &= 2\left[P(E|A)\frac{7}{8} + P(E|A^c)\frac{1}{8}\right] \\ &= 2\left[\frac{1}{3}\frac{7}{8}\right] = 7/12\end{aligned}$$

76. Condition on outcome of initial trial

$$\begin{aligned}P(E \text{ before } F) &= P(E \text{ b } F \mid E)P(E) + P(E \text{ b } F \mid F)P(F) \\ &\quad + P(E \text{ b } F \mid \text{neither } E \text{ or } F)[1 - P(E) - P(F)] \\ &= P(E) + P(E \text{ b } F)(1 - P(E) - P(F)].\end{aligned}$$

Hence,

$$P(E \text{ b } F) = \frac{P(E)}{P(E)+P(F)}.$$

77. (a) This is equal to the conditional probability that the first trial results in outcome 1 ($F_1$) given that it results in either 1 or 2, giving the result 1/2. More formally, with $L_3$ being the event that outcome 3 is the last to occur

$$P(F_1 \mid L_3) = \frac{P(L_3|F_1)P(F_1)}{P(L_3)} = \frac{(1/2)(1/3)}{1/3} = 1/2$$

(b) With $S_1$ being the event that the second trial results in outcome 1, we have

$$P(F_1S_1 \mid L_3) = \frac{P(L_3|F_1S_1)P(F_1S_1)}{P(L_3)} = \frac{(1/2)(1/9)}{1/3} = 1/6$$

78. (a) Because there will be 4 games if each player wins one of the first two games and then one of them wins the next two, $P(4 \text{ games}) = 2p(1-p)[p^2 + (1-p)^2]$.

(b) Let $A$ be the event that $A$ wins. Conditioning on the outcome of the first two games gives

$$\begin{aligned}P(A &= P(A \mid a, a)p^2 + P(A \mid a, b)p(1-p) + P(A \mid b, a)(1-p)p + P(A \mid b, b)(1-p)^2 \\ &= p^2 + P(A)2p(1-p)\end{aligned}$$

where the notation $a$, $b$ means, for instance, that $A$ wins the first and $B$ wins the second game. The final equation used that $P(A \mid a, b) = P(A \mid b, a) = P(A)$. Solving, gives

$$P(A) = \frac{p^2}{1-2p(1-p)}$$

79. Each roll that is either a 7 or an even number will be a 7 with probability

$$p = \frac{P(7)}{P(7)+P(\text{even})} = \frac{1/6}{1/6+1/2} = 1/4$$

Hence, from Example 4*i* we see that the desired probability is

$$\sum_{i=2}^{7}\binom{7}{i}(1/4)^i(3/4)^{7-i} = 1 - (3/4)^7 - 7(3/4)^6(1/4)$$

80. (a) $P(A_i) = (1/2)^i$, if $i < n$
$= (1/2)^{n-1}$, if $i = n$

(b) $\dfrac{\sum_{i=1}^{n} i(1/2)^i + n(1/2)^{n-1}}{2^n - 1} = \dfrac{1}{2^{n-1}}$

(c) Condition on whether they initially play each other. This gives

$$P_n = \frac{1}{2^n-1} + \frac{2^n-2}{2^n-1}\left(\frac{1}{2}\right)^2 P_{n-1}$$

where $\left(\frac{1}{2}\right)^2$ is the probability they both win given they do not play each other.

(d) There will be $2^n - 1$ losers, and thus that number of games.

(e) Since the 2 players in game $i$ are equally likely to be any of the $\binom{2^n}{2}$ pairs it follows that

$P(B_i) = 1\Big/\binom{2^n}{2}$.

(f) Since the events $B_i$ are mutually exclusive

$$P(\cup B_i) = \sum P(B_i) = (2^n-1)\Big/\binom{2^n}{2} = (1/2)^{n-1}$$

81. $\dfrac{1-(9/11)^{15}}{1-(9/11)^{30}}$

82. (a) $P(A) = P_1^2 + (1-P_1^2)\left[(1-P_2^2)P(A)\right]$ or $P(A) = \dfrac{P_1^2}{P_1^2+P_2^2-P_1^2P_2^2}$

(c) similar to (a) with $P_i^3$ replacing $P_i^2$.

(b) and (d) Let $P_{ij}(\overline{P}_{ij})$ denote the probability that $A$ wins when $A$ needs $i$ more and $B$ needs $j$ more and $A(B)$ is to flip. Then

$$P_{ij} = P_1 P_{i-1,j} + (1 - P_1)\overline{P}_{ij}$$
$$\overline{P}_{ij} = P_2 \overline{P}_{i,j-1} + (1 - P_2)P_{ij}\,.$$

These equations can be recursively solved starting with

$$P_{01} = 1,\ P_{1,0} = 0.$$

83. (a) Condition on the coin flip

$$P\{\text{throw } n \text{ is red}\} = \frac{1}{2}\frac{4}{6} + \frac{1}{2}\frac{2}{6} = \frac{1}{2}$$

(b) $$P\{r \mid rr\} = \frac{P\{rrr\}}{P\{rr\}} = \frac{\frac{1}{2}\left(\frac{2}{3}\right)^3 + \frac{1}{2}\left(\frac{1}{3}\right)^3}{\frac{1}{2}\left(\frac{2}{3}\right)^2 + \frac{1}{2}\left(\frac{1}{3}\right)^2} = \frac{3}{5}$$

(c) $$P\{A \mid rr\} = \frac{P\{rr \mid A\}P(A)}{P\{rr\}} = \frac{\left(\frac{2}{3}\right)^2\frac{1}{2}}{\left(\frac{2}{3}\right)^2\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)^2\frac{1}{2}} = 4/5$$

84. (b) $$P(A \text{ wins}) = \frac{4}{12} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{4}{9} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{4}{6} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}$$
$$P(B \text{ wins}) = \frac{8}{12}\frac{4}{11} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{4}{5}$$
$$P(C \text{ wins}) = \frac{8}{12}\frac{7}{11}\frac{4}{10} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{4}{7} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{1}{5}$$

85. Part (a) remains the same. The possibilities for part (b) become more numerous.

86. Using the hint

$$P\{A \subset B\} = \sum_{i=0}^{n}(2^i/2^n)\binom{n}{i}\bigg/2^n = \sum_{i=0}^{n}\binom{n}{i}2^i/4^n = (3/4)^n$$

where the final equality uses

$$\sum_{i=0}^{n}\binom{n}{i}2^i 1^{n-i} = (2+1)^n$$

(b) $P(AB = \phi) = P(A \subset B^c) = (3/4)^n$, by part (a), since $B^c$ is also equally likely to be any of the subsets.

87. $$P\{i^{\text{th}} \mid \text{all heads}\} = \frac{(i/k)^n}{\sum_{j=0}^{k}(j/k)^n}.$$

88. No—they are conditionally independent given the coin selected.

89. (a) $P(J_3 \text{ votes guilty} \mid J_1 \text{ and } J_2 \text{ vote guilty}\}$

$= P\{J_1, J_2, J_3 \text{ all vote guilty}\}/P\{J_1 \text{ and } J_2 \text{ vote guilty}\}$

$$= \frac{\frac{7}{10}(.7)^3 + \frac{3}{10}(.2)^3}{\frac{7}{10}(.7)^2 + \frac{3}{10}(.2)^2} = \frac{97}{142}.$$

(b) $P(J_3 \text{ guilty} \mid \text{one of } J_1, J_2 \text{ votes guilty}\}$

$$= \frac{\frac{7}{10}(.7)2(.7)(.3) + \frac{3}{10}(2.)2(.2)(.8)}{\frac{7}{10}2(.7)(.3) + \frac{3}{10}2(.2)(.8)} = \frac{15}{26}.$$

(c) $P\{J_3 \text{ guilty} \mid J_1, J_2 \text{ vote innocent}\}$

$$= \frac{\frac{7}{10}(.7)(.3)^2 + \frac{3}{10}(.2)(.8)^2}{\frac{7}{10}(.3)^2 + \frac{3}{10}(.8)^2} = \frac{33}{102}.$$

$E_i$ are conditionally independent given the guilt or innocence of the defendant.

90. Let $N_i$ denote the event that none of the trials result in outcome $i$, $i = 1, 2$. Then

$$\begin{aligned} P(N_1 \cup N_2) &= P(N_1) + P(N_2) - P(N_1N_2) \\ &= (1 - p_1)^n + (1 - p_2)^n - (1 - p_1 - p_2)^n \end{aligned}$$

Hence, the probability that both outcomes occur at least once is
$1 - (1 - p_1)^n - (1 - p_2)^n + (p_0)^n$.

## Theoretical Exercises

1. $$P(AB \mid A) = \frac{P(AB)}{P(A)} \ge \frac{P(AB)}{P(A \cup B)} = P(AB \mid A \cup B)$$

2. If $A \subset B$

$$P(A \mid B) = \frac{P(A)}{P(B)},\ P(A \mid B^c) = 0, \qquad P(B \mid A) = 1, \qquad P(B \mid A^c) = \frac{P(BA^c)}{P(A^c)}$$

3. Let $F$ be the event that a first born is chosen. Also, let $S_i$ be the event that the family chosen in method $a$ is of size $i$.

$$P_a(F) = \sum_i P(F \mid S_i)P(S_i) = \sum_i \frac{1}{i}\frac{n_i}{m}$$

$$P_b(F) = \frac{m}{\sum_i in_i}$$

Thus, we must show that

$$\sum_i in_i \sum_i n_i / i \ge m^2$$

or, equivalently,

$$\sum_i in_i \sum_j n_j / j \ge \sum_i n_i \sum_j n_j$$

or,

$$\sum\sum_{i \ne j} \frac{i}{j} n_i n_j \ge \sum\sum_{i \ne j} n_i n_j$$

Considering the coefficients of the term $n_i n_j$, shows that it is sufficient to establish that

$$\frac{i}{j} + \frac{j}{i} \ge 2$$

or equivalently

$$i^2 + j^2 \ge 2ij$$

which follows since $(i - j)^2 \ge 0$.

4. Let $N_i$ denote the event that the ball is not found in a search of box $i$, and let $B_j$ denote the event that it is in box $j$.

$$P(B_j \mid N_i) = \frac{P(N_i|B_j)P(B_j)}{P(N_i|B_i)P(B_i) + P(N_i|B_i^c)P(B_i^c)}$$

$$= \frac{P_j}{(1-\alpha_i)P_i + 1 - P_i} \quad \text{if } j \neq i$$

$$= \frac{(1-\alpha_i)P_i}{(1-\alpha_i)P_i + 1 - P_i} \quad \text{if } j = i$$

5. None are true.

6. $P\left(\bigcup_1^n E_i\right) = 1 - P\left(\bigcap_1^n E_i^c\right) = 1 - \prod_1^n [1 - P(E_i)]$

7. (a) They will all be white if the last ball withdrawn from the urn (when all balls are withdrawn) is white. As it is equally likely to by any of the $n + m$ balls the result follows.

(b) $P(RBG) = \dfrac{g}{r+b+g} P(RBG \mid G \text{ last}) = \dfrac{g}{r+b+g}\dfrac{b}{r+b}$.

Hence, the answer is $\dfrac{bg}{(r+b)(r+b+g)} + \dfrac{b}{r+b+g}\dfrac{g}{r+g}$.

8. (a) $P(A) = P(A \mid C)P(C) + P(A \mid C^c)P(C^c) > P(B \mid C)P(C) + P(B \mid C^c)P(C^c) = P(B)$

(b) For the events given in the hint

$$P(A \mid C) = \frac{P(C|A)P(A)}{3/36} = \frac{(1/6)(1/6)}{3/36} = 1/3$$

Because $1/6 = P(A$ is a weighted average of $P(A \mid C)$ and $P(A \mid C^c)$, it follows from the result $P(A \mid C) > P(A)$ that $P(A \mid C^c) < P(A)$. Similarly,

$$1/3 = P(B \mid C) > P(B) > P(B \mid C^c)$$

However, $P(AB \mid C) = 0 < P(AB \mid C^c)$.

9. $P(A) = P(B) = P(C) = 1/2$, $P(AB) = P(AC) = P(BC) = 1/4$. But, $P(ABC) = 1/4$.

10. $P(A_{i,j}) = 1/365$. For $i \neq j \neq k$, $P(A_{i,j}A_{j,k}) = 365/(365)^3 = 1/(365)^2$. Also, for $i \neq j \neq k \neq r$, $P(A_{i,j}A_{k,r}) = 1/(365)^2$.

11. $1 - (1 - p)^n \geq 1/2$, or, $n \geq -\dfrac{\log(2)}{\log(1-p)}$

12. $a_i\prod_{j=1}^{i-1}(1-a_j)$ is the probability that the first head appears on the $i^{\text{th}}$ flip and $\prod_{i=1}^{\infty}(1-a_i)$ is the probability that all flips land on tails.

13. Condition on the initial flip. If it lands on heads then $A$ will win with probability $P_{n-1,m}$ whereas if it lands tails then $B$ will win with probability $P_{m,n}$ (and so $A$ will win with probability $1 - P_{m,n}$).

14. Let $N$ go to infinity in Example 4$j$.

15. $P\{r$ successes before $m$ failures$\}$
$= P\{r^{\text{th}}$ success occurs before trial $m + r\}$
$= \sum_{n=r}^{m+r-1}\binom{n-1}{r-1}p^r(1-p)^{n-r}$ .

16. If the first trial is a success, then the remaining $n - 1$ must result in an odd number of successes, whereas if it is a failure, then the remaining $n - 1$ must result in an even number of successes.

17. $P_1 = 1/3$
$P_2 = (1/3)(4/5) + (2/3)(1/5) = 2/5$
$P_3 = (1/3)(4/5)(6/7) + (2/3)(4/5)(1/7) + (1/3)(1/5)(1/7) = 3/7$
$P_4 = 4/9$

(b) $P_n = \dfrac{n}{2n+1}$

(c) Condition on the result of trial $n$ to obtain

$$P_n = (1 - P_{n-1})\frac{1}{2n+1} + P_{n-1}\frac{2n}{2n+1}$$

(d) Must show that

$$\frac{n}{2n+1} = \left[1-\frac{n-1}{2n-1}\right]\frac{1}{2n+1} + \frac{n-1}{2n-1}\frac{2n}{2n+1}$$

or equivalently, that

$$\frac{n}{2n+1} = \frac{n}{2n-1}\frac{1}{2n+1} + \frac{n-1}{2n-1}\frac{2n}{2n+1}$$

But the right hand side is equal to

$$\frac{n+2n(n-1)}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

18. Condition on when the first tail occurs.

19. $P_{n,i} = p^{P}_{n-1,i+1} + (1-p)P_{n-1,i-1}$

20. $\alpha_{n+1} = \alpha_n p + (1 - \alpha_n)(1 - p^1)$
$P_n = \alpha_n p + (1 - \alpha_n)p^1$

21. (b) $P_{n,1} = P\{A \text{ receives first 2 votes}\} = \dfrac{n(n-1)}{(n+1)n} = \dfrac{n-1}{n+1}$

$P_{n,2} = P\{A \text{ receives first 2 and at least 1 of the next 2}\}$

$$= \frac{n}{n+2}\frac{n-1}{n+1}\left\{1 - \frac{2\cdot 1}{n(n-1)}\right\} = \frac{n-2}{n+2}$$

(c) $P_{n,m} = \dfrac{n-m}{n+m}$, $n \geq m$.

(d) $P_{n,m} = P\{A \text{ always ahead}\}$

$$= P\{A \text{ always} \mid A \text{ receives last vote}\}\ \frac{n}{n+m}$$

$$+ P\{A \text{ always} \mid B \text{ receives last vote}\}\ \frac{m}{n+m}$$

$$= \frac{n}{n+m}P_{n-1,m} + \frac{m}{n+m}P_{n,m-1}$$

(e) The conjecture of (c) is true when $n + m = 1$ ($n = 1$, $m = 0$).
Assume it when $n + m = k$. Now suppose that $n + m = k + 1$. By (d) and the induction hypothesis we have that

$$P_{n,m} = \frac{n}{n+m}\frac{n-1-m}{n-1+m} + \frac{m}{n+m}\frac{n-m+1}{n+m-1} = \frac{n-m}{n+m}$$

which completes the proof.

22. $P_n = P_{n-1}p + (1 - P_{n-1})(1 - p)$

$= (2p - 1)P_{n-1} + (1 - p)$

$= (2p-1)\left[\dfrac{1}{2} + \dfrac{1}{2}(2p-1)^{n-1}\right] + 1 - p$ by the induction hypothesis

$= \dfrac{2p-1}{2} + \dfrac{1}{2}(2p-1)^n + 1 - p$

$= \dfrac{1}{2} + \dfrac{1}{2}(2p-1)^n$.

23. $P_{1,1} = 1/2$. Assume that $P_{a,b} = 1/2$ when $k \ge a + b$ and now suppose $a + b = k + 1$. Now

$$P_{a,b} = P\{\text{last is white} \mid \text{first } a \text{ are white}\} \frac{1}{\binom{a+b}{a}}$$

$$+ P\{\text{last is white} \mid \text{first } b \text{ are black}\} \frac{1}{\binom{b+a}{b}}$$

$$+ P\{\text{last is white} \mid \text{neither first } a \text{ are white nor first } b \text{ are black}\}$$

$$\left[1 - \frac{1}{\binom{a+b}{a}} - \frac{1}{\binom{b+a}{b}}\right] = \frac{a!b!}{(a+b)!} + \frac{1}{2}\left[1 - \frac{a!b!}{(a+b)!} - \frac{a!b!}{(a+b)!}\right] = \frac{1}{2}$$

where the induction hypothesis was used to obtain the final conditional probability above.

24. The probability that a given contestant does not beat all the members of some given subset of $k$ other contestants is, by independence, $1 - (1/2)^k$. Therefore $P(B_i)$, the probability that none of the other $n - k$ contestants beats all the members of a given subset of $k$ contestants, is $[1 - (1/2)^k]^{n-k}$. Hence, Boole's inequality we have that

$$P(\cup B_i) \le \binom{n}{k}[1 - (1/2)^k]^{n-k}$$

Hence, if $\binom{n}{k}[1 - (1/2)^k]^{n-k} < 1$ then there is a positive probability that none of the $\binom{n}{k}$ events $B_i$ occur, which means that there is a positive probability that for every set of $k$ contestants there is a contestant who beats each member of this set.

25. $P(E \mid F) = P(EF)/P(F)$

$$P(E \mid FG)P(G \mid F) = \frac{P(EFG)}{P(FG)} \frac{P(FG)}{P(F)} = \frac{P(EFG)}{P(F)}$$

$$P(E \mid FG^c)P(G^c \mid F) = \frac{P(EFG^c)}{P(F)}.$$

The result now follows since

$$P(EF) = P(EFG) + P(EFG^c)$$

27. $E_1, E_2, \ldots, E_n$ are conditionally independent given $F$ if for all subsets $i_1, \ldots, i_r$ of $1, 2, \ldots, n$

$$P\left(E_{i_1} \ldots E_{i_r} \middle| F\right) = \prod_{j=1}^{r} P\left(E_{i_j \mid F}\right).$$

28. Not true. Let $F = E_1$.

29. $P\{\text{next } m \text{ heads} \mid \text{first } n \text{ heads}\}$

$= P\{\text{first } n + m \text{ are heads}\}/P(\text{first } n \text{ heads}\}$

$$= \int_0^1 p^{n+m}dp \Big/ \int_0^1 p^n dp = \frac{n+1}{n+m+1}.$$

# Chapter 4

## Problems

1. $P\{X=4\} = \dfrac{\binom{4}{2}}{\binom{14}{2}} = \dfrac{6}{91}$ $\qquad$ $\mathrm{P}\{X=0\} = \dfrac{\binom{2}{2}}{\binom{14}{2}} = \dfrac{1}{91}$

$P\{X=2\} = \dfrac{\binom{4}{2}\binom{2}{1}}{\binom{14}{2}} = \dfrac{8}{91}$ $\qquad$ $P\{X=-1\} = \dfrac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \dfrac{16}{91}$

$P\{X=1\} = \dfrac{\binom{4}{1}\binom{8}{1}}{\binom{14}{2}} = \dfrac{32}{91}$ $\qquad$ $P\{X=-2\} = \dfrac{\binom{8}{2}}{\binom{14}{2}} = \dfrac{28}{91}$

2.

| | | | | |
|---|---|---|---|---|
| $p(1) = 1/36$ | $p(5) = 2/36$ | $p(9) = 1/36$ | $p(15) = 2/36$ | $p(24) = 2/36$ |
| $p(2) = 2/36$ | $p(6) = 4/36$ | $p(10) = 2/36$ | $p(16) = 1/36$ | $p(25) = 1/36$ |
| $p(3) = 2/36$ | $p(7) = 0$ | $p(11) = 0$ | $p(18) = 2/36$ | $p(30) = 2/36$ |
| $p(4) = 3/36$ | $p(8) = 2/36$ | $p(12) = 4/36$ | $p(20) = 2/36$ | $p(36) = 1/36$ |

4. $P\{X=1\} = 1/2$, $P\{X=2\} = \dfrac{5}{10}\dfrac{5}{9} = \dfrac{5}{18}$, $P\{X=3\} = \dfrac{5}{10}\dfrac{4}{9}\dfrac{5}{8} = \dfrac{5}{36}$,

$P\{X=4\} = \dfrac{5}{10}\dfrac{4}{9}\dfrac{3}{8}\dfrac{5}{7} = \dfrac{10}{168}$, $P\{X=5\} = \dfrac{5\cdot4\cdot3\cdot2}{10\cdot9\cdot8\cdot7}\dfrac{5}{6} = \dfrac{5}{252}$,

$P\{X=6\} = \dfrac{5\cdot4\cdot3\cdot2\cdot1}{10\cdot9\cdot8\cdot7\cdot6} = \dfrac{1}{252}$

5. $n - 2i$, $i = 0, 1, \ldots, n$

6. $P(X=3\} = 1/8$, $P\{X=1\} = 3/8$, $P\{X=-1\} = 3/8$, $P\{X=-3\} = 1/8$

8. (a) $p(6) = 1 - (5/6)^2 = 11/36$, $p(5) = 2\ 1/6\ \ 4/6 + (1/6)^2 = 9/36$
$p(4) = 2\ 1/6\ \ 3/6 + (1/6)^2 = 7/36$, $p(3) = 2\ 1/6\ \ 2/6 + (1/6)^2 = 5/36$
$p(2) = 2\ 1/6\ \ 1/6 + (1/6)^2 = 3/36$, $p(1) = 1/36$

(d) $p(5) = 1/36$, $p(4) = 2/36$, $p(3) = 3/36$, $p(2) = 4/36$, $p(1) = 5/36$
$p(0) = 6/36$, $p(-j) = p(j)$, $j > 0$

11. (a) $P\{\text{divisible by 3}\} = \dfrac{333}{1000}$ $\quad P\{\text{divisible by 105}\} = \dfrac{9}{1000}$

$P\{\text{divisible by 7}\} = \dfrac{142}{1000}$

$P\{\text{divisible by 15}\} = \dfrac{66}{1000}$

In limiting cases, probabilities converge to 1/3, 1/7, 1/15, 1/10

(b) $P\{\mu(N) \neq 0\} = P\{N \text{ is not divisible by } p_i^2,\ i \geq 1\}$

$= \prod_i P\{N \text{ is not divisible by } p_i^2\}$

$= \prod_i (1 - 1/p_i^2) = 6/\pi^2$

13. $p(0) = P\{\text{no sale on first and no sale on second}\}$
$= (.7)(.4) = .28$
$p(500) = P\{\text{1 sale and it is for standard}\}$
$= P\{\text{1 sale}\}/2$
$= [P\{\text{sale, no sale}\} + P\{\text{no sale, sale}\}]/2$
$= [(.3)(.4) + (.7)(.6)]/2 = .27$

$p(1000) = P\{\text{2 standard sales}\} + P\{\text{1 sale for deluxe}\}$
$= (.3)(.6)(1/4) + P\{\text{1 sale}\}/2$
$= .045 + .27 = .315$

$p(1500) = P\{\text{2 sales, one deluxe and one standard}\}$
$= (.3)(.6)(1/2) = .09$

$p(2000) = P\{\text{2 sales, both deluxe}\} = (.3)(.6)(1/4) = .045$

14. $P\{X = 0\} = P\{\text{1 loses to 2}\} = 1/2$

$P\{X = 1\} = P\{\text{of 1, 2, 3: 3 has largest, then 1, then 2}\}$
$= (1/3)(1/2) = 1/6$

$P\{X = 2\} = P\{\text{of 1, 2, 3, 4: 4 has largest and 1 has next largest}\}$
$= (1/4)(1/3) = 1/12$

$P\{X = 3\} = P\{\text{of 1, 2, 3, 4, 5: 5 has largest then 1}\}$
$= (1/5)(1/4) = 1/20$

$P\{X = 4\} = P\{\text{1 has largest}\} = 1/5$

15. $P\{X = 1\} = 11/66$

$$P\{X=2\} = \sum_{j=2}^{11}\left(\frac{12-j}{66}\right)\left(\frac{11}{54+j}\right)$$

$$P\{X=3\} = \sum_{\substack{k\neq 1\\ k\neq j}}\sum_{j=2}\left(\frac{12-j}{66}\right)\left(\frac{12-k}{54+j}\right)\left(\frac{11}{42+j+k}\right)$$

$$P\{X=4\} = 1 - \sum_{i=1}^{3} P\{X=1\}$$

16. $P\{Y_1 = i\} = \dfrac{12-i}{66}$

$$P\{Y_2=i\} = \sum_{j\neq i}\left(\frac{12-j}{66}\right)\left(\frac{12-i}{54+j}\right)$$

$$P\{Y_3=i\} = \sum_{\substack{k\neq j\\ k\neq i}}\sum_{j\neq i}\left(\frac{12-j}{66}\right)\left(\frac{12-k}{54+j}\right)\left(\frac{11}{42+k+j}\right)$$

All sums go from 1 to 11, except for prohibited values.

20. (a) $P\{x > 0\} = P\{\text{win first bet}\} + P\{\text{lose, win, win}\}$
$= 18/38 + (20/38)(18/38)^2 \approx .5918$

(b) No, because if the gambler wins then he or she wins \$1.
However, a loss would either be \$1 or \$3.

(c) $E[X] = 1[18/38 + (20/38)(18/38)^2] - [(20/38)2(20/38)(18/38)] - 3(20/38)^3 \approx -.108$

21. (a) $E[X]$ since whereas the bus driver selected is equally likely to be from any of the 4 buses, the student selected is more likely to have come from a bus carrying a large number of students.

(b) $P\{X = i\} = i/148$, $i = 40, 33, 25, 50$

$E[X] = [(40)^2 + (33)^2 + (25)^2 + (50)^2]/148 \approx 39.28$
$E[Y] = (40 + 33 + 25 + 50)/4 = 37$

22. Let $N$ denote the number of games played.

(a) $E(N) = 2[p^2 + (1-p)^2] + 3[2p(1-p)] = 2 + 2p(1-p)$

The final equality could also have been obtained by using that $N = 2 + ]$ where $I$ is 0 if two games are played and 1 if three are played. Differentiation yields that

$$\frac{d}{dp}E[N] = 2 - 4p$$

and so the minimum occurs when $2 - 4p = 0$ or $p = 1/2$.

(b) $E[N] = 3[p^3 + (1-p)^3 + 4[3p^2(1-p)p + 3p(1-p)^2(1-p)]$
$+ 5[6p^2(1-p)^2 = 6p^4 - 12p^3 + 3p^2 + 3p + 3$

Differentiation yields

$$\frac{d}{dp}E[N] = 24p^3 - 36p^2 + 6p + 3$$

Its value at $p = 1/2$ is easily seen to be 0.

23. (a) Use all your money to buy 500 ounces of the commodity and then sell after one week. The expected amount of money you will get is

$$E[\text{money}] = \frac{1}{2}500 + \frac{1}{2}2000 = 1250$$

(b) Do not immediately buy but use your money to buy after one week. Then

$$E[\text{ounces of commodity}] = \frac{1}{2}1000 + \frac{1}{2}250 = 625$$

24. (a) $p-(1-p)\frac{3}{4} = \frac{7}{4}p - 3/4$, (b) $-\frac{3}{4}p + (1-p)2 = -\frac{11}{4}p + 2$

$\frac{7}{4}p - 3/4 = -\frac{11}{4}p + 2 \Rightarrow p = 11/18$, maximum value = 23.72

(c) $q - \frac{3}{4}(1-q)$ (d) $-\frac{3}{4}q + 2(1-q)$, minimax value = 23/72 attained when $q = 11/18$

25. (a) $\frac{1}{10}(1+2+\ldots+10) = \frac{11}{2}$

(b) after 2 questions, there are 3 remaining possibilities with probability 3/5 and 2 with probability 2/5. Hence.

$$E[\text{Number}] = \frac{2}{5}(3) + \frac{3}{5}\left[2 + \frac{1}{3} + 2\frac{2}{3}\right] = \frac{17}{5}.$$

The above assumes that when 3 remain, you choose 1 of the 3 and ask if that is the one.

27. $C - Ap = \frac{A}{10} \Rightarrow C = A\left(p + \frac{1}{10}\right)$

28. $3 \cdot \frac{4}{20} = 3/5$

29. If check 1, then (if desired) 2: Expected Cost = $C_1 + (1 - p)C_2 + pR_1 + (1 - p)R_2$;
if check 2, then 1: Expected Cost = $C_2 + pC_1 + pR_1 + (1 - p)R_2$ so 1, 2, best if
$C_1 + (1 - p)C_2 \le C_2 + pC_1$, or $C_1 \le \dfrac{p}{1-p}C_2$

30. $E[X] = \sum_{n=1}^{\infty} 2^n (1/2)^n = \infty$

(a) probably not

(b) yes, if you could play an arbitrarily large number of games

31. $E[\text{score}] = p^*[1 - (1 - P)^2 + (1 - p^*)(1 - p^2)$

$$\frac{d}{dp} = 2(1 - p)p^* - 2p(1 - p^*)$$
$$= 0 \Rightarrow p = p^*$$

32. If $T$ is the number of tests needed for a group of 10 people, then

$$E[T] = (.9)^{10} + 11[1 - (.9)^{10}] = 11 - 10(.9)^{10}$$

35. If $X$ is the amount that you win, then

$$P\{X = 1.10\} = 4/9 = 1 - P\{X = -1\}$$
$$E[X] = (1.1)4/9 - 5/9 = -.6/9 \approx =-.067$$
$$\text{Var}(X) = (1.1)^2(4/9) + 5/9 - (.6/9)^2 \approx 1.089$$

36. Using the representation

$$N = 2 + I$$

where $I$ is 0 if the first two games are won by the same team and 1 otherwise, we have that

$$\text{Var}(N) = \text{Var}(I) = E[I]^2 - E^2[I]$$

Now, $E[I]^2 = E[I\} = P\{I = 1\} = 2p\{1 - p\}$ and so
$\text{Var}(N) = 2p(1 - p)[1 - 2p(1 - p)] = 8p^3 - 4p^4 - 6p^2 + 2p$

Differentiation yields

$$\frac{d}{dp}\text{Var}(N) = 24p^2 - 16p^3 - 12p + 2$$

and it is easy to verify that this is equal to 0 when $p = 1/2$.

37. $E[X^2] = [(40)^3 + (33)^3 + (25)^3 + (50)^3]/148 \approx 1625.4$

$$\text{Var}(X = E[X^2] - (E[X])^2 \approx 82.2$$

$E[Y^2] = = [(40)^2 + (33)^2 + (25)^2 + (50)^2]/4 = 1453.5$, Varr($Y$) = 84.5

38. (a) $E[(2 + X)^2] = \text{Var}(2 + X) + (E[2 + X])^2 = \text{Var}(X) + 9 = 14$

(b) $\text{Var}(4 + 3X) = 9\ \text{Var}(X) = 45$

39. $\binom{4}{2}(1/2)^4 = 3/8$

40. $\binom{5}{4}(1/3)^4(2/3)^1 + (1/3)^5 = 11/243$

41. $\sum_{i=7}^{10}\binom{10}{i}(1/2)^{10}$

42. $\binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p) + p^5 \geq \binom{3}{2}p^2(1-p) + p^3$

$\Leftrightarrow 6p^3 - 15p^2 + 12p - 3 \geq 0$

$\Leftrightarrow 6(p - 1/2)(p - 1)^2 \geq 0$

$\Leftrightarrow p \geq 1/2$

43. $\binom{5}{3}(.2)^3(.8)^2 + \binom{5}{4}(.2)^4(.8) + (.2)^5$

44. $\alpha\sum_{i=k}^{n}\binom{n}{i}p_1^i(1-p_1)^{n-i} + (1-\alpha)\sum_{i=k}^{n}\binom{n}{i}p_2^i(1-p_2)^{n-i}$

45. with 3: $P\{\text{pass}\} = \frac{1}{3}\left[\binom{3}{2}(.8)^2(.2) + (.8)^3\right] + \frac{2}{3}\left[\binom{3}{2}(.4)^2(.6) + (.4)^3\right]$

$= .533$

with 5: $P\{\text{pass}\} = \frac{1}{3}\sum_{i=3}^{5}\binom{5}{i}(.8)^i(.2)^{5-i} + \frac{2}{3}\sum_{i=3}^{5}\binom{5}{i}(.4)^i(.6)^{5-i}$

$= .3038$

47. (a) and (b): (i) $\sum_{i=5}^{9}\binom{9}{i}p^i(1-p)^{9-i}$, (ii) $\sum_{i=5}^{8}\binom{8}{i}p^i(1-p)^{8-i}$,

(iii) $\sum_{i=4}^{7}\binom{7}{i}p^i(1-p)^{7-i}$ where $p = .7$ in (a) and $p = .3$ in (b).

48. The probability that a package will be returned is $p = 1 - (.99)^{10} - 10(.99)^9(.01)$. Hence, if someone buys 3 packages then the probability they will return exactly 1 is $3p(1-p)^2$.

49. (a) $\frac{1}{2}\binom{10}{7}.4^7.6^3 + \frac{1}{2}\binom{10}{7}.7^7.3^3$

(b) $\dfrac{\frac{1}{2}\binom{9}{6}.4^7.6^3 + \frac{1}{2}.7^7.3^3}{.55}$

50. (a) $P\{H, T, T \mid 6 \text{ heads}\}$ $= P(H, T, T \text{ and } 6 \text{ heads}\}/P\{6 \text{ heads}\}$
$= P\{H, T, T\}P\{6 \text{ heads} \mid H, T, T\}/P\{6 \text{ heads}\}$
$= pq^2\binom{7}{5}p^5q^2 \Big/ \binom{10}{6}p^6q^4$
$=1/10$

(b) $P\{T, H, T \mid 6 \text{ heads}\}$ $= P(T, H, T \text{ and } 6 \text{ heads}\}/P\{6 \text{ heads}\}$
$= P\{T, H, T\}P\{6 \text{ heads} \mid T, H, T\}/P\{6 \text{ heads}\}$
$= q^2p\binom{7}{5}p^5q^2 \Big/ \binom{10}{6}p^6q^4$
$=1/10$

51. (a) $e^{-.2}$ (b) $1 - e^{-.2} - .2e^{-.2} = 1 - 1.2e^{-.2}$
Since each letter has a small probability of being a typo, the number of errors should approximately have a Poisson distribution.

52. (a) $1 - e^{-3.5} - 3.5e^{-3.5} = 1 - 4.5e^{-3.5}$

(b) $4.5e^{-3.5}$

Since each flight has a small probability of crashing it seems reasonable to suppose that the number of crashes is approximately Poisson distributed.

53. (a) The probability that an arbitrary couple were both born on April 30 is, assuming independence and an equal chance of having being born on any given date, $(1/365)^2$. Hence, the number of such couples is approximately Poisson with mean $80{,}000/(365)^2 \approx .6$. Therefore, the probability that at least one pair were both born on this date is approximately $1 - e^{-.6}$.

(b) The probability that an arbitrary couple were born on the same day of the year is 1/365. Hence, the number of such couples is approximately Poisson with mean $80{,}000/365 \approx 219.18$. Hence, the probability of at least one such pair is $1 - e^{-219.18} \approx 1$.

54. (a) $e^{-2.2}$ (b) $1 - e^{-2.2} - 2.2e^{-2.2} = 1 - 3.2e^{-2.2}$

55. $\frac{1}{2}e^{-3} + \frac{1}{2}e^{-4.2}$

56. The number of people in a random collection of size $n$ that have the same birthday as yourself is approximately Poisson distributed with mean $n/365$. Hence, the probability that at least one person has the same birthday as you is approximately $1 - e^{-n/365}$. Now, $e^{-x} = 1/2$ when $x = \log(2)$. Thus, $1 - e^{-n/365} \geq 1/2$ when $n/365 \geq \log(2)$. That is, there must be at least 365 log(2) people.

57. (a) $1 - e^{-3} - 3e^{-3} - e^{-3}\frac{3^2}{2} = 1 - \frac{17}{2}e^{-3}$

(b) $P\{X \geq 3 \mid X \geq 1\} = \frac{P\{X \geq 3\}}{P\{X \geq 1\}} = \frac{1 - \frac{17}{2}e^{-3}}{1 - e^{-3}}$

59. (a) $1 - e^{-1/2}$

(b) $\frac{1}{2}e^{-1/2}$

(c) $1 - e^{-1/2} = \frac{1}{2}e^{-1/2}$

60. $$P\{\text{beneficial} \mid 2\} = \frac{P\{2 \mid \text{beneficial}\}3/4}{P\{2 \mid \text{beneficial}\}3/4 + P\{2 \mid \text{not beneficial}\}1/4}$$

$$= \frac{e^{-3}\frac{3^2}{2}\frac{3}{4}}{e^{-3}\frac{3^2}{2}\frac{3}{4} + e^{-5}\frac{5^2}{2}\frac{1}{4}}$$

61. $1 - e^{-1.4} - 1.4e^{-1.4}$

62. If $A_i$ is the event that couple number $i$ are seated next to each other, then these events are, when $n$ is large, roughly independent. As $P(A_i = 2/(2n - 1)$ it follows that, for $n$ large, the number of wives that sit next to their husbands is approximately Poisson with mean $2n/(2n - 1) \approx 1$. Hence, the desired probability is $e^{-1} = .368$ which is not particularly close to the exact solution of .2656 provided in Example 5$n$ of Chapter 2, thus indicating that $n = 10$ is not large enough for the approximation to be a good one.

63. (a) $e^{-2.5}$

(b) $1 - e^{-2.5} - 2.5e^{-2.5} - \frac{(2.5)^2}{2}e^{-2.5} - \frac{(2.5)^3}{3!}e^{-2.5}$

64. (a) $1 - \sum_{i=0}^{7} e^{-4} 4^i / i! \equiv p$

(b) $1 - (1 - p)^{12} - 12p(1 - p)^{11}$

(c) $(1 - p)^{i-1} p$

65. (a) $1 - e^{-1/2}$

(b) $P\{X \geq 2 \mid X \geq 1\} = \dfrac{1 - e^{-1/2} - \frac{1}{2} e^{-1/2}}{1 - e^{-1/2}}$

(c) $1 - e^{-1/2}$

(d) $1 - \exp\{-500 - i)/1000\}$

66. Assume $n > 1$.

(a) $\dfrac{2}{2n-1}$

(b) $\dfrac{2}{2n-2}$

(c) $\exp\{-2n/(2n - 1)\} \approx e^{-1}$

67. Assume $n > 1$.

(a) $\dfrac{2}{n}$

(b) Conditioning on whether the man of couple $j$ sits next to the woman of couple $i$ gives the result: $\dfrac{1}{n-1}\dfrac{1}{n-1} + \dfrac{n-2}{n-1}\dfrac{2}{n-1} = \dfrac{2n-3}{(n-1)^2}$

(c) $e^{-2}$

68. $\exp(-10e^{-5}\}$

69. With $P_j$ equal to the probability that 4 consecutive heads occur within $j$ flips of a fair coin, $P_1 = P_2 = P + 3 = 0$, and

$P_4 = 1/16$
$P_5 = (1/2)P_4 + 1/16 = 3/32$
$P_6 = (1/2)P_5 + (1/4)P_4 + 1/16 = 1/8$
$P_7 = (1/2)P_6 + (1/4)P_5 + (1/8)P_4 + 1/16 = 5/32$
$P_8 = (1/2)P_7 + (1/4)P_6 + (1/8)P_5 + (1/16)P_4 + 1/16 = 6/32$
$P_9 = (1/2)P_8 + (1/4)P_7 + (1/8)P_6 + (1/16)P_5 + 1/16 = 111/512$
$P_{10} = (1/2)P_9 + (1/4)P_8 + (1/8)P_7 + (1/16)P_6 + 1/16 = 251/1024 = .2451$

The Poisson approximation gives

$$P_{10} \approx 1 - \exp\{-6/32 - 1/16\} = 1 - e^{-.25} = .2212$$

70. $e^{-\lambda t} + (1 - e^{-\lambda t})p$

71. (a) $\left(\frac{26}{38}\right)^5$

(b) $\left(\frac{26}{38}\right)^3 \frac{12}{38}$

72. $P\{\text{wins in } i \text{ games}\} = \binom{i-1}{3}(.6)^4(.4)^{i-4}$

73. Let $N$ be the number of games played. Then

$P\{N = 4\} = 2(1/2)^4 = 1/8$, $\quad$ $P\{N = 5\} = 2\binom{4}{1}(1/2)(1/2)^4 = 1/4$

$P\{N = 6\} = 2\binom{5}{2}(1/2)^2(1/2)^4 = 5/16$, $\quad$ $P\{N = 7\} = 5/16$

$E[N] = 4/8 + 5/4 + 30/16 + 35.16 = 93/16 = 5.8125$

74. (a) $\left(\frac{2}{3}\right)^5$

(b) $\binom{8}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^3 + \binom{8}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^2 + \binom{8}{7}\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^8$

(c) $\binom{5}{4}\left(\frac{2}{3}\right)^5\frac{1}{3}$

(d) $\binom{6}{4}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^2$

76. $\binom{N_1 + N_2 - k}{N_1}(1/2)^{N_1+N_2-k}(1/2) + \binom{N_1 + N_2 - k}{N_2}(1/2)^{N_1+N_2-k}(1/2)$

77. $2\binom{2N-k}{N}(1/2)^{2N-k}$

$2\binom{2N-k-1}{N-1}(1/2)^{2N-k-1}(1/2)$

79. (a) $P\{X = 0\} = \dfrac{\binom{94}{10}}{\binom{100}{10}}$

(b) $P\{X > 2\} = 1 - \dfrac{\binom{94}{10} + \binom{94}{9}\binom{6}{1} + \binom{94}{8}\binom{6}{2}}{\binom{100}{10}}$

80. $P\{\text{rejected} \mid 1 \text{ defective}\} = 3/10$

$P\{\text{rejected} \mid 4 \text{ defective}\} = 1 - \binom{6}{3}\Big/\binom{10}{3} = 5/6$

$P\{4 \text{ defective} \mid \text{rejected}\} = \dfrac{\frac{5}{6}\frac{3}{10}}{\frac{5}{6}\frac{3}{10} + \frac{3}{10}\frac{7}{10}} = 75/138$

81. $P\{\text{rejected}\} = 1 - (.9)^4$

## Theoretical Exercises

1. Let $E_i$ = {no type $i$ in first $n$ selections}

$$P\{T > n\} = P\left(\bigcup_{i=1}^{N} E_i\right)$$
$$= \sum_i (1-P_i)^n - \sum\sum_{I<J}(1-P_i-P_j)^n + \sum\sum\sum_{i<j<k}(1-p_J-p_k)^n$$
$$\ldots + (-1)^N \sum_i P_i^n$$

$$P\{T = n\} = P\{\mathrm{T} > n-1\} - P\{T > n\}$$

3. $1 - \lim_{h\to 0} F(a-h)$

4. Not true. Suppose $P\{X = b\} = \varepsilon > 0$ and $b_n = b + 1/n$. Then $\lim_{b_n \to b} P(X < b_n\} = P\{X \le b\} \ne$ $P\{X < b\}$.

5. When $\alpha > 0$

$$P\{\alpha X + \beta \le x\} = P\left\{x \le \frac{x-\beta}{\alpha}\right\} = F\left(\frac{x-\beta}{\alpha}\right)$$

When $\alpha < 0$

$$P\{\alpha X + \beta \le x\} = P\left\{X \ge \frac{x-\beta}{\alpha}\right\} = 1 - \lim_{h\to 0^+} F\left(\frac{x-\beta}{\alpha} - 1\right).$$

6. $$\sum_{i=1}^{\infty} P\{N \ge i\} = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} P\{N = k\}$$
$$= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} P\{N = K\}$$
$$= \sum_{k=1}^{\infty} kP\{N = k\} = E[N].$$

7. $$\sum_{i=0}^{\infty} i\, P\{N > i\} = \sum_{i=0}^{\infty} i \sum_{k=i+1}^{\infty} P\{N = k\}$$
$$= \sum_{k=1}^{\infty} P\{N = k\} \sum_{i=0}^{k-1} i$$
$$= \sum_{k=1}^{\infty} P\{N = k\}(k-1)k/2$$
$$= \left(\sum_{k=1}^{\infty} k^2 P\{N = k\} - \sum_{k=1}^{\infty} kP\{N = k\}\right)\Big/ 2$$

8. $E[c^X] = cp + c^{-1}(1-p)$

Hence, $1 = E[c^X]$ if

$cp + c^{-1}(1-p) = 1$

or, equivalently

$pc^2 - c + 1 - p = 0$

or

$(pc - 1 + p)(c - 1) = 0$

Thus, $c = (1-p)/p$.

9. $E[Y] = E[X/\sigma - \mu/\sigma] = \dfrac{1}{\sigma}E[X] - \mu/\sigma = \mu/\sigma - \mu/\sigma = 0$

$\text{Var}(Y) = (1/\sigma)^2\ \text{Var}(X) = \sigma^2/\sigma^2 = 1.$

10.
$$
\begin{aligned}
E[1/(X+1)] &= \sum_{i=0}^{n} \frac{1}{i+1}\frac{n!}{(n-i)!i!}p^i(1-p)^{n-i} \\
&= \sum_{i=0}^{n} \frac{n!}{(n-i)!(i+1)!}p^i(1-p)^{n-i} \\
&= \frac{1}{(n+1)p}\sum_{i=0}^{n}\binom{n+1}{i+1}p^{i+1}(1-p)^{n-i} \\
&= \frac{1}{(n+1)p}\sum_{j=1}^{n+1}\binom{n+1}{j}p^{j}(1-p)^{n+1-j} \\
&= \frac{1}{(n+1)p}\left[1-\binom{n+1}{0}p^0(1-p)^{n+1-0}\right] \\
&= \frac{1}{(n+1)p}[1-(1-p)^{n+1}]
\end{aligned}
$$

11. For any given arrangement of $k$ successes and $n - k$ failures:

$$P\{\text{arrangement} \mid \text{total of } k \text{ successes}\}$$

$$= \frac{P\{\text{arrangement}\}}{P\{k \text{ successes}\}} = \frac{p^k(1-p)^{n-k}}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{1}{\binom{n}{k}}$$

12. Condition on the number of functioning components and then use the results of Example 4c of Chapter 1:

$$\text{Prob} = \sum_{i=0}^{n}\binom{n}{i}p^i(1-p)^{n-i}\left[\binom{i+1}{n-i}\Big/\binom{n}{i}\right]$$

where $\binom{i+1}{n-i} = 0$ if $n - i > i + 1$. We are using the results of Exercise 11.

13. Easiest to first take log and then determine the $p$ that maximizes $\log P\{X = k\}$.

$$\log P\{X = k\} = \log\binom{n}{k} + k \log p + (n - k) \log (1 - p)$$

$$\frac{\partial}{\partial p}\log P\{x = k\} = \frac{k}{p} - \frac{n-k}{1-p}$$
$$= 0 \Rightarrow p = k/n \text{ maximizes}$$

14. (a) $1 - \sum_{n=1}^{\infty} \alpha p^n = 1 - \frac{\alpha p}{1-p}$

(b) Condition on the number of children: For $k > 0$

$$P\{k \text{ boys}\} = \sum_{n=1}^{\infty} P\{k | n \text{ children}\}\alpha p^n$$
$$= \sum_{n=k}^{\infty}\binom{n}{k}(1/2)^n \alpha p^n$$

$$P\{0 \text{ boys}\} = 1 - \frac{\alpha p}{1-p} + \sum_{n=1}^{\infty} \alpha p^n (1/2)^n$$

17. (a) If $X$ is binomial $(n, p)$ then, from exercise 15,

$$P\{X \text{ is even}\} = [1 + (1 - 2p)^n]/2$$
$$= [1 + (1 - 2\lambda/n)^n]/2 \quad \text{when } \lambda = np$$
$$\to (1 + e^{-2\lambda})/2 \quad \text{as } n \text{ approaches infinity}$$

(b) $P\{X \text{ is even}\} = e^{-\lambda}\sum_n \lambda^{2n}/(2n)! = e^{-\lambda}(e^{\lambda} + e^{-\lambda})/2$

18. $\log P\{X = k\} = -\lambda + k \log \lambda - \log (k!)$

$$\frac{\partial}{\partial \lambda}\log P\{X = k\} = -1 + \frac{k}{\lambda}$$
$$= 0 \Rightarrow \lambda = k$$

19. $E[X^n] = \sum_{i=0}^{\infty} i^n e^{-\lambda} \lambda^i / i!$

$$= \sum_{i=1}^{\infty} i^n e^{-\lambda} \lambda^i / i!$$

$$= \sum_{i=1}^{\infty} i^{n-1} e^{-\lambda} \lambda^i / (i-1)!$$

$$= \sum_{j=0}^{\infty} (j+1)^{n-1} e^{-\lambda} \lambda^{j+1} / j!$$

$$= \lambda \sum_{j=0}^{\infty} (j+1)^{n-1} e^{-\lambda} \lambda^j / j!$$

$$= \lambda E[(X+1)^{n-1}]$$

Hence $[X^3] = \lambda E(X+1)^2]$

$$= \lambda \sum_{i=0}^{\infty} (i+1)^2 e^{-\lambda} \lambda^i / i!$$

$$= \lambda \left[ \sum_{i=0}^{\infty} i^2 e^{-\lambda} \lambda^i / i! + 2 \sum_{i=0}^{\infty} i e^{-\lambda} \lambda^i / i! + \sum_{i=0}^{\infty} e^{-\lambda} \lambda^i / i! \right]$$

$$= \lambda [E[X^2] + 2E[X] + 1)$$

$$= \lambda(\mathrm{Var}(X) = E^2[X] + 2E[X] + 1)$$

$$= \lambda(\lambda + \lambda^2 + 2\lambda + 1) = \lambda(\lambda^2 + 3\lambda + 1)$$

20. Let $S$ denote the number of heads that occur when all $n$ coins are tossed, and note that $S$ has a distribution that is approximately that of a Poisson random variable with mean $\lambda$. Then, because $X$ is distributed as the conditional distribution of $S$ given that $S > 0$,

$$P\{X = 1\} = P\{S = 1 \mid S > 0\} = \frac{P\{S = 1\}}{P\{S > 0\}} \approx \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}}$$

21. (i) 1/365
(ii) 1/365
(iii) 1 The events, though independent in pairs, are not independent.

22. (i) Say that trial $i$ is a success if the $i^{\text{th}}$ pair selected have the same number. When $n$ is large trials 1, …, $k$ are roughly independent.
(ii) Since, $P\{\text{trial } i \text{ is a success}\} = 1/(2n-1)$ it follows that, when $n$ is large, $M_k$ is approximately Poisson distributed with mean $k/(2n-1)$. Hence,

$$P\{M_k = 0\} \approx \exp[-k/(2n-1)]$$

(iii) and (iv) $P\{T > \alpha n\} = P\{M_{\alpha n} = 0\} \approx \exp[-\alpha n/(2n-1)] \to e^{-\alpha/2}$

23. (a) $P(E_i) = 1 - \sum_{j=0}^{2} \binom{365}{j} (1/365)^j (364/365)^{365-j}$

(b) $\exp(-365P(E_1)\}$

24. (a) There will be a string of $k$ consecutive heads within the first $n$ trials either if there is one within the first $n - 1$ trials, or if the first such string occurs at trial $n$; the latter case is equivalent to the conditions of 2.

(b) Because cases 1 and 2 are mutually exclusive

$$P_n = P_{n-1} + (1 - P_{n-k-1})(1 - P)p^k$$

25. $$\begin{aligned} P(m \text{ counted}) &= \sum_n P(m \mid n \text{ events}) e^{-\lambda} \lambda^n / n! \\ &= \sum_{n=m}^{\infty} \binom{n}{m} p^m (1-p)^{n-m} e^{-\lambda} \lambda^n / n! \\ &= e^{-\lambda p} \frac{(\lambda p)^m}{m!} \sum_{n=m}^{\infty} \frac{[\lambda(1-p)]^{n-m}}{(n-m)!} e^{-\lambda(1-p)} \\ &= e^{-\lambda p} \frac{(\lambda p)^m}{m!} \end{aligned}$$

Intuitively, the Poisson $\lambda$ random variable arises as the approximate number of successes in $n$ (large) independent trials each having a small success probability $\alpha$ (and $\lambda \; n\alpha$). Now if each successful trial is counted with probability $p$, than the number counted is Binomial with parameters $n$ (large) and $\alpha p$ (small) which is approximately Poisson with parameter $\alpha pn = \lambda p$.

27. $$\begin{aligned} P\{X = n + k \mid X > n\} &= \frac{P\{X = n + k\}}{P\{X > n\}} \\ &= \frac{p(1-p)^{n+k-1}}{(1-p)^n} \\ &= p(1-p)^{k-1} \end{aligned}$$

If the first $n$ trials are fall failures, then it is as if we are beginning anew at that time.

28. The events $\{X > n\}$ and $\{Y < r\}$ are both equivalent to the event that there are fewer than $r$ successes in the first $n$ trials; hence, they are the same event.

29. $$\begin{aligned} \frac{P\{X = k+1\}}{P\{X = k\}} &= \frac{\binom{Np}{k+1}\binom{N-np}{n-k-1}}{\binom{Np}{k}\binom{N-Np}{n-k}} \\ &= \frac{(Np-k)(n-k)}{(k+1)(N-Np-n+k+1)} \end{aligned}$$

30. $P\{Y=j\} = \binom{j-1}{n-1}\bigg/\binom{N}{n},\ n \le j \le N$

$$E[Y] = \sum_{j=n}^{N}\binom{j-1}{n-1}\bigg/\binom{N}{n}$$

$$= \frac{n}{\binom{N}{n}}\sum_{j=n}^{N}\binom{j}{n}$$

$$= \frac{n}{\binom{N}{n}}\sum_{i=n+1}^{N+1}\binom{i-1}{n+1-1}$$

$$= \frac{n}{\binom{N}{n}}\binom{N+1}{n+1}$$

$$= \frac{n(N+1)}{n+1}$$

31. Let $Y$ denote the largest of the remaining $m$ chips. By exercise 28

$$P\{Y=j\} = \binom{j-1}{m-1}\bigg/\binom{m+n}{m},\ m \le j \le n+m$$

Now, $X = n + m - Y$ and so

$$P\{X=i\} = P\{Y=m+n-i\} = \binom{m+n-i-1}{m-1}\bigg/\binom{m+n}{m},\ i \le n$$

32. $P\{X=k\} = \frac{k-1}{n}\prod_{i=0}^{k-2}\frac{n-i}{n},\ k > 1$

34. $E[X] = \sum_{k=0}^{n}\frac{k\binom{n}{k}}{2^n-1} = \frac{n2^{n-1}}{2^n-1}$

$$E[X^2] = \sum_{k=0}^{n}\frac{k^2\binom{n}{k}}{2^n-1} = \frac{2^{n-2}n(n+1)}{2^n-1}$$

$$\text{Var}(X) = E[X^2] - \{E[X]\}^2 = \frac{n2^{2n-2} - n(n+1)2^{n-2}}{(2^n-1)^2}$$

$$\sim \frac{n2^{2n-2}}{2^{2n}} = \frac{n}{4}$$

$$E[Y] = \frac{n+1}{2},\ E[Y^2] = \sum_{i=1}^{n} i^2/n \sim \int_1^{n+1} \frac{\mathrm{x}^2\,dx}{n} \sim \frac{n^2}{3}$$

$$\mathrm{Var}(Y) \sim \frac{n^2}{3} - \left(\frac{n+1}{2}\right)^2 \sim \frac{n^2}{12}$$

35. (a) $P\{X > i\} = \frac{1}{2}\frac{2}{3}\cdots\frac{i}{i+1} = \frac{1}{i+1}$

(b) $P(X < \infty\} = \lim_{i\to\infty} P\{X \le i\}$

$= \lim_i (1 - 1/(i+1)) = 1$

(c) $E[X] = \sum_i iP\{X = i\}$

$= \sum_i i(P\{X > i-1\} - P\{X > i\}$

$= \sum_i i\left(\frac{1}{i} - \frac{1}{i+1}\right)$

$= \sum_i \frac{1}{i+1}$

$= \infty$

# Chapter 5

## Problems

1. (a) $c\int_{-1}^{1}(1-x^2)dx = 1 \Rightarrow c = 3/4$

   (b) $F(x) = \frac{3}{4}\int_{-1}^{x}(1-x^2)dx = \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right), -1 < x < 1$

2. $\int xe^{-x/2}dx = -2xe^{-x/2} - 4e^{-x/2}$. Hence,

$$c\int_{0}^{\infty} xe^{-x/2}dx = 1 \Rightarrow c = 1/4$$

$$P\{X > 5\} = \frac{1}{4}\int_{5}^{\infty} xe^{-x/2}dx = \frac{1}{4}[10e^{-5/2} + 4e^{-5/2}]$$

$$= \frac{14}{4}e^{-5/2}$$

3. No. $f(5/2) < 0$

4. (a) $\int_{20}^{\infty}\frac{10}{x^2}dx = \frac{-10}{x}\Big|_{20}^{\infty} = 1/2$.

   (b) $F(y) = \int_{10}^{y}\frac{10}{x^2}dx = 1 - \frac{10}{y}$, $y > 10$. $F(y) = 0$ for $y < 10$.

   (c) $\sum_{i=3}^{6}\binom{6}{i}\left(\frac{2}{3}\right)^i\left(\frac{1}{3}\right)^{6-i}$ since $\overline{F}(15) = \frac{10}{15}$. Assuming independence of the events that the devices exceed 15 hours.

5. Must choose $c$ so that

$$.01 = \int_{c}^{1} 5(1-x)^4 dx = (1-c)^5$$

so $c = 1 - (.01)^{1/5}$.

6. (a) $E[X] = \dfrac{1}{4}\displaystyle\int_0^\infty x^2 e^{-x/2}dx = 2\int_0^\infty y^2 e^{-y}dx = 2\Gamma(3) = 4$

(b) By symmetry of $f(x)$ about $x = 0$, $E[X] = 0$

(c) $E[X] = \displaystyle\int_5^\infty \frac{5}{x}dx = \infty$

7. $\displaystyle\int_0^1 (a + bx^2)dx = 1$ or $a + \dfrac{b}{3} = 1$

$\displaystyle\int_0^1 x(a + bx^2)dx = \frac{3}{5}$ or $\dfrac{a}{2} + \dfrac{b}{4} = 3/5$. Hence,

$a = \dfrac{3}{5}$, $b = \dfrac{6}{5}$

8. $E[X] = \displaystyle\int_0^\infty x^2 e^{-x}dx = \Gamma(3) = 2$

9. If $s$ units are stocked and the demand is $X$, then the profit, $P(s)$, is given by

$$
\begin{aligned}
P(s) &= bX - (s - X)P \qquad && \text{if } X \le s \\
&= sb && \text{if } X > s
\end{aligned}
$$

Hence

$$
\begin{aligned}
E[P(s)] &= \int_0^s (bx - (s-x)\ell)f(x)dx + \int_s^\infty sbf(x)dx \\
&= (b+\ell)\int_0^s xf(x)dx - s\ell\int_0^s f(x)dx + sb\left[1 - \int_0^s f(x)dx\right] \\
&= sb + (b+\ell)\int_0^s (x-s)f(x)dx
\end{aligned}
$$

Differentiation yields

$$
\begin{aligned}
\frac{d}{ds}E[P(s)] &= b + (b+\ell)\frac{d}{ds}\left[\int_0^s xf(x)dx - s\int_0^s f(x)dx\right] \\
&= b + (b+\ell)\left[sf(s) - sf(s) - \int_0^s f(s)dx\right] \\
&= b - (b+\ell)\int_0^s f(x)dx
\end{aligned}
$$

Equating to zero shows that the maximal expected profit is obtained when $s$ is chosen so that

$$F(s) = \frac{b}{b+\ell}$$

where $F(s) = \int_0^s f(x)dx$ is the cumulative distribution of demand.

10. (a) $P\{\text{goes to } A\} = P\{5 < X < 15 \text{ or } 20 < X < 30 \text{ or } 35 < X < 45 \text{ or } 50 < X < 60\}$.
$= 2/3$ since $X$ is uniform (0, 60).

(b) same answer as in (a).

11. $X$ is uniform on $(0, L)$.

$$P\left\{\min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) < 1/4\right\}$$
$$= 1 - P\left\{\min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) > 1/4\right\}$$
$$= 1 - P\left\{\frac{X}{L-X} > 1/4, \frac{L-X}{X} > 1/4\right\}$$
$$= 1 - P\{X > L/5, X < 4L/5\}$$
$$= 1 - P\left\{\frac{L}{5} < X < 4L/5\right\}$$
$$= 1 - \frac{3}{5} = \frac{2}{5}.$$

13. $P\{X > 10\} = \frac{2}{3}$, $P\{X > 25 \mid X > 15\} = \frac{P\{X > 25}{P\{X > 15\}} = \frac{5/30}{15/30} = 1/3$
where $X$ is uniform (0, 30).

14. $E[X^n] = \int_0^1 x^n dx = \frac{1}{n+1}$

$P\{X^n \le x\} = P\{X \le x^{1/n}\} = x^{1/n}$

$E[X^n] = \int_0^1 x \frac{1}{n} x^{\left(\frac{1}{n}-1\right)} dx = \frac{1}{n}\int_0^1 x^{1/n} dx = \frac{1}{n+1}$

15. (a) $\Phi(.8333) = .7977$
(b) $2\Phi(1) - 1 = .6827$
(c) $1 - \Phi(.3333) = .3695$
(d) $\Phi(1.6667) = .9522$
(e) $1 - \Phi(1) = .1587$

16. $P\{X > 50\} = P\left\{\frac{X-40}{4} > \frac{10}{4}\right\} = 1 - \Phi(2.5) = 1 - .9938$

Hence, $(P\{X < 50\})^{10} = (.9938)^{10}$

17. $E[\text{Points}] = 10(1/10) + 5(2/10) + 3(2/10) = 2.6$

18. $.2 = P\left\{\frac{X-5}{\sigma} > \frac{9-5}{\sigma}\right\} = P\{Z > 4/\sigma\}$ where $Z$ is a standard normal. But from the normal table $P\{Z < .84) \approx .80$ and so

$$.84 \approx 4/\sigma \text{ or } \sigma \approx 4.76$$

That is, the variance is approximately $(4.76)^2 = 22.66$.

19. Letting $Z = (X - 12)/2$ then $Z$ is a standard normal. Now, $.10 = P\{Z > (c - 12)/2\}$. But from Table 5.1, $P\{Z < 1.28\} = .90$ and so

$$(c - 12)/2 = 1.28 \quad \text{or} \quad c = 14.56$$

20. Let $X$ denote the number in favor. Then $X$ is binomial with mean 65 and standard deviation $\sqrt{65(.35)} \approx 4.77$. Also let $Z$ be a standard normal random variable.

(a) $P\{X \geq 50\} = P\{X \geq 49.5\} = P\{X - 65\}/4.77 \geq -15.5/4.77$
$\approx P\{Z \geq -3.25\} \approx .9994$

(b) $P\{59.5 \leq X \leq 70.5\} \approx P\{-5.5/4.77 \leq Z \leq 5.5/4.77\}$
$= 2P\{Z \leq 1.15\} - 1 \approx .75$

(c) $P\{X \leq 74.5\} \approx P\{Z \leq 9.5/4.77\} \approx .977$

22. (a) $P\{.9000 - .005 < X < .9000 + .005\}$

$$= P\left\{-\frac{.005}{.003} < Z < \frac{.005}{.003}\right\}$$
$$= P\{-1.67 < Z < 1.67\}$$
$$= 2\Phi(1.67) - 1 = .9050.$$

Hence 9.5 percent will be defective (that is each will be defective with probability $1 - .9050 = .0950$).

(b) $P\left\{-\frac{.005}{\sigma} < Z < \frac{.005}{\sigma}\right\} = 2\Phi\left(\frac{.005}{\sigma}\right) - 1 = .99$ when

$$\Phi\left(\frac{.005}{\sigma}\right) = .995 \Rightarrow \frac{.005}{\sigma} = 2.575 \Rightarrow \sigma = .0019\,.$$

23. (a) $P\{149.5 < X < 200.5\} = P\left\{\frac{149.5 - \frac{1000}{6}}{\sqrt{1000\frac{1}{6}\frac{5}{6}}} < Z < \frac{200.5 - \frac{1000}{6}}{\sqrt{1000\frac{1}{6}\frac{5}{6}}}\right\}$

$$= \Phi\left(\frac{200.5 - 166.7}{\sqrt{5000/36}}\right) - \Phi\left(\frac{149.5 - 166.7}{\sqrt{5000/36}}\right)$$

$$\approx \Phi(2.87) + \Phi(1.46) - 1 = .9258.$$

(b) $P\{X < 149.5\} = P\left\{Z < \frac{149.5 - 800(1/5)}{\sqrt{800\frac{1}{5}\frac{4}{5}}}\right\}$

$$= P\{Z < -.93\}$$
$$= 1 - \Phi(.93) = .1762.$$

24. With $C$ denoting the life of a chip, and $\phi$ the standard normal distribution function we have

$$P\{C < 1.8 \times 10^6\} = \phi\left(\frac{1.8\times10^6 - 1.4\times10^6}{3\times10^5}\right)$$
$$= \phi(1.33)$$
$$= .9082$$

Thus, if $N$ is the number of the chips whose life is less than $1.8 \times 10^6$ then $N$ is a binomial random variable with parameters (100, .9082). Hence,

$$P\{N > 19.5\} \approx 1 - \phi\left(\frac{19.5 - 90.82}{90.82(.0918)}\right) = 1 - \phi(-24.7) \approx 1$$

25. Let $X$ denote the number of unacceptable items among the next 150 produced. Since $X$ is a binomial random variable with mean 150(.05) = 7.5 and variance 150(.05)(.95) = 7.125, we obtain that, for a standard normal random variable $Z$.

$$\begin{aligned} P\{X \le 10\} &= P\{X \le 10.5\} \\ &= P\left\{\frac{X - 7.5}{\sqrt{7.125}} \le \frac{10.5 - 7.5}{\sqrt{7.125}}\right\} \\ &\approx P\{Z \le 1.1239\} \\ &= .8695 \end{aligned}$$

The exact result can be obtained by using the text diskette, and (to four decimal places) is equal to .8678.

27. $P\{X > 5{,}799.5\} = P\left\{Z > \frac{799.5}{\sqrt{2{,}500}}\right\}$

$$= P\{Z > 15.99\} = \text{negligible}.$$

28. Let $X$ equal the number of lefthanders. Assuming that $X$ is approximately distributed as a binomial random variable with parameters $n = 200$, $p = .12$, then, with $Z$ being a standard normal random variable,

$$P\{X > 19.5\} = P\left\{\frac{X - 200(.12)}{\sqrt{200(.12)(.88)}} > \frac{19.5 - 200(.12)}{\sqrt{200(.12)(.88)}}\right\}$$
$$\approx P\{Z > -.9792\}$$
$$\approx .8363$$

29. Let $s$ be the initial price of the stock. Then, if $X$ is the number of the 1000 time periods in which the stock increases, then its price at the end is

$$su^X d^{1000\text{-}X} = sd^{1000}\left(\frac{u}{d}\right)^X$$

Hence, in order for the price to be at least 1.3$s$, we would need that

$$d^{1000}\left(\frac{u}{d}\right)^X > 1.3$$

or

$$X > \frac{\log(1.3) - 1000\log(d)}{\log(u/d)} = 469.2$$

That is, the stock would have to rise in at least 470 time periods. Because $X$ is binomial with parameters 1000, .52, we have

$$P\{X > 469.5\} = P\left\{\frac{X - 1000(.52)}{\sqrt{1000(.52)(.48)}} > \frac{469.5 - 1000(.52)}{\sqrt{1000(.52)(.48)}}\right\}$$
$$\approx P\{Z > -3.196\}$$
$$\approx .9993$$

30. $$P\{\text{in black}\} = \frac{P\{5 \mid \text{black}\}\alpha}{P\{5 \mid \text{black}\}\alpha + P\{5 \mid \text{white}\}(1-\alpha)}$$

$$= \frac{\frac{1}{2\sqrt{2\pi}}e^{-(5-4)^2/8}\alpha}{\frac{1}{2\sqrt{2\pi}}e^{-(5-4)^2/8}\alpha + (1-\alpha)\frac{1}{3\sqrt{2\pi}}e^{-(5-6)^2/18}}$$

$$= \frac{\frac{\alpha}{2}e^{-1/8}}{\frac{\alpha}{2}e^{-1/8} + \frac{(1-\alpha)}{3}e^{-1/8}}$$

$\alpha$ is the value that makes preceding equal 1/2

31. (a) $E[|X-a|] = \int_a^A (x-a)\frac{dx}{A} + \int_0^a (a-x)\frac{dx}{A} = \frac{A}{2} - \left(a - \frac{a^2}{A}\right)$

$$\frac{d}{da}(\quad) = \frac{2a}{A} - 1 = 0 \Rightarrow a = A/2$$

(b) $E[|X-a|] = \int_0^a (a-x)\lambda e^{-\lambda x}dx + \int_a^\infty (x-a)\lambda e^{-\lambda x}dx$

$$= a(1-e^{-\lambda a}) + ae^{-\lambda a} + \frac{e^{-\lambda a}}{\lambda} - \frac{1}{\lambda} + ae^{-\lambda a} + \frac{e^{-\lambda a}}{\lambda} - ae^{-\lambda a}$$

Differentiation yields that the minimum is attained at $\bar{a}$ where

$e^{-\lambda\bar{a}} = 1/2$ or $\bar{a} = \log 2/\lambda$

(c) Minimizing $a$ = median of $F$

32. (a) $e^{-1}$
(b) $e^{-1/2}$

33. $e^{-1}$

34. (a) $P\{X > 20\} = e^{-1}$

(b) $P\{X > 30 | X > 10 = \frac{P\{X > 30\}}{P\{X > 10\}} = \frac{1/4}{3/4} = 1/3$

35. (a) $\exp\left[-\int_{40}^{50} \lambda(t)dt\right] = e^{-.35}$

(b) $e^{-1.21}$

36. (a) $1 - F(2) = \exp\left[-\int_0^2 t^3 dt\right] = e^{-4}$

(b) $\exp[-(.4)^4/4] - \exp[-(1.4)^4/4]$

(c) $\exp\left[-\int_1^2 t^3 dt\right] = e^{-15/4}$

37. (a) $P\{|X| > 1/2\} = P\{X > 1/2\} + P\{X < -1/2\} = 1/2$

(b) $P\{|X| \le a\} = P\{-a \le X \le a\} = a$, $0 < a < 1$. Therefore,
$f_{|X|}(a) = 1,\ 0 < a < 1$
That is, $|X|$ is uniform on (0, 1).

38. For both roots to be real the discriminant $(4Y)^2 - 44(Y + 2)$ must be $\geq 0$. That is, we need that $Y^2 \geq Y + 2$. Now in the interval $0 < Y < 5$.

$$Y^2 \geq Y + 2 \Leftrightarrow Y \geq 2 \text{ and so}$$
$$P\{Y^2 \geq Y + 2\} = P\{Y \geq 2\} = 3/5.$$

39. $F_Y(y) = P\{\log X \leq y\}$
$= P\{X \leq e^y\} = F_X(e^y)$

$f_Y(y) = f_X(e^y)e^y = e^y e^{-e^y}$

40. $F_Y(y) = P\{e^X \leq y\}$
$= F_X(\log y)$

$$f_Y(y) = f_X(\log y)\frac{1}{y} = \frac{1}{y}, \; 1 < y < e$$

## Theoretical Exercises

1. The integration by parts formula $\int u dv = uv - \int v du$ with $dv = -2bxe^{-bx^2}$, $u = -x/2b$ yields that

$$\int_0^\infty x^2 e^{-bx^2} dx = \frac{-xe^{-bx^2}}{2b}\Big|_0^\infty + \frac{1}{2b}\int_0^\infty e^{-bx^2} dx$$

$$= \frac{1}{(2b)^{3/2}}\int_0^\infty e^{-y^2/2} dy \text{ by } y = x\sqrt{2b}$$

$$= \frac{\sqrt{2\pi}}{2}\frac{1}{(2b)^{3/2}} = \frac{\sqrt{\pi}}{4b^{3/2}}$$

where the above uses that $\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-y^2/2} dy = 1/2$. Hence, $a = \frac{4b^{3/2}}{\sqrt{\pi}}$

2. $$\int_0^\infty P\{Y < -y\} dy = \int_0^\infty \int_{-\infty}^{-y} f_Y(x) dx\, dy$$

$$= \int_{-\infty}^{0} \int_0^{-x} f_Y(x) dy\, dx = -\int_{-\infty}^{0} x f_Y(x)\, dx$$

Similarly,

$$\int_0^\infty P\{Y > y\} dy = \int_0^\infty x f_Y(x)\, dx$$

Subtracting these equalities gives the result.

4. $$E[aX + b] = \int (ax+b) f(x)\, dx = a\int x f(x) dx + b\int f(x) dx$$

$$= aE[X] + b$$

5. $$E[X^n] = \int_0^\infty P\{X^n > t\} dt$$

$$= \int_0^\infty P\{X^n > x^n\} n x^{n-1} dx \text{ by } t = x^n,\ dt = nx^{n-1}dx$$

$$= \int_0^\infty P\{X > x\} n x^{n-1} dx$$

6. Let $X$ be uniform on (0, 1) and define $E_a$ to be the event that $X$ is unequal to $a$. Since $\bigcap_a E_a$ is the empty set, it must have probability 0.

7. $SD(aX + b) = \sqrt{\text{Var}(aX + b)} = \sqrt{a^2\sigma^2} = |a|\sigma$

8. Since $0 \le X \le c$, it follows that $X^2 \le cX$. Hence,

$$\begin{aligned}
\text{Var}(X) &= E[X^2] - (E[X])^2 \\
&\le E[cX - (E[X])^2 \\
&= cE[X] - (E[X])^2 \\
&= E[X](c - E[X]) \\
&= c^2[\alpha(1 - \alpha)] \quad \text{where } \alpha = E[X]/c \\
&\le c^2/4
\end{aligned}$$

where the last inequality first uses the hypothesis that $P\{0 \le X \le c\} = 1$ to calculate that $0 \le \alpha \le 1$ and then uses calculus to show that $\underset{0 \le \alpha \le 1}{\text{maximum}}\ \alpha(1 - \alpha) = 1/4$.

9. The final step of parts (a) and (b) use that $-Z$ is also a standard normal random variable.

(a) $P\{Z > x\} = P\{-Z < -x\} = P\{Z < -x\}$

(b) $P\{|Z| > x\} = P\{Z > x\} + P\{Z < -x\} = P\{Z > x\} + P\{-Z > x\}$
$= 2P\{Z > x\}$

(c) $P\{|Z| < x\} = 1 - P\{|Z| > x\} = 1 - 2P\{Z > x\}$ by (b)
$= 1 - 2(1 - P\{Z < x\})$

10. With $c = 1/\left(\sqrt{2\pi}\sigma\right)$ we have

$$f(x) = ce^{-(x-\mu)^2/2\sigma^2}$$
$$f'(x) = -ce^{-(x-\mu)^2/2\sigma^2}(x - \mu)/\sigma^2$$
$$f''(x) = c\sigma^{-4}e^{-(x-\mu)^2/2\sigma^2}(x - \mu)^2 - c\sigma^{-2}e^{-(x-\mu)^2/2\sigma^2}$$

Therefore,

$$f''(\mu + \sigma) = f''(\mu - \sigma) = c\sigma^{-2}e^{-1/2} - c\sigma^{-2}e^{-1/2} = 0$$

11. $E[X^2] = \int_0^\infty P\{X > x\}2x^{2-1}dx = 2\int_0^\infty xe^{-\lambda x}dx = \frac{2}{\lambda}E[X] = 2/\lambda^2$

12. (a) $\frac{b + a}{2}$

(b) $\mu$

(c) $1 - e^{-\lambda m} = 1/2$ or $m = \frac{1}{\lambda}\log 2$

13. (a) all values in (a, b)

(b) $\mu$

(c) 0

14. $P\{cX < x\} = P\{X < x/c\} = 1 - e^{-\lambda x/c}$

15. $\lambda(t) = \dfrac{f(t)}{\overline{F}(t)} = \dfrac{1/a}{(a-t)/a} = \dfrac{1}{a-t}$, $0 < t < a$

16. If $X$ has distribution function $F$ and density $f$, then for $a > 0$

$$F_{aX}(t) = P\{aX \le t\} = F(t/a)$$

and

$$f_{ax} = \frac{1}{a} f(t/a)$$

Thus,

$$\lambda_{aX}(t) = \frac{\frac{1}{a} f(t/a)}{1 - F(t/a)} = \frac{1}{a}\lambda_X(t/a) .$$

18. $E[X^k] = \int_0^\infty x^k \lambda e^{-\lambda x} dx = \lambda^{-k} \int_0^\infty \lambda e^{-\lambda x} (\lambda x)^k dx$

$$= \lambda^{-k}\, \Gamma(\mathrm{k}+1) = \mathrm{k}!/\lambda^k$$

19. $E[X^k] = \dfrac{1}{\Gamma(t)} \int_0^\infty x^k \lambda e^{-\lambda x} (\lambda x)^{t-1} dx$

$$= \frac{\lambda^{-k}}{\Gamma(t)} \int_0^\infty \lambda e^{-\lambda x} (\lambda x)^{t+k-1} dx$$

$$= \frac{\lambda^{-k}}{\Gamma(t)} \Gamma(t+k)$$

Therefore,

$E[X] = t/\lambda$,
$E[X^2] = = \lambda^{-2}\Gamma(t+2)/\Gamma(t) = (t+1)t/\lambda^2$

and thus

$$\mathrm{Var}(X) = (t+1)t/\lambda^2 - t^2/\lambda^2 = t/\lambda^2$$

20. $\Gamma(1/2) = \int_0^\infty e^{-x}x^{-1/2}dx$

$= \sqrt{2}\int_0^\infty e^{-y^2/2}dy$ by $x = y^2/2$, $dx = ydy = \sqrt{2x}\ dy$

$= 2\sqrt{\pi}\int_0^\infty (2\pi)^{-1/2}e^{-y^2/2}dy$

$= 2\sqrt{\pi}P\{Z > 0\}$ where $Z$ is a standard normal

$= \sqrt{\pi}$

21. $1/\lambda(s) = \int_{x \ge s} \lambda e^{-\lambda x}(\lambda x)^{t-1}dx / \lambda e^{-\lambda s}(\lambda s)^{t-1}$

$= \int_{x \ge s} e^{-\lambda(x-s)}(x/s)^{t-1}dx$

$= \int_{y \ge 0} e^{-\lambda y}(1 + y/s)^{t-1}dy$ by letting $y = x - s$

As the above, equal to the inverse of the hazard rate function, is clearly decreasing in $s$ when $t \ge 1$ and increasing when $t \le 1$ the result follows.

22. $\lambda(s) = c(s - v)^{\beta-1}$, $s > v$ which is clearly increasing when $\beta \ge 1$ and decreasing otherwise.

23. $F(\alpha) = 1 - e^{-1}$

24. Suppose $X$ is Weibull with parameters $v$, $\alpha$, $\beta$. Then

$$P\left\{\left(\frac{X-v}{\alpha}\right)^\beta \le x\right\} = P\left\{\frac{X-v}{\alpha} \le x^{1/\beta}\right\}$$
$$= P\{X \le v + \alpha x^{1/\beta}\}$$
$$= 1 - \exp\{-x\}.$$

25. We use Equation (6.3).

$$E[X] = B(a + 1, b)/B(A, b) = \frac{\Gamma(a+1)}{\Gamma(a+b+1)}\frac{\Gamma(a+b)}{\Gamma(a)} = \frac{a}{a+b}$$

$$E[X^2] = B(a + 2, b)/B(a, b) = \frac{\Gamma(a+2)}{\Gamma(a+b+2)}\frac{\Gamma(a+b)}{\Gamma(a)} = \frac{(a+1)a}{(a+b+1)(a+b)}$$

Thus,

$$\text{Var}(X) = \frac{(a+1)a}{(a+b+1)(a+b)} - \frac{a^2}{(a+b)^2} = \frac{ab}{(a+b+1)(a+b)^2}$$

26. $(X - a)/(b - a)$

28. $P\{F(X \le x\} = P\{X \le F^{-1}(x)\}$
$= F(F^{-1}(x))$
$= x$

29. $F_Y(x) = P\{aX + b \le x\}$

$= P\left\{X \le \dfrac{x-b}{a}\right\}$ when $a > 0$

$= F_X((x-b)/a)$ when $a > 0$.

$f_Y(x) = \dfrac{1}{a} f_X((x-b)/a)$ if $a > 0$.

When $a < 0$, $F_Y(x) = P\left\{X \ge \dfrac{x-b}{a}\right\} = 1 - F_X\left(\dfrac{x-b}{a}\right)$ and so

$f_Y(x) = -\dfrac{1}{a} f_X\left(\dfrac{x-b}{a}\right)$.

30. $F_Y(x) = P\{e^X \le x\}$
$= P\{X \le \log x\}$
$F_X(\log x)$

$f_Y(x) = f_X(\log x)/x$

$$= \frac{1}{x\sqrt{2\pi}\sigma} e^{-(\log x - \mu)^2 / 2\sigma^2}$$

# Chapter 6

## Problems

2. (a) $p(0, 0) = \dfrac{8 \cdot 7}{13 \cdot 12} = 14/39$,

$p(0, 1) = p(1, 0) = \dfrac{8 \cdot 5}{13 \cdot 12} = 10/39$

$p(1, 1) = \dfrac{5 \cdot 4}{13 \cdot 12} = 5/39$

(b) $p(0, 0, 0) = \dfrac{8 \cdot 7 \cdot 6}{13 \cdot 12 \cdot 11} = 28/143$

$p(0, 0, 1) = p(0, 1, 0) = p(1, 0, 0) = \dfrac{8 \cdot 7 \cdot 5}{13 \cdot 12 \cdot 11} = 70/429$

$p(0, 1, 1) = p(1, 0, 1) = p(1, 1, 0) = \dfrac{8 \cdot 5 \cdot 4}{13 \cdot 12 \cdot 11} = 40/429$

$p(1, 1, 1) = \dfrac{5 \cdot 4 \cdot 3}{13 \cdot 12 \cdot 11} = 5/143$

3. (a) $p(0, 0) = (10/13)(9/12) = 15/26$

$p(0, 1) = p(1, 0) = (10/13)(3/12) = 5/26$

$p(1, 1) = (3/13)(2/12) = 1/26$

(b) $p(0, 0, 0) = (10/13)(9/12)(8/11) = 60/143$

$p(0, 0, 1) = p(0, 1, 0) = p(1, 0, 0) = (10/13)(9/12)(3/11) = 45/286$

$p(i, j, k) = (3/13)(2/12)(10/11) = 5/143 \quad$ if $i + j + k = 2$

$p(1, 1, 1) = (3/13)(2/12)(1/11) = 1/286$

4. (a) $p(0, 0) = (8/13)^2$, $p(0, 1) = p(1, 0) = (5/13)(8/13)$, $p(1, 1) = (5/13)^2$

(b) $p(0, 0, 0) = (8/13)^3$

$p(i, j, k) = (8/13)^2(5/13)$ if $i + j + k = 1$

$p(i, j, k) = (8/13)(5/13)^2$ if $i + j + k = 2$

5. $p(0, 0) = (12/13)^3(11/12)^3$

$p(0, 1) = p(1, 0) = (12/13)^3[1 - (11/12)^3]$

$p(1, 1) = (2/13)[(1/13) + (12.13)(1/13)] + (11/13)(2/13)(1/13)$

8. $f_Y(y) = c\int_{-y}^{y}(y^2 - x^2)e^{-y}dx$

$= \frac{4}{3}cy^3e^{-y}$, $-0 < y < \infty$

$\int_0^{\infty} f_Y(y)dy = 1 \Rightarrow c = 1/8$ and so $f_Y(y) = \frac{y^3e^{-y}}{6}$, $0 < y < \infty$

$f_X(x) = \frac{1}{8}\int_{|x|}^{\infty}(y^2 - x^2)e^{-y}dy$

$= \frac{1}{4}e^{-|x|}(1 + |x|)$ upon using $-\int y^2e^{-y} = y^2e^{-y} + 2ye^{-y} + 2e^{-y}$

9. (b) $f_X(x) = \frac{6}{7}\int_0^2\left(x^2 + \frac{xy}{2}\right)dy = \frac{6}{7}(2x^2 + x)$

(c) $P\{X > Y\} = \frac{6}{7}\int_0^1\int_0^x\left(x^2 + \frac{xy}{2}dydx\right) = \frac{15}{56}$

(d) $P\{Y > 1/2 \mid X < 1/2\} = P\{Y > 1/2, X < 1/2\}/P\{X < 1/2\}$

$$= \frac{\int_{1/2}^{2}\int_{0}^{1/2}\left(x^2 + \frac{xy}{2}dxdy\right)}{\int_0^{1/2}(2x^2 + x)dx}$$

10. (a) $f_X(x) = e^{-x}$ , $f_Y(y) = e^{-y}$, $0 < x < \infty$, $0 < y < \infty$

$P\{X < Y\} = 1/2$

(b) $P\{X < a\} = 1 - e^{-a}$

11. $\frac{5!}{2!1!2!}(.45)^2(.15)(.40)^2$

12. $e^{-5} + 5e^{-5} + \frac{5^2}{2!}e^{-5} + \frac{5^3}{3!}e^{-5}$

14. Let $X$ and $Y$ denoted respectively the locations of the ambulance and the accident of the moment the accident occurs.

$$P\{\,|Y - X| < a\} = P\{Y < X < Y + a\} + P\{X < Y < X + a\}$$

$$= \frac{2}{L^2}\int_0^L \int_y^{\min(y+a,L)} dx\,dy$$

$$= \frac{2}{L^2}\left[\int_0^{L-a}\int_y^{y+a} dx\,dy + \int_{L-a}^{L}\int_y^{L} dx\,dy\right]$$

$$= 1 - \frac{L-a}{L} + \frac{a}{L^2}(L-a) = \frac{a}{L}\left(2 - \frac{a}{L}\right),\ \ 0 < a < L$$

15. (a) $1 = \iint f(x,y)\,dy\,dx = \iint_{(x,y)\in R} c\ dy\,dx\ = cA(R)$

where $A(R)$ is the area of the region $R$.

(b) $f(x, y) = 1/4,\ -1 \le x, y \le 1$
$= f(x)f(y)$
where $f(v) = 1/2,\ -1 \le v \le 1$.

(c) $P\{X^2 + Y^2 \le 1\} = \dfrac{1}{4}\iint_c dy\,dx\ =$ (area of circle)/4 = $\pi/4$.

16. (a) $A = \cup A_i$,
(b) yes
(c) $P(A) = \sum P(A_i)\ = n(1/2)^{n-1}$

17. $\dfrac{1}{3}$ since each of the 3 points is equally likely to be the middle one.

18. $P\{Y - X > L/3\} = \displaystyle\iint_{\substack{y-x>L/3 \\ \frac{L}{2}<y<L \\ 0<x<\frac{L}{2}}} \frac{4}{L^2}\,dy\,dx$

$$= \frac{4}{L^2}\left[\int_0^{L/6}\int_{L/2}^{L} dy\,dx + \int_{L/6}^{L/2}\int_{x+L/3}^{L} dy\,dx\right]$$

$$= \frac{4}{L^2}\left[\frac{L^2}{12} + \frac{5L^2}{24} - \frac{7L^2}{72}\right] = 7/9$$

19. $\int_0^1 \int_0^x \frac{1}{x}\, dydx = \int_0^1 dx = 1$

(a) $\int_y^1 \frac{1}{x} dx = -\ln(y)$, $0 < y < 1$

(b) $\int_0^x \frac{1}{x} dy = 1, \quad 0 < y < 1$

(c) $\frac{1}{2}$

(d) Integrating by parts gives that

$$\int_0^1 y \ \ln(y)dy = -1 - \int_0^1 (y\ln(y) - y)dy$$

yielding the result

$$E[Y] = -\int_0^1 y\ln(y)dy = 1/4$$

20. (a) yes: $f_X(x) = xe^{-x}, f_Y(y) = e^{-y}$, $0 < x < \infty$, $0 < y < \infty$

(b) no: $f_X(x) = \int_x^1 f(x,y)dy = 2(1-x)$, $0 < x < 1$

$$f_Y(y) = \int_0^y f(x,y)dx = 2y, \ 0 < y < 1$$

21. (a) We must show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$. Now,

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= \int_0^1\int_0^{1-y} 24xy\ dxdy \\ &= \int_0^1 12y(1-y)^2 dy \\ &= \int_0^1 12(y - 2y^2 + y^3)dy \\ &= 12(1/2 - 2/3 + 1/4) = 1\end{aligned}$$

(b)
$$\begin{aligned}E[X] &= \int_0^1 xf_X(x)dx \\ &= \int_0^1 x\int_0^{1-x} 24xy\ dydx \\ &= \int_0^1 12x^2(1-x)^2 dx = 2/5\end{aligned}$$

(c) 2/5

22. (a) No, since the joint density does not factor.

(b) $f_X(x) = \int_0^1 (x+y)dy = x + 1/2, \ 0 < x < 1.$

(c) $P\{X+Y<1\} = \int_0^1 \int_0^{1-x} (x+y)dydx$

$= \int_0^1 [x(1-x)+(1-x)^2/2]dx = 1/3$

23. (a) yes

$f_X(x) = 12x(1-x)\int_0^1 ydy = 6x(1-x), \ 0 < x < 1$

$f_Y(y) = 12y\int_0^1 x(1-x)dx = 2y, \ 0 < y < 1$

(b) $E[X] = \int_0^1 6x^2(1-x)dx = 1/2$

(c) $E[Y] = \int_0^1 2y^2 dy = 2/3$

(d) $\text{Var}(X) = \int_0^1 6x^3(1-x)dx - 1/4 = 1/20$

(e) $\text{Var}(Y) = \int_0^1 2y^3 dy - 4/9 = 1/18$

24. $P\{N=n\} = p_0^{n-1}(1-p_0)$

(b) $P\{X=j\} = p_j/(1-p_0)$

(c) $P\{N=n, X=j\} = p_0^{n-1}p_j$

25. $\dfrac{e^{-1}}{i!}$ by the Poisson approximation to the binomial.

26. (a) $F_{A,B,C}(a, b, c) = abc \quad 0 < a, b, c < 1$

(b) The roots will be real if $B^2 \geq 4AC$. Now

$$P\{AC \le x\} = \iint\limits_{\substack{c \le x/a \\ 0 \le a \le 1 \\ 0 \le c \le 1}} dadc = \int_0^x \int_0^1 dcda + \int_x^1 \int_0^{x/a} dcda$$

$= x - x \log x.$

Hence, $F_{AC}(x) = x - x \log x$ and so

$f_{AC}(x) = -\log x, \ 0 < x < 1$

$$P\{B^2/4 \geq AC\} = -\int_0^1 \int_0^{b^2/4} \log x dx db$$

$$= \int_0^1 \left[\frac{b^2}{4} - \frac{b^2}{4}\log(b^2/4)\right] db$$

$$= \frac{\log 2}{6} + \frac{5}{36}$$

where the above uses the identity

$$\int x^2 \log x dx = \frac{x^3 \log x}{3} - \frac{x^3}{9}.$$

27. (a) $P\{X + Y \leq a\} = \int_0^a \int_0^{a-x} e^{-y} dy dx = a - 1 + e^{-a}, a < 1$

$$= \int_0^1 \int_0^{a-x} e^{-y} dy dx = 1 - e^{-a}(e - 1), a > 1$$

(b) $P\{Y > X/a\} = \int_0^1 \int_{x/a}^{\infty} e^{-y} dy dx = a(1 - e^{-1/a})$

28. $P\{X_1/X_2 < a\} = \int_0^{\infty} \int_0^{ay} \lambda_1 e^{-\lambda} 1^x \lambda_2 e^{-\lambda_2 y} dx dy$

$$= \int_0^{\infty} \left(1 - e^{-\lambda_1 a y}\right) \lambda_2 e^{-\lambda_2 y} dy$$

$$= 1 - \frac{\lambda_2}{\lambda_2 + \lambda_1 a} = \frac{\lambda_1 a}{a\lambda_1 + \lambda_2}$$

$$P\{X_1/X_2 < 1\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

29. $P\{I^2 R \leq w\} = \int\int_{\substack{x^2 y \leq w \\ 0 \leq x \leq 1 \\ 0 \leq y \leq 1}} 6x(1 - x)2y dy dx$

$$= \int_0^{\sqrt{w}} \int_0^1 12x(1 - x) y dy dx + \int_{\sqrt{w}}^1 \int_0^{w/x^2} 12x(1 - x) y dy dx$$

$= 3w - 2w^{3/2} = 6w(1 + (\log w)/2 - \sqrt{w}\,)$

$= 4w^{3/2} - 3w(1 + \log w), 0 < w < 1$

30. (a) $e^{-2}$

(b) $1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2}$
The number of typographical errors on each page should approximately be Poisson distributed and the sum of independent Poisson random variables is also a Poisson random variable.

31. (a) $1 - e^{-2.2} - 2.2e^{-2.2} - e^{-2.2}(2.2)^2/2!$

(b) $1 - \sum_{i=0}^{4} e^{-4.4}(4.4)^i / i!$, (c) $1 - \sum_{i=0}^{5} e^{-6.6}(6.6)^i / i!$

The reasoning is the same as in Problem 26.

32. (a) If $W = X_1 + X_2$ is the sales over the next two weeks, then $W$ is normal with mean 4,400 and standard deviation $\sqrt{2(230)^2} = 325.27$. Hence, with $Z$ being a standard normal, we have

$$
\begin{aligned}
P\{W > 5000\} &= P\left\{Z > \frac{5000 - 4400}{325.27}\right\} \\
&= P\{Z > 1.8446\} = .0326
\end{aligned}
$$

(b)
$$
\begin{aligned}
P\{X > 2000\} &= P\{Z > (2000 - 2200)/230\} \\
&= P\{Z > -.87\} = P\{Z < .87\} = .8078
\end{aligned}
$$

Hence, the probability that weekly sales exceeds 2000 in at least 2 of the next 3 weeks $p^3 + 3p^2(1 - p)$ where $p = .8078$.

We have assumed that the weekly sales are independent.

33. Let $X$ denote Jill's score and let $Y$ be Jack's score. Also, let $Z$ denote a standard normal random variable.

(a)
$$
\begin{aligned}
P\{Y > X\} &= P\{Y - X > 0\} \\
&\approx P\{Y - X > .5\} \\
&= P\left\{\frac{Y - X - (160 - 170)}{\sqrt{(20)^2 + (15)^2}} > \frac{.5 - (160 - 170)}{\sqrt{(20)^2 + (15)^2}}\right\} \\
&\approx P\{Z > .42\} \approx .3372
\end{aligned}
$$

(b)
$$
\begin{aligned}
P\{X + Y > 350\} &= P\{X + Y > 350.5\} \\
&= P\left\{\frac{X + Y - 330}{\sqrt{(20)^2 + (15)^2}} > \frac{20.5}{\sqrt{(20)^2 + (15)^2}}\right\} \\
&\approx P\{Z > .82\} \approx .2061
\end{aligned}
$$

34. Let $X$ and $Y$ denote, respectively, the number of males and females in the sample that never eat breakfast. Since

$$E[X] = 50.4, \text{Var}(X) = 37.6992, \ E[Y] = 47.2, \text{Var}(Y) = 36.0608$$

it follows from the normal approximation to the binomial that is approximately distributed as a normal random variable with mean 50.4 and variance 37.6992, and that $Y$ is approximately distributed as a normal random variable with mean 47.2 and variance 36.0608. Let $Z$ be a standard normal random variable.

(a) $$\begin{aligned} P\{X+Y \geq 110\} &= P\{X+Y \geq 109.5\} \\ &= P\left\{\frac{X+Y-97.6}{\sqrt{73.76}} \geq \frac{109.5-97.6}{\sqrt{73.76}}\right\} \\ &\approx P\{Z > 1.3856\} \approx .0829 \end{aligned}$$

(b) $$\begin{aligned} P\{Y \geq X\} &= P\{Y - X \geq -.5\} \\ &= P\left\{\frac{Y-X-(-3.2)}{\sqrt{73.76}} \geq \frac{-.5-(-3.2)}{\sqrt{73.76}}\right\} \\ &\approx P\{Z \geq .3144\} \approx .3766 \end{aligned}$$

35. (a) $P\{X_1 = 1 \mid X_2 = 1\} = 4/12 = 1 - P\{X_1 = 0 \mid X_2 = 1\}$

(b) $P\{X_1 = 1 \mid X_2 = 0\} = 5/12 = 1 - P\{X_1 = 0 \mid X_2 = 0\}$

36. (a) $P\{X_1 = 1 \mid X_2 = 1\} = 5/13 = 1 - P\{X_1 = 0 \mid X_2 = 1\}$

(b) same as in (a)

37. (a) $P\{Y_1 = 1 \mid Y_2 = 1\} = 2/12 = 1 - P\{Y_1 = 0 \mid Y_2 = 1\}$

(b) $P\{Y_1 = 1 \mid Y_2 = 0\} = 3/12 = 1 - P\{Y_1 = 0 \mid Y_2 = 0\}$

38. (a) $P\{Y_1 = 1 \mid Y_2 = 1\} = p(1, 1)/[1 - (12/13)^3] = 1 - P\{Y_1 = 0 \mid Y_2 = 1\}$

(b) $P\{Y_1 = 1 \mid Y_2 = 0\} = p(1, 0)/(12/13)^3 = 1 - P\{Y_1 = 0 \mid Y_2 = 0\}$
where $p(1, 1)$ and $p(1, 0)$ are given in the solution to Problem 5.

39. (a) $P\{X = j, Y = i\} = \dfrac{1}{5}\dfrac{1}{j}$, $j = 1, \ldots, j$, $i = 1, \ldots, j$

(b) $P\{X = j \mid Y = i\} = \dfrac{1}{5j} \Big/ \sum_{k=i}^{5} 1/5\,k = \dfrac{1}{j} \Big/ \sum_{k=i}^{5} 1/k$, $5 \geq j \geq i$.

(c) No.

40. For $j = i$: $P\{Y = i \mid X = i\} = \dfrac{P\{Y = i, X = i\}}{P\{X = i\}} = \dfrac{1}{36P\{X = i\}}$

For $j < i$: $P\{Y = j \mid X = i\} = \dfrac{2}{36P\{X = i\}}$

Hence

$$1 = \sum_{j=1}^{i} P\{Y = j \mid X = i\} = \frac{2(i-1)}{36P\{X = i\}} + \frac{1}{36P\{X = i\}}$$

and so, $P\{X = i\} = \dfrac{2i-1}{36}$ and

$$P\{Y = j \mid X = i\} = \begin{cases} \dfrac{1}{2i-i} & j = i \\ \dfrac{2}{2i-1} & j < i \end{cases}$$

42. (a) $f_{X|Y}(x \mid y) = \dfrac{xe^{-x(y+1)}}{\int xe^{-x(y+1)}dx} = (y + 1)^2 xe^{-x(y+1)}$, $0 < x$

(b) $f_{Y|X}(y \mid x) = \dfrac{xe^{-x(y+1)}}{\int xe^{-x(y+1)}dy} = xe^{-xy}$, $0 < y$

$$P\{XY < a\} = \int_0^\infty \int_0^{a/x} xe^{-x(y+1)}dydx$$

$$= \int_0^\infty (1 - e^{-a})e^{-x}dx = 1 - e^{-a}$$

$f_{XY}(a) = e^{-a}$, $0 < a$

43. $f_{Y|X}(y \mid x) = \dfrac{(x^2 - y^2)e^{-x}}{\int_{-x}^{x} (x^2 - y^2)e^{-x}dx}$

$$= \frac{3}{4x^3}(x^2 - y^2), \quad -x < y < x$$

$$F_{Y|X}(y \mid x) = \frac{3}{4x^3}\int_{-x}^{y} (x^2 - y^2)dy$$

$$= \frac{3}{4x^3}(x^2 y - y^3/3 + 2x^3/3), \quad -x < y < x$$

44. $$f(\lambda \mid n) = \frac{P\{N = n|\lambda\}g(\lambda)}{P\{N = n\}}$$
$$= C_1 e^{-\lambda}\lambda^n \alpha e^{-\alpha\lambda}(\alpha\lambda)^{s-1}$$
$$= C_2 e^{-(\alpha+1)\lambda}\lambda^{n+s-1}$$

where $C_1$ and $C_2$ do not depend on $\lambda$. But from the preceding we can conclude that the conditional density is the gamma density with parameters $\alpha + 1$ and $n + s$. The conditional expected number of accidents that the insured will have next year is just the expectation of this distribution, and is thus equal to $(n + s)/(\alpha + 1)$.

45. $P\{X_1 > X_2 + X_3\} + P\{X_2 > X_1 + X_3\} + P\{X_3 > X_1 + X_2\}$

$$= 3P\{X_1 > X_2 + X_3\}$$

$$= 3\iiint\limits_{\substack{x_1 > x_2 > x_3 \\ 0 \le x_i \le 1 \\ i = 1,2,3}} dx_1 dx_2 dx_3 \qquad \text{(take } a = 0,\ b = 1\text{)}$$

$$= 3\int_0^1 \int_0^{1-x_3} \int_{x_2+x_3}^1 dx_1 dx_2 dx_3 = 3\int_0^1 \int_0^{1-x_3} (1 - x_2 - x_3) dx_2 dx_3$$
$$= 3\int_0^1 \frac{(1-x_3)^2}{2} dx_3 = 1/2 .$$

46. $$f_{X_{(3)}}(x) = \frac{5!}{2!2!}\left[\int_0^x xe^{-x}dx\right]^2 xe^{-x}\left[\int_x^\infty xe^{-x}dx\right]^2$$
$$= 30(x + 1)^2 e^{-2x} xe^{-x}[1 - e^{-x}(x + 1)]^2$$

47. $$\left(\frac{L-2d}{L}\right)^3$$

48. $$\int_{1/4}^{3/4} f_{X_{(3)}}(x)dx = \frac{5!}{2!2!}\int_{1/4}^{3/4} x^2(1-x)^2 dx$$

49. (a) $P\{\min X_i \le a\} = 1 - P\{\min X_i > a\} = 1 - \prod P\{X_i > a\} = 1 - e^{-5\lambda a}$

(b) $P\{\max X_i \le a\} = \prod P\{X_i \le a\} = (1 - e^{-\lambda a})^5$

50. $$f_{X_{(1)},X_{(4)}}(x, y) = \frac{4!}{2!}2x\left(\int_X^Y 2zdz\right)^2 2y,\ x < y$$
$$= 48xy(y^2 - x^2).$$

$$P(X_{(4)} - X_{(1)} \le a\} = \int_0^{1-a}\int_0^{a+x} 48xy(y^2 - x^2)dydx$$
$$+ \int_{1-a}^{1}\int_0^{1} 48xy(y^2 - x^2)dydx$$

51. $f_{R_1}(r,\theta) = \frac{r}{\pi} = 2r\frac{1}{2\pi}$, $0 \le r \le 1$, $0 \le \theta < 2\pi$.

Hence, $R$ and $\theta$ are independent with $\theta$ being uniformly distributed on $(0, 2\pi)$ and $R$ having density $f_R(\mathrm{r}) = 2r$, $0 < r < 1$.

52. $f_{R,\theta}(r,\theta) = r$, $0 < r \sin\theta < 1$, $0 < r\cos\theta < 1$, $0 < \theta < \pi/2$, $0 < r < \sqrt{2}$

53. $$J = \begin{vmatrix} \frac{1}{2}x^{-1/2}\cos u\,\sqrt{2} & \frac{1}{2}z^{-1/2}\sin u\,\sqrt{2} \\ -\sqrt{2z}\sin u & \sqrt{2z}\cos u \end{vmatrix} = \cos^2 u + \sin^2 u = 1$$

$f_{u,z}(u, z)$ - $\frac{1}{2\pi}e^{-z}$ . But $x^2 + y^2 = 2z$ so

$$f_{X,Y}(x, y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$$

54. (a) If $u = xy$, $v = xy$, then $J = \begin{vmatrix} y & x \\ \frac{1}{y} & \frac{-x}{y^2} \end{vmatrix} = -2\frac{x}{y}$ and

$y = \sqrt{u/v}$ , $x = \sqrt{vu}$ . Hence,

(b) $f_{u,v}(u, v) = \frac{1}{2v}f_{X,Y}\left(\sqrt{vy}, \sqrt{u/v}\right) = \frac{1}{2vu^2}$ , $u \ge 1$, $\frac{1}{u} < v < u$

$$f_u(u) = \int_{1/u}^{u} \frac{1}{2vu^2}dv = \frac{1}{u^2}\log u \text{ , } u \ge 1.$$

For $v > 1$

$$f_V(v) = \int_v^{\infty} \frac{1}{2vu^2}du = \frac{1}{2v^2} \text{ , } v > 1$$

For $v < 1$

$$f_V(v) = \int_{1/2}^{\infty} \frac{1}{2vu^2}du = \frac{1}{2} \text{ , } 0 < v < 1.$$

55. (a) $u = x + y,\ v = x/y \Rightarrow y = \dfrac{u}{v+1},\ x = \dfrac{uv}{v+1}$

$$J = \begin{vmatrix} 1 & 1 \\ 1/y & -x/y^2 \end{vmatrix} = -\left(\frac{x}{y^2} + \frac{1}{y}\right) = \frac{-1}{y^2}(x+y) = \frac{-(v+1)^2}{u}$$

$$f_{u,v}(u, v) = \frac{u}{(v+1)^2},\ 0 < uv < 1 + v,\ 0 < u < 1 + v$$

57. $y_1 = x_1 + x_2,\ y_2 = e^{x_1}.\ J = \begin{vmatrix} 1 & 1 \\ e^{x_1} & 0 \end{vmatrix} = -e^{x_1} = -y_2$

$x_1 = \log y_2,\ x_2 = y_1 - \log y_2$

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{y_2}\lambda e^{-\lambda \log y_2}\lambda e^{-\lambda(y_1 - \log y_2)}$$
$$= \frac{1}{y_2}\lambda^2 e^{-\lambda y_1},\ 1 \le y_2,\ y_1 \ge \log y_2$$

58. $u = x + y,\ \ v = x + z,\ \ w = y + z \Rightarrow z = \dfrac{v + w - u}{2},\ x = \dfrac{v - w + u}{2},\ y = \dfrac{w - v + u}{2}$

$$J = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2$$

$$f(u, v, w) = \frac{1}{2}\exp\left\{-\frac{1}{2}(u + v + w)\right\},\ u + v > w,\ \ u + w > v,\ \ v + w + u$$

59. $P(Y_j = i_j, j = 1, \ldots, k + 1\} = P\{Y_j = i_j, j = 1, \ldots, k\}\ P(Y_{k+1} = i_{k+1} \mid Y_j = i_j, j = 1, \ldots, k\}$

$$= \frac{k!(n-k)!}{n!} P\{n + 1 - \sum_{i=1}^{k} Y_i = i_{k+1} | Y_j = i_j, j = 1, \ldots, k\}$$

$$= \begin{cases} k!(n-k)!/n!, & \text{if } \sum_{j=1}^{k+1} i_j = n + 1 \\ 0, & \text{otherwise} \end{cases}$$

Thus, the joint mass function is symmetric, which proves the result.

60. The joint mass function is

$$P\{X_i = x_i,\ i = 1, \ldots, n\} = 1/\binom{n}{k},\ x_i \in \{0, 1\},\ i = 1, \ldots, n,\ \sum_{i=1}^{n} x_i = k$$

As this is symmetric in $x_1, \ldots, x_n$ the result follows.

## Theoretical Exercises

1. $$\begin{aligned} P\{X \le a_2,\ Y \le b_2\} = {} & P\{a_1 < X \le a_2,\ b_1 < Y \le b_2\} \\ & + P\{X \le a_1,\ b_1 < Y \le b_2\} \\ & + P\{a_1 < X \le a_2,\ Y \le b_1\} \\ & + P\{X \le a_1,\ Y \le b_1\}. \end{aligned}$$

The above following as the left hand event is the union of the 4 mutually exclusive right hand events. Also,

$$\begin{aligned} P\{X \le a_1, Y \le b_2\} = {} & P\{X \le a_1,\ b_1 < Y \le b_2\} \\ & + P\{X \le a_1,\ Y \le b_1\} \end{aligned}$$

and similarly,

$$\begin{aligned} P\{X \le a_2, Y \le b_1\} = {} & P\{a_1 \le X \le a_2, < Y \le b_1\} \\ & + P\{X \le a_1,\ Y \le b_1\}. \end{aligned}$$

Hence, from the above

$$\begin{aligned} F(a_2, b_2) = {} & P\{a_1 < X \le a_2, b_1 < Y \le b_2\} + F(a_1, b_2) - F(a_1, b_1) \\ & + F(a_2, b_1) - F(a_1, b_1) + F(a_1, b_1). \end{aligned}$$

2. Let $X_i$ denote the number of type $i$ events, $i = 1, \ldots, n$.

$$\begin{aligned} P\{X_1 = r_1, \ldots, X_n = r_n\} &= P\left\{X_1 = r_1, \ldots, X_n = r_n \,\middle|\, \sum_{1}^{n} r_i \text{ events}\right\} \\ &\quad \times e^{-\lambda} \lambda^{\sum_1^n r_i} \Big/ \left(\sum_1^n r_i\right)! \\ &= \frac{\left(\sum_1^n r_i\right)!}{r_1! \ldots r_n!} P_1^{r_1} \ldots p_n^{r_n} \frac{e^{-\lambda} \lambda^{\sum_1^n r_i}}{\left(\sum_1^n r_i\right)!} \\ &= \prod_{i=1}^{n} e^{-\lambda P_i} (\lambda_{p_i})^{r_i} \big/ r_i! \end{aligned}$$

3. Throw a needle on a table, ruled with equidistant parallel lines a distance $D$ apart, a large number of times. Let $L$, $L < D$, denote the length of the needle. Now estimate $\pi$ by $\dfrac{2L}{fD}$ where $f$ is the fraction of times the needle intersects one of the lines.

5. (a) For $a > 0$

$$F_Z(a) = P\{X \le aY\}$$
$$= \int_0^\infty \int_0^{a/y} f_X(x) f_Y(y)\,dx\,dy$$
$$= \int_0^\infty F_X(ay) f_Y(y)\,dy$$
$$f_Z(a) = \int_0^\infty f_X(ay)\, y f_Y(y)\,dy$$

(b)
$$F_Z(a) = P\{XY < a\}$$
$$= \int_0^\infty \int_0^{a/y} f_X(x) f_Y(y)\,dx\,dy$$
$$= \int_0^\infty F_X(a/y) f_Y(y)\,dy$$
$$f_Z(a) = \int_0^\infty f_X(a/y)\frac{1}{y} f_Y(y)\,dy$$

If $X$ is exponential with rate $\lambda$ and $Y$ is exponential with rate $\mu$ then (a) and (b) reduce to

(a) $F_Z(a) = \int_0^\lambda \lambda e^{-\lambda a y}\, y \mu e^{-\mu y}\,dy$

(b) $F_Z(a) = \int_0^\infty \lambda e^{-\lambda a/y}\frac{1}{y}\mu e^{-\mu y}\,dy$

6. Interpret $X_i$ as the number of trials needed after the $(i-1)^{\text{st}}$ success until the $i^{\text{th}}$ success occurs, $i = 1, \ldots, n$, when each trial is independent and results in a success with probability $p$. Then each $X_i$ is an identically distributed geometric random variable and $\sum_{i=1}^{n} X_i$, representing the number of trials needed to amass $n$ successes, is a negative binomial random variable.

7. (a) $P\{cX \le a\} = P\{X \le a/c\}$ and differentiation yields

$$f_{cX}(a) = \frac{1}{c} f_X(a/c) = \frac{\lambda}{c} e^{-\lambda\, a/c}(\lambda\, a/c)^{t-1}\Gamma(t).$$

Hence, $cX$ is gamma with parameters $(t, \lambda/c)$.

(b) A chi-squared random variable with $2n$ degrees of freedom can be regarded as being the sum of $n$ independent chi-square random variables each with 2 degrees of freedom (which by Example is equivalent to an exponential random variable with parameter $\lambda$). Hence by Proposition $X_{2n}^2$ is a gamma random variable with parameters $(n, 1/2)$ and the result now follows from part (a).

8. (a) $P\{W \le t\} = 1 - P\{W > t\} = 1 - P\{X > t, Y > t\} = 1 - [1 - F_X(t)]\,[1 - F_Y(t)]$

(b) $f_W(t) = f_X(t)[1 - F_Y(t)] + f_Y(t)\,[1 - F_X(t)]$

Dividing by $[1 - F_X(t)][1 - F_Y(t)]$ now yields

$$\lambda_W(t) = f_X(t)/[1 - F_X(t)] + f_Y(t)/[1 - F_Y(t)] = \lambda_X(t) + \lambda_Y(t)$$

9. $P\{\min(X_1, \ldots, X_n) > t\} = P\{X_1 > t, \ldots, X_n > t\}$
$= e^{-\lambda t} \ldots e^{-\lambda t} = e^{-n\lambda t}$

thus showing that the minimum is exponential with rate $n\lambda$.

10. If we let $X_i$ denote the time between the $i^{\text{th}}$ and $(i + 1)^{\text{st}}$ failure, $i = 0, \ldots, n - 2$, then it follows from Exercise 9 that the $X_i$ are independent exponentials with rate $2\lambda$. Hence, $\sum_{i=0}^{n-2} X_i$ the amount of time the light can operate is gamma distributed with parameters $(n - 1, 2\lambda)$.

11.
$$\begin{aligned}
I &= \underset{x_1 < x_2 > x_3 < x_4 > x_5}{\int\!\!\int\!\!\int\!\!\int\!\!\int} f(x_1) \ldots f(x_5)dx_1 \ldots dx_5 \\
&= \underset{\substack{u_1 < u_2 > u_3 < u_4 > u_5 \\ 0 < u_i < 1}}{\int\!\!\int\!\!\int\!\!\int\!\!\int} du_1 \ldots du_5 \qquad \text{by } u_i = F(x_i),\ i = 1, \ldots, 5 \\
&= \int\!\!\int\!\!\int\!\!\int u_2 du_2 \ldots du_5 \\
&= \int\!\!\int\!\!\int (1 - u_3^2)/2 \quad du_3 \ldots \\
&= \int\!\!\int [u_4 - u_4^3/3]/2 du_4 du_5 \\
&= \int_0^1 [u^2 - u^4/3]/2 du = 2/15
\end{aligned}$$

12. Assume that the joint density factors as shown, and let

$$C_i = \int_{-\infty}^{\infty} g_i(x)dx,\ i = 1, \ldots, n$$

Since the $n$-fold integral of the joint density function is equal to 1, we obtain that

$$1 = \prod_{i=1}^{n} C_i$$

Integrating the joint density over all $x_i$ except $x_j$ gives that

$$f_{X_j}(x_j) = g_j(x_j) \prod_{i \ne j} C_i = g_j(x_j)/C_j$$

If follows from the preceding that

$$f(x_1, \ldots, x_n) = \prod_{j=1}^{n} f_{X_j}(x_j)$$

which shows that the random variables are independent.

13. No. Let $X_i = \begin{cases} 1 & \text{if trial } i \text{ is a success} \\ 0 & -- \end{cases}$. Then

$$f_{X|X_1,\ldots,X_{n+m}}(x|x_1,\ldots,x_{n+m}) = \frac{P\{x_1,\ldots,x_{n+m}|X = x\}}{P\{x_1,\ldots,x_{n+m}\}} f_X(x)$$
$$= cx^{\sum x_i}(1-x)^{n+m-\sum x_i}$$

and so given $\sum_{1}^{n+m} X_i = n$ the conditional density is still beta with parameters $n + 1$, $m + 1$.

14. $P\{X = i \mid X + Y = n\} = P\{X = i,\ Y = n - i\}/P\{X + Y = n\}$

$$= \frac{p(1-p)^{i-1}p(1-p)^{n-i-1}}{\binom{n-1}{1}p^2(1-p)^{n-2}} = \frac{1}{n-1}$$

15. $$P\{X = k \mid X + Y = m\} = \frac{P\{X = k, X + Y = m\}}{P\{X + Y = m\}}$$
$$= \frac{P\{X = k, Y = m - k\}}{P\{X + Y = m\}}$$
$$= \frac{\binom{n}{k}p^k(1-p)^{n-k}\binom{n}{m-k}p^{m-k}(1-p)^{n-m+k}}{\binom{2n}{m}p^m(1-p)^{2n-m}}$$
$$= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}$$

16. $$P(X = n,\ Y = m) = \sum_i P(X = n, Y = m|X_2 = i)P(X_2 = i)$$
$$= e^{-(\lambda_1+\lambda_2+\lambda_3)} \sum_{i=0}^{\min(n,m)} \frac{\lambda_1^{n-i}}{(n-1)!}\frac{\lambda_3^{m-i}}{(m-i)!}\frac{\lambda_2^{i}}{i!}$$

17. (a) $P\{X_1 > X_2 \mid X_1 > X_3\} = \dfrac{P\{X_1 = \max(X_1, X_2, X_3)\}}{P\{X_1 > X_3\}} = \dfrac{1/3}{1/2} = 2/3$

(b) $P\{X_1 > X_2 \mid X_1 < X_3\} = \dfrac{P\{X_3 > X_1 > X_2\}}{P\{X_1 < X_3\}} = \dfrac{1/3!}{1/2} = 1/3$

(c) $P\{X_1 > X_2 \mid X_2 > X_3\} = \dfrac{P\{X_1 > X_2 > X_3\}}{P\{X_2 > X_3\}} = \dfrac{1/3!}{1/2} = 1/3$

(d) $P\{X_1 > X_2 \mid X_2 < X_3\} = \dfrac{P\{X_2 = \min(X_1, X_2, X_3)\}}{P\{X_2 < X_3\}} = \dfrac{1/3}{1/2} = 2/3$

18. $P\{U > s \mid U > a\} = P\{U > s\}/P\{U > a\}$

$$= \frac{1-s}{1-a},\ a < s < 1$$

$P\{U < s \mid U < a\} = P\{U < s\}/P\{U < a\}$

$= s/a,\ 0 < s < \text{a}$

Hence, $U \mid U > a$ is uniform on $(a, 1)$, whereas $U \mid U < a$ is uniform over $(0, a)$.

19. $$f_{W|N}(w \mid n) = \frac{P\{N = n \mid W = w\} f_W(w)}{P\{N = n\}}$$

$$= Ce^{-w}\frac{w^n}{n!}\beta e^{-\beta w}(\beta w)^{t-1}$$

$$= C_1 e^{-(\beta+1)w} w^{n+t-1}$$

where $C$ and $C_1$ do not depend on $w$. Hence, given $N = n$, $W$ is gamma with parameters $(n + t, \beta + 1)$.

20. $$f_{W|X_i, i=1,\ldots,n}(w \mid x_1,\ldots,x_n) = \frac{f(x_1,\ldots,x_n \mid w) f_w(w)}{f(x_1,\ldots,x_n)}$$

$$= C\prod_{\mathrm{i}=1}^{\mathrm{n}} we^{-wx_i} e^{-\beta w}(\beta w)^{t-1}$$

$$= Ke^{-w\left(\beta + \sum_1^n x_i\right)} w^{n+t-1}$$

21. Let $X_{ij}$ denote the element in row $i$, column $j$.

$P\{X_{ij}$ is s saddle point$\}$

$$= P\left\{\min_{k=1,\ldots,m} X_{ik} > \max_{k\neq i} X_{kj}, X_{ij} = \min_k X_{ik}\right\}$$

$$= P\left\{\min_k X_{ik} > \max_{k\neq i} X_{kj}\right\} P\left\{X_{ij} = \min_k X_{ik}\right\}$$

where the last equality follows as the events that every element in the $i^{\text{th}}$ row is greater than all elements in the $j^{\text{th}}$ column excluding $X_{ij}$ is clearly independent of the event that $X_{ij}$ is the smallest element in row $i$. Now each size ordering of the $n + m - 1$ elements under consideration is equally likely and so the probability that the $m$ smallest are the ones in row $i$ is $1\Big/\binom{n+m-1}{m}$. Hence

$$P\{X_{ij} \text{ is a saddlepoint}\} = \frac{1}{\binom{n+m-1}{m}}\frac{1}{m} = \frac{(m-1)!(n-1)!}{(n+m-1)!}$$

and so

$$\begin{aligned} P\{\text{there is a saddlepoint}\} &= P\left(\bigcup_{i,j}\{X_{ij} \text{ is a saddlepoint}\}\right) \\ &= \sum_{\text{i,j}} P\{X_{ij} \text{ is a saddlepoint}\} \\ &= \frac{m!n!}{(n+m-1)!} \end{aligned}$$

22. For $0 < x < 1$

$$P([X] = n, X - [X] < x) = P(n < X < n + x) = e^{-n\lambda} - e^{-(n+x)\lambda} = e^{-n\lambda}(1 - e^{-x\lambda})$$

Because the joint distribution factors, they are independent. $[X] + 1$ has a geometric distribution with parameter $p = 1 - e^{-\lambda}$ and $x - [X]$ is distributed as an exponential with rate $\lambda$ conditioned to be less than 1.

23. Let $Y = \max(X_1, \ldots, X_n)$ , $Z = \min(X_1, \ldots, X_n)$

$$P\{Y \le x\} = P\{X_i \le x, i = 1, \ldots, n\} = \prod_1^n P\{X_i \le x\} = F^n(x)$$

$$P\{Z > x\} = P\{X_i > x, i = 1, \ldots, n\} = \prod_1^n P\{X_i > x\} = [1 - F(x)]^n.$$

24. (a) Let $d = D/L$. Then the desired probability is

$$n!\int_0^{1-(n-1)d}\int_{x_1+d}^{1-(n-2)d}\cdots\int_{x_{n-3}+d}^{1-2d}\int_{x_{n-2}+d}^{1-d}\int_{x_{n-1}+d}^{1} dx_n dx_{n-1}\ldots dx_2 dx_1$$

$= [1 - (n-1)\mathrm{d}]^n.$

(b) 0

25. $$F_{x_{(j)}}(x) = \sum_{i=j}^{n}\binom{n}{i}F^i(x)[1-F(x)]^{n-i}$$

$$f_{X_{(j)}}(x) = \sum_{i=j}^{n}\binom{n}{i}iF^{i-1}(x)f(x)[1-F(x)]^{n-i}$$

$$-\sum_{i=j}^{n}\binom{n}{i}F^i(x)(n-i)[1-F(x)]^{n-i-1}f(x)$$

$$= \sum_{i=j}^{n}\frac{n!}{(n-i)!(i-1)!}F^{i-1}(x)f(x)[1-F(x)]^{n-i}$$

$$-\sum_{k=j+1}^{n}\frac{n!}{(n-k)!(k-1)!}F^{k-1}(x)f(x)[1-F(x)]^{n-k} \text{ by } k = i+1$$

$$= \frac{n!}{(n-j)!(j-1)!}F^{j-1}(x)f(x)[1-F(x)]^{n-j}$$

26. $$f_{X(n+1)}(x) = \frac{(2n+1)!}{n!n!}x^n(1-x)^n$$

27. In order for $X_{(i)} = x_i$ , $X_{(j)} = x_j$ , $i < j$ , we must have

(i) $i - 1$ of the $X$'s less than $x_i$

(ii) 1 of the $X$'s equal to $x_i$

(iii) $j - i - 1$ of the $X$'s between $x_i$ and $x_j$

(iv) 1 of the $X$'s equal to $x_j$

(v) $n - j$ of the $X$'s greater than $x_j$

Hence,

$$f_{x_{(i)},X_{(j)}}(x_i,x_j)$$

$$= \frac{n!}{(i-1)!1!(j-i-1)!1!(n-j)!}F^{i-1}(x_i)f(x_i)[F(x_j)-F(x_i)]^{j-i-1}f(x_j)\times[1-F(x_j)]^{n-j}$$

29. Let $X_1, \ldots, X_n$ be $n$ independent uniform random variables over $(0, a)$. We will show by induction on $n$ that

$$P\{X_{(k)} - X_{(k-1)} > t\} = \begin{cases} \left(\dfrac{a-t}{a}\right)^n & \text{if } t < a \\ 0 & \text{if } t > a \end{cases}$$

It is immediate when $n = 1$ so assume for $n - 1$. In the $n$ case, consider

$$P\{X_{(k)} - X_{(k-1)} > t \,|\, X_{(n)} = s\}.$$

Now given $X_{(n)} = s$, $X_{(1)}, \ldots, X_{(n-1)}$ are distributed as the order statistics of a set of $n - 1$ uniform $(0, s)$ random variables. Hence, by the induction hypothesis

$$P\{X_{(k)} - X_{(k-1)} > t \,|\, X_{(n)} = s\} = \begin{cases} \left(\dfrac{s-t}{s}\right)^{n-1} & \text{if } t < s \\ 0 & \text{if } t > s \end{cases}$$

and thus, for $t < a$,

$$P\{X_{(k)} - X_{(k-1)} > t = \int_t^a \left(\frac{s-t}{s}\right)^{n-1} \frac{ns^{n-1}}{a^n} ds = \left(\frac{a-t}{a}\right)^n$$

which completes the induction. (The above used that $f_{X_{(n)}}(s) = n\left(\dfrac{s}{a}\right)^{n-1} \dfrac{1}{a} = \dfrac{ns^{n-1}}{a^n}$).

30. (a) $P\{X > X_{(n)}\} = P\{X$ is largest of $n + 1\} = 1/(n + 1)$

(b) $P\{X > X_{(1)}\} = P\{X$ is not smallest of $n + 1\} = 1 - 1/(n + 1) = n/(n + 1)$

(c) This is the probability that $X$ is either the $(i + 1)^{\text{st}}$ or $(i + 2)^{\text{nd}}$ or … $j^{\text{th}}$ smallest of the $n + 1$ random variables, which is clearly equal to $(j - 1)/(n + 1)$.

33. The Jacobian of the transformation is

$$J = \begin{vmatrix} 1 & 1/y \\ 0 & -x/y^2 \end{vmatrix} = -x/y^2$$

Hence, $|J|^{-1} = y^2/|x|$. Therefore, as the solution of the equations $u = x$, $v = x/y$ is $x = u$, $y = u/v$, we see that

$$f_{u,v}(u, v) = \frac{|u|}{v^2} f_{X,Y}(u, u/v) = \frac{|u|}{v^2} \frac{1}{2\pi} e^{-(u^2 + u^2/v^2)/2}$$

Hence,

$$
\begin{aligned}
f_{V(u)} &= \frac{1}{2\pi v^2}\int_{-\infty}^{\infty}|u|e^{-u^2(1+1/v^2)/2}du \\
&= \frac{1}{2\pi v^2}\int_{-\infty}^{\infty}|u|e^{-u^2/2\sigma^2}du \text{ , where } \sigma^2 = v^2/(1+v^2) \\
&= \frac{1}{\pi v^2}\int_{0}^{\infty}ue^{-u^2/2\sigma^2}du \\
&= \frac{1}{\pi v^2}\sigma^2\int_{0}^{\infty}e^{-y}dy \\
&= \frac{1}{\pi(1+v^2)}
\end{aligned}
$$

# Chapter 7

## Problems

1. Let $X = 1$ if the coin toss lands heads, and let it equal 0 otherwise. Also, let $Y$ denote the value that shows up on the die. Then, with $p(i, j) = P\{X = i, Y = j\}$

$$\begin{aligned} E[\text{return}] &= \sum_{j=1}^{6} 2jp(1, j) + \sum_{j=1}^{6} \frac{j}{2} p(0, j) \\ &= \frac{1}{12}(42 + 10.5) = 52.5/12 \end{aligned}$$

2. (a) $6 \cdot 6 \cdot 9 = 324$

(b) $X = (6 - S)(6 - W)(9 - R)$

(c) $E[X] = 6(6)(6)P\{S = 0, W = 0, R = 3\} + 6(3)(9)P\{S = 0, W = 3, R = 0\}$
$+ 3(6)(9)P\{S = 3, W = 0, R = 0\} + 6(5)(7)P\{S = 0, W = 1, R = 2\}$
$+ 5(6)(7)P\{S = 1, W = 0, R = 2\} + 6(4)(8)P\{S = 0, W = 2, R = 1\}$
$+ 4(6)(8)P\{S = 2, W = 0, R = 1\} + 5(4)(9)P\{S = 1, W = 2, R = 0\}$
$+ 4(5)(9)P\{S = 2, W = 1, R = 0\} + 5(5)(8)P\{S = 1, W = 1, R = 1\}$

$$= \frac{1}{\binom{21}{3}}\left[216\binom{9}{3} + 324\binom{6}{3} + 420 \cdot 6\binom{9}{2} + 384\binom{6}{2}9 + 360\binom{6}{2}6 + 200(6)(6)(9)\right]$$

$\approx 198.8$

3. $E[|X - Y|^a] = \int_0^1\int_0^1 |x - y|^a dydx$. Now

$$\begin{aligned} \int_0^1 |x - y|^a dy &= \int_0^x (x - y)^a dy + \int_x^1 (y - x)^a dy \\ &= \int_0^x u^a du + \int_0^{1-x} u^a du \\ &= [x^{a+1} + (1 - x)^{a+1}]/(a + 1) \end{aligned}$$

Hence,

$$\begin{aligned} E[|X - Y|^a] &= \frac{1}{a + 1}\int_0^1 [x^{a+1} + (1 - x)^{a+1}]dx \\ &= \frac{2}{(a + 1)(a + 2)} \end{aligned}$$

4. $E[|X-Y|] = \frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}|i-j|$. Now,

$$\sum_{j=1}^{m}|i-j| = \sum_{j=1}^{i}(i-j) + \sum_{j=i+1}^{m}(j-i)$$
$$= [i(i-1) + (m-i)(m-i+1)]/2$$

Hence, using the identity $\sum_{j=1}^{m} j^2 = m(m+1)(2m+1)/6$, we obtain that

$$E[|X-Y|] = \left[\frac{1}{m^2}\frac{m(m+1)(2m+1)}{6} - \frac{m(m+1)}{2}\right] = \frac{(m+1)(m-1)}{3m}$$

5. The joint density of the point $(X, Y)$ at which the accident occurs is

$$f(x, y) = \frac{1}{9},\ -3/2 < x, y < 3/2$$
$$= f(x)\, f(y)$$

where

$$f(a) = 1/3,\ -3/2 < a < 3/2.$$

Hence we may conclude that $X$ and $Y$ are independent and uniformly distributed on $(-3/2, 3/2)$ Therefore,

$$E[\,|X| + |Y|\,] = 2\int_{-3/2}^{3/2}\frac{1}{3}x\,dx = \frac{4}{3}\int_{0}^{3/2}xdx = 3/2\,.$$

6. $E\left[\sum_{i=1}^{10} X_i\right] = \sum_{i=1}^{10} E[X_i] = 10(7/2) = 35.$

8. $E$[number of occupied tables] $= E\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} E[X_i]$

Now,

$E[X_i] = P\{i^{\text{th}}$ arrival is not friends with any of first $i-1\}$
$= (1-p)^{i-1}$

and so

$$E[\text{number of occupied tables}] = \sum_{i=1}^{N}(1-p)^{i-1}$$

7. Let $X_i$ equal 1 if both choose item $i$ and let it be 0 otherwise; let $Y_i$ equal 1 if neither $A$ nor $B$ chooses item $i$ and let it be 0 otherwise. Also, let $W_i$ equal 1 if exactly one of $A$ and $B$ choose item $i$ and let it be 0 otherwise. Let

$$X = \sum_{i=1}^{10} X_i\,, \quad Y = \sum_{i=1}^{10} Y_i\,, \quad W = \sum_{i=1}^{10} W_i$$

(a) $$E[X] = \sum_{i=1}^{10} E[X_i] = 10(3/10)^2 = .9$$

(b) $$E[Y] = \sum_{i=1}^{10} E[Y_i] = 10(7/10)^2 = 4.9$$

(c) Since $X + Y + W = 10$, we obtain from parts (a) and (b) that

$$E[W] = 10 - .9 - 4.9 = 4.2$$

Of course, we could have obtained $E[W]$ from

$$E[W] = \sum_{i=1}^{10} E[W_i] = 10(2)(3/10)(7/10) = 4.2$$

9. Let $X_j$ equal 1 if urn $j$ is empty and 0 otherwise. Then

$E[X_j] = P\{\text{ball } i \text{ is not in urn } j,\ i \geq j\} = \prod_{i=j}^{n} (1 - 1/i)$

Hence,

(a) $E[\text{number of empty urns}] = \sum_{j=1}^{n}\sum_{i=j}^{n} (1 - 1/i)$

(b) $$P\{\text{none are empty}\} = P\{\text{ball } j \text{ is in urn } j, \text{ for all } j\} = \prod_{j=1}^{n} 1/j$$

10. Let $X_i$ equal 1 if trial $i$ is a success and 0 otherwise.

(a) .6. This occurs when $P\{X_1 = X_2 = X_3\} = 1$. It is the largest possible since $1.8 = \sum P\{X_i = 1\} = 3P\{X_i = 1\}$. Hence, $P\{X_i = 1\} = .6$ and so

$$P\{X = 3\} = P\{X_1 = X_2 = X_3 = 1\} \leq P\{X_i = 1\} = .6.$$

(b) 0. Letting

$$X_1 = \begin{cases} 1 \text{ if } U \leq .6 \\ 0 \text{ otherwise} \end{cases}, \qquad X_2 = \begin{cases} 1 \text{ if } U \leq .4 \\ 0 \text{ otherwise} \end{cases}, \qquad X_3 = \begin{cases} 1 \text{ if } U \leq .3 \\ 0 \text{ otherwise} \end{cases}$$

Hence, it is not possible for all $X_i$ to equal 1.

11. Let $X_i$ equal 1 if a changeover occurs on the $i^{\text{th}}$ flip and 0 otherwise. Then

$$\begin{aligned} E[X_i] &= P\{i-1 \text{ is } H, i \text{ is } T\} + P\{i-1 \text{ is } T, i \text{ is } H\} \\ &= 2(1-p)p, \quad i \geq 2. \end{aligned}$$

$$E[\text{number of changeovers}] = E\left[\sum X_i\right] = \sum_{i=1}^{n} E[X_i] = 2(n-1)(1-p)$$

12. (a) Let $X_i$ equal 1 if the person in position $i$ is a man who has a woman next to him, and let it equal 0 otherwise. Then

$$E[X_i] = \begin{cases} \dfrac{1}{2}\dfrac{n}{2n-1}, & \text{if } i = 1, 2\text{n} \\ \dfrac{1}{2}\left[1 - \dfrac{(n-1)(n-2)}{(2n-1)(2n-2)}\right], & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} E\left[\sum_{i=1}^{n} X_i\right] &= \sum_{i=1}^{2n} E[X_i] \\ &= \frac{1}{2}\left(\frac{2n}{2n-1} + (2n-2)\frac{3n}{4n-2}\right) \\ &= \frac{3n^2 - n}{4n-2} \end{aligned}$$

(b) In the case of a round table there are no end positions and so the same argument as in part (a) gives the result

$$n\left[1 - \frac{(n-1)(n-2)}{(2n-1)(2n-2)}\right] = \frac{3n^2}{4n-2}$$

where the right side equality assumes that $n > 1$.

13. Let $X_i$ be the indicator for the event that person $i$ is given a card whose number matches his age. Because only one of the cards matches the age of the person $i$

$$E\left[\sum_{i=1}^{1000} X_i\right] = \sum_{i=1}^{1000} E[X_i] = 1$$

14. The number of stages is a negative binomial random variable with parameters $m$ and $1 - p$. Hence, its expected value is $m/(1-p)$.

15. Let $X_{i,j}$ , $i \neq j$ equal 1 if $i$ and $j$ form a matched pair, and let it be 0 otherwise.

Then

$$E[X_{i,j}] = P\{i, j \text{ is a matched pair}\} = \frac{1}{n(n-1)}$$

Hence, the expected number of matched pairs is

$$E\left[\sum_{i<j} X_{i,j}\right] = \sum_{i<j} E[X_{i,j}] = \binom{n}{2}\frac{1}{n(n-1)} = \frac{1}{2}$$

16. $$E[X] = \int_{y>x} y\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

17. Let $I_i$ equal 1 if guess $i$ is correct and 0 otherwise.

(a) Since any guess will be correct with probability $1/n$ it follows that

$$E[N] = \sum_{i=1}^{n} E[I_i] = n/n = 1$$

(b) The best strategy in this case is to always guess a card which has not yet appeared. For this strategy, the $i^{\text{th}}$ guess will be correct with probability $1/(n - i + 1)$ and so

$$E[N] = \sum_{i=1}^{n} 1/(n-i+1)$$

(c) Suppose you will guess in the order 1, 2, …, $n$. That is, you will continually guess card 1 until it appears, and then card 2 until it appears, and so on. Let $J_i$ denote the indicator variable for the event that you will eventually be correct when guessing card $i$; and note that this event will occur if among cards 1 thru $i$, card 1 is first , card 2 is second, …, and card $i$ is the last among these $i$ cards. Since all $i!$ orderings among these cards are equally likely it follows that

$$E[J_i] = 1/i! \text{ and thus } E[N] = E\left[\sum_{i=1}^{n} J_i\right] = \sum_{i=1}^{n} 1/i!$$

18. $$E[\text{number of matches}] = E\left[\sum_{1}^{52} I_i\right], \quad I_i = \begin{cases} 1 & \text{match on card } i \\ 0 & \text{---} \end{cases}$$

$$= 52\frac{1}{13} = 4 \quad \text{since } E[I_i] = 1/13$$

19. (a) $E$[time of first type 1 catch] − 1 $= \frac{1}{p_1} - 1$ using the formula for the mean of a geometric random variable.

(b) Let

$$X_j = \begin{cases} 1 & \text{a type } j \text{ is caught before a type 1} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$E\left[\sum_{j \neq 1} X_j\right] = \sum_{j \neq 1} E[X_j]$$
$$= \sum_{j \neq 1} P\{\text{type } j \text{ before type 1}\}$$
$$= \sum_{j \neq 1} P_j/(P_j + P_1),$$

where the last equality follows upon conditioning on the first time either a type 1 or type $j$ is caught to give.

$$P\{\text{type } j \text{ before type 1}\} = P\{j \,|\, j \text{ or } 1\} = \frac{P_j}{P_j + P_1}$$

20. Similar to (b) of 19. Let

$$X_j = \begin{cases} 1 & \text{ball } j \text{ removed before ball 1} \\ 0 & \text{---} \end{cases}$$

$$E\left[\sum_{j \neq 1} X_j\right] = \sum_{j \neq 1} E[X_j] = \sum_{j \neq 1} P\{\text{ball } j \text{ before ball 1}\}$$
$$= \sum_{j \neq 1} P\{j \,|\, j \text{ or } 1\}$$
$$= \sum_{j \neq 1} W(j)/W(1) + W(j)$$

21. (a) $365\binom{100}{3}\left(\frac{1}{365}\right)^3\left(\frac{364}{365}\right)^{97}$

(b) Let $X_j = \begin{cases} 1 & \text{if day } j \text{ is someones birthday} \\ 0 & \text{---} \end{cases}$

$$E\left[\sum_1^{365} X_j\right] = \sum_1^{365} E[X_j] = 365\left[1 - \left(\frac{364}{365}\right)^{100}\right]$$

22. From Example 3g, $1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + 6$

23. $E\left[\sum_1^5 X_i + \sum_1^8 Y_i\right] = \sum_1^5 E[X_i] + \sum_1^8 E(Y_i)$

$$= 5\frac{2}{11}\frac{3}{20} + 8\frac{3}{120} = \frac{147}{110}$$

24. Number the small pills, and let $X_i$ equal 1 if small pill $i$ is still in the bottle after the last large pill has been chosen and let it be 0 otherwise, $i = 1, \ldots, n$. Also, let $Y_i$, $i = 1, \ldots, m$ equal 1 if the $i^{\text{th}}$ small pill created is still in the bottle after the last large pill has been chosen and its smaller half returned.

Note that $X = \sum_{i=1}^{n} X_i + \sum_{i=1}^{m} Y_i$. Now,

$E[X_i] = P\{$small pill $i$ is chosen after all $m$ large pills$\}$
$= 1/(m + 1)$

$E[Y_i] = P\{i^{\text{th}}$ created small pill is chosen after $m - i$ existing large pills$\}$
$= 1/(m - i + 1)$

Thus,

(a) $E[X] = n/(m + 1) + \sum_{i=1}^{m} 1/(m - i + 1)$

(b) $Y = n + 2m - X$ and thus

$E[Y] = n + 2m - E[X]$

25. $P\{N \geq n\}\ P\{X_1 \geq X_2 \geq \ldots \geq X_n\} = \frac{1}{n!}$

$$E[N] = \sum_{n=1}^{\infty} P\{N \geq n\} = \sum_{n=1}^{\infty} \frac{1}{n!} = e$$

26. (a) $E[\max] = \int_0^1 P\{\max > t\}dt$

$$= \int_0^1 (1 - P\{\max \le t)\}dt$$

$$= \int_0^1 (1 - t^n / dt = \frac{n}{n+1}$$

(b) $\mathrm{E}[\min] = \int_0^1 p\{\min > t\}4t$

$$= \int_0^1 (1-t)^n dt = \frac{1}{n+1}$$

27. Let $X$ denote the number of items in a randomly chosen box. Then, with $X_i$ equal to 1 if item $i$ is in the randomly chosen box

$$E[X] = E\left[\sum_{i=1}^{101} X_i\right] = \sum_{i=1}^{101} E[X_i] = \frac{101}{10} > 10$$

Hence, $X$ can exceed 10, showing that at least one of the boxes must contain more than 10 items.

28. We must show that for any ordering of the 47 components there is a block of 12 consecutive components that contain at least 3 failures. So consider any ordering, and randomly choose a component in such a manner that each of the 47 components is equally likely to be chosen. Now, consider that component along with the next 11 when moving in a clockwise manner and let $X$ denote the number of failures in that group of 12. To determine $E[X]$, arbitrarily number the 8 failed components and let, for $i = 1, \ldots, 8$,

$$X_i = \begin{cases} 1, & \text{if failed component } i \text{ is among the group of 12 components} \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$X = \sum_{i=1}^{8} X_i$$

and so

$$E[X] = \sum_{i=1}^{8} E[X_i]$$

Because $X_i$ will equal 1 if the randomly selected component is either failed component number $i$ or any of its 11 neighboring components in the counterclockwise direction, it follows that $E[X_i] = 12/47$. Hence,

$$E[X] = 8(12/47) = 96/47$$

Because $E[X] > 2$ it follows that there is at least one possible set of 12 consecutive components that contain at least 3 failures.

29. Let $X_{ii}$ be the number of coupons one needs to collect to obtain a type $i$. Then

$$\begin{aligned}
E[X_{i)}] &= 8, \quad i = 1,2 \\
E\{X_i] &= 8/3, \quad i = 3,4 \\
E[\min(X_1, X_2)] &= 4 \\
E[\min(X_i, X_j)] &= 2, \quad i = 1,2, \quad j = 3,4 \\
E[\min(X_3, X_4)] &= 4/3 \\
E[\min(X_1, X_2, X_j)] &= 8/5, \quad j = 3,4 \\
E[\min(X_i, X_{3,} X_4)] &= 8/7, \quad i = 1,2 \\
E[\min(X_1, X_2, X_3, X_4] &= 1
\end{aligned}$$

(a) $E[\max X_i] = 2 \cdot 8 + 2 \cdot 8/3 - (4 + 4 \cdot 2 + 4/3) + (2 \cdot 8/5 + 2 \cdot 8/7) - 1 = \dfrac{437}{35}$

(b) $E[\max(X_1, X_2)] = 8 + 8 - 4 = 12$

(c) $E[\max(X_3, X_4)] = 8/3 + 8/3 - 4/3 = 4$

(d) Let $Y_1 = \max(X_1, X_2)$, $Y_2 = \max(X_3, X_4)$. Then

$$E[\max(Y_1, Y_2)] = E[Y_1] + E[Y_2] - E[\min(Y_1, Y_2)]$$

giving that

$$E[\min(Y_1, Y_2)] = 12 + 4 - \frac{437}{35} = \frac{123}{35}$$

30. $E[(X - Y)]^2 = \text{Var}(X - Y) = \text{Var}(X) + \text{Var}(-Y) = 2\sigma^2$

31. $\text{Var}\left(\sum_{i=1}^{10} X_i\right) = 10 \text{ Var}(X_1)$. Now

$$\begin{aligned}
\text{Var}(X_1) &= E[X_1^2] - (7/2)^2 \\
&= [1 + 4 + 9 + 16 + 25 + 36]/6 - 49/4 \\
&= 35/12
\end{aligned}$$

and so $\text{Var}\left(\sum_{i=1}^{10} X_i\right) = 350/12$.

32. Use the notation in Problem 9,

$$X = \sum_{j=1}^{n} X_j$$

where $X_j$ is 1 if box $j$ is empty and 0 otherwise. Now, with

$E[X_j] = P\{X_j = 1\} = \prod_{i=j}^{n}(1-1/i)$, we have that

$$\text{Var}(X_j) = E[X_j](1 - E[X_j]).$$

Also, for $j < k$

$$E[X_jX_k] = \prod_{i=j}^{k-1}(1-1/i)\prod_{i=k}^{n}(1-2/i)$$

Hence, for $j < k$,

$$\text{Cov}(X_j, X_k) = \prod_{i=j}^{k-1}(1-1/i)\prod_{i=k}^{n}(1-2/i) - \prod_{i=j}^{n}(1-1/i)\prod_{i=k}^{n}(1-1/i)$$

$$\text{Var}(X) = \sum_{j=1}^{n} E[X_j](1-E[X_j]) + 2\text{Cov}(X_j, X_k)$$

33. (a) $E[X^2 + 4X + 4] = E[X^2] + 4E[X] + 4 = \text{Var}(X) + E^2[X] + 4E[X] + 4 = 14$

(b) $\text{Var}(4 + 3X) = \text{Var}(3X) = 9\text{Var}(X) = 45$

34. Let $X_j = \begin{cases} 1 & \text{if couple } j \text{ are seated next to each other} \\ 0 & \text{otherwise} \end{cases}$

(a) $E\left[\sum_{1}^{10} X_j\right] = 10\frac{2}{19} = \frac{20}{19}$; $P\{X_j = 1\} = \frac{2}{19}$ since there are 2 people seated next to wife $j$ and so the probability that one of them is her husband is $\frac{2}{19}$.

(b) For $i \neq j$, $E[X_iX_j] = P\{X_i = 1, X_j = 1\}$

$= P\{X_i = 1\}P\{X_j = 1 \mid X_i = 1\}$

$= \frac{2}{19}\frac{2}{18}$ since given $X_i = 1$ we can regard couple $i$ as a single entity.

$$\text{Var}\left(\sum_{j=1}^{10} X_j\right) = 10\frac{2}{19}\left(1-\frac{2}{19}\right) + 10\cdot 9\left[\frac{2}{19}\frac{2}{18} - \left(\frac{2}{19}\right)^2\right]$$

35. (a) Let $X_1$ denote the number of nonspades preceding the first ace and $X_2$ the number of nonspades between the first 2 aces. It is easy to see that

$$P\{X_1 = i,\ X_2 = j\} = P\{X_1 = j,\ X_2 = i\}$$

and so $X_1$ and $X_2$ have the same distribution. Now $E[X_1] = \frac{48}{5}$ by the results of Example 3j and so $E[2 + X_1 + X_2] = \frac{106}{5}$.

(b) Same method as used in (a) yields the answer $5\left(\frac{39}{14}+1\right) = \frac{265}{14}$.

(c) Starting from the end of the deck the expected position of the first (from the end) heart is, from Example 3j, $\frac{53}{14}$. Hence, to obtain all 13 hearts we would expect to turn over $52 - \frac{53}{14} + 1 = \frac{13}{14}(53)$.

36. Let $X_i = \begin{cases} 1 & \text{roll } i \text{ lands on } 1 \\ 0 & \text{otherwise} \end{cases}$, $\quad Y_i = \begin{cases} 1 & \text{roll } i \text{ lands on } 2 \\ 0 & \text{otherwise} \end{cases}$

$$\text{Cov}(X_i, Y_j) = E[X_i Y_j] - E[X_i]E[Y_j]$$
$$= \begin{cases} -\frac{1}{36} & i = j \text{ (since } X_iY_j = 0 \text{ when } i = j \\ \frac{1}{36} - \frac{1}{36} = 0 & i \neq j \end{cases}$$

$$\text{Cov}\sum_i X_i, \sum_j Y_j = \sum_i \sum_j \text{Cov}(X_i, Y_j)$$
$$= -\frac{n}{36}$$

37. Let $W_i$, $i = 1, 2$, denote the $i^{\text{th}}$ outcome.

$$\text{Cov}(X, Y) = \text{Cov}(W_1 + W_2,\ W_1 - W_2)$$
$$= \text{Cov}(W_1, W_1) - \text{Cov}(W_2, W_2)$$
$$= \text{Var}(W_1) - \text{Var}(W_2) = 0$$

38. $E[XY] = \int_0^\infty \int_0^x y 2e^{-2x} dy dx$

$$= \int_0^\infty x^2 e^{-2x} dx = \frac{1}{8}\int_0^\infty y^2 e^{-y} dy = \frac{\Gamma(3)}{8} = \frac{1}{4}$$

$$E[X] = \int_0^\infty x f_x(x)dx,\ f_x(x) = \int_0^x \frac{2e^{-2x}}{x}dy\ = 2e^{-2x}$$
$$= \frac{1}{2}$$

$$E[Y] = \int_0^\infty y f_Y(y)dy,\ f_Y(y) = \int_0^\infty \frac{2e^{-2x}}{x}dx$$
$$= \int_0^\infty \int_y^\infty y \frac{2e^{-2x}}{x}dxdy$$
$$= \int_0^\infty \int_0^x y \frac{2e^{-2x}}{x}dydx$$
$$= \int_0^\infty xe^{-2x}dx = \frac{1}{4}\int ye^{-2}dy = \frac{\Gamma(2)}{4} = \frac{1}{4}$$

$$\text{Cov}(X, Y) = \frac{1}{4} - \frac{1}{2}\frac{1}{4} = \frac{1}{8}$$

39. $\text{Cov}(Y_n, Y_n) = \text{Var}(Y_n) = 3\sigma^2$
$\text{Cov}(Y_n, Y_{n+1}) = \text{Cov}(X_n + X_{n+1} + X_{n+2},\ X_{n+1} + X_{n+2} + X_{n+3})$
$= \text{Cov}(X_{n+1} + X_{n+2},\ X_{n+1} + X_{n+2}) = \text{Var}(X_{n+1} + X_{n+2}) = 2\sigma^2$
$\text{Cov}(Y_n, Y_{n+2}) = \text{Cov}(X_{n+2}, X_{n+2}) = \sigma^2$
$\text{Cov}(Y_n, Y_{n+j}) = 0$ when $j \geq 3$

40. $f_Y(y) = e^{-y}\int \frac{1}{y}e^{-x/y}dx = e^{-y}$. In addition, the conditional distribution of $X$ given that $Y = y$ is exponential with mean $y$. Hence,

$E[Y] = 1,\ E[X] = E[E[X \mid Y]] = E[Y] = 1$

Since, $E[XY] = E[E[XY \mid Y]] = E[YE[X \mid Y]] = E[Y^2] = 2$ (since $Y$ is exponential with mean 1, it follows that $E[Y^2] = 2$). Hence, $\text{Cov}(X, Y) = 2 - 1 = 1$.

41. The number of carp is a hypergeometric random variable.

$$E[X] = \frac{60}{10} = 6$$

$\text{Var}(X) = \dfrac{20(80)}{99}\dfrac{3}{10}\dfrac{7}{10} = \dfrac{336}{99}$ from Example 5c.

42. (a) Let $X_i = \begin{cases} 1 & \text{pair } i \text{ consists of a man and a woman} \\ 0 & \text{otherwise} \end{cases}$

$$E[X_i] = P\{X_i = 1\} = \frac{10}{19}$$

$$E[X_iX_j] = P\{X_i = 1, X_j = 1\} = P\{X_i = 1\}P\{X_j = 1 \mid X_2 = 1\}$$

$$= \frac{10}{19}\frac{9}{17},\ i \neq \text{j}$$

$$E\left[\sum_1^{10} X_i\right] = \frac{100}{19}$$

$$\text{Var}\left(\sum_1^{10} X_i\right) = 10\frac{10}{19}\left(1-\frac{10}{19}\right) + 10\cdot 9\left[\frac{10}{19}\frac{9}{17} - \left(\frac{10}{19}\right)^2\right] = \frac{900}{(19)^2}\frac{18}{17}$$

(b) $X_i = \begin{cases} 1 & \text{pair } i \text{ consists of a married couple} \\ 0 & \text{otherwise} \end{cases}$

$$E[X_i] = \frac{1}{19},\ E[X_iX_j] = P\{X_i = 1\}P\{X_j = 1 \mid X_i = 1\} = \frac{1}{19}\frac{1}{17},\ i \neq j$$

$$E\left[\sum_1^{10} X_i\right] = \frac{10}{19}$$

$$\text{Var}\left(\sum_1^{10} X_i\right) = 10\frac{1}{19}\frac{15}{19} + 10\cdot 9\left[\frac{1}{19}\frac{1}{17} - \left(\frac{1}{19}\right)^2\right] = \frac{180}{(19)^2}\frac{18}{17}$$

43. $E[R] = n(n + m + 1)/2$

$$\text{Var}(R) = \frac{nm}{n+m-1}\left[\frac{\sum_{i=1}^{n+m} i^2}{n+m} - \left(\frac{n+m+1}{2}\right)^2\right]$$

The above follows from Example 3d since when $F = G$, all orderings are equally likely and the problem reduces to randomly sampling $n$ of the $n + m$ values 1, 2, …, $n + m$.

44. From Example 8l $\frac{n}{n+m}+\frac{nm}{n+m}$. Using the representation of Example 2l the variance can be computed by using

$$E[I_1 I_{l+j}] = \begin{cases} 0 & , \quad j=1 \\ \frac{n}{n+m}\frac{m}{n+m-1}\frac{n-1}{n+m-2} & , \quad n-1 \le j < 1 \end{cases}$$

$$E[I_i I_{i+j}] = \begin{cases} 0 & , \quad j=1 \\ \frac{mn(m-1)(n-1)}{(n+m)(n+m-1)(n+m-2)(n+m-3)} & , \quad n-1 \le j < 1 \end{cases}$$

45. (a) $\frac{\mathrm{Cov}(X_1+X_2, X_2+X_3)}{\sqrt{\mathrm{Var}(X_1+X_2)}\sqrt{\mathrm{Var}(X_2+X_3)}} = \frac{1}{2}$

(b) 0

46. 
$$\begin{aligned} E[I_1 I_2] &= \sum_{i=2}^{12} E[I_1 I_2 \mid \text{bank rolls } i] P\{\text{bank rolls } i\} \\ &= \sum_i (P\{\text{roll is greater than } i\})^2 P\{\text{bank rolls } i\} \\ &= E[I_1^2] \\ &\ge (E[I_1])^2 \\ &= E[I_1]\, E[I_2] \end{aligned}$$

47. (a) It is binomial with parameters $n - 1$ and $p$.

(b) Let $x_{i,j}$ equal 1 if there is an edge between vertices $i$ and $j$, and let it be 0 otherwise. Then, $D_i = \sum_{k\ne i} X_{i,k}$, and so, for $i \ne j$

$$\begin{aligned} \mathrm{Cov}(D_i, D_j) &= \mathrm{Cov}\left(\sum_{k\ne i} X_{i,k}, \sum_{r\ne j} X_{r,j}\right) \\ &= \sum_{k\ne i}\sum_{r\ne j} Cov(X_{i,k}, X_{r,j}) \\ &= \mathrm{Cov}(X_{i,j}, X_{i,j}) \\ &= \mathrm{Var}(X_{i,j}) \\ &= p(1-p) \end{aligned}$$

where the third equality uses the fact that except when $k = j$ and $r = i$, $X_{i,k}$ and $X_{r,j}$ are independent and thus have covariance equal to 0. Hence, from part (a) and the preceding we obtain that for $i \ne j$,

$$\rho(D_i, D_j) = \frac{p(1-p)}{(n-1)p(1-p)} = \frac{1}{n-1}$$

48. (a) $E[X] = 6$

(b) $E[X \mid Y = 1] = 1 + 6 = 7$

(c) $1\frac{1}{5} + 2\frac{4}{5}\frac{1}{5} + 3\left(\frac{4}{5}\right)^2\frac{1}{5} + 4\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^4(5+6)$

49. Let $C_i$ be the event that coin $i$ is being flipped (where coin 1 is the one having head probability .4), and let $T$ be the event that 2 of the first 3 flips land on heads. Then

$$P(C_1 \mid T) = \frac{P(T \mid C_1)P(C_1)}{P(T \mid C_1)P(C_1) + P(T \mid C_2)P(C_2)}$$
$$= \frac{3(.4)^2(.6)}{3(.4)^2(.6) + 3(.7)^2(.3)} = .395$$

Now, with $N_j$ equal to the number of heads in the final $j$ flips, we have

$E[N_{10} \mid T] = 2 + E[N_7 \mid T]$

Conditioning on which coin is being used, gives

$E[N_7 \mid T] = E[N_7 \mid TC_1]P(C_1T) + E[N_7TC_2]P(C_2 \mid T) = 2.8(.395) + 4.9(.605) = 4.0705$

Thus, $E[N_{10} \mid T] = 6.0705$.

50. $$f_{X|Y}(x \mid y) = \frac{e^{-x/y}e^{-y}/y}{\int_0^\infty e^{-x/y}e^{-y}/y\, dx} = \frac{1}{y}e^{-x/y}, \quad 0 < x < \infty$$

Hence, given $Y = y$, $X$ is exponential with mean $y$, and so

$E[X^2 \mid Y = y] = 2y^2$

51. $$f_{X|Y}(x \mid y) = \frac{e^{-y}/y}{\int_0^y e^{-y}/y\, dx} = \frac{1}{y}, \quad 0 < x < y$$

$$E[X^3 \mid Y = y] = \int_0^y x^3 \frac{1}{y} dx = y^3/4$$

52. The average weight, call it $E[W]$, of a randomly chosen person is equal to average weight of all the members of the population. Conditioning on the subgroup of that person gives

$$E[W] = \sum_{i=1}^r E\{W \mid \text{member of subgroup } i]p_i = \sum_{i=1}^r w_i p_i$$

53. Let $X$ denote the number of days until the prisoner is free, and let $I$ denote the initial door chosen. Then

$$E[X] = E[X \mid I = 1](.5) + E[X \mid I = 2](.3) + E[X \mid I = 3](.2)$$
$$= \quad (2 + E[X])(.5) + (4 + E[X])(.3) + .2$$

Therefore,

$$E[X] = 12$$

54. Let $R_i$ denote the return from the policy that stops the first time a value at least as large as $i$ appears. Also, let $X$ be the first sum, and let $p_i = P\{X = i\}$. Conditioning on $X$ yields

$$E[R_5] = \sum_{i=2}^{12} E[R_5 \mid X = i\} p_i$$
$$= E[R_5](p_2 + p_3 + p_4) + \sum_{i=5}^{12} ip_i \; - 7p_7$$
$$= \frac{6}{36}E[R_5] \; + 5(4/36) + 6(5/36) + 8(5/36) + 9(4/36) + 10(3/36) + 11(2/36) + 12(1/36)$$
$$= \frac{6}{36}E[R_5] \; + 190/36$$

Hence, $E[R_5] = 19/3 \approx 6.33$. In the same fashion, we obtain that

$$E[R_6] = \frac{10}{36}E[R_6] + \frac{1}{36}[30 + 40 + 36 + 30 + 22 + 12]$$

implying that

$$E[R_6] = 170/26 \approx 6.54$$

Also,

$$E[R_8] = \frac{15}{36}E[R_8] + \frac{1}{36}(140)$$

or,

$$E[R_8] = 140/21 \approx 6.67$$

In addition,

$$E[R_9] = \frac{20}{26}E[R_9] + \frac{1}{36}(100)$$

or

$$E[R_9] = 100/16 = 6.25$$

And

$$E[R_{10}] = \frac{24}{36}E[R_{10}] + \frac{1}{36}(64)$$

or

$$E[R_{10}] = 64/12 \approx 5.33$$

The maximum expected return is $E[R_8]$.

55. Let $N$ denote the number of ducks. Given $N = n$, let $I_1, \ldots, I_n$ be such that

$$I_i = \begin{cases} 1 & \text{if duck } i \text{ is hit} \\ 0 & \text{otherwise} \end{cases}$$

$$E[\text{Number hit} \mid N = n] = E\left[\sum_{i=1}^{n} I_i\right]$$

$$= \sum_{i=1}^{n} E[I_i] = n\left[1 - \left(1 - \frac{.6}{n}\right)^{10}\right], \text{ since given}$$

$N = n$, each hunter will independently hit duck $i$ with probability $.6/n$.

$$E[\text{Number hit}] = \sum_{n=0}^{\infty} n\left(1 - \frac{.6}{n}\right)^{10} e^{-6}6^n / n!$$

56. Let $I_i = \begin{cases} 1 & \text{elevator stops at floor } i \\ 0 & \text{otherwise} \end{cases}$. Let $X$ be the number that enter on the ground floor.

$$E\left[\sum_{i=1}^{N} I_i \middle| X = k\right] = \sum_{i=1}^{N} E[I_i \mid X = k] = N\left[1 - \left(\frac{N-1}{N}\right)^k\right]$$

$$E\left[\sum_{i=1}^{N} I_i\right] = N - N\sum_{k=0}^{\infty}\left(\frac{N-1}{N}\right)^k e^{-10}\frac{(10)^k}{k!}$$

$$= N - Ne^{-10/N} = N(1 - e^{-10/N})$$

57. $E\left[\sum_{i=1}^{N} X_i\right] = E[N]E[X] = 12.5$

58. Let $X$ denote the number of flips required. Condition on the outcome of the first flip to obtain.

$$\begin{aligned} E[X] &= E[X \mid \text{heads}]p + E[x \mid \text{tails}](1-p) \\ &= [1 + 1/(1-p)]p + [1 + 1/p](1-p) \\ &= 1 + p/(1-p) + (1-p)/p \end{aligned}$$

59. (a) $E$[total prize shared] = $P$\{someone wins\} = $1 - (1 - p)^{n+1}$

(b) Let $X_i$ be the prize to player $i$. By part (a)

$$E\left[\sum_{i=1}^{n+1} X_i\right] = 1 - (1-p)^{n+1}$$

But, by symmetry all $E[X_i]$ are equal and so

$$E[X] = [1 - (1 - p)^{n+1}]/(n + 1)$$

(c) $E[X] = p\,E[1/(1 + B)]$ where $B$, which is binomial with parameters $n$ and $p$, represents the number of other winners.

60. (a) Since the sum of their number of correct predictions is $n$ (one for each coin) it follows that one of them will have more than $n/2$ correct predictions. Now if $N$ is the number of correct predictions of a specified member of the syndicate, then the probability mass function of the number of correct predictions of the member of the syndicate having more than $n/2$ correct predictions is

$$\begin{aligned} P\{i \text{ correct}\} &= P\{N = i\} + P(N = n - i\} \quad i > n/2 \\ &= 2P\{N = i\} \\ &= P\{N = i \mid N > n/2\} \end{aligned}$$

(b) $X$ is binomial with parameters $m$, 1/2.

(c) Since all of the $X + 1$ players (including one from the syndicate) that have more than $n/2$ correct predictions have the same expected return we see that

$$(X + 1) \cdot \text{Payoff to syndicate} = m + 2$$

implying that

$$E[\text{Payoff to syndicate}] = (m + 2)\, E[(X + 1)^{-1}]$$

(d) This follows from part (b) above and (c) of Problem 56.

61. (a) $P(M \le x) = \sum_{n=1}^{\infty} P(M \le x \mid N = n)P(N = n) = \sum_{n=1}^{\infty} F^n(x)p(1-p)^{n-1} = \dfrac{pF(x)}{1-(1-p)F(x)}$

(b) $P(M \le x \mid N = 1) = F(x)$

(c) $P(M \le x \mid N > 1) = F(x)P(M \le x)$

(d) $$\begin{aligned} P(M \le x) &= P(M \le x \mid N = 1)P(N = 1) + P(M \le x \mid N > 1)P(N > 1) \\ &= F(x)p + F(x)P(M \le x)(1 - p) \end{aligned}$$

again giving the result

$$P(M \le x) = \frac{pF(x)}{1-(1-p)F(x)}$$

62. The result is true when $n = 0$, so assume that

$$P\{N(x) \geq n\} = x^n/(n-1)!$$

Now,

$$\begin{aligned} P\{N(x) \geq n+1\} &= \int_0^1 P\{N(x) \geq n+1 \mid U_1 = y\}dy \\ &= \int_0^x P\{N(x-y) \geq n\}dy \\ &= \int_0^x P\{N(u) \geq n\}du \\ &= \int_0^x u^{n-1}/(n-1)!\ du \text{ by the induction hypothesis} \\ &= x^n/n! \end{aligned}$$

which completes the proof.

(b) $E[N(x)] = \sum_{n=0}^{\infty} P\{N(x) > n = \sum_{n=0}^{\infty} P\{N(x) \geq n+1\} = \sum_{n=0}^{\infty} x^n/n! = e^x$

63. (a) Number the red balls and the blue balls and let $X_i$ equal 1 if the $i^{\text{th}}$ red ball is selected and let it by 0 otherwise. Similarly, let $Y_j$ equal 1 if the $j^{\text{th}}$ blue ball is selected and let it be 0 otherwise.

$$\text{Cov}\left(\sum_i X_i, \sum_j Y_j\right) = \sum_i \sum_j \text{Cov}(X_i, Y_j)$$

Now,

$E[X_i] = E[Y_j] = 12/30$

$E[X_iY_j] = P\{\text{red ball } i \text{ and blue ball } j \text{ are selected}\} = \binom{28}{10}\Big/\binom{30}{12}$

Thus,

$$\text{Cov}(X, Y) = 80\left[\binom{28}{10}\Big/\binom{30}{12} - (12/30)^2\right] = -96/145$$

(b) $E[XY|X] = XE[Y|X] = X(12 - X)8/20$

where the above follows since given $X$, there are 12-X additional balls to be selected from among 8 blue and 12 non-blue balls. Now, since $X$ is a hypergeometric random variable it follows that

$E[X] = 12(10/30) = 4$ and $E[X^2] = 12(18)(1/3)(2/3)/29 + 4^2 = 512/29$

As $E[Y] = 8(12/30) = 16/5$, we obtain

$$E[XY] = \frac{2}{5}(48 - 512/29) = 352/29,$$

and

$$\text{Cov}(X, Y) = 352/29 - 4(16/5) = -96/145$$

64. (a) $E[X] = E[X|\text{type 1}]p + E[X|\text{type 2}](1 - p) = p\mu_1 + (1 - p)\mu_2$

(b) Let $I$ be the type.

$E[X|I] = \mu_I$, $\text{Var}(X|I) = \sigma_I^2$

$\text{Var}(X) = E[\sigma_I^2] + \text{Var}(\mu_I)$

$= p\sigma_1^2 + (1-p)\sigma_2^2 + p\mu_1^2 + (1-p)\mu_2^2 - [p\mu_1 + (1-p)\mu_2]^2$

65. Let $X$ be the number of storms, and let $G(B)$ be the events that it is a good (bad) year. Then

$$E[X] = E[X|G]P(G) + E[X|B]P(B) = 3(.4) + 5(.6) = 4.2$$

If $Y$ is Poisson with mean $\lambda$, then $E[Y^2] = \lambda + \lambda^2$. Therefore,

$$E[X^2] = E[X^2|G]P(G) + E[X^2|B]P(B) = 12(.4) + 30(.6) = 22.8$$

Consequently,

$$\text{Var}(X) = 22.8 - (4.2)^2 = 5.16$$

66. $E[X^2] = \frac{1}{3}\{E[X^2|Y=1] + E[X^2|Y=2] + E[X^2|Y=3]\}$

$= \frac{1}{3}\{9 + E[(5+X)^2] + E[(7+X)^2]\}$

$= \frac{1}{3}\{83 + 24E[X] + 2E[X^2]\}$

$= \frac{1}{3}\{443 + 2E[X^2]\}$ since $E[X] = 15$

Hence,

$\text{Var}(X) = 443 - (15)^2 = 218.$

67. Let $F_n$ denote the fortune after $n$ gambles.

$$
\begin{aligned}
E[F_n] = E[E[F_n \mid F_{n-1}]] &= E[2(2p-1)F_{n-1}p + F_{n-1} - (2p-1)F_{n-1}] \\
&= (1 + (2p-1)^2)E[F_{n-1}] \\
&= [1 + (2p-1)^2]^2 E[F_{n-2}] \\
&\ \vdots \\
&= [1 + (2p-1)^2]^n E[F_0]
\end{aligned}
$$

68. (a) $.6e^{-2} + .4e^{-3}$

(b) $.6e^{-2}\dfrac{2^3}{3!} + .4e^{-3}\dfrac{3^3}{3!}$

(c) $P\{3 \mid 0\} = \dfrac{P\{3,0\}}{P\{0\}} = \dfrac{.6e^{-2}e^{-2}\dfrac{2^3}{3!} + .4e^{-3}e^{-3}\dfrac{3^3}{3!}}{.6e^{-2} + .4e^{-3}}$

69. (a) $\displaystyle\int_0^\infty e^{-x}e^{-x}dx = \frac{1}{2}$

(b) $\displaystyle\int_0^\infty e^{-x}\frac{x^3}{3!}e^{-x}dx = \frac{1}{96}\int_0^\infty e^{-y}y^3dy = \frac{\Gamma(4)}{96} = \frac{1}{16}$

(c) $\dfrac{\displaystyle\int_0^\infty e^{-x}e^{-x}\frac{x^3}{3\,!}e^{-x}dx}{\displaystyle\int_0^\infty e^{-x}e^{-x}dx} = \dfrac{2}{3^4} = \dfrac{2}{81}$

70. (a) $\displaystyle\int_0^1 pdp = 1/2$

(b) $\displaystyle\int_0^1 p^2dp = 1/3$

71. $\displaystyle P\{X = i\} = \int_0^1 P\{X = i \mid p\}dp = \int_0^1 \binom{n}{i} p^i(1-p)^{n-i}dp$

$$= \binom{n}{i}\frac{i!(n-i)!}{(n+1)!} = 1/(n+1)$$

72. (a) $P\{N \ge i\} = \int_0^1 P\{N \ge i | p\}dp = \int_0^1 (1-p)^{i-1}dp = 1/i$

(b) $P\{N = i\} = P\{N \ge i\} - P\{N \ge i + 1\} = \dfrac{1}{i(i+1)}$

(c) $E[N] = \sum_{i=1}^{\infty} P\{N \ge i\} = \sum_{i=1}^{\infty} 1/i = \infty$ .

73. (a) $E[R] = E[E[R \mid S]] = E[S] = \mu$

(b) $\text{Var}(R \mid S) = 1$, $E[R \mid S] = S$
$\text{Var}(R) = 1 + \text{Var}(S) = 1 + \sigma^2$

(c) $f_R(r) = \int f_S(s) F_{R|S}(r \mid s) ds$

$$= C\int e^{-(s-\mu)^2/2\sigma^2} e^{-(r-s)^2/2} ds$$

$$= K\int \exp\left\{-\left(S - \frac{\mu + r\sigma^2}{1+\sigma^2}\right)\Big/ 2\left(\frac{\sigma^2}{1+\sigma^2}\right)\right\} ds \exp\{-(ar^2 + br)\}$$

Hence, $R$ is normal.

(d) $E[RS] = E[E[RS \mid S]] = E[SE[R \mid S]] = E[S^2] = \mu^2 + \sigma^2$

$\text{Cov}(R, S) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

75. $X$ is Poisson with mean $\lambda = 2$ and $Y$ is Binomial with parameters 10, 3/4. Hence

(a) $P\{X + Y = 2\} = P\{X = 0)P\{Y = 2\} + P\{X = 1\}P\{Y = 1\} + P\{X = 2\}P\{Y = 0\}$

$$= e^{-2}\binom{10}{2}(3/4)^2(1/4)^8 + 2e^{-2}\binom{10}{1}(3/4)(1/4)^9 + 2e^{-2}(1/4)^{10}$$

(b) $P\{XY = 0\} = P\{X = 0\} + P\{Y = 0\} - P\{X = Y = 0\}$
$= e^{-2} + (1/4)^{10} - e^{-2}(1/4)^{10}$

(c) $E[XY] = E[X]E[Y] = 2 \cdot 10 \cdot \dfrac{3}{4} = 15$

77. The joint moment generating function, $E[e^{tX+sY}]$ can be obtained either by using

$$E[e^{tX+sY}] = \int\int e^{tX+sY} f(x,y)dy\ dx$$

or by noting that $Y$ is exponential with rate 1 and, given $Y$, $X$ is normal with mean $Y$ and variance 1. Hence, using this we obtain

$$E[e^{tX+sY} \mid Y] = e^{sY}E[E^{tX} \mid Y] = e^{sY}e^{Yt+t^2/2}$$

and so

$$\begin{aligned} E[e^{tX+sY}] &= e^{t^2/2}E[e^{(s+t)Y}] \\ &= e^{t^2/2}(1-s-t)^{-1},\ s+t<1 \end{aligned}$$

Setting first $s$ and then $t$ equal to 0 gives

$$E[e^{tX}] = e^{t^2/2}(1-t)^{-1},\ t<1$$
$$E[e^{sY}] = (1-s)^{-1},\ s<1$$

78. Conditioning on the amount of the initial check gives

$$\begin{aligned} E[\text{Return}] &= E[\text{Return} \mid A]/2 + E[\text{Return} \mid \text{B}]/2 \\ &= \{AF(A) + B[1-F(A)]\}/2 + \{BF(B) + A[1-F(B)]\}/2 \\ &= \{A + B + [B-A][F(B)-F(A)]\}/2 \\ &> (A+B)/2 \end{aligned}$$

where the inequality follows since $[B - A]$ and $[F(B) - F(A)$ both have the same sign.

(b) If $x < A$ then the strategy will accept the first value seen: if $x > B$ then it will reject the first one seen; and if $x$ lies between $A$ and $B$ then it will always yield return B. Hence,

$$E[\text{Return of } x\text{-strategy}] = \begin{array}{ll} B & \text{if } A<x<B \\ (A+B)/2 & \text{otherwise} \end{array}$$

(c) This follows from (b) since there is a positive probability that $X$ will lie between $A$ and $B$.

79. Let $X_i$ denote sales in week $i$. Then

$$\begin{aligned} E[X_1+X_2] &= 80 \\ \text{Var}(X_1+X_2) &= \text{Var}(X_1) + \text{Var}(X_2) + 2\ \text{Cov}(X_1, X_2) \\ &= 72 + 2[.6(6)(6)] = 93.6 \end{aligned}$$

(a) With $Z$ being a standard normal

$$\begin{aligned} P(X_1+X_2>90) &= P\left(Z > \frac{90-80}{\sqrt{93.6}}\right) \\ &= P(Z > 1.034) \approx .150 \end{aligned}$$

(b) Because the mean of the normal $X_1 + X_2$ is less than 90 the probability that it exceeds 90 is increased as the variance of $X_1 + X_2$ increases. Thus, this probability is smaller when the correlation is .2.

(c) In this case,

$$P(X_1 + X_2 > 90) = P\left(Z > \frac{90-80}{\sqrt{72+2[.2(6)(6)]}}\right)$$
$$= P(Z > 1.076) \approx .141$$

## Theoretical Exercises

1. Let $\mu = E[X]$. Then for any $a$

$$\begin{aligned} E[(X-a)^2 &= E[(X-\mu+\mu-a)^2] \\ &= E[(X-\mu)^2] + (\mu-a)^2 + 2E[(x-\mu)(\mu-a)] \\ &= E[(X-\mu)^2] + (\mu-a)^2 + 2(\mu-a)E[(X-\mu)] \\ &= E[(X-\mu)^2 + (\mu-a)^2 \end{aligned}$$

2. $$\begin{aligned} E[\,|X-a|\,] &= \int_{x<a}(a-x)f(x)dx + \int_{x>a}(x-a)f(x)dx \\ &= aF(a) - \int_{x<a}xf(x)dx + \int_{x>a}xf(x)dx - a[1-F(a)] \end{aligned}$$

Differentiating the above yields
derivative = $2af(a) + 2F(a) - af(a) - af(a) - 1$
Setting equal to 0 yields that $2F(a) = 1$ which establishes the result.

3. $$\begin{aligned} E[g(X, Y)] &= \int_0^\infty P\{g(X,Y) > a\}da \\ &= \int_0^\infty \iint_{\substack{x,y: \\ g(x,y)>a}} f(x,y)dydxda = \iint \int_0^{g(x,y)} daf(x,y)dydx \\ &= \iint g(x,y)dydx \end{aligned}$$

4. $$\begin{aligned} g(X) &= g(\mu) + g'(\mu)(X-\mu) + g''(\mu)\ \frac{(X-\mu)^2}{2} + \ldots \\ &\approx g(\mu) + g'(\mu)(X-\mu) + g''(\mu)\ \frac{(X-\mu)^2}{2} \end{aligned}$$

Now take expectations of both sides.

5. If we let $X_k$ equal 1 if $A_k$ occurs and 0 otherwise then

$$X = \sum_{k=1}^n X_k$$

Hence,

$$E[X] = \sum_{k=1}^n E[X_k] = \sum_{k=1}^n P(A_k)$$

But

$$E[X] = \sum_{k=1}^n P\{X \ge k\} = \sum_{k=1}^n P(C_k)\ .$$

6. $X = \int_0^\infty X(t)dt$ and taking expectations gives

$$E[X] = \int_0^\infty E[X(t)]\, dt = \int_0^\infty P\{X > t\}dt$$

7. (a) Use Exercise 6 to obtain that

$$E[X] = \int_0^\infty P\{X > t\}dt \geq \int_0^\infty P\{Y > t\}dt \ = E[Y]$$

(b) It is easy to verify that

$X^+ \geq_{\text{st}} Y^+$ and $Y^- \geq_{\text{st}} X^-$

Now use part (a).

8. Suppose $X \geq_{\text{st}} Y$ and $f$ is increasing. Then

$$\begin{aligned} P\{f(X) > a\} &= P\{X > f^{-1}(a)\} \\ &\geq P\{Y > f^{-1}(a)\} \quad \text{since } x \geq_{st} Y \\ &= P\{f(Y) > a\} \end{aligned}$$

Therefore, $f(X) \geq_{st} f(Y)$ and so, from Exercise 7, $E[f(X)] \geq E[f(Y)]$.

On the other hand, if $E[f(X)] \geq E[f(Y)]$ for all increasing functions $f$, then by letting $f$ be the increasing function

$$f(x) = \begin{matrix} 1 & \text{if } x > t \\ 0 & \text{otherwise} \end{matrix}$$

then

$$P\{X > t\} = E[f(X)] \geq E[f(Y)] = P\{Y > t\}$$

and so X $>_{\text{st}} Y$.

9. Let

$$I_j = \begin{cases} 1 & \text{if a run of size } k \text{ begins at the } j^{\text{th}} \text{ flip} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\text{Number of runs of size } k = \sum_{j=1}^{n-k+1} I_j$$

$$E[\text{Number of runs of size } k = E\left[\sum_{j=1}^{n-k+1} I_j\right]$$

$$= P(I_1 = 1) + \sum_{j=2}^{n-k} P(I_j = 1) + P(I_{n-k+1} = 1)$$

$$= p^k(1-p) + (n-k-1)p^k(1-p)^2 + p^k(1-p)$$

10. $$1 = E\left[\sum_1^n X_i \Big/ \sum_1^n X_i\right] = \sum_1^n E\left[X_i \Big/ \sum_1^n X_i\right] = nE\left[X_1 \Big/ \sum_1^n X_i\right]$$

Hence,

$$E\left[\sum_1^k X_i \Big/ \sum_1^n X_i\right] = k/n$$

11. Let

$$I_j = \begin{cases} 1 & \text{outcome } j \text{ never occurs} \\ 0 & \text{otherwise} \end{cases}$$

Then $X = \sum_1^r I_j$ and $E[X] = \int_{j=1}^{r} (1-p_j)^n$

12. Let

$$I_j = \begin{cases} 1 & \text{success on trial } j \\ 0 & \text{otherwise} \end{cases}$$

$$E\left[\sum_1^n I_j\right] = \sum_1^n P_j \quad \text{independence not needed}$$

$$\text{Var}\left(\sum_1^n I_j\right) = \sum_1^n p_j(1-p_j) \quad \text{independence needed}$$

13. Let

$$I_j = \begin{cases} 1 & \text{record at } j \\ 0 & \text{otherwise} \end{cases}$$

$$E\left[\sum_1^n I_j\right] = \sum_1^n E[I_j] = \sum_1^n P\{X_j \text{ is largest of } X_1, \ldots, X_j\} = \sum_1^n 1/j$$

$$\text{Var}\left(\sum_1^n I_j\right) = \sum_1^n \text{Var}(I_j) = \sum_1^n \frac{1}{j}\left(1 - \frac{1}{j}\right)$$

15. $\mu = \sum_{i=1}^{n} p_i$ by letting Number $= \sum_{i=1}^{n} X_i$ where $X_i = \begin{cases} 1 & i \text{ is success} \\ 0 & \text{---} \end{cases}$

Var(Number) $= \sum_{i=1}^{n} p_i(1 - p_i)$

maximization of variance occur when $p_i \equiv \mu/n$

minimization of variance when $p_i = 1$, $i = 1, \ldots, [\mu]$, $p_{[\mu]+1} = \mu - [\mu]$

To prove the maximization result, suppose that 2 of the $p_i$ are unequal—say $p_i \neq p_j$. Consider a new $p$-vector with all other $p_k$, $k \neq i, j$, as before and with $\overline{p}_i = \overline{p}_j = \dfrac{p_i + p_j}{2}$. Then in the variance formula, we must show

$$2\left(\frac{p_i + p_j}{2}\right)\left(1 - \frac{p_i + p_j}{2}\right) \geq p_i(1 - p_i) + p_j(1 - p_j)$$

or equivalently,

$$p_i^2 + p_j^2 - 2p_i p_j = (p_i - p_j)^2 \geq 0.$$

The maximization is similar.

16. Suppose that each element is, independently, equally likely to be colored red or blue. If we let $X_i$ equal 1 if all the elements of $A_i$ are similarly colored, and let it be 0 otherwise, then $\sum_{i=1}^{r} X_i$ is the number of subsets whose elements all have the same color. Because

$$E\left[\sum_{i=1}^{r} X_i\right] = \sum_{i=1}^{r} E[X_i] = \sum_{i=1}^{r} 2(1/2)^{|A_i|}$$

it follows that for at least one coloring the number of monocolored subsets is less than or equal to $\sum_{i=1}^{r} (1/2)^{|A_i|-1}$

17. $\text{Var}\big(\lambda X_1 + (1-\lambda)X_2\big) = \lambda^2\sigma_1^2 + (1-\lambda)^2\sigma_2^2$

$\dfrac{d}{d\lambda}(\quad) = 2\lambda\sigma_1^2 - 2(1-\lambda)\sigma_2^2 = 0 \Rightarrow \lambda = \dfrac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$

As Var($\lambda X_1 + (1 - \lambda)X_2$) = $E\left[(\lambda X_1 + (1-\lambda)X_2 - \mu)^2\right]$ we want this value to be small.

18. (a. Binomial with parameters $m$ and $P_i + P_j$.

(b) Using (a) we have that $\text{Var}(N_i + N_j) = m(P_i + P_j)(1 - P_i - P_j)$ and thus

$$m(P_i + P_j)(1 - P_i - P_j) = mP_i(1 - P_i) + mP_j(1 - P_j) + 2\,\text{Cov}(N_i, N_j)$$

Simplifying the above shows that

$$\text{Cov}(N_i, N_j) = -mP_iP_j.$$

19. $$\begin{aligned}\text{Cov}(X + Y, X - Y) &= \text{Cov}(X, X) + \text{Cov}(X, -Y) + \text{Cov}(Y, X) + \text{Cov}(Y, -Y)\\ &= \text{Var}(X) - \text{Cov}(X, Y) + \text{Cov}(Y, X) - \text{Var}(Y)\\ &= \text{Var}(X) - \text{Var}(Y) = 0.\end{aligned}$$

20. (a) $\text{Cov}(X, Y|Z)$

$$\begin{aligned}&= E[XY - E[X|Z]Y - XE[Y|Z] + E[X|Z]E[Y|Z]\ [Z]\\ &= E[XY|Z] - E[X|Z]\,E[Y|Z] - E[X|Z]E[Y|Z] + E[X|Z]E[Y|Z]\\ &= E[XY|Z] - E[X|Z]E[Y|Z]\end{aligned}$$

where the next to last equality uses the fact that given $Z$, $E[X|Z]$ and $E[Y|Z]$ can be treated as constants.

(b) From (a)

$$E[\text{Cov}(X, Y|Z)] = E[XY] - E[E[X|Z]E[Y|Z]]$$

On the other hand,

$$\text{Cov}(E[X|Z], E[Y|Z] = E[E[X|Z]E[Y|Z]] - E[X]E[Y]$$

and so

$$\begin{aligned}E[\text{Cov}(X, Y|Z)] + \text{Cov}(E[X|Z], E[Y|Z]) &= E[XY] - E[X]E[Y]\\ &= \text{Cov}(X, Y)\end{aligned}$$

(c) Noting that $\text{Cov}(X, X|Z) = \text{Var}(X|Z)$ we obtain upon setting $Y = Z$ that

$$\text{Var}(X) = E[\text{Var}(X|Z)] + \text{Var}(E[X|Z])$$

21. (a) Using the fact that $f$ integrates to 1 we see that

$c(n, i) \equiv \int_0^1 x^{i-1}(1-x)^{n-i}\,dx = (i - 1)!(n - i)!/n!$. From this we see that

$$E[X_{(i)}] = c(n + 1, i + 1)/c(n, i) = i/(n + 1)$$

$$E[X_{(i)}^2] = c(n + 2, i + 2)/c(n, i) = \frac{i(i+1)}{(n+2)(n+1)}$$

and thus

$$\text{Var}(X_{(i)}) = \frac{i(n+1-i)}{(n+1)^2(n+2)}$$

(b) The maximum of $i(n + 1 - i)$ is obtained when $i = (n + 1)/2$ and the minimum when $i$ is either 1 or $n$.

22. $\text{Cov}(X, Y) = b\ \text{Var}(X)$, $\text{Var}(Y) = b^2\ \text{Var}(X)$

$$\rho(X,Y) = \frac{b\ \text{Var}(X)}{\sqrt{b^2}\text{Var}(X)} = \frac{b}{|b|}$$

26. Follows since, given $X$, $g(X)$ is a constant and so

$$E[g(X)Y \mid X] = g(X)E[Y \mid X]$$

27.
$$\begin{aligned} E[XY] &= E[E[XY \mid X]] \\ &= E[XE[Y \mid X]] \end{aligned}$$

Hence, if $E[Y \mid X] = E[Y]$, then $E[XY] = E[X]E[Y]$. The example in Section 3 of random variables uncorrelated but not independent provides a counterexample to the converse.

28. The result follows from the identity

$E[XY] = E[E[XY \mid X]] = E[XE[Y \mid X]]$ which is obtained by noting that, given $X$, $X$ may be treated as a constant.

29.
$$\begin{aligned} x = E[X_1 + \ldots + X_n \mid X_1 + \ldots + X_n = x] &= E\left[X_1 \middle| \sum X_i = x\right] + \ldots + E\left[X_n \middle| \sum X_i = x\right] \\ &= nE\left[X_1 \middle| \sum X_i = x\right] \end{aligned}$$

Hence, $E[X_1 \mid X_1 + \ldots + X_n = x] = x/n$

30. $E[N_iN_j \mid N_i] = N_iE[N_j \mid N_i] = N_i(n - N_i)\dfrac{p_j}{1-p_i}$ since each of the $n - N_i$ trials no resulting in outcome $i$ will independently result in $j$ with probability $p_j/(1 - p_i)$. Hence,

$$\begin{aligned} E[N_iN_j] &= \frac{p_j}{1-p_i}\left(nE[N_i] - E\left[N_i^2\right]\right) = \frac{p_j}{1-p_i}\left[n^2p_i - n^2p_i^2 - np_i(1-p_i)\right] \\ &= n(n-1)p_ip_j \end{aligned}$$

and

$$\text{Cov}(N_i, N_j) = n(n-1)p_ip_j - n^2p_ip_j = -np_ip_j$$

31. By induction: true when $t = 0$, so assume for $t - 1$. Let $N(t)$ denote the number after stage $t$.

$$E[N(t) \mid N(t-1)] = N(t-1) - E[\text{number selected}]$$
$$= N(t-1) - N(t-1)\frac{r}{b+w+r}$$
$$E[N(t) \mid N(t-1)] = N(t-1)\frac{b+w}{b+w+r}$$
$$E[N(t)] = \left(\frac{b+w}{b+w+r}\right)^t w$$

32. $E[X_1X_2 \mid Y = y] = E[X_1 \mid Y = y]E[X_2 \mid Y = y] = y^2$

Therefore, $E[X_1X_2 \mid Y] = Y^2$. As $E[X_i \mid Y] = Y$, this gives that

$$E[X_1X_2] = E[E[X_1X_2 \mid Y]] = Ei/[Y^2], \quad E[X_i] = E[E[X_i \mid Y]] = E[Y]$$

Consequently,

$$\text{Cov}(X_1, X_2) = E[X_1X_2] - E[X_1]E[X_2] = \text{Var}(Y)$$

34. (a) $E[T_r \mid T_{r-1}] = T_{r-1} + 1 + (1-p)E[T_r]$

(b) Taking expectations of both sides of (a) gives

$$E[T_r] = E[T_{r-1}] + 1 + (1-p)E[T_r]$$

or

$$E[T_r] = \frac{1}{p} + \frac{1}{p}E[T_{r-1}]$$

(c) Using the result of part (b) gives

$$E[T_r] = \frac{1}{p} + \frac{1}{p}E[T_{r-1}]$$
$$= \frac{1}{p} + \frac{1}{p}\left(\frac{1}{p} + \frac{1}{p}E[T_{r-2}]\right)$$
$$= 1/p + (1/p)^2 + (1/p)^2E[T_{r-2}]$$
$$= 1/p + (1/p)^2 + (1/p)^3 + (1/p)^3E[T_{r-3}]$$
$$= \sum_{i=1}^{r}(1/p)^i + (1/p)^r E[T_0]$$
$$= \sum_{i=1}^{r}(1/p)^i \quad \text{since } E[T_0] = 0.$$

35. $P(Y > X) = \sum_j P(Y > X \mid X = j) p_j$

$$= \sum_j P(Y > j \mid X = j) p_j$$

$$= \sum_j P(Y > j) p_j$$

$$= \sum_j (1-p)^j p_j$$

36. Condition on the first ball selected to obtain

$$M_{a,b} = \frac{a}{a+b} M_{a-1,b} + \frac{b}{a+b} M_{a,b-1}, \; a, b > 0$$

$$M_{a,0} = a, \qquad M_{0,b} = b, \qquad M_{a,b} = M_{b,a}$$

$$M_{2,1} = \frac{4}{3}, \qquad M_{3,1} = \frac{7}{4}, \qquad M_{3,2} = 3/2$$

37. Let $X_n$ denote the number of white balls after the $n^{\text{th}}$ drawing

$$E[X_{n+1} \mid X_n] = X_n \frac{X_n}{a+b} + (X_n + 1)\left(1 - \frac{X_n}{a+b}\right) = \left(1 - \frac{1}{a+b}\right) X_n + 1$$

Taking expectations now yields (a).

To prove (b), use (a) and the boundary condition $M_0 = a$

(c) $P\{(n + 1)\text{st is white}\} = E[P\{(n + 1)\text{st is white} \mid X_n\}]$

$$= E\left[\frac{X_n}{a+b}\right] = \frac{M_n}{a+b}$$

40. For (a) and (c), see theoretical Exercise 18 of Chapter 6. For (c)

$$E[XY] = E[E[XY \mid X]] = E[XE[Y \mid X]]$$

$$= E\left[X\left(\mu_y + \rho\frac{\sigma_y}{\sigma_x}(X - \mu_x\right)\right]$$

$$= \mu_x\mu_y - \rho\frac{\sigma_y}{\sigma_x}\mu_x^2 + \rho\frac{\sigma_y}{\sigma_x}\left(\mu_x^2 + \sigma_x^2\right)$$

and so

$$\text{Corr}(X, Y) = \frac{\rho\sigma_y\sigma_x}{\sigma_y\sigma_x} = \rho$$

41. (a) No

(b) Yes, since $f_Y(x \mid I = 1) = f_X(x) = f_X(-x) = f_Y(x \mid I = 0)$

(c) $f_Y(x) = \frac{1}{2} f_X(x) + \frac{1}{2} f_X(-x) = f_X(x)$

(d) $E[XY] = E[E[XY \mid X]] = E[XE[Y \mid X]] = 0$

(e) No, since $X$ and $Y$ are not jointly normal.

42. If $E[Y \mid X]$ is linear in $X$, then it is the best linear predictor of $Y$ with respect to $X$.

43. Must show that $E[Y^2] = E[XY]$. Now

$$\begin{aligned} E[XY] &= E[XE[X \mid Z]] \\ &= E[E[XE[X \mid Z] \mid Z]] \\ &= E[E^2[X \mid Z]] = E[Y^2] \end{aligned}$$

44. Write $X_n = \sum_{i=1}^{X_{n-1}} Z_i$ where $Z_i$ is the number of offspring of the $i$th individual of the $(n-1)$st generation. Hence,

$$E[X_n] = E[E[X_n \mid X_{n-1}]] = E[\mu X_{n-1}] = \mu E[X_{n-1}]$$

so,

$$E[X_n] = \mu E[X_{n-1}] = \mu^2 E[X_{n-2}] \ldots = \mu^n E[X_0] = \mu^n$$

(c) Use the above representation to obtain

$$E[X_n \mid X_{n-1}] = \mu X_{n-1}, \ \mathrm{Var}(X_n \mid X_{n-1}) = \sigma^2 X_{n-1}$$

Hence, using the conditional Variance Formula,

$$\mathrm{Var}(X_n) = \mu^2 \,\mathrm{Var}(X_{n-1}) + \sigma^2 \mu^{n-1}$$

(d) $\pi = P\{\text{dies out}\}$

$= \sum_j P\{\text{dies out} \mid X_i = j\} p_j$

$= \sum_j \pi^j p_j$, since each of the $j$ members of the first generation can be thought of as starting their own (independent) branching process.

46. It is easy to see that the $n^{\text{th}}$ derivative of $\sum_{j=0}^{\infty} (t^2/2)^j / j!$ will, when evaluated at $t = 0$, equal 0 whenever $n$ is odd (because all of its terms will be constants multiplied by some power of $t$). When $n = 2j$ the $n^{\text{th}}$ derivative will equal $\frac{d^n}{dt^n}\{t^n\}/(j!2^j)$ plus constants multiplied by powers of $t$. When evaluated at 0, this gives that

$$E[Z^{2j}] - (2j)!/(j!2^j)$$

47. Write $X = \sigma Z + \mu$ where $Z$ is a standard normal random variable. Then, using the binomial theorem,

$$E[X^n] = \sum_{i=0}^{n} \binom{n}{i} \sigma^i E[Z^i] \mu^{n-i}$$

Now make use of theoretical exercise 46.

48. $\phi_Y(t) = E[e^{tY}] = E[e^{t(aX+b)}] = e^{tb}E[e^{taX}] = e^{tb}\phi_X(ta)$

49. Let $Y = \log(X)$. Since $Y$ is normal with mean $\mu$ and variance $\sigma^2$ it follows that its moment generating function is

$$M(t) = E[e^{tY}] = e^{\mu t + \sigma^2 t^2/2}$$

Hence, since $X = e^Y$, we have that

$$E[X] = M(1) = e^{\mu + \sigma^2/2}$$

and

$$E[X^2] = M(2) = e^{2\mu + 2\sigma^2}$$

Therefore,

$$\text{Var}(X) = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$$

50. $\psi(t) = \log \phi(t)$

$$\psi'(t) = \phi'(t)/\phi(t)$$

$$\psi''(t) = \frac{\phi(t)\phi''(t) - (\phi'(t))^2}{\phi^2(t)}$$

$$\psi''(t)\Big|_{t=0} = E[X^2] - (E[X])^2 = \text{Var}(X).$$

51. Gamma ($n$, $\lambda$)

52. Let $\phi(s, t) = E[e^{sX+tY}]$

$$\frac{\partial^2}{\partial s \partial t}\phi(s,t)\bigg|_{\substack{s=0\\t=0}} = E[XYe^{sX+tY}]\bigg|_{\substack{s=0\\t=0}} = E[XY]$$

$$\frac{\partial}{\partial s}\phi(s,t)\bigg|_{\substack{s=0\\t=0}} = E[X], \quad \frac{\partial}{\partial t}\phi(s,t)\bigg|_{\substack{s=0\\t=0}} = E[Y]$$

53. Follows from the formula for the joint moment generating function.

54. By symmetry, $E[Z^3] = E[Z] = 0$ and so $\text{Cov}(Z,Z^3) = 0$.

55. (a) This follows because the conditional distribution of $Y + Z$ given that $Y = y$ is normal with mean $y$ and variance 1, which is the same as the conditional distribution of $X$ given that $Y = y$.

(b) Because $Y + Z$ and $Y$ are both linear combinations of the independent normal random variables $Y$ and $Z$, it follows that $Y + Z$, $Y$ has a bivariate normal distribution.

(c)
$$\mu_x = E[X] = E[Y + Z] = \mu$$
$$\sigma_x^2 = \text{Var}(X) = \text{Var}(Y + Z) = \text{Var}(Y) + \text{Var}(Z) = \sigma^2 + 1$$
$$\rho = \text{Corr}(X, Y) = \frac{\text{Cov}(Y+Z,Y)}{\sigma\sqrt{\sigma^2+1}} = \frac{\sigma}{\sqrt{\sigma^2+1}}$$

(d) and (e) The conditional distribution of $Y$ given $X = x$ is normal with mean

$$E[Y|X=x] = \mu + \rho\frac{\sigma}{\sigma_x}(x-\mu_x) = \mu + \frac{\sigma^2}{1+\sigma^2}(x-\mu)$$

and variance

$$\text{Var}(Y|X=x) = \sigma^2\left(1-\frac{\sigma^2}{\sigma^2+1}\right) = \frac{\sigma^2}{\sigma^2+1}$$

# Chapter 8

## Problems

1. $P\{0 \le X \le 40\} = 1 - P\{|X - 20| > 20\} \ge 1 - 20/400 = 19/20$

2. (a) $P\{X \ge 85\} \le E[X]/85 = 15/17$

   (b) $P\{65 \le X \le 85) = 1 - P\{|X - 75| > 10\} \ge 1 - 25/100$

   (c) $P\left\{\left|\sum_{i=1}^{n} X_i / n - 75\right| > 5\right\} \le \dfrac{25}{25n}$ so need $n = 10$

3. Let $Z$ be a standard normal random variable. Then,

$$P\left\{\left|\sum_{i=1}^{n} X_i / n - 75\right| > 5\right\} \approx P\{|Z| > \sqrt{n}\} \le .1 \text{ when } n = 3$$

4. (a) $P\left\{\sum_{i=1}^{20} X_i > 15\right\} \le 20/15$

   (b) 
$$\begin{aligned} P\left\{\sum_{i=1}^{20} X_i > 15\right\} &= P\left\{\sum_{i=1}^{20} X_i > 15.5\right\} \\ &\approx P\left\{Z > \frac{15.5 - 20}{\sqrt{20}}\right\} \\ &= P\{Z > -1.006\} \\ &\approx .8428 \end{aligned}$$

5. Letting $X_i$ denote the $i^{\text{th}}$ roundoff error it follows that $E\left[\sum_{i=1}^{50} X_i\right] = 0$,

$\text{Var}\left(\sum_{i=1}^{50} X_i\right) = 50\ \text{Var}(X_1) = 50/12$, where the last equality uses that $.5 + X$ is uniform (0, 1) and so $\text{Var}(X) = \text{Var}(.5 + X) = 1/12$. Hence,

$$\begin{aligned} P\left\{\left|\sum X_i\right| > 3\right\} &\approx P\{|N(0, 1)| > 3(12/50)^{1/2}\} \text{ by the central limit theorem} \\ &= 2P\{N(0, 1) > 1.47 = .1416 \end{aligned}$$

6. If $X_i$ is the outcome of the $i^{\text{th}}$ roll then $E[X_i] = 7/2$ $\text{Var}(X_i) = 35/12$ and so

$$\begin{aligned} P\left\{\sum_{i=1}^{79} X_i \le 300\right\} &= P\left\{\sum_{i=1}^{79} X_i \le 300.5\right\} \\ &\approx P\left\{N(0,1) \le \frac{300.5 - 79(7/2)}{(79 \times 35/12)^{1/2}}\right\} = P\{N(0,1) \le 1.58\} = .9429 \end{aligned}$$

7. $$P\left\{\sum_{i=1}^{100} X_i > 525\right\} \approx P\left\{N(0,1) > \frac{525-500}{\sqrt{(100\times 25)}}\right\} = P\{N(0,1) > .5\} = .3085$$

where the above uses that an exponential with mean 5 has variance 25.

8. If we let $X_i$ denote the life of bulb $i$ and let $R_i$ be the time to replace bulb $i$ then the desired probability is $P\left\{\sum_{i=1}^{100} X_i + \sum_{i=1}^{99} R_i \le 550\right\}$. Since $\sum X_i + \sum R_i$ has mean $100 \times 5 + 99 \times .25 =$ 524.75 and variance 2500 + 99/48 = 2502 it follows that the desired probability is approximately equal to $P\{N(0, 1) \le [550 - 524.75]/(2502)^{1/2}\} = P\{N(0, 1) \le .505\} = .693$
It should be noted that the above used that

$$\text{Var}(R_i) = \text{Var}\left(\frac{1}{2}\text{Unif}[0,1]\right) = 1/48$$

9. Use the fact that a gamma $(n, 1)$ random variable is the sum of $n$ independent exponentials with rate 1 and thus has mean and variance equal to $n$, to obtain:

$$\begin{aligned} P\left\{\left|\frac{X-n}{n}\right| > .01\right\} &= P\left\{|X-n|/\sqrt{n} > .01\sqrt{n}\right\} \\ &\approx P\left\{|N(0,1)| > .01\sqrt{n}\right\} \\ &= 2P\left\{N(0,1) > .01\sqrt{n}\right\} \end{aligned}$$

Now $P\{N(0, 1) > 2.58\} = .005$ and so $n = (258)^2$.

10. If $W_n$ is the total weight of $n$ cars and $A$ is the amount of weight that the bridge can withstand then $W_n - A$ is normal with mean $3n - 400$ and variance $.09n + 1600$. Hence, the probability of structural damage is

$$P\{W_n - A \ge 0\} \approx P\left\{Z \ge (400-3n)/\sqrt{.09n+1600}\right\}$$

Since $P\{Z \ge 1.28\} = .1$ the probability of damage will exceed .1 when $n$ is such that

$$400 - 3n \le 1.28\sqrt{.09n+1600}$$

The above will be satisfied whenever $n \ge 117$.

12. Let $L_i$ denote the life of component $i$.

$$E\left[\sum_{i=1}^{100} L_i\right] = 1000 + \frac{1}{10}50(101) = 1505$$

$$\text{Var}\left(\sum_{i=1}^{100} L_i\right) = \sum_{i=1}^{100}\left(10 + \frac{i}{10}\right)^2 = (100)^2 + (100)(101) + \frac{1}{100}\sum_{i=1}^{100} i^2$$

Now apply the central limit theorem to approximate.

13. (a) $P\{\overline{X} > 80\} = P\left\{\dfrac{\overline{X} - 74}{14/5} > 15/7\right\} \approx PPZ > 2.14\} \approx .0162$

(b) $P\{\overline{Y} > 80\} = P\left\{\dfrac{\overline{Y} - 74}{14/8} > 24/7\right\} \approx P\{Z > 3.43\} \approx .0003$

(c) Using that $SD(\overline{Y} - \overline{X}) = \sqrt{196/64 + 196/25} \approx 3.30$ we have

$$P\{\overline{Y} - \overline{X} > 2.2\} = P\{\overline{Y} - \overline{X}\}/3.30 > 2.2/3.30\}$$
$$\approx P\{Z > .67\} \approx .2514$$

(d) same as in (c)

14. Suppose $n$ components are in stock. The probability they will last for at least 2000 hours is

$$p = P\left\{\sum_{i=1}^{n} X_i \geq 2000\right\} \approx P\left\{Z \geq \frac{2000 - 100n}{30\sqrt{n}}\right\}$$

where $Z$ is a standard normal random variable. Since

$.95 = P\{Z \geq -1.64\}$ it follows that $p \geq .95$ if

$$\frac{2000 - 100n}{30\sqrt{n}} \leq -1.64$$

or, equivalently,

$$(2000 - 100n)/\sqrt{n} \leq -49.2$$

and this will be the case if $n \geq 23$.

15. $P\left\{\sum_{i=1}^{10,000} X_i > 2{,}700{,}000\right\} \approx P\{Z \geq (2{,}700{,}000 - 2{,}400{,}000)/(800 \cdot 100)\} = P\{Z \geq 3.75\} \approx 0$

18. Let $Y_i$ denote the additional number of fish that need to be caught to obtain a new type when there are at present $i$ distinct types. Then $Y_i$ is geometric with parameter $\dfrac{4-i}{4}$.

$$E[Y] = E\left[\sum_{i=0}^{3} Y_i\right] = 1 + \frac{4}{3} + \frac{4}{2} + 4 = \frac{25}{3}$$

$$\text{Var}[Y] = \text{Var}\left(\sum_{i=0}^{3} Y_i\right) = \frac{4}{9} + 2 + 12 = \frac{130}{9}$$

Hence,

$$P\left\{\left|Y-\frac{25}{3}\right|>\frac{25}{3}\sqrt{\frac{1300}{9}}\right\}\le\frac{1}{10}$$

and so we can take $a=\dfrac{25-\sqrt{1300}}{3}$, $b=\dfrac{25+\sqrt{1300}}{3}$.

Also,

$$P\left\{Y-\frac{25}{3}>a\right\}\le\frac{130}{130+9a^2}=\frac{1}{10}\text{ when }a=\frac{\sqrt{1170}}{3}.$$

Hence $P\left\{Y>\dfrac{25+\sqrt{1170}}{3}\right\}\le .1$.

20. $g(x)=x^{n(n-1)}$ is convex. Hence, by Jensen's Inequality

$E[Y^{n/(n-1)}]\ge E[Y])^{n/(n-1)}$ Now set $Y=X^{n-1}$ and so
$E[X^n]\ge(E[X^{n-1}])^{n/(n-1)}$ or $(E[X^n])^{1/n}\ge(E[X^{n-1}])^{1/(n-1)}$

21. No

22. (a) $20/26\approx .769$

(b) $20/(20+36)=5/14\approx .357$

(d) $p\approx P\{Z\ge(25.5-20)/\sqrt{20}\}\approx P\{Z\ge 1.23\}\approx .1093$

(e) $p=.112184$

## Theoretical Exercises

1. This follows immediately from Chebyshev's inequality.

2. $P\{D > \alpha\} = P\{|X - \mu| > \alpha\mu\} \le \dfrac{\varsigma^2}{\alpha^2\mu^2} = \dfrac{1}{\alpha^2 r^2}$

3. (a) $\dfrac{\lambda}{\sqrt{\lambda}} = \sqrt{\lambda}$

   (b) $\dfrac{np}{\sqrt{np(1-p)}} = \sqrt{np/(1-p)}$

   (c) answer = 1

   (d) $\dfrac{1/2}{\sqrt{1/12}} = \sqrt{3}$

   (e) answer = 1

   (d) answer = $|\mu|/\sigma$

4. For $\varepsilon > 0$, let $\delta > 0$ be such that $|g(x) - g(c)| < \varepsilon$ whenever $|x - c| \le \delta$. Also, let $B$ be such that $|g(x)| < B$. Then,

$$E[g(Z_n)] = \int_{|x-c|\le\delta} g(x)dF_n(x) + \int_{|x-c|>\delta} g(x)dF_n(x)$$
$$\le (\varepsilon + g(c))P\{|Z_n - c| \le \delta\} + BP\{|Z_n - c| > \delta\}$$

In addition, the same equality yields that

$$E[g(Z_n)] \ge (g(c) - \varepsilon)P\{|Z_n - c| \le \delta\} - BP\{|Z_n - c| > \delta\}$$

Upon letting $n \to \infty$, we obtain that

$$\limsup E[g(Z_n)] \le g(c) + \varepsilon$$
$$\liminf E[g(Z_n)] \ge g(c) - \varepsilon$$

The result now follows since $\varepsilon$ is arbitrary.

5. Use the notation of the hint. The weak law of large numbers yields that

$$\lim_{n\to\infty} P\{|(X_1 + ... + X_n)/n - c| > \varepsilon\} = 0$$

Since $X_1 + \ldots + X_n$ is binomial with parameters $n$, $x$, we have

$$E\left[f\left(\frac{X_1+\ldots+X_n}{n}\right)\right]=\sum_{k=1}^{n} f(k/n)\binom{n}{k}x^k(1-x)^{n-k}$$

The result now follows from Exercise 4.

6. $$E[X] = \sum_{i=1}^{k} i\,P\{X=i\} + \sum_{i=k+1}^{\infty} i\,P\{X=i\}$$
$$\geq \sum_{i=1}^{k} i\,P\{X=k\}$$
$$= P\{X=k\}^{k(k+1)/2}$$
$$\geq \frac{k^2}{2}P\{X=k\}$$

7. Take logs and apply the central limit theorem

8. It is the distribution of the sum of $t$ independent exponentials each having rate $\lambda$.

9. 1/2

10. Use the Chernoff bound: $e^{-ti}M(t) = e^{\lambda(e^t-1)-ti}$ will obtain its minimal value when $t$ is chosen to satisfy

$\lambda e^t = i$, and this value of $t$ is negative provided $i < \lambda$.

Hence, the Chernoff bound gives

$P\{X \leq i\} \leq e^{i-\lambda}(\lambda/i)^i$

11. $e^{-ti}M(t) = (pe^t + q)^n e^{-ti}$ and differentiation shows that the value of $t$ that minimizes it is such that

$npe^t = i(pe^t + q)$ or $e^t = \dfrac{iq}{(n-i)p}$

Using this value of $t$, the Chernoff bound gives that

$$P\{X \geq i\} \leq \left(\frac{iq}{n-i}+q\right)^n (n-i)^i p^i/(iq)^i$$
$$= \frac{(nq)^n(n-i)^i p^i}{i^i q^i (n-i)^n}$$

12. $1 = E[e^{\theta X}] \geq e^{\theta E[X]}$ by Jensen's inequality.

Hence, $\theta E[X] \leq 0$ and thus $\theta > 0$.

# Chapter 9

## Problems and Theoretical Exercises

1. (a) $P$(2 arrivals in (0, $s$) $|$ 2 arrivals in (0, 1)}

$=P\{2 \text{ in } (0, s), 0 \text{ in } (s, 1)\}/e^{-\lambda}\lambda^2/2)$
$= [e^{-\lambda s}(\lambda s)^2/2][e^{-(1-s)\lambda}]/(e^{-\lambda}\lambda^2/2) = s^2 = 1/9$ when $s = 1/3$

(b) $1 - P\{$both in last 40 minutes$) = 1 - (2/3)^2 = 5/9$

2. $e^{-3s/60}$

3. $e^{-3s/60} + (s/20)e^{-3s/60}$

8. The equations for the limiting probabilities are:

$$\prod_c = .7\prod_c + .4\prod_s + .2\prod_g$$
$$\prod_s = .2\prod_x + .3\prod_s + .4\prod_g$$
$$\prod_g = .1\prod_c + .3\prod_s + .4\prod_g$$
$$\prod_c + \prod_s + \prod_g = 1$$

and the solution is: $\prod_c = 30/59$, $\prod_s = 16/59$, $\prod_g = 13/59$. Hence, Buffy is cheerful 3000/59 percent of the time.

9. The Markov chain requires 4 states:

0 = RR = Rain today and rain yesterday

1 = RD = Dry today, rain yesterday

2 = DR = Rain today, dry yesterday

3 = DD = Dry today and dry yesterday

with transition probability matrix

$$\underline{P} = \begin{vmatrix} .8 & .2 & 0 & 0 \\ 0 & 0 & .3 & .7 \\ .4 & .6 & 0 & 0 \\ 0 & 0 & .2 & .8 \end{vmatrix}$$

The equations for the limiting probabilities are:

$$\Pi_0 = .8\Pi_0 + .4\Pi_2$$
$$\Pi_1 = .2\Pi_0 + .6\Pi_2$$
$$\Pi_2 = .3\Pi_1 + .2\Pi_3$$
$$\Pi_3 = .7\Pi_1 + .8\Pi_3$$
$$\Pi_0 + \Pi_1 + \Pi_2 + \Pi_3 = 1$$

which gives

$$\Pi_0 = 4/15,\ \Pi_1 = \Pi_2 = 2/15,\ \Pi_3 = 7/15.$$

Since it rains today when the state is either 0 or 2 the probability is 2/5.

10. Let the state be the number of pairs of shoes at the door he leaves from in the morning. Suppose the present state is $i$, where $i > 0$. Now after his return it is equally likely that one door will have $i$ and the other $5 - i$ pairs as it is that one will have $i - 1$ ant the other $6 - i$. Hence, since he is equally likely to choose either door when he leaves tomorrow it follows that

$$P_{i,i} = P_{i,5-i} = P_{i,i-1} = P_{i,6-i} = 1/4$$

provided all the states $i$, $5 - i$, $i - 1$, $6 - i$ are distinct. If they are not then the probabilities are added. From this it is easy to see that the transition matrix $P_{ij}$, $i, j = 0, 1, \ldots, 5$ is as follows:

$$\underline{P} = \begin{matrix} 1/2 & 0 & 0 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \end{matrix}$$

Since this chain is doubly stochastic (the column sums as well as the row sums all equal to one) it follows that $\Pi_i = 1/6$, $i = 0, \ldots, 5$, and thus he runs barefooted one-sixth of the time.

11. (b) 1/2

(c) Intuitively, they should be independent.

(d) From (b) and (c) the (limiting) number of molecules in urn 1 should have a binomial distribution with parameters ($M$, 1/2).

# Chapter 10

1. (a) After stage $k$ the algorithm has generated a random permutation of 1, 2, …, $k$. It then puts element $k + 1$ in position $k + 1$; randomly chooses one of the positions 1, …, $k + 1$ and interchanges the element in that position with element $k + 1$.

   (b) The first equality in the hint follows since the permutation given will be the permutation after insertion of element $k$ if the previous permutation is $i_1, \ldots, i_{j-1}, i, i_j, \ldots, i_{k-2}$ and the random choice of one of the $k$ positions of this permutation results in the choice of position $j$.

2. Integrating the density function yields that that distribution function is

$$F(x) = \begin{matrix} e^{2x}/2 \quad , & x > 0 \\ 1 - e^{-2s}/2, & x > 0 \end{matrix}$$

which yields that the inverse function is given by

$$F^{-1}(u) = \begin{matrix} \log(2u)/2 & \text{if } u < 12 \\ -\log(2[1-u])/2 & \text{if } u > 1/2 \end{matrix}$$

Hence, we can simulate $X$ from $F$ by simulating a random number $U$ and setting $X = F^{-1}(U)$.

3. The distribution function is given by

$$F(x) = \begin{matrix} x^2/4 - x + 1, & 2 \le x \le 3, \\ x - x^2/12 - 2, & 3 \le x \le 6 \end{matrix}$$

Hence, for $u \le 1/4$, $F^{-1}(u)$ is the solution of

$$x^2/4 - x + 1 = u$$

that falls in the region $2 \le x \le 3$. Similarly, for $u \ge 1/4$, $F^{-1}(u)$ is the solution of

$$x - x^2/12 - 2 = u$$

that falls in the region $3 \le x \le 6$. We can now generate $X$ from $F$ by generating a random number $U$ and setting $X = F^{-1}(U)$.

4. Generate a random number $U$ and then set $X = F^{-1}(U)$. If $U \le 1/2$ then $X = 6U - 3$, whereas if $U \ge 1/2$ then $X$ is obtained by solving the quadratic $1/2 + X^2/32 = U$ in the region $0 \le X \le 4$.

5. The inverse equation $F^{-1}(U) = X$ is equivalent to

or

$$1 - e^{-\alpha X^{\beta}} = U$$
$$X = \{-\log(1 - U)/\alpha\}^{1/\beta}$$

Since $1 - U$ has the same distribution as $U$ we can generate from $F$ by generating a random number $U$ and setting $X = \{-\log(U)/\alpha\}^{1/\beta}$.

6. If $\lambda(t) = ct^n$ then the distribution function is given by

$$1 - F(t) = \exp\{-kt^{n+1}\},\ t \ge 0 \quad \text{where } k = c/(n + 1)$$

Hence, using the inverse transform method we can generate a random number $U$ and then set $X$ such that

$$\exp\{-kX^{n+1}\} = 1 - U$$

or

$$X = \{-\log(1 - U)/k\}^{1/(n+1)}$$

Again $U$ can be used for $1 - U$.

7. (a) The inverse transform method shows that $U^{1/n}$ works.

(b) $P\{\text{Max}U_i \le v\} = P\{U_1 \le x, \ldots, U_n \le x\}$
$= \prod P\{U_i \le x\}$ by independence
$= x^n$

(c) Simulate $n$ random numbers and use the maximum value obtained.

8. (a) If $X_i$ has distribution $F_i$ , $i = 1, \ldots, n$, then, assuming independence, $F$ is the distribution of $\text{Max}X_i$. Hence, we can simulate from $F$ by simulating $X_i$, $i = 1, \ldots, n$ and setting $X = \text{Max}X_i$.

(b) Use the method of (a) replacing Max by Min throughout.

9. (a) Simulate $X_i$ from $F_i$, $i = 1, 2$. Now generate a random number $U$ and set $X$ equal to $X_1$ if $U < p$ and equal to $X_2$ if $U > p$.

(b) Note that

$$F(x) = \frac{1}{3}F_1(x) + \frac{2}{3}F_2(x)$$

where

$$F_1(x) = 1 - e^{-3x}, \quad x > 0, \;\; F_2(x) = x, \quad 0 < x < 1$$

Hence, using (a) let $U_1$, $U_2$, $U_3$ be random numbers and set

$$X = \begin{array}{ll} -\log(U_1)/3 & \text{if } U_3 < 1/3 \\ U_2 & \text{if } U_3 > 1/3 \end{array}$$

where the above uses that $-\log(U_1)/3$ is exponential with rate 3.

10. With $g(x) = \lambda e^{-\lambda x}$

$$\frac{f(x)}{g(x)} = \frac{2e^{-x^2/2}}{\lambda(2\pi)^{1/2}e^{-\lambda x}} = \frac{2}{\lambda(2\pi)^{1/2}}\exp\{-[(x-\lambda)^2 - \lambda^2]/2\}$$
$$\frac{2e^{\lambda^2/2}}{\lambda(2\pi)^{1/2}}\exp\{-(x-\lambda)^2/2\}$$

Hence, $c = 2e^{\lambda^2/2}/[\lambda(2\pi)^{1/2}]$ and simple calculus shows that this is minimized when $\lambda = 1$.

11. Calculus yields that the maximum value of $f(x)/g(x) = 60x^3(1-x)^2$ is attained when $x = 3/5$ and is thus equal to 1296/625. Hence, generate random numbers $U_1$ and $U_2$ and set $X = U_1$ if $U_2 \le 3125U_1^3(1-U_1)^2/108$. If not, repeat.

12. Generate random numbers $U_1, \ldots, U_n$, and approximate the integral by $[k(U_1) + \ldots + k(U_n)]/n$. This works by the law of large numbers since $E[k(U)] = \int_0^1 k(x)dx$.

16. $E[g(X)/f(X)] = \int [g(x)/f(x)]f(x)dx = \int g(x)dx$

Corrections to Ross, A FIRST COURSE IN PROBABILITY, seventh ed.

- p. 79, line 2: $P(\sum_{j=1}^{n} R_j) \rightarrow P(\bigcup_{j=1}^{n} R_j)$
- p. 95, centered eq. on line 12: $P_{n,m-1,m} \rightarrow P_{n,m-1}$
- p. 288, l. 2: change "when $j > r$." to "when $j < 0$."
- p. 309, first line of Example 8b: change "let $Y_i$ denote the selection " to "let $Y_i$ denote the selection "
- p. 373, line -8: on the centered equation following "the preceding equation yields" add a right paren at the very end. That is,
  $(1 - (1 - p_i)^n \rightarrow (1 - (1 - p_i)^n))$
- p. 416, line 1: change "Let $X_1, \ldots, X_n$ be independent" to "Let $X_1, \ldots$ be independent"
- p. 509, lines 6 and 7: 73. should be 83. and 74. should be 84.
- p. 509, Solution to Problem 68 of Chapter 4: change
  $(1 - e^{-5})^{80}$ to $(1 - e^{-5})^{10}$