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$$\varepsilon y'' - x^2 y' - y = 0 \qquad y(0) = y(1) = 1$$

Εξωτερικό Ανάπτυγμα $\varepsilon = 0$

$$-x^2 y_0' - y = 0 \Rightarrow -x^2 \frac{dy_0}{dx} = y_0$$

$$\Rightarrow \frac{1}{y_0} dy_0 = -\frac{1}{x^2} dx \Rightarrow \ln y_0 = \frac{1}{x} + C$$

$$\Rightarrow y_0(x) = e^{\frac{1}{x}+C} \Rightarrow y_0(x) = C_0 e^{\frac{1}{x}}, \quad C_0 = e^C.$$

Άρα $y_0(x) = 0$.

$$\lim_{x \to 0^+} y_0(x) = \lim_{x \to 0} C_0 e^{\frac{1}{x}} = \lim_{u \to +\infty} C_0 e^u = +\infty.$$

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

Αφού $0 \le y(x) \le 1 \Rightarrow C_0 = 0$.

Εσωτερικό Ανάπτυγμα έτσι $x=1$

$-x^4 \approx -1$ $\quad \eta = \frac{1-x}{\varepsilon} \Rightarrow \eta\varepsilon = 1-x \Rightarrow x = 1-\varepsilon\eta.$ $\quad \frac{d\eta}{dx} = -\frac{1}{\varepsilon}$

$$Y(x) = y\left(\frac{1-x}{\varepsilon}\right) \Rightarrow \dot{Y}(x) = \dot{y}\left(\frac{1-x}{\varepsilon}\right)\left(-\frac{1}{\varepsilon}\right) = -\frac{1}{\varepsilon}\dot{y}\left(\frac{1-x}{\varepsilon}\right) = -\frac{1}{\varepsilon}\dot{Y}(\eta)$$

$$\Rightarrow \ddot{Y}(x) = \ddot{y}\left(\frac{1-x}{\varepsilon}\right)\left(-\frac{1}{\varepsilon}\right)^2 = \frac{1}{\varepsilon^2}\ddot{y}\left(\frac{1-x}{\varepsilon}\right) = \frac{1}{\varepsilon^2}\ddot{Y}(\eta)$$

Άρα $\quad \frac{1}{\varepsilon}\ddot{y} + (1-\varepsilon\eta)^2\frac{1}{\varepsilon}\dot{y} - Y = 0 \Rightarrow \ddot{Y} + (1-\varepsilon\eta)^2\dot{y} - \varepsilon Y = 0$

$\varepsilon = 0 \Rightarrow \ddot{Y} + \dot{Y} = 0 \Rightarrow \frac{d\dot{Y}}{d\eta} = -\dot{Y} \Rightarrow \frac{1}{\dot{Y}}d\dot{Y} = -d\eta \Rightarrow \ln\dot{Y} = -\eta + C$

$$\Rightarrow \dot{Y} = e^{-\eta+C} \Rightarrow \dot{Y} = C_1 e^{-\eta}$$

$$\frac{dY}{d\eta} = C_1 e^{-\eta} \Rightarrow dY = C_1 e^{-\eta}\,d\eta \Rightarrow Y = C_1 e^{-\eta} + C_2$$

$$Y(0) = 1 \Rightarrow C_1 + C_2 = 1.$$

Συναρτηση στο $x=1$

$$\lim_{n\to+\infty} Y(n) = 0 \Rightarrow \lim_{n\to+\infty} c_1 e^{-n} + c_2 = 0 \Rightarrow c_2 = 0 \Rightarrow c_1 = 1$$

Αρα $Y(n) = e^{-n}$

Εξωτερικό Ανάπτυγμα στο $x=0$

$\eta = \frac{x}{\delta} \Rightarrow x = \delta\eta$ $\qquad$ $Y(x) = y\left(\frac{x}{\delta}\right) \Rightarrow \dot{Y}(x) = \dot{y}\left(\frac{x}{\delta}\right)\frac{1}{\delta} = \frac{1}{\delta}\dot{y}(\eta)$

$Y(0) = 1$ $\qquad$ $\Rightarrow \ddot{Y}(x) = \ddot{y}\left(\frac{x}{\delta}\right)\frac{1}{\delta^2} = \frac{1}{\delta^2}\ddot{y}(\eta)$

Άρα $\frac{\varepsilon}{\delta^2}\ddot{y} - \delta^2\eta^2\dot{y} - y = 0$

Έστω ότι $\frac{\varepsilon}{\delta^2} \sim \delta^2 \Rightarrow \delta^2(\ddot{y} - \eta^2\dot{y}) - y = 0 \Rightarrow y = 0$. Μη αποδεκτή.

Έστω ότι $\delta^2 \sim 1 \Rightarrow \varepsilon\ddot{y} - \eta^2\dot{y} - y = 0$ Συμπίπτει με την εξωτερική

Μη αποδεκτή.

Άρα $\frac{\varepsilon}{\delta^2} \sim 1 \Rightarrow \ddot{y} - \varepsilon\eta^2\dot{y} - y = 0$

$\varepsilon = 0 \Rightarrow \ddot{y} - y = 0$ $\qquad$ $m^2 - 1 = 0 \Rightarrow m_1 = 1$, $m_2 = -1$

Άρα $y(\eta) = c_1 e^{\eta} + c_2 e^{-\eta}$

$Y(0) = 1 \Rightarrow c_1 + c_2 = 1$ $\qquad$ Αφού $0 \le y(\eta) \le 1 \Rightarrow c_1 = 0$ και $c_2 = 1$

Συνάρτηση έτσι $x=0$

$$\lim_{n\to+\infty} y(n) = 0 \iff \lim_{n\to+\infty} e^{-n} = 0$$ ισχύει.

Ομοιόμορφη Προσέγγιση

$$y_{ομ}(x) = e^{-\frac{x}{\sqrt{\varepsilon}}} + e^{-\frac{1-x}{\varepsilon}}$$