Van der Pol – averaging

$$\ddot{x} + x = \varepsilon \dot{x}(1 - x^2), \quad x(0) = a, \quad \dot{x}(0) = 0$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \varepsilon x_2 (1 - x_1^2) \end{cases}$$

για $\varepsilon \to 0$ $\quad \ddot{x} + x = 0 \qquad x_0 = r e^{i(\theta - t)}$

μεταφορά σε μεταβλητές (r, θ) : $x_1 = r(t)\cos(\theta(t) - t)$

$x_2 = r(t)\sin(\theta(t) - t)$

$$\dot{r}(t)\cos(\theta(t) - t) - r\sin(\theta(t) - t)(\dot{\theta}(t) - 1) = r(t)\sin(\theta(t) - t)$$

$$\dot{r}(t)\sin(\theta(t) - t) + r\cos(\theta(t) - t)(\dot{\theta}(t) - 1) = -r(t)\cos(\theta(t) - t) + \varepsilon r(t)\sin(\theta(t) - t)\cdot(1 - r^2(t)\cos^2(\theta(t) - t))$$

$$(+)\begin{cases} \dot{r}(t)\cdot\cos^2(\theta(t) - t) - r\cos(\theta(t) - t)\sin(\theta(t) - t)(\dot{\theta}(t) - 1) = r(t)\cos(\theta(t) - t)\sin(\theta(t) - t) \\ \dot{r}(t)\sin^2(\theta(t) - t) + r\cos(\theta(t) - t)\sin(\theta(t) - t)(\dot{\theta}(t) - 1) = -r(t)\cos(\theta(t) - t)\sin(\theta(t) - t) + \varepsilon r(t)\sin^2(\theta(t) - t)\cdot(1 - r^2(t)\cos^2(\theta(t) - t)) \end{cases}$$

$$\dot{r}(t) = \varepsilon r(t)\sin^2(\theta(t) - t)(1 - r^2(t)\cos^2(\theta(t) - t))$$

$$(-)\begin{cases} \dot{r}(t)\cos(\theta(t) - t)\cdot\sin(\theta(t) - t) - r\sin^2(\theta(t) - t)(\dot{\theta}(t) - 1) = r(t)\sin^2(\theta(t) - t) \\ \dot{r}(t)\sin(\theta(t) - t)\cdot\cos(\theta(t) - t) + r\cos^2(\theta(t) - t)(\dot{\theta}(t) - 1) = -r(t)\cos^2(\theta(t) - t) + \varepsilon r(t)\cos(\theta(t) - t)\cdot\sin(\theta(t) - t)\cdot(1 - r^2(t)\cos^2(\theta(t) - t)) \end{cases}$$

$$r\cdot(\dot{\theta}(t) - 1) = -r(t) + \varepsilon r(t)\cos(\theta(t) - t)\cdot\sin(\theta(t) - t)(1 - r^2(t)\cdot\cos^2(\theta(t) - t))$$

$$\dot{\theta}(t) = \varepsilon\cdot\cos(\theta(t) - t)\cdot\sin(\theta(t) - t)\cdot(1 - r^2(t)\cdot\cos^2(\theta(t) - t))$$

$$\begin{cases} \dfrac{dr}{dt} = \varepsilon\cdot r\cdot\sin^2(\theta - t)(1 - r^2\cos^2(\theta - t)) \\ \dfrac{d\theta}{dt} = \varepsilon\cdot\cos(\theta - t)\cdot\sin(\theta - t)\cdot(1 - r^2\cdot\cos^2(\theta - t)) \end{cases}$$

Σε μια περίοδο τα $r(t), \theta(t)$ μπορούν να θεωρηθούν σχεδόν σταθερά, αφού είναι συναρτήσεις του t αργά μεταβαλλόμενες, οπότε αντικαθιστούμε στο δεξί μέλος με averages, $\bar{r}$ να $\int_0^{2\pi} \varphi\, d\varphi$

$$\begin{cases} \dfrac{d\bar{r}}{dt} = \dfrac{\bar{r}\varepsilon}{2\pi}\displaystyle\int_0^{2\pi} \sin^2\varphi\,(1-\bar{r}^2\cos^2\varphi)\,d\varphi \\ \dfrac{d\bar{\theta}}{dt} = \dfrac{\varepsilon}{2\pi}\displaystyle\int_0^{2\pi} (1-\bar{r}^2\cos^2\varphi)\sin\varphi\cdot\cos\varphi\,d\varphi = 0 \quad \text{λόγω συμμετρίας} \end{cases}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \bar{r}\sin^2\varphi(1-\bar{r}^2\cos^2\varphi)\,d\varphi = \frac{1}{2\pi}\left[\int_0^{2\pi}\bar{r}\sin^2\varphi\,d\varphi - \int_0^{2\pi}\bar{r}^3\sin^2\varphi\cos^2\varphi\,d\varphi\right] =$$

$$\frac{\bar{r}}{2\pi}\int_0^{2\pi}\frac{1-\cos 2\varphi}{2}\,d\varphi - \frac{\bar{r}^3}{2\pi}\int_0^{2\pi}\cos^2\varphi\,d\varphi + \frac{\bar{r}^3}{2\pi}\int_0^{2\pi}\cos^4\varphi\,d\varphi =$$

$$\frac{\bar{r}}{2} - \frac{\bar{r}^3}{2\pi}\int_0^{2\pi}\frac{\cos 2\varphi+1}{2}\,d\varphi + \frac{\bar{r}^3}{2\pi}\int_0^{2\pi}\frac{1}{4}\left[\frac{\cos 2\varphi}{2} + 2\cos\varphi + 1\right]d\varphi =$$

$$\frac{\bar{r}}{2} - \frac{\bar{r}^3}{2} + \frac{3\bar{r}^3}{8} = \frac{\bar{r}}{2} - \frac{\bar{r}^3}{8} = \frac{\bar{r}}{2}\left(1-\frac{\bar{r}^2}{4}\right)$$

$$\begin{cases} \dfrac{d\bar{r}}{dt} = \varepsilon\dfrac{\bar{r}}{2}\left(1-\dfrac{\bar{r}^2}{4}\right) \\ \dfrac{d\bar{\theta}}{dt} = 0 \end{cases}$$

$$r' = \varepsilon\frac{r}{2} - \frac{r^3}{8}\varepsilon \Rightarrow r^{-3}r' - \frac{r^{-2}}{2}\varepsilon = -\varepsilon\frac{1}{8} \Leftrightarrow \quad v = r^{-2} \quad v' = -2r^{-3}r'$$

$$-2r^{-3}r' + r^{-2}\varepsilon = \frac{1}{4}\varepsilon \Rightarrow v' + v\varepsilon = \frac{1}{4}\varepsilon \Rightarrow (v e^{\varepsilon t})' = \left(\frac{1}{4}e^{\varepsilon t}\right)' \Leftrightarrow v = \frac{1}{4} + Ce^{-\varepsilon t}$$

$$\begin{cases} \bar{r}^2 = \dfrac{4}{1-Ae^{-\varepsilon t}} \\ \bar{\theta} = \theta_0 \end{cases} \qquad \text{Άρα} \quad x(t) = \frac{2}{\sqrt{1-Ae^{-\varepsilon t}}}\cos(t+\theta_0) + O(\varepsilon)$$

$$\frac{dx(0)}{dt} = 0, \quad x(0) = a \Rightarrow \theta_0 = 0 \quad \text{και} \quad A = 1 - \frac{4}{a^2}$$

β) Από το $\dfrac{d\bar{r}}{dt} = \varepsilon\dfrac{\bar{r}}{2}\left(1-\dfrac{\bar{r}^2}{4}\right)$ βλέπουμε ότι το πλάτος είναι σταθερό για $\bar{r}=0$ ή $\bar{r}=2$, όπως είναι ότι βρήκαμε και στην άσκηση 3.

3.14) α) $\varepsilon y'' + 2y' + y = 0 \quad y(0)=0,\ y(1)=1$

boundary layer: $x=0$, για $\varepsilon=0$ $\quad 2y'+y=0 \iff ye^{\frac{1}{2}x} = c \iff y = ce^{-\frac{1}{2}x}$

$y(1)=1 \Rightarrow \quad ce^{-\frac{1}{2}} = 1 \iff c = e^{\frac{1}{2}} \qquad y_0 = \sqrt{e}\cdot e^{-\frac{1}{2}x} = e^{(1-x)/2}$

$\zeta = \frac{x}{\delta(\varepsilon)} \quad Y(\zeta) = y(\delta(\varepsilon)\cdot\zeta) \qquad \frac{\varepsilon}{\delta(\varepsilon)^2}Y''(\zeta) + \frac{2}{\delta(\varepsilon)}Y'(\zeta) + Y(\zeta) = 0$

$\frac{\varepsilon}{\delta^2(\varepsilon)} \sim \frac{2}{\delta(\varepsilon)} \Rightarrow \delta(\varepsilon) = O(\varepsilon) \qquad \frac{\varepsilon}{\delta^2(\varepsilon)} \sim 1 \Rightarrow \delta(\varepsilon) = O(\sqrt{\varepsilon})$ τότε το $\frac{2}{\delta(\varepsilon)}$ δεν είναι μικρό σε σχέση με το 1 άρα ασυνέπεια

$Y''(\zeta) + 2Y'(\zeta) + \varepsilon Y(\zeta) = 0 \overset{\varepsilon=0}{\Longrightarrow} Y''(\zeta) + 2Y'(\zeta) = 0 \Rightarrow Y'(\zeta)\cdot e^{2\zeta} = c_1$

$Y'(\zeta) = c_1 e^{-2\zeta} \Rightarrow \quad Y(\zeta) = -\frac{1}{2}c_1 e^{-2\zeta} + c_2 \qquad Y(0) = 0 \Rightarrow \quad c_1 = 2c_2$

matching: $\lim_{x\to 0^+} y_0(x) = \lim_{\zeta\to\infty} Y_i(\zeta) \iff \sqrt{e} = c_2 \qquad c_1 = 2\sqrt{e}$

$Y_i(x) = \sqrt{e}\cdot e^{-2\frac{x}{\varepsilon}} + \sqrt{e},\ x = O(\varepsilon) \qquad y_0(x) = \sqrt{e}\cdot e^{-\frac{1}{2}x},\ x = O(1)$

β) $\varepsilon y'' + y' + y^2 = 0 \quad y(0) = \frac{1}{4},\ y(1) = \frac{1}{2}$

bound. layer: $x=0$, για $\varepsilon=0$ $\quad y'+y^2=0 \iff y^{-2}y'+1=0 \quad u = y^{-1} \quad u' = -y^{-2}y'$

$+u'-1=0 \Rightarrow \quad u = x+c \qquad y = \frac{1}{x+c} \qquad y(1)=\frac{1}{2} \iff \frac{1}{2} = \frac{1}{1+c} \iff c=1$

$y_0 = \frac{1}{x+1}. \qquad \zeta = \frac{x}{\delta(\varepsilon)} \quad Y(\zeta) = y(\delta(\varepsilon)\cdot\zeta) \Rightarrow \frac{\varepsilon}{\delta^2(\varepsilon)}Y''(\zeta) + \frac{1}{\delta(\varepsilon)}Y'(\zeta) + Y^2(\zeta) = 0$

$\frac{\varepsilon}{\delta^2(\varepsilon)} \sim \frac{1}{\delta(\varepsilon)} \Rightarrow \delta(\varepsilon) = O(\varepsilon) \qquad \frac{\varepsilon}{\delta^2(\varepsilon)} \sim 1 \Rightarrow \delta(\varepsilon) = \sqrt{\varepsilon}$ απορρίπτεται

$\overset{\delta(\varepsilon)=\varepsilon}{\Longrightarrow} \quad Y''(\zeta) + Y'(\zeta) + \varepsilon Y^2(\zeta) = 0 \overset{\varepsilon=0}{\Longrightarrow} Y''(\zeta) + Y'(\zeta) = 0 \quad (Y'(\zeta)\cdot e^{\zeta})' = c_1 \iff$

$Y'(\zeta) = c_1 e^{-\zeta} \Rightarrow \quad Y(\zeta) = -c_1 e^{-\zeta} + c_2 \qquad Y(0) = \frac{1}{4} \iff c_2 = c_1 + \frac{1}{4}$

Matching: $\lim_{\zeta\to\infty} Y(\zeta) = \lim_{x\to 0^+} y_0(x) \iff c_2 = 1 \Rightarrow c_1 = \frac{3}{4}$

$y_0(x) = \frac{1}{x+1},\ x = O(1) \qquad y_i(x) = -\frac{3}{4}e^{-\frac{x}{\varepsilon}} + 1,\ x = O(\varepsilon)$

γ) $\varepsilon y'' + (1+t)y' = 1 \qquad y(0)=0,\ y(1)=1+\ln 2$

boundary layer: $x=0$, για $\varepsilon=0$ $\qquad (1+t)y'=1 \Leftrightarrow y'=\frac{1}{1+t} \Rightarrow y=\ln(1+t)+c$

$y(1)=1+\ln 2 \Rightarrow c=1 \qquad y_0(t)=\ln(1+t)+1$

$\xi=\frac{x}{\delta(\varepsilon)} \qquad \frac{\varepsilon}{\delta^2(\varepsilon)}Y''(\xi)+\frac{(1+t)}{\delta(\varepsilon)}Y'(\xi)=1 \xrightarrow{t=0} \frac{\varepsilon}{\delta^2(\varepsilon)}\sim\frac{1}{\delta(\varepsilon)} \Rightarrow \delta(\varepsilon)=O(\varepsilon)$

$\frac{\varepsilon}{\delta^2(\varepsilon)}\sim 1 \Rightarrow \delta(\varepsilon)=O(\sqrt{\varepsilon})$ απορρίπτεται $\quad$ άρα για $\delta(\varepsilon)=\varepsilon$

$Y''(\xi)+(1+\xi\varepsilon)Y'(\xi)-\varepsilon=0 \overset{\varepsilon=0}{\Longrightarrow} Y''(\xi)+Y'(\xi)=0 \Rightarrow Y(\xi)=-c_1e^{-\xi}+c_2$

$Y(0)=0 \Rightarrow c_2=c_1 \qquad \lim\limits_{\xi\to\infty}Y(\xi)=\lim\limits_{x\to0^+}y_0(x) \Leftrightarrow c_2=1 \Rightarrow c_1=c$

$y_i(t)=-e^{-\frac{x}{\varepsilon}}+1,\ x=O(\varepsilon) \qquad y_0(t)=\ln(t+1)+1,\ x=O(1)$

δ) $\varepsilon y''+(t+1)y'+y=0 \qquad y(0)=0,\ y(1)=1$

boundary layer: $x=0$, για $\varepsilon=0$ $\qquad (t+1)y'+y=0 \Leftrightarrow \frac{y'}{y}=-\frac{1}{t+1} \Leftrightarrow \ln y=-\ln(t+1)+c_1$

$y=\frac{1}{t+1}c_1' \qquad y(1)=1 \Leftrightarrow 1=\frac{1}{2}c_1 \Rightarrow c_1=2 \qquad y_0(t)=\frac{2}{t+1}$

$\xi=\frac{t}{\delta(\varepsilon)} \Rightarrow \frac{\varepsilon}{\delta(\varepsilon)^2}Y''(\xi)+\frac{(t+1)}{\delta(\varepsilon)}Y'(\xi)+Y(\xi)=0 \qquad \frac{\varepsilon}{\delta^2(\varepsilon)}\sim\frac{1}{\delta(\varepsilon)} \Rightarrow \delta(\varepsilon)=O(\varepsilon)$

$Y''(\xi)+(\xi\cdot\varepsilon+1)Y'(\xi)+\varepsilon Y(\xi)=0 \overset{\varepsilon=0}{\Longrightarrow} Y''(\xi)+Y'(\xi)=0 \Rightarrow Y(\xi)=-c_1e^{-\xi}+c_2$

$Y(0)=0 \Rightarrow c_2=c_1 \qquad \lim\limits_{\xi\to\infty}Y(\xi)=\lim\limits_{t\to0^+}y^0(t) \Rightarrow c_2=2 \Rightarrow c_1=2$

$y_i(t)=-2e^{-t/\varepsilon}+2,\ x=O(\varepsilon) \qquad y_0(t)=\frac{2}{t+1},\ x=O(1)$