$$\frac{dr}{dt} = -\varepsilon\,(r\cos(\ ))^3 \sin(\ ) \qquad (\ ) = \theta(t)-t$$

$$\frac{d\theta}{dt} = -\frac{\varepsilon}{r}(r\cos(\ ))^3 \cos(\ )$$

$$\boxed{t \to -t}$$

$$(2)\begin{cases} \dfrac{dr}{dt} = \varepsilon\,(r\cos(t+\theta))^3 \sin(t+\theta) \\ \dfrac{d\theta}{dt} = \dfrac{\varepsilon}{r}(r\cos(t+\theta))^3 \cos(t+\theta) \end{cases}$$

$\theta(t), r(t)$ μεταβάλλονται αργά $\sim O(\varepsilon)$
(r φραγμένο μακριά από $r=0$, $r=\infty$)

<u>Θεώρημα</u>
Έστω $x(t), y(t) \in \mathbb{R}^n$

$$(3)\begin{cases} \dfrac{dx}{dt} = \varepsilon f(t,x), & x(0)=x_0 \\ \dfrac{dy}{dt} = \varepsilon f_0(y), & y(0)=x_0 \end{cases}$$

ανοιχτό συνεκτικό

$(t,x) \in [0,\infty) \times D$

$f(t+T,x) = f(t,x)$

$$f_0(y) := \frac{1}{T}\int_0^T f(t,y)\,dt$$

$\Rightarrow$

$$(4) \qquad |x(t)-y(t)| \le C\varepsilon \;, \quad \boxed{0 \le t \le \frac{C}{\varepsilon}} \; (*)$$

C σταθερά ανεξάρτητη του ε.

(*) Υπενθυμίζουμε την $u_{\text{προσ}}(t) = \cos t + \varepsilon\left(\frac{1}{32}(\cos 3t - \cos t) - \frac{3}{8}\boxed{t\cos t}\right)$

Απ

Ορίζουμε

5) $w(t) := y(t) + \varepsilon u(t, y(t))$

6) $u(t,\psi) := \int_0^t \{f(\tau,\psi) - f_0(\psi)\}\, d\tau$, $\psi \in \mathbb{R}^n$, ανεξ. του t.

$$\frac{1}{T}\int_0^T \left(f(\tau,\psi) - f_0(\psi)\right) d\tau = f_0(\psi) - f_0(\psi) = 0$$

$\Rightarrow$ $t \to u(t,\psi)$ T-περιοδική, $\psi \in \mathbb{R}^n$ φιξαρισμένη.

7) $x(t) - w(t) = \int_0^t \left(\frac{dx}{dt} - \frac{dw}{dt}\right) d\tau$

$\nabla_\psi u$

8) $$\frac{dx}{dt} - \frac{dw}{dt} = \varepsilon f(t,x(t)) - \frac{dy}{dt} - \varepsilon\left[\frac{\partial u}{\partial t} + \frac{\partial u}{\partial \psi}\cdot\frac{dy}{dt}\right]$$

$$= \varepsilon f(t,x(t)) - \cancel{\varepsilon f_0(y(t))} - \varepsilon\left[f(t,y(t)) - \cancel{f_0(y(t))}\right] - \varepsilon\frac{\partial u}{\partial \psi}\cdot\left(\varepsilon f_0(y)\right)$$

$$= \varepsilon\{f(t,x(t)) - f(t,y(t))\} - \varepsilon^2 \nabla_\psi u \cdot f_0(y(t))$$

$$= \varepsilon\{f(t,x(t)) - f(t,w(t))\} + \varepsilon\{f(t,w(t)) - f(t,y(t))\} - \varepsilon^2 \nabla_\psi u \cdot f_0(y(t))$$

L = Lipschitz σταθερά της f

$|f(t,x_A) - f(t,x_B)| \le L|x_A - x_B|$

$\Rightarrow$

9) $$\left|\frac{dx}{dt} - \frac{dw}{dt}\right| \le \varepsilon L|x(t) - w(t)| + \varepsilon L|w(t) - y(t)| + C\varepsilon^2$$

$$\le \varepsilon L|x(t) - w(t)| + C\varepsilon^2.$$

$\overset{(7)}{\Longrightarrow}$

$$|x(t)-w(t)| \leq \varepsilon L \int_0^t |x(\tau)-w(\tau)|\,d\tau + C_\varepsilon{}^{2}\, t$$

$\Leftrightarrow$

$$(10) \quad |x(t)-w(t)| + \frac{C_\varepsilon}{L} \leq \varepsilon L \int_0^t \left\{ |x(\tau)-w(\tau)| + \frac{C_\varepsilon}{L} \right\} d\tau + \frac{C_\varepsilon}{L}$$

Gronwall (Θεωρημα 2.2 A + Καλογεροπουλος, σ 100)

$$(11) \quad |x(t)-w(t)| + \frac{C_\varepsilon}{L} \leq \frac{C_\varepsilon}{L} e^{\varepsilon L t}$$

$$(11), (5) \Longrightarrow \boxed{\varepsilon t \leq C}$$

$$|x(t)-y(t)-\varepsilon u(t,y(t))| \leq K\varepsilon$$

$$\Longrightarrow |x(t)-y(t)| \leq K\varepsilon, \quad \boxed{\varepsilon t \leq C}$$

□

Εφαρμογη στην Duffing

$$\frac{1}{2\pi}\int_0^{2\pi} (r\cos(t+\theta))^3 \sin(t+\theta)\,dt = -\int_0^{2\pi} (r\cos(t+\theta))^3\, d(\cos(t+\theta)) = 0$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\varepsilon}{r}(r\cos(t+\theta))^3 \cos(t+\theta)\,dt = \frac{\varepsilon r^2}{2\pi}\int_0^{2\pi} \cos^4(t+\theta)\,dt = \frac{3}{8}\varepsilon r^2$$

$$\left.\begin{aligned} \frac{d\bar r}{dt} &= 0 \\ \frac{d\bar\theta}{dt} &= \frac{3}{8}\varepsilon \bar r^2 \end{aligned}\right\} \Longrightarrow y(t) = \alpha \cos\left\{\left(1+\frac{3}{8}\varepsilon^2\alpha^2\right)t\right\}.$$

□

Διάλεξη 7 Μέθοδος Μέσων Όρων (Averaging)

(1) $\ddot{u} + u + \varepsilon u^3 = 0$ (Αρχικές Συνθήκες)

$u(0) = \alpha, \quad \dot{u}(0) = 0$

$L = \frac{d^2}{dt^2} + 1$

μέτρο ταλάντωσης

$L u_0 = 0 \Rightarrow u_0 = r e^{i(-t+\theta)}$ $(t \to -t)$

$\underline{\varepsilon = 0} \Rightarrow r, \theta$ σταθερές στον χρόνο παράμετροι, καθορίζονται από Α.Σ.

$\underline{\varepsilon > 0} \Rightarrow r = r(t), \ \theta = \theta(t)$ (βραδεία μεταβολή στον χρόνο φάσης πλάτους)

$(1) \Leftrightarrow \quad \dot{u} = v$

$\dot{v} = -u - \varepsilon u^3$

$(u(t), v(t)) = (r(t)\cos(\theta(t) - t), \ r(t)\sin(\theta(t) - t))$

$\cos \ | \ \dot{r}\cos(\ ) - r\sin(\ )(\dot{\theta} - 1) = r\sin(\ )$

$\sin \ | \ \dot{r}\sin(\ ) + r\cos(\ )(\dot{\theta} - 1) = -r\cos(\ ) - \varepsilon(r\cos(\ ))^3$

$\dot{r}\cos^2 - r\sin(\ )\cos(\ )(\dot{\theta} - 1) = r\sin(\ )\cos(\ )$

$\dot{r}\sin^2 + r\sin(\ )\cos(\ )(\dot{\theta} - 1) = -r\sin(\ )\cos(\ ) - \varepsilon(r\cos(\ ))^3\sin(\ )$

(i) $\boxed{\dot{r} = -\varepsilon(r\cos(\ ))^3\sin(\ )}$

$\sin \ | \ \dot{r}\cos(\ ) - r\sin^2(\ )(\dot{\theta} - 1) = r\sin^2(\ )$

$\cos \ | \ \dot{r}\sin(\ ) + r\cos^2(\ )(\dot{\theta} - 1) = -r\cos^2(\ ) - \varepsilon(r\cos(\ ))^3\cos$

(ii) $\boxed{r(\dot{\theta} - \cancel{1}) = -\cancel{r} - \varepsilon(r\cos(\ ))^3\cos(\ )}$