# FURTHER INSIGHTS INTO THE EMBEDDING PROPERTIES OF HADAMARD MATRICES AND D-OPTIMAL DESIGNS

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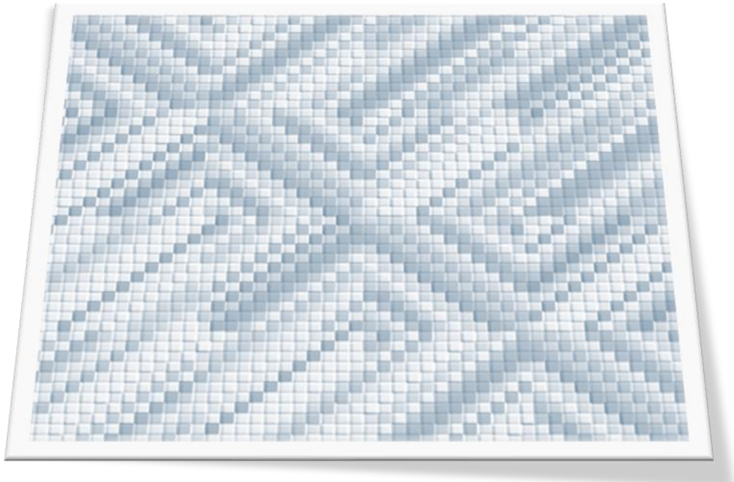

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# Further insights into the embedding properties of Hadamard matrices and D-optimal designs



**Topics:**

1. Hadamard matrices – Basic properties and applications.
2. Remarks on the construction of Hadamard matrices.
3. Minors of $(\pm 1)$-matrices and the spectrum of the determinant function.
4. Hadamard equivalence and D-optimal designs.
5. Embeddability of Hadamard matrices of order $n-k$ in Hadamard matrices of order $n$.
6. Embeddability of D-optimal designs of order $k$ in Hadamard matrices of order $n$.
7. Embeddability of D-optimal designs of order $n-1$ in D-optimal designs of order $n$.
8. Conclusions and further research.

# Hadamard matrices



Hadamard was interested in finding the maximal determinant of square matrices with entries from the unit disc.

He showed (*Bull. Sciences Math.* 1893) that this maximal determinant, $n^{n/2}$, was achieved by matrices $X = [x_{ij}]_{n\times n}$ with entries $\pm 1$ which satisfied the equality of the inequality:

$$|\det X|^2 \leq \prod_{i=1}^{n} \sum_{j=1}^{n} |x_{ij}|^2$$

or $X X^{\mathsf{T}} = I_n$

Jacques Salomon Hadamard
1865 – 1963

# Hadamard matrices



A square matrix with elements ±1 and size $n$, whose distinct row vectors are orthogonal is an ***Hadamard matrix of order n***.

| $n = 1$ | $n = 2$ | $n = 4$ |
|---|---|---|
| $H_1$ | $H_2$ | $H_4$ |
| $[1]$, | $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, | $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ |

Jacques Salomon Hadamard
1865 – 1963

**Basic properties of Hadamard matrices:**

a) $H H^{\mathsf{T}} = n I_n$ b) $|\det H| = n^{n/2}$ c) $H H^{\mathsf{T}} = H^{\mathsf{T}} H$

# The Hadamard conjecture



Furthermore, Hadamard observed that such matrices could exist only if $n$ was 1, 2, or a multiple of 4.

This observation has formed the basis of one of the greatest unsolved mathematical problems.

***The Hadamard Conjecture***
***There is a Hadamard matrix of order n for any natural number n multiple of 4.***

Jacques Salomon Hadamard
1865 – 1963

Despite the efforts of several mathematicians, Hadamard's observation remains unproven, even though it is widely believed that it is true.

# Sylvester-Hadamard matrices

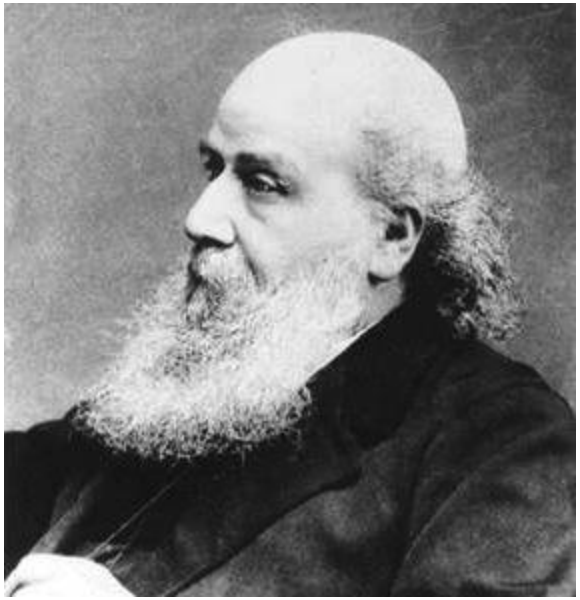


However, such matrices were first studied by Sylvester (*Phil. Mag.* 1867) who observed that if $H$ is an Hadamard-type matrix, then

$$\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

is also an Hadamard-type matrix.

The matrices of order $\mathbf{2^k}$ constructed using Sylvester's construction are usually referred to as ***Sylvester-Hadamard*** matrices.



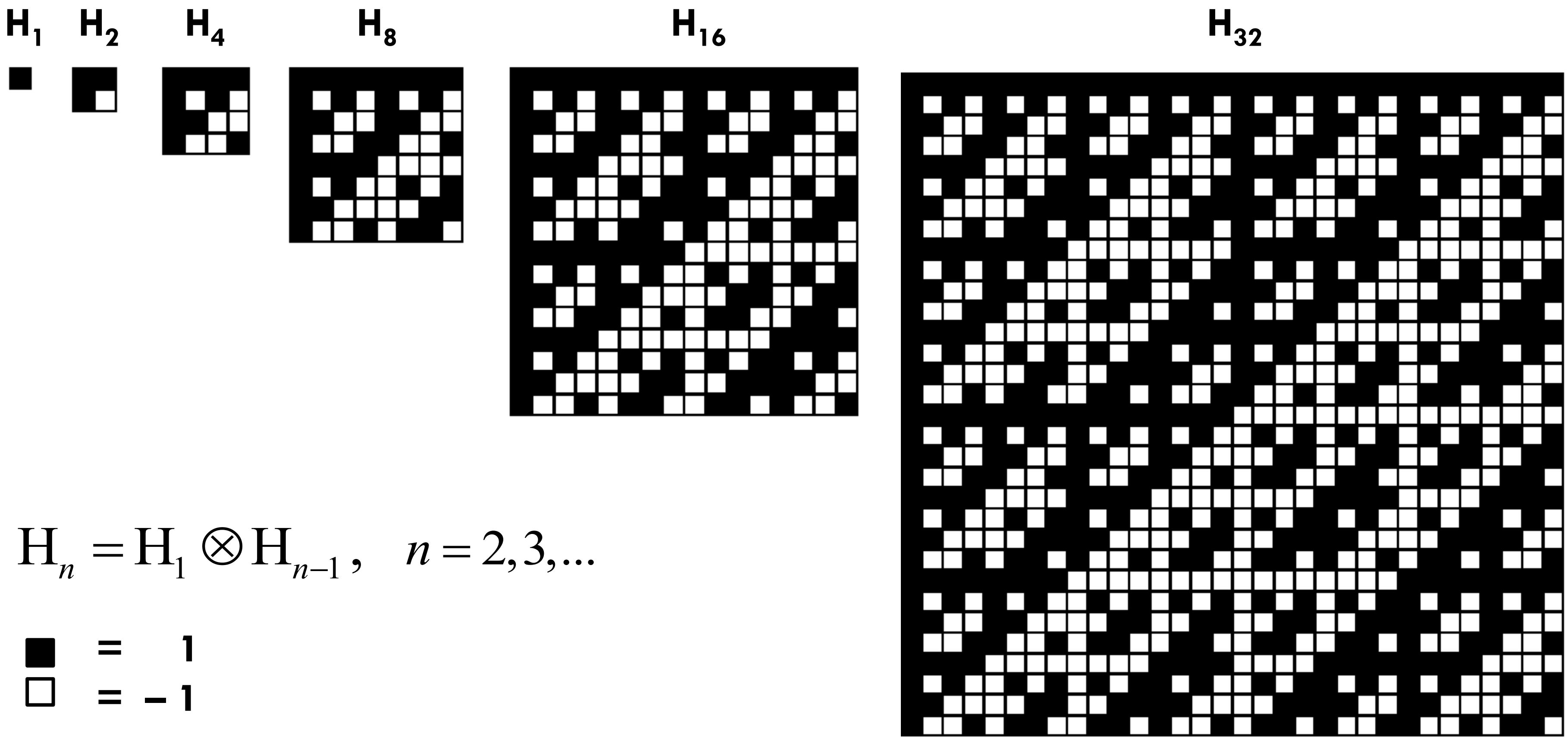
James Joseph Sylvester
1814 – 1897

***Lemma* (*Sylvester* 1867)**

***There is an Hadamard-type matrix of order $2^k$ for all natural numbers k.***

# Visualization of Sylvester-Hadamard matrices






$$H_n = H_1 \otimes H_{n-1}, \quad n = 2, 3, \ldots$$

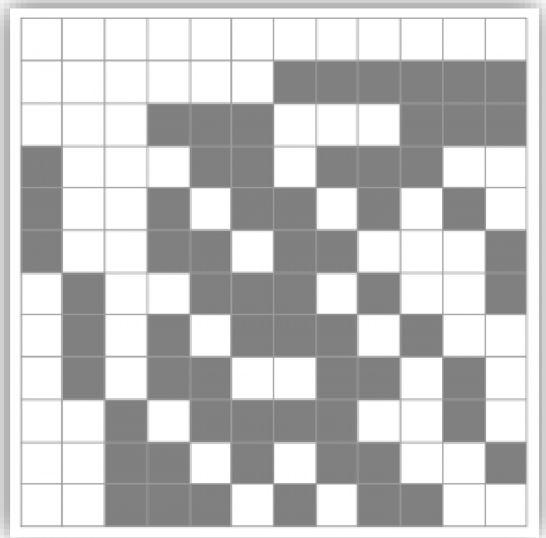
■ = 1
□ = – 1

# Hadamard’s matrices

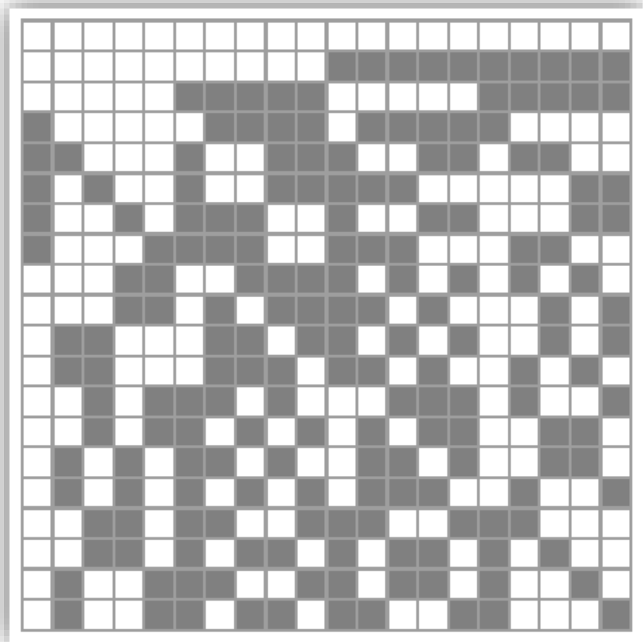


Sylvester's construction (1867) yields Hadamard matrices of order 1, 2, 4, 8, 16, 32, etc. Hadamard matrices of orders 12 and 20 were subsequently constructed by Hadamard in 1893.

12 × 12 Hadamard matrix



20 × 20 Hadamard matrix



**White square = 1**
**Gray square = –1**

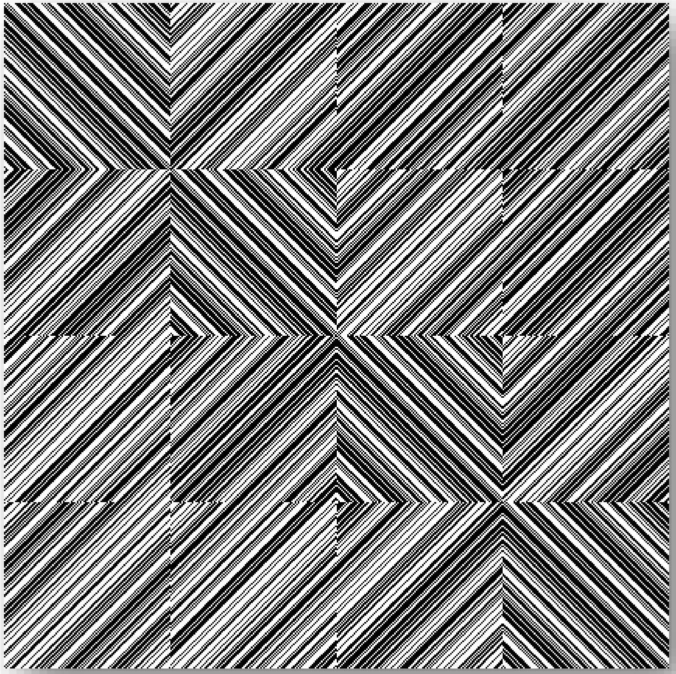
# Construction of Hadamard matrices



Different ***construction techniques*** of Hadamard matrices have been developed for a wide variety of applications:

- ***Sylvester's*** technique (1867)
- ***Paley's*** technique (1933)
- ***Williamson's*** technique (1944)
- ***Ahmed & Rao's*** technique (1975)
- ***Henderson's*** technique (1978)
- ***Golay's*** technique (1982)
- ***Lee & Kaveh's*** technique (1986)

…and others



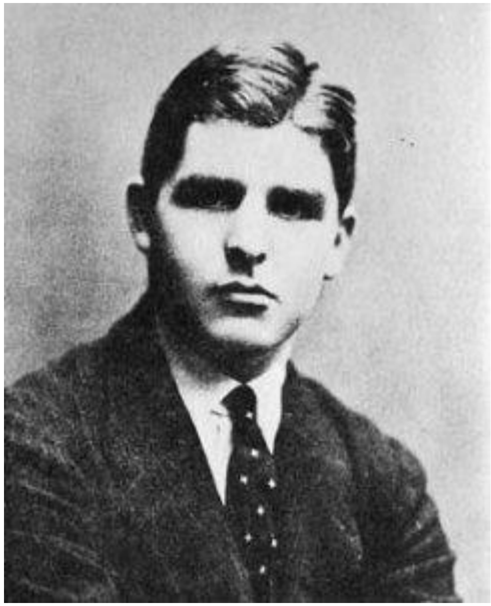
428 × 428 Hadamard matrix
*H. Kharaghani and B. Tayfeh-Rezaie, 2005*

# Construction of Hadamard matrices



In 1933, Raymond Paley discovered a construction that produces a Hadamard matrix of order $q + 1$ when $q$ is any prime power that is congruent to (3 mod 4) and that produces a Hadamard matrix of order $2(q+1)$ when $q$ is a prime power that is congruent to (1 mod 4).
His method uses finite fields.

The Hadamard conjecture should probably be attributed to Paley.



Raymond Paley
1907– 1933

The smallest order that cannot be constructed by a combination of Sylvester's and Paley's methods is 92. An Hadamard matrix of this order was found using a computer by Baumert, Golomb, and Hall in 1962 at JPL. They used a construction, due to Williamson, that has yielded many additional orders.

# Construction of Hadamard matrices

**Facts for Hadamard matrices:**

- ✓ In 2005, Hadi Kharaghani and Behruz Tayfeh-Rezaie published their construction of an Hadamard matrix of order 428. As a result, the smallest order for which no Hadamard matrix is presently known is 668.

- ✓ As of 2008, there are 12 multiples of 4 less than or equal to 2000 for which no Hadamard matrix of that order is known. They are:
  668, 716, 892, 1132, 1244, 1388, 1436, 1676, 1772, 1916, 1948, and 1964.

- ✓ Every Hadamard matrix of order $n > 4$ contains a submatrix equivalent to:

- ✓ For any order $n$, $\boldsymbol{H_4 \in H_n}$ and $\boldsymbol{H_n \in H_{2n}}$ .

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ \hline 1 & 1 & - & - \\ 1 & - & - & 1 \end{array}\right]$$

# Applications of Hadamard matrices



- **Signal processing, Coding and Cryptography**
  - Design of experiments
  - Object recognition
  - Coding of digital signals (CDMA telecommunications)
- **Spectral analysis or signal separation**
  - Mass spectroscopy
  - Polymer chemistry
  - Signal and information processing
  - Geophysics
  - Acoustics
  - Nuclear medicine and nuclear physics
- **Other novel applications**

  Digital logic design, pattern recognition, data compression, magnetic resonance imaging, neuroscience and quantum computing

# Hadamard equivalence and D-optimal designs



**Definition 1 (H-equivalence).** *We call two Hadamard matrices Hadamard equivalent or H-equivalent if one can be obtained from the other by a sequence of the operations:*

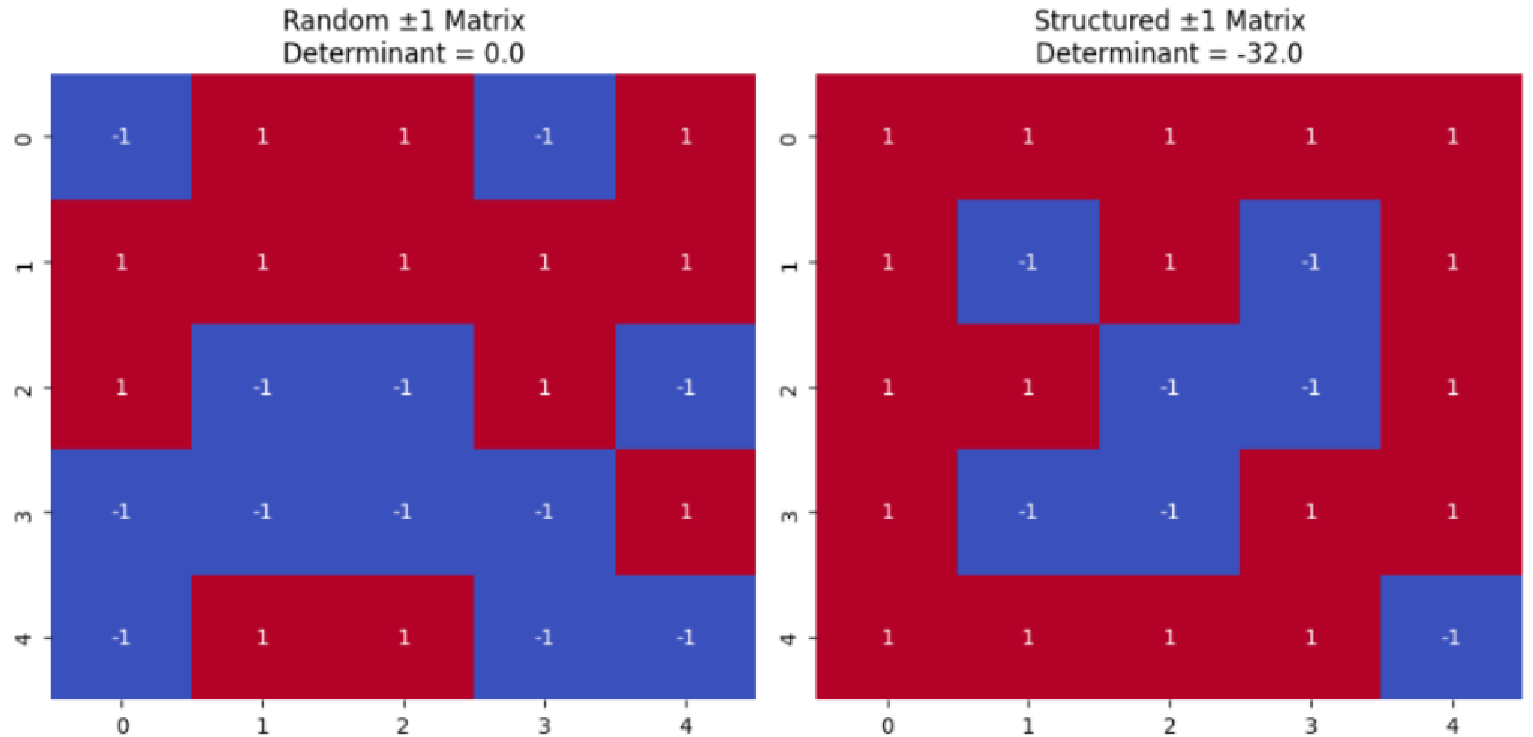
*i) interchange any pair of rows and/or columns,*
*ii) multiply any row and/or column through by-1.*

**Definition 2.** *A* $k \times k$ $(\pm 1)$*-matrix with maximum determinant (in magnitude) is called a D-optimal design of order* $k$ *and is denoted* $D_k$*. We also denote its determinant by* $d_k = |det(D_k)|$*. If the first row and the first column of* $D_k$ *consists of all 1's is said to be a normalized form.*

# Hadamard equivalence and D-optimal designs



Random (±1)-matrices have determinants of moderate value, that vary unpredictably. Structured matrices (similar to a Hadamard-like design) have a significantly higher determinant, indicating better spread and orthogonality among its rows.



# Hadamard equivalence and D-optimal designs



A D-optimal design of order $\boldsymbol{k}$ **multiple of 4** is a Hadamard matrix of the same order, i.e., $\boldsymbol{D_k} \triangleq \boldsymbol{H_k}$ .Therefore, it achieves the maximum determinant value $\boldsymbol{d_k} = \boldsymbol{k^{\frac{k}{2}}}$.

$$D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix}, \; d_4 = 16 \qquad D_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & 1 & - & - & - \\ 1 & 1 & - & 1 & - & 1 & - & - \\ 1 & 1 & - & - & - & - & 1 & 1 \\ 1 & - & 1 & 1 & - & - & 1 & - \\ 1 & - & 1 & - & - & 1 & - & 1 \\ 1 & - & - & 1 & 1 & - & - & 1 \\ 1 & - & - & - & 1 & 1 & 1 & - \end{bmatrix}, \; d_8 = 4096$$

# Hadamard equivalence and D-optimal designs



A D-optimal design is a design that maximizes the determinant of the information matrix $X^T X$, where $X$ is the design matrix.

Maximizing the determinant leads to minimizing the generalized variance of the estimated regression coefficients. Thus, the estimates will be as precise as possible, given the constraints of the design.

$$D_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & - \\ 1 & 1 & - & - & - \\ 1 & - & - & 1 & - \\ 1 & - & - & - & 1 \end{bmatrix}, \ D_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & - & 1 \\ 1 & 1 & - & - & - & 1 \\ 1 & - & - & 1 & - & 1 \\ 1 & 1 & 1 & 1 & - & - \\ 1 & - & - & - & 1 & - \end{bmatrix}, \ D_7 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & - & 1 & 1 \\ 1 & - & 1 & - & - & 1 & 1 \\ 1 & - & - & 1 & 1 & - & 1 \\ 1 & - & - & 1 & 1 & 1 & - \\ 1 & 1 & 1 & - & 1 & - & - \\ 1 & 1 & 1 & 1 & - & - & \end{bmatrix}$$

$$d_5 = 48, \quad d_6 = 160, \quad d_7 = 576$$

# Minors of $(\pm 1)$-matrices



**Lemma** (*Day & Peterson*, 1988)

*Let A be an* $n \times n$ *matrix with elements* $\pm 1$.
*It holds that:*

*a)* *detA is an integer and* $\mathbf{2^{n-1}}$ *divides detA*
*b)* *when* $n \leq 6$, *the only possible values for detA are the following, and they do all occur:*

| $n$ | det$A$ |
|---|---|
| 1 | 1 |
| 2 | 0, 2 |
| 3 | 0, 4 |
| 4 | 0, 8, 16 |
| 5 | 0, 16, 32, 48 |
| 6 | 0, 32, 64, 96, 128, 160 |

If $M_m$ denotes the absolute value of an $m \times m$ minor of a $(\pm 1)$-matrix of order $n \geq m$, then $\boldsymbol{M_m = p\, 2^{m-1}}$, where $p$ is either a positive integer, or zero.

**Definition 3.** *The* spectrum of the determinant function for $(\pm 1)$-matrices *is defined to be the set of values taken by* $p = 2^{1-m}\,|\det(R)|$ *as the matrix* $R$ *ranges over all* $m \times m$ $(\pm 1)$*-matrices. We denote this by* spec(m).

# Spectrum of the determinant function



| $m$ | **Confirmed spectrum spec**$\{m\}$ |
|---|---|
| 1 | 1 |
| 2 | [0, 1] |
| 3 | [0, 1] |
| 4 | [0, 2] |
| 5 | [0, 3] |
| 6 | [0, 5] |
| 7 | [0, 9] |
| 8 | [0, 18], 20, 24, 32 |
| 9 | [0, 40], 42, [44, 45], 48, 56 |
| 10 | [0, 102], [104, 105], 108, 110, 112, [116, 117], 120, 125, 128, 144 |
| 11 | [0, 268], [270, 276], [278, 280], [282, 286], 288, 291, [294, 297], 304, 312, 315, 320 |
| 13 | [0, 2172], [2174, 2185], [2187, 2196], [2199, 2202], 2205, 2208, 2210, 2211, [2214, 2218], [2220, 2226], 2228, 2229, 2230, 2232, 2233, 2235, 2238, 2240, 2241, [2243, 2245], 2247, 2248, 2250, 2253, 2256, [2258, 2260], 2262, 2264, 2265, 2267, 2268, 2271, 2272, 2274, 2277, 2280, 2283, 2286, 2288, 2292, 2295, 2296, 2304, 2307, 2312, 2313, 2316, 2319, 2320, 2322, 2325, 2328, 2331, 2334, 2336, 2340, 2343, 2344, 2349, 2352, 2355, 2360, 2361, 2367, 2368, 2370, 2373, 2376, 2385, 2394, 2400, 2403, 2406, 2421, 2430, 2432, 2439, 2457, 2472, 2484, 2496, 2511, 2520, 2538, 2560, 2583, 2592, 2619, 2646, 2673, 2835, 2916, 3159, 3645 |

$$\max p = \hat{p} = 2\left(\frac{m}{4}\right)^{\frac{m}{2}}$$

For $m = 8$, $\max p = 2 \cdot 2^4 = 32$

$M_8 = |H_8| = 32 \cdot 2^7 = 4096$

# Spectrum of the determinant function



| $n$ | $d_n$ | $n$ | $d_n$ |
|---|---|---|---|
| 1 | $1$ | 26 | $24^{12} \cdot 50$ |
| 2 | $2$ | 27 | $\geq 2^{26} \cdot 6^{11} \cdot 546$ |
| 3 | $4$ | 28 | $28^{14}$ |
| 4 | $4^2$ | 29 | $\geq 2^{28} \cdot 7^{12} \cdot 320$ |
| 5 | $4^2 \cdot 3$ | 30 | $28^{14} \cdot 58$ |
| 6 | $4^2 \cdot 10$ | 31 | $\geq 2^{30} \cdot 7^{13} \cdot 784$ |
| 7 | $2^6 \cdot 9$ | 32 | $2^{80}$ |
| 8 | $8^4$ | 33 | $\geq 2^{32} \cdot 8^{14} \cdot 441$ |
| 9 | $2^9 \cdot 28$ | 34 | $\geq 2^{33} \cdot 8^{15} \cdot 256$ |
| 10 | $8^4 \cdot 18$ | 35 | $\geq 2^{34} \cdot 8^{15} \cdot 1064$ |
| 11 | $2^{16} \cdot 5$ | 36 | $36^{18}$ |
| 12 | $12^6$ | 37 | $2^{36} \cdot 9^{17} \cdot 72$ |
| 13 | $12^6 \cdot 5$ | 38 | $36^{18} \cdot 74$ |
| 14 | $12^6 \cdot 26$ | 39 | $\geq 2^{38} \cdot 9^{17} \cdot 1440$ |
| 15 | $2^{14} \cdot 3^5 \cdot 105$ | 40 | $40^{20}$ |
| 16 | $16^8$ | 41 | $40^{20} \cdot 9$ |
| 17 | $16^7 \cdot 80$ | 42 | $40^{20} \cdot 82$ |
| 18 | $16^8 \cdot 34$ | 43 | $\geq 2^{42} \cdot 10^{19} \cdot 1890$ |
| 19 | $2^{18} \cdot 4^6 \cdot 833$ | 44 | $44^{22}$ |
| 20 | $20^{10}$ | 45 | $\geq 2^{44} \cdot 11^{21} \cdot 89$ |
| 21 | $20^9 \cdot 116$ | 46 | $44^{22} \cdot 90$ |
| 22 | $2^{21} \cdot 5^9 \cdot 100$ | 47 | $\geq 2^{46} \cdot 11^{18} \cdot 3037500$ |
| 23 | $\geq 2^{22} \cdot 5^6 \cdot 42411$ | 48 | $48^{24}$ |
| 24 | $24^{12}$ | 49 | $\geq 2^{48} \cdot 12^{23} \cdot 96$ |
| 25 | $24^{12} \cdot 7$ | 50 | $48^{24} \cdot 98$ |

Bounds for the maximum determinant $d_n$ of all $n \times n$ $(\pm 1)$-matrices.

$$|\det H_n| = n^{\frac{n}{2}}$$

$$d_n \geq p\, 2^{n-1}$$

**Example:**

- $d_{26} = 24^{12} \cdot 50 = 6^{12} \cdot 25 \cdot 2^{25}$
- $d_{28} = 28^{14} = 28^{\frac{28}{2}}$

(*Orrick and Solomon, 2010*)

# Minors of Hadamard matrices



The current approach is based on the analysis of the result derived from the next proposition by simply using calculus techniques.

**Proposition** (*Williamson 1944, Szöllősi 2010*)
*Let $M_k$ the absolute value of a $k \times k$ minor of the Hadamard matrix $H_n$ of order n,*

$$M_k = \left|\det H_{k,n}\right|$$

*where $H_{k,n}$ denotes the $k \times k$ submatrix of $H_n$.*
*Then, there is a one-to-one correspondence between the minors of size **k** and **n-k** described by the equation:*

$$M_{n-k} = n^{\frac{n}{2}-k} M_k$$

The above result readily applies to D-optimal designs of order $n > 1$.

# Minors of Hadamard matrices



*Proof.* We consider an orthogonal $n \times n$ matrix $U$ of the form $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

where $A$ is a $k \times k$ orthogonal matrix and $D$ is a $(n-k) \times (n-k)$ orthogonal matrix for $1 < k < n$. Then,

$$U\,U^\top = I_n \quad \text{and} \quad U^{-1} = U^\top = \begin{bmatrix} A^\top & C^\top \\ B^\top & D^\top \end{bmatrix}.$$

Consequently, $\det(U) = 1$ and by the orthogonality property we have

$$\det(D) = \det(D^\top).$$

Using Jacobi's determinant identity for $U$, it follows:

$$\det(A) = \det(U) \cdot \det(D^\top) = 1 \cdot \det(D) = \det(D)$$

# Embedded Hadamard matrices



**Theorem** (*Cohn, 1965*)

*If an $H_{n+m}$ exists and $n > m$, then no n-rowed minor is an $H_n$ .*

**Example:** $n = 20$ and $m = 8$, then $H_{20}$ is not embedded in $H_{28}$

**Theorem** (*Brent & Osborn, 2013*)

*Let $H_n$ be an Hadamard matrix of order n having a Hadamard submatrix M of order $m < n$. Then, $m \leq \frac{n}{2}$.*

**Example:** $n = 20$ and $m = 8$, then $H_8$ is embedded in $H_{20}$

# Embedded Hadamard matrices and D-optimal designs
# Problem motivation



**Problem 1:**

*Can an Hadamard matrix (D-optimal design) of order* $\boldsymbol{n} - \mathbf{4}$ or $\boldsymbol{n} - \mathbf{8}$ *exist embedded in an Hadamard matrix of order n, for* $\boldsymbol{n} = \mathbf{4}\boldsymbol{t}$ *with integer* $\boldsymbol{t} > \mathbf{2}$ *?*

$$H_8 \in H_{12} \;,\; H_{12} \in H_{16} \;,\; H_{16} \in H_{20} \;,\ldots,\; H_{n-4} \in H_n$$

$$H_{12} \in H_{20} \;,\; H_{16} \in H_{24} \;,\; H_{20} \in H_{28} \;,\ldots,\; H_{n-8} \in H_n$$

**Problem 2** (Generalization)

*Can an Hadamard matrix (D-optimal design) of order* $\boldsymbol{n} - \boldsymbol{k}$ *exist embedded in an Hadamard matrix of order n, for* $\boldsymbol{n} = \mathbf{4}\boldsymbol{t}$ and $\boldsymbol{k} = \mathbf{4}\boldsymbol{r}$ with $\mathbf{0} < \mathbf{2}\boldsymbol{r} < \boldsymbol{t}$ *?*

*What are the characteristics of such an embedding property?*

# Is $H_{n\text{-}4}$ embedded in $H_n$?



**Research problem 1:**

*Can an Hadamard matrix of order* $\mathbf{n-4}$ *be embedded in the Hadamard matrix of order n, for* $\mathbf{n=4t}$ *with integer* $\mathbf{t>2}$*?*

$$H_8 \in H_{12} \quad , \quad H_{12} \in H_{16} \quad , \quad H_{16} \in H_{20} \quad , \ldots , \quad H_{n-4} \in H_n$$

Why integer $t > 2$ ?
Because

- ✓ for $t = 1$, $n - 8 = 0$
- ✓ for $t = 2$, $n = 8$ and $n - 4 = 4$. So, we have $H_4 \in H_8$ which is true.

# Is $H_{n\text{-}4}$ embedded in $H_n$?



- **Case I :** For $n = 4t$ with integer $t > 2$ and $M_4 = 8 = 2^3$.

$$\left|\det H_{n-4}\right| = n^{\frac{n}{2}-4} \cdot 2^3 \overset{\text{Using Calculus}}{\Longleftrightarrow} f(t-1) - f(t) = \frac{-\ln 2}{2(t-1)(t-2)}, \quad \text{where } f(t) = \frac{\ln t}{t-1}$$

For any integer $t > 2$ it can be proved that: $\begin{cases} f(t-1) - f(t) > 0 \\ \dfrac{-\ln 2}{2(t-1)(t-2)} < 0 \end{cases}$

**Argument is invalid**

- **Case II :** For $n = 4t$ with integer $t > 2$ and $M_4 = 16 = 2^4$.

$$\left|\det H_{n-4}\right| = n^{\frac{n}{2}-4} \cdot 2^4 \overset{\text{Using Calculus}}{\Longleftrightarrow} f(t-1) - f(t) = 0$$

**Argument is invalid**

# Is $H_{n-4}$ embedded in $H_n$?



**Result 1**

For every $n = 4t$ with integer $t > 2$,

$$\left|\det H_{n-4}\right| \neq n^{\frac{n}{2}-4} \cdot M_4$$

Therefore, a Hadamard matrix of order $n - 4$ cannot be embedded in an Hadamard matrix of order $n > 4$.

$$H_{n-4} \notin H_n$$

# Is $H_{n\text{-}8}$ embedded in $H_n$?



**Research problem 1:**

*Can an Hadamard matrix of order* $\mathbf{n-8}$ *be embedded in the Hadamard matrix of order n, for* $\mathbf{n = 4t}$ *with integer* $\mathbf{t > 4}$*?*

$$H_{12} \in H_{20} \quad , \quad H_{16} \in H_{24} \quad , \quad H_{20} \in H_{28} \quad , \ldots , \quad H_{n-8} \in H_n$$

Why integer $t > 4$ ?

Because

- ✓ for $t = 1$ and $t = 2$, $n - 8 \leq 0$
- ✓ for $t = 3$, $n = 12$ and $n - 8 = 4$. So, we have $H_4 \in H_{12}$ which is true.
- ✓ for $t = 4$, $n = 16$ and $n - 8 = 8$. So, we have $H_8 \in H_{16}$ which is true.

# Is $H_{n\text{-}8}$ embedded in $H_n$?



It is known that $\boldsymbol{M_8 = p \cdot 2^7}$ ( *Lemma Day & Peterson, 1988* ) and for the 8×8 case it has been confirmed that the possible existing values for the integer $p$ are (*Orrick & Solomon, 2010*)

1, 2,…, 18, 20, 24, and 32

If $n = 4t$ with integer $t > 4$ and $\boldsymbol{M_8 = p \cdot 2^7}$ , then we investigate the existence of a Hadamard matrix of order $n - 8$ embedded in an Hadamard matrix of order $n$ considering the following relation:

$$\left|\det H_{n-8}\right| = n^{\frac{n}{2}-8} \cdot p \cdot 2^7 \quad \overset{\substack{n=4t \\ t>4}}{\Leftrightarrow} \quad p = 2t^4 \left(\frac{t-2}{t}\right)^{2t-4}$$

# Is $H_{n\text{-}8}$ embedded in $H_n$?



The equation $p = 2t^4 \left( \frac{t-2}{t} \right)^{2t-4}$ holds for the following pairs of integer values:

**$( t, p ) = ( 3, 18 )$** and **$( t, p ) = ( 4, 32 )$**

which correspond to the known cases $H_4 \in H_{12}$ and $H_8 \in H_{16}$ , respectively.

However,

**There are no integer values for $t > 4$ and $p > 0$ satisfying the above equation.**

The above result can be proved by studying the properties of the function $p(t)$, but a proof based on number theory is also available.

# Is $H_{n-8}$ embedded in $H_n$?



**Result 2**

For every $n = 4t$ with integer $t > 4$,

$$\left|\det H_{n-8}\right| \neq n^{\frac{n}{2}-8} \cdot M_8$$

Therefore, a Hadamard matrix of order $n - 8$ cannot be embedded in an Hadamard matrix of order $n > 16$.

$$H_{n-8} \notin H_n$$

# Generalization



**Results**

- ✓ *For every order* $\boldsymbol{n} = \mathbf{4}\boldsymbol{t}$ *with integer* $\boldsymbol{t} > \mathbf{2}$, *it holds* $H_{n-4} \notin H_n$
- ✓ *For every order* $\boldsymbol{n} = \mathbf{4}\boldsymbol{t}$ *with integer* $\boldsymbol{t} > \mathbf{4}$, *it holds* $H_{n-8} \notin H_n$

**Research problem 2:**

Is it possible that

$$H_{n-k} \notin H_n$$

for every order $\boldsymbol{n}$ and integer $\boldsymbol{k}$, such that $\boldsymbol{n} = \mathbf{4}\boldsymbol{t}$ and $\boldsymbol{k} = \mathbf{4}\boldsymbol{r}$ with $\boldsymbol{r} > \mathbf{0}$ and $\boldsymbol{t} > \mathbf{2}\boldsymbol{r}$ ?

# Is $H_{n\text{-}k}$ embedded in $H_n$?



Given a positive integer $k$, it is known that $M_k \le k^{\frac{k}{2}}$ and $M_k = p\,2^{k-1}$.

Therefore, if $k = 4r$, where $r > 0$, the maximum value of the integer $p$, denoted by $\hat{p}$ is given by

$$\hat{p} = 2\left(\frac{k}{4}\right)^{\frac{k}{2}} = 2r^{2r}$$

For $\boldsymbol{n = 4t}$, $\boldsymbol{k = 4r}$ for integers $\boldsymbol{r > 0}$ and $\boldsymbol{t > r}$ it holds:

$$\left|\det H_{n-k}\right| = n^{\frac{n}{2}-k} \cdot p \cdot 2^{k-1} \quad \Leftrightarrow \quad p = 2t^{2r}\left(\frac{t-r}{t}\right)^{2(t-r)}$$

A necessary condition for the general embedding problem is the following:

$$p \le \hat{p}$$

# Is $H_{n\text{-}k}$ embedded in $H_n$?



Let $\theta = \frac{r}{t} = \frac{4r}{4t} = \frac{k}{n}$. Since $n > k$, then $0 < \theta < 1$ and $0 < 1 - \theta < 1$. Hence,

$$p \leq \hat{p} \;\Leftrightarrow\; 2t^{2r}\left(\frac{t-r}{t}\right)^{2(t-r)} \leq 2r^{2r} \;\overset{\text{Using Calculus}}{\Leftrightarrow}\; (1-\theta)\ln(1-\theta) - \theta\ln\theta \leq 0$$

Studying the sign of the real function

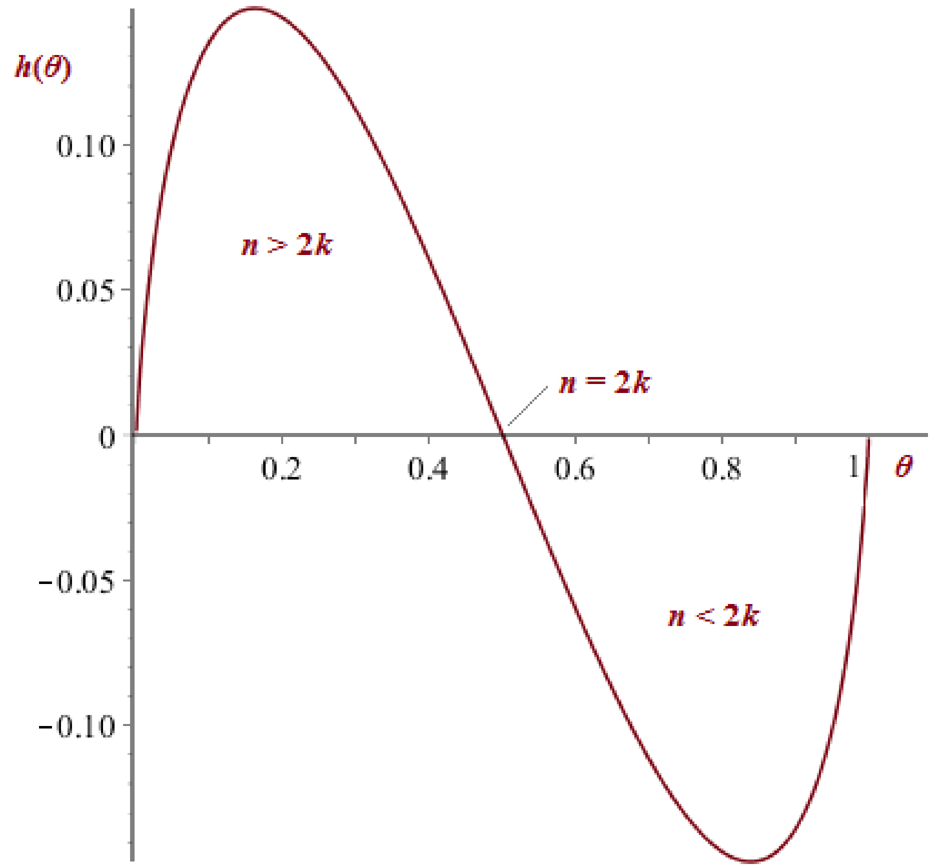
$$h(\theta) = (1-\theta)\ln(1-\theta) - \theta\ln\theta$$

provides very important information about the behavior of $p$ for the various values of the integers $n$ and $k$ when $n > k$.

# Is $H_{n\text{-}k}$ embedded in $H_n$?



$$h(\theta)=(1-\theta)\ln(1-\theta)-\theta\ln\theta$$




- $n>2k \iff h(\theta)>0 \iff p>\hat{p}$

  $H_{n-k} \notin H_n$ ✓

- $n=2k \iff h(\theta)=0 \iff p=\hat{p}$

  $H_k \in H_{2k}$ ✓

- $n<2k \iff h(\theta)<0 \iff p<\hat{p}$

  $H_{n-k} \in H_n$ !

But for what values of $p$ is true?

# Is $H_{n\text{-}k}$ embedded in $H_n$?



Embeddability of Hadamard matrices $H_{n\text{-}k}$ for $4 \leq k \leq 20$ and $8 \leq n \leq 40$.

| Order | $k = 4$ | $k = 8$ | $k = 12$ | $k = 16$ | $k = 20$ |
|---|---|---|---|---|---|
| $n = 8$ | $H_4 \in H_8$ | | | | |
| $n = 12$ | $H_8 \notin H_{12}$ | $H_4 \in H_{12}$ | | | |
| $n = 16$ | $H_{12} \notin H_{16}$ | $H_8 \in H_{16}$ | $H_4 \in H_{16}$ | | |
| $n = 20$ | $H_{16} \notin H_{20}$ | $H_{12} \notin H_{20}$ | $H_8 \in H_{20}$ | $H_4 \in H_{20}$ | |
| $n = 24$ | $H_{20} \notin H_{24}$ | $H_{16} \notin H_{24}$ | $H_{12} \in H_{24}$ | $H_8 \in H_{24}$ | $H_4 \in H_{24}$ |
| $n = 28$ | $H_{24} \notin H_{28}$ | $H_{20} \notin H_{28}$ | $H_{16} \notin H_{28}$ | $H_{12} \in H_{28}$ | $H_8 \in H_{28}$ |
| $n = 32$ | $H_{28} \notin H_{32}$ | $H_{24} \notin H_{32}$ | $H_{20} \notin H_{32}$ | $H_{16} \in H_{32}$ | $H_{12} \in H_{32}$ |
| $n = 36$ | $H_{32} \notin H_{36}$ | $H_{28} \notin H_{36}$ | $H_{24} \notin H_{36}$ | $H_{20} \notin H_{36}$ | $H_{16} \in H_{36}$ |
| $n = 40$ | $H_{36} \notin H_{40}$ | $H_{32} \notin H_{40}$ | $H_{28} \notin H_{40}$ | $H_{24} \notin H_{40}$ | $H_{20} \in H_{40}$ |

# Is $H_{n\text{-}k}$ embedded in $H_n$?



Embeddability of Hadamard matrices $H_{n\text{-}k}$ for $4 \le k \le 20$ and $8 \le n \le 40$.

| Order | $k = 4$ | $k = 8$ | $k = 12$ | $k = 16$ | $k = 20$ |
|---|---|---|---|---|---|
| $n = 8$ | $H_4 \in H_8$ | | | | |
| $n = 12$ | $H_8 \notin H_{12}$ | $H_4 \in H_{12}$ | | | |
| $n = 16$ | $H_{12} \notin H_{16}$ | $H_8 \in H_{16}$ | $H_4 \in H_{16}$ | | |
| $n = 20$ | $H_{16} \notin H_{20}$ | $H_{12} \notin H_{20}$ | $H_8 \in H_{20}$ | $H_4 \in H_{20}$ | |
| $n = 24$ | $H_{20} \notin H_{24}$ | $H_{16} \notin H_{24}$ | $H_{12} \in H_{24}$ | $H_8 \in H_{24}$ | $H_4 \in H_{24}$ |
| $n = 28$ | $H_{24} \notin H_{28}$ | $H_{20} \notin H_{28}$ | $H_{16} \notin H_{28}$ | $H_{12} \in H_{28}$ | $H_8 \in H_{28}$ |
| $n = 32$ | $H_{28} \notin H_{32}$ | $H_{24} \notin H_{32}$ | $H_{20} \notin H_{32}$ | $H_{16} \in H_{32}$ | $H_{12} \in H_{32}$ |
| $n = 36$ | $H_{32} \notin H_{36}$ | $H_{28} \notin H_{36}$ | $H_{24} \notin H_{36}$ | $H_{20} \notin H_{36}$ | $H_{16} \in H_{36}$ |
| $n = 40$ | $H_{36} \notin H_{40}$ | $H_{32} \notin H_{40}$ | $H_{28} \notin H_{40}$ | $H_{24} \notin H_{40}$ | $H_{20} \in H_{40}$ |

# When is $H_{n-k}$ embedded in $H_n$?



**Proposition:**

The discrete function

$$\mathcal{P}(n,k) = 2\left(\frac{n}{4}\right)^{\frac{k}{2}}\left(\frac{n-k}{n}\right)^{\frac{n-k}{2}} \quad \text{for} \quad \frac{n}{2} \leq k < n \quad \text{and} \quad \begin{cases} n = 8, 12, 16, \ldots \\ k = 4, 8, 12, \ldots \end{cases}$$

provides the values for the parameter $p$ which satisfies the equations:

$$|\det H_{n-k}| = n^{\frac{n}{2}-k} M_k \quad \text{or} \quad |\det H_{n-k}| = 2^{(n-k)-1}\left(\frac{n}{4}\right)^{\frac{n}{2}-k} p$$

# Conclusions on Hadamard embeddability



**Theorem :** *An Hadamard matrix of order $n-k$ cannot be embedded in an Hadamard matrix of order $n$ for any positive integers $n$ and $k$ multiples of 4 when $k < \frac{n}{2}$. That is*

$$H_{n-k} \notin H_n, \quad 4 \leq k < \frac{n}{2}$$

**Conjecture :** Consider a Hadamard matrix $H_n$. If $H_n^{(k)}$ is a $k \times k$ submatrix of $H_n$, where $n \geq 8$ and $k \geq 4$ are integers multiples of 4 such that $\frac{n}{2} \leq k < n$, and $|\det H_n^{(k)}| = p\,2^{k-1}$ with $p = \mathcal{P}(n,k)$, then an Hadamard matrix of order $n-k$ may exist embedded in the Hadamard matrix of order $n$, i.e.,

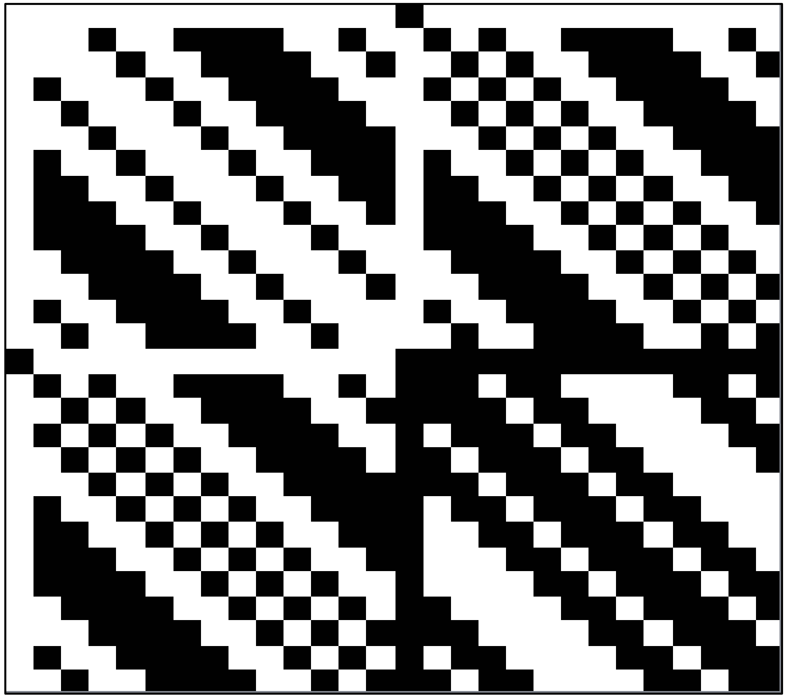
$$H_{n-k} \in H_n, \quad 4 \leq \frac{n}{2} \leq k < n$$

# Example 1: $H_8 \in H_{28}$



$$n = 28,\ k = 20, \text{ and } p = \mathcal{P}(28, 20) = 3764768$$

$H_{28}$ (2nd *Paley type*)



**Argument:** If $p = 3764768$ exists in the spectrum, meaning that $H_{28}$ has a $20 \times 20$ submatrix with minor:

$$|\det H_{28}^{(20)}| = 3764768 \cdot 2^{19} = 1973822685184$$

then it may $H_8$ exist embedded in $H_{28}$.

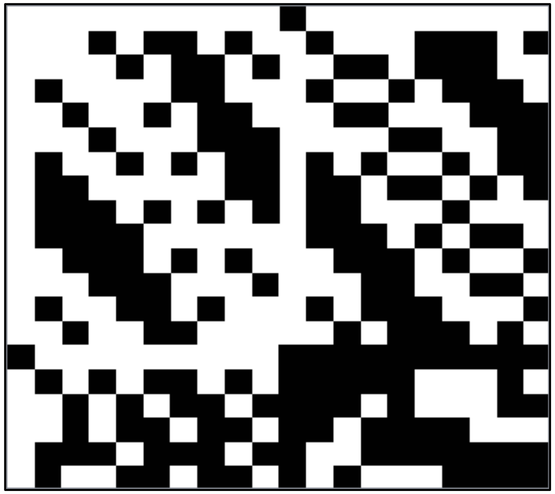
□ = 1
■ = – 1

N. J. A. Sloane. A library of Hadamard matrices. http://neilsloane.com/hadamard/index.html

# Example 1: $H_8 \in H_{28}$



$H_{28}^{(20)}$ *submatrix of* $H_{28}$



$n = 28$, $k = 20$, and $p = \mathcal{P}(28, 20) = 3764768$

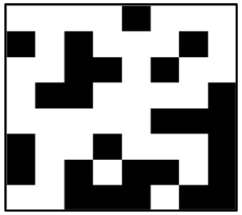
$H_{28}^{(20)} = [a_{ij}]$ of $H_{28}$ where

$i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 23, 27\}$,

$j \in \{1, 2, 3, 4, 5, 7, 9, 11, 13, 14, 15, 16, 17, 19, 20, 22, 23, 24, 26, 27\}$

$|\det H_{28}^{(20)}| = 3764768 \cdot 2^{19} = 1973822685184$

$H_8 \in H_{28}$



$A = [a_{ij}]$ of $H_{28}$ where $i \in \{1, 2, 3, 4, 5, 6, 16, 17\}$, $j \in \{4, 6, 9, 14, 15, 19, 21, 26\}$

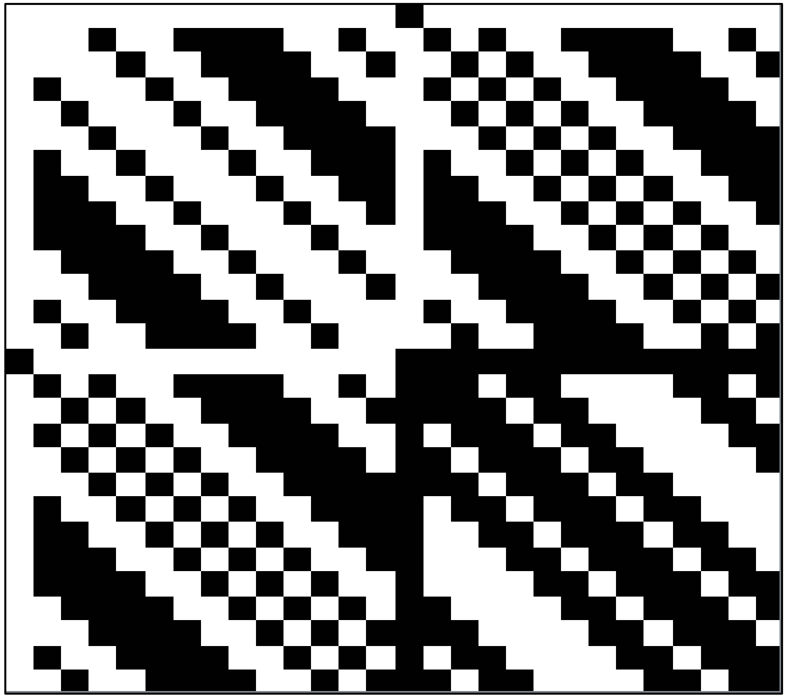
$$|\det A| = 2^{(28-20)-1} \cdot \left(\frac{28}{4}\right)^{\frac{28}{2}-20} \cdot 3764768 = 4096 = |\det H_8|$$

# Example 2: $H_{12} \in H_{28}$



$$n = 28,\ k = 16, \text{ and } p = \mathcal{P}(28, 16) = 71442$$

$H_{28}$ (2nd *Paley type*)



**Argument:** If $p = 71442$ exists in the spectrum, meaning that $H_{28}$ has a $16 \times 16$ submatrix with minor:

$$|\det H_{28}^{(16)}| = 71442 \cdot 2^{15} = 2341011456$$

then it may $H_{12}$ exist embedded in $H_{28}$.

□ = 1

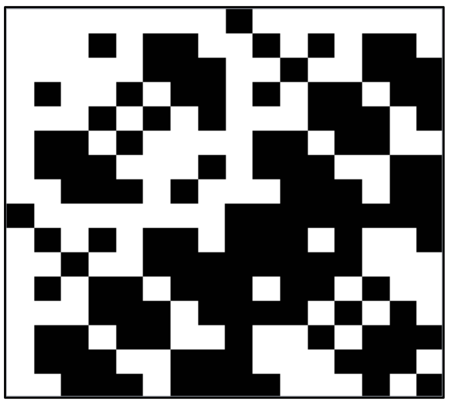
■ = – 1

N. J. A. Sloane. A library of Hadamard matrices. http://neilsloane.com/hadamard/index.html

# Example 2: $H_{12} \in H_{28}$



$H_{28}^{(16)}$ *submatrix of* $H_{28}$



$n = 28$, $k = 16$, and $p = \mathcal{P}(28, 16) = 71442$

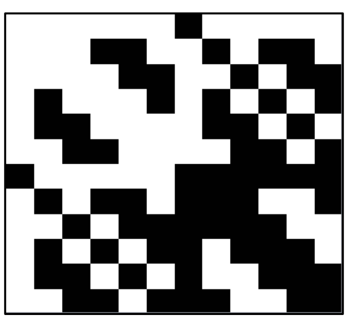
$H_{28}^{(16)} = [a_{ij}]$ of $H_{28}$ where

$i \in \{1, 2, 3, 4, 6, 8, 9, 11, 15, 16, 17, 18, 20, 22, 23, 25\}$,

$j \in \{1, 2, 3, 4, 6, 8, 9, 11, 15, 16, 17, 18, 20, 22, 23, 25\}$

$$|\det H_{28}^{(16)}| = 71442 \cdot 2^{15} = 2341011456$$

$H_{12} \in H_{28}$



$A = [a_{ij}]$ of $H_{28}$ where $i \in \{1, 2, 3, 4, 8, 11, 15, 16, 17, 18, 22, 25\}$, $j \in \{1, 2, 3, 4, 8, 11, 15, 16, 17, 18, 22, 25\}$

$$|\det A| = 2^{(28-16)-1} \cdot \binom{28}{4}^{\frac{28}{2}-16} \cdot 71442 = 2985984 = |\det H_{12}|$$

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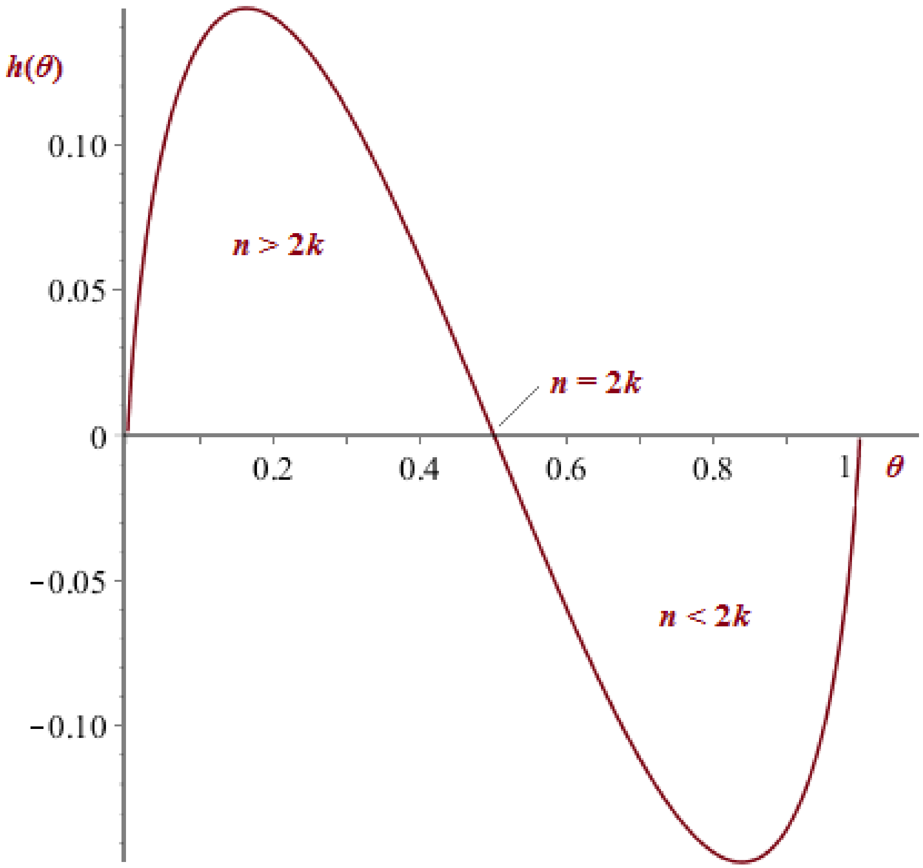
# Is $D_k$ embedded in $H_n$?



$$h(\theta) = (1-\theta)\ln(1-\theta) - \theta\ln\theta$$




$$\theta = \frac{r}{t} = \frac{4r}{4t} = \frac{k}{n} \in (0,1)$$

| $t$ | $r$ | $\theta$ | Result |
|---|---|---|---|
| 3 | 2 | 0.67 | $D_8 \notin H_{12}$ |
| 4 | 3 | 0.75 | $D_{12} \notin H_{16}$ |
| 5 | 3 | 0.60 | $D_{12} \notin H_{20}$ |
| 5 | 4 | 0.80 | $D_{16} \notin H_{20}$ |
| 6 | 4 | 0.67 | $D_{16} \notin H_{24}$ |

# Is $D_k$ embedded in $H_n$?



**Theorem:** *For positive integers $n, k$ multiples of 4 such that $4 < k < n$, a D-optimal design $D_k$ of order $k$ cannot exist embedded in a Hadamard matrix of order $n$ when $n < 2k$. That is*

$$D_k \notin H_n, \ \textit{for } n < 2k \ \textit{and } n, k \equiv 0 \pmod 4$$

**Conjecture:** Consider a Hadamard matrix $H_n$. If $D_k$ is a $k \times k$ D-optimal design, where $n \geq 8$ and $k \geq 4$ are integers multiples of 4 such that $k < n$, then $D_k$ may exist embedded in $H_n$ when $n > 2k$.

$$D_k \in H_n, \ \text{for } 4 \leq k < \frac{n}{2} \ \text{and } n, k \equiv 0 \pmod 4$$

# Embedding D-optimal designs of order $k > 4$



Let us consider a D-optimal design $D_k$ of order $k > 4$. If it can exist embedded in an Hadamard matrix of order $n > k$, then:

$$M_{n-k} = n^{\frac{n-2k}{2}} d_k \Leftrightarrow$$

$$p_{n-k}\, 2^{n-k-1} = n^{\frac{n-2k}{2}} \hat{p}_k\, 2^{k-1} \Leftrightarrow$$

$$p_{n-k} = 2^{k-2n}\, n^{\frac{n-2k}{2}} \hat{p}_k$$

This equation holds if $p_{n-k} \in spec\{m\}$. For positive integers $n$, $k$ such that $4 < k < n$, we denote the computed value of $p_{n-k}$ as $p^*$ and form the following equation:

$$p^* = \left(\frac{\sqrt{n}}{2}\right)^{n-2k} \hat{p}_k \qquad \textbf{[ Eq. (20) ]}$$

# Is $D_k$ embedded in $H_n$?



| $n$ | $k$ | $\hat{p}_k$ | $m = n - k$ | $p^*$ | $p^* \in \text{spec}\{m\}$ | Result |
|---|---|---|---|---|---|---|
| 8 | 5 | 3 | 3 | 1.5 | NO | $D_5 \notin H_8$ |
| 8 | 6 | 5 | 2 | 1.25 | NO | $D_6 \notin H_8$ |
| 8 | 7 | 9 | 1 | 1.125 | NO | $D_7 \notin H_8$ |
| 12 | 5 | 3 | 7 | 9 | YES | $D_5 \in H_{12}$ |
| 12 | 6 | 5 | 6 | 5 | YES | $D_6 \in H_{12}$ |
| 12 | 7 | 9 | 5 | 3 | YES | $D_7 \in H_{12}$ |
| 12 | 8 | 32 | 4 | 3.5556 | NO | $D_8 \notin H_{12}$ |
| 12 | 9 | 56 | 3 | 2.0741 | NO | $D_9 \notin H_{12}$ |
| 12 | 10 | 144 | 2 | 1.7778 | NO | $D_{10} \notin H_{12}$ |
| 12 | 11 | 320 | 1 | 1.3169 | NO | $D_{11} \notin H_{12}$ |

# Is $D_k$ embedded in $H_n$?



| $n$ | $k$ | $\hat{p}_k$ | $m = n - k$ | $p^*$ | $p^* \in \text{spec}\{m\}$ | Result |
|---|---|---|---|---|---|---|
| 16 | 5 | 3 | 11 | 192 | YES | $D_5 \in H_{16}$ |
| 16 | 6 | 5 | 10 | 80 | YES | $D_6 \in H_{16}$ |
| 16 | 7 | 9 | 9 | 36 | YES | $D_7 \in H_{16}$ |
| 16 | 8 | 32 | 8 | 32 | YES | $D_8 \in H_{16}$ |
| 16 | 9 | 56 | 7 | 14 | NO | $D_9 \notin H_{16}$ |
| 16 | 10 | 144 | 6 | 9 | NO | $D_{10} \notin H_{16}$ |
| 16 | 11 | 320 | 5 | 5 | NO | $D_{11} \notin H_{16}$ |
| 16 | 12 | 1458 | 4 | 5.6953 | NO | $D_{12} \notin H_{16}$ |
| 16 | 13 | 3645 | 3 | 3.5596 | NO | $D_{13} \notin H_{16}$ |
| 16 | 14 | 9477 | 2 | 2.3137 | NO | $D_{14} \notin H_{16}$ |
| 16 | 15 | 25515 | 1 | 1.5573 | NO | $D_{15} \notin H_{16}$ |

# Is $D_k$ embedded in $H_n$?



| $n$ | $k$ | $\hat{p}_k$ | $m = n - k$ | $p^*$ | $p^* \in \text{spec}\{m\}$ | Result |
|---|---|---|---|---|---|---|
| 20 | 7 | 9 | 13 | 1125 | YES | $D_7 \in H_{20}$ |
| 20 | 8 | 32 | 12 | 800 | YES | $D_8 \in H_{20}$ |
| 20 | 9 | 56 | 11 | 280 | YES | $D_9 \in H_{20}$ |
| 20 | 10 | 144 | 10 | 144 | YES | $D_{10} \in H_{20}$ |
| 20 | 11 | 320 | 9 | 64 | NO | $D_{11} \notin H_{20}$ |
| 20 | 12 | 1458 | 8 | 58.32 | NO | $D_{12} \notin H_{20}$ |
| 20 | 13 | 3645 | 7 | 29.16 | NO | $D_{13} \notin H_{20}$ |
| 20 | 14 | 9477 | 6 | 15.163 | NO | $D_{14} \notin H_{20}$ |
| 20 | 15 | 25515 | 5 | 8.1648 | NO | $D_{15} \notin H_{20}$ |
| 20 | 16 | 131070 | 4 | 8.3886 | NO | $D_{16} \notin H_{20}$ |
| 20 | 17 | 327680 | 3 | 4.1943 | NO | $D_{17} \notin H_{20}$ |
| 20 | 18 | 1114100 | 2 | 2.8521 | NO | $D_{18} \notin H_{20}$ |
| 20 | 19 | 3412000 | 1 | 1.7469 | NO | $D_{19} \notin H_{20}$ |

# Embedding D-optimal designs of order $n-1$ or $n+1$



An intriguing direction for further investigation is the potential existence of Hadamard matrices embedded within higher-order D-optimal designs.

Understanding such embeddings could reveal deeper structural properties and symmetries in optimal design theory.

**Lemma 2 .** *For* $n = 2, 3, \ldots, 7$, *if an* $n \times n$ *maximal-determinant* $(\pm 1)$*-matrix is* $D_n$, *then a* $D_{n-1}$ *must be embedded in it, i.e.* $D_{n-1} \in D_n$.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**Immediate results:** $H_4 \in D_5$ , $D_5 \in D_6$ , $D_6 \in D_7$

# Embedding D-optimal designs of order $n-1$ or $n+1$



**Definition 4.** *If $H$ is a Hadamard matrix of order $m$, then the sum of all its entries is called* excess *of $H$ and is denoted by $\sigma(H)$. The* maximal excess *of Hadamard matrices of order $m$ is denoted by $\sigma(m)$.*

$\sigma(2)=4$ , $\sigma(4)=8$ , $\sigma(8)=20$

$(K.\ W.\ Schmidt\ and\ E.\ T.\ H.\ Wang, 1977)$

$\sigma(12)=36$ , $\sigma(16)=64$ , $\sigma(20)=80$ , $\sigma(24)=112$ , $\sigma(28)=140$

$(M.R.\ Best, 1977)$

$\sigma(40)=244$ , $\sigma(44)=280$ , $\sigma(48)=324$ , $\sigma(52)=364$ , $\sigma(80)=704$ , $\sigma(84)=756$

$(N.\ Farmakis\ and\ S.\ Kounias, 1987)$

# Embedding D-optimal designs of order $n-1$ or $n+1$



In 1987, Farmakis and Kounias studied the problem of maximizing the excess of $H$, trying to find a $\sigma(m)$ for all Hadamard matrices $H$ of a specific order $m$. They focused on finding a Hadamard matrix of order $m$ so that the determinant of the $(m+1) \times (m+1)$ matrix $R$ is maximized.

$$R = \begin{bmatrix} 1 & -e^\top \\ e & H \end{bmatrix}$$

Then, the determinant of $R$ was proved to be equal to

$$\det(R) = m^{\frac{m}{2}-1}(m + e^\top H e) \qquad \textbf{[ Eq. (23) ]}$$

where $e$ is a row-vector of 1's of size $m$ and $e^\top He$ is the excess of $H$.

# Embedding D-optimal designs of order $n-1$ or $n+1$



**Proposition 2.** *A D-optimal design of order $n = 9$ or 17 has a Hadamard matrix of order $n-1$ embedded in it, i.e.,*

$$H_8 \in D_9 \text{ and } H_{16} \in D_{17}.$$

*Proof.* For $n = 9$, we know that

$$d_9 = 56 \cdot 2^{9-1} = 56 \cdot 256 = 14336.$$

$$\det(R) = 8^{\frac{8}{2}-1}(8+\sigma(8)) = 8^3 \cdot 28 = 512 \cdot 28 = 14336.$$

Similarly, for $n = 17$, $d_{17} = 327680 \cdot 2^{16} = 21474836480$

$$\det(R) = 16^{\frac{16}{2}-1}(16+\sigma(16)) = 16^7 \cdot 80 = 21474836480.$$

# Embedding D-optimal designs of order $n-1$ or $n+1$



| $n$ | Result for $n-1$ | Proof | $n$ | Result for $n-1$ | Proof |
|---|---|---|---|---|---|
| 4 | $D_3 \in H_4$ | Lemma 2 | 5 | $H_4 \in D_5$ | Lemma 2 |
| 6 | $D_5 \in D_6$ | Lemma 2 | 7 | $D_6 \in D_7$ | Lemma 2 |
| 8 | $D_7 \notin H_8$ | Eq. (20) | 9 | $H_8 \in D_9$ | Prop. 2 |
| 10 | $D_9 \; ? \; D_{10}$ | N/A | 11 | $D_{10} \; ? \; D_{11}$ | N/A |
| 12 | $D_{11} \notin H_{12}$ | Eq. (20) | 13 | $H_{12} \notin D_{13}$ | Eq. (23) |
| 14 | $D_{13} \; ? \; D_{14}$ | N/A | 15 | $D_{14} \; ? \; D_{15}$ | N/A |
| 16 | $D_{15} \notin H_{16}$ | Eq. (20) | 17 | $H_{16} \in D_{17}$ | Prop. 2 |
| 18 | $D_{17} \; ? \; D_{18}$ | N/A | 19 | $D_{18} \; ? \; D_{19}$ | N/A |
| 20 | $D_{19} \notin H_{20}$ | Eq. (20) | 21 | $H_{20} \notin D_{21}$ | Eq. (23) |
| 22 | $D_{21} \; ? \; D_{22}$ | N/A | 23 | $D_{22} \; ? \; D_{23}$ | N/A |
| 24 | $D_{23} \notin H_{24}$ | Eq. (20) | 25 | $H_{24} \notin D_{25}$ | Eq. (23) |
| 26 | $D_{25} \; ? \; D_{26}$ | N/A | 27 | $D_{26} \; ? \; D_{27}$ | N/A |
| 28 | $D_{27} \notin H_{28}$ | Eq. (20) | 29 | $H_{28} \notin D_{29}$ | Eq. (23) |

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# Embedding D-optimal designs of order $n-1$ or $n+1$



**Research problem:**

Construct the D-optimal designs $D_{10}, D_{11}, D_{14}$, and $D_{15}$ and search for submatrices that may satisfy the following embedding relations:

a) $D_9 \in D_{10}$ and $D_{10} \in D_{11}$

b) $D_{13} \in D_{14}$ and $D_{14} \in D_{15}$

If such submatrices exist, determine:

- how many arise for each case,
- the size of their determinant,
- the parameter $p$ associated with their determinant.

$$
\begin{aligned}
d_9 &= 58 \cdot 2^8 \\
d_{10} &= 144 \cdot 2^9 \\
d_{11} &= 320 \cdot 2^{10} \\
d_{13} &= 3645 \cdot 2^{12} \\
d_{14} &= 9477 \cdot 2^{13} \\
d_{15} &= 25515 \cdot 2^{15} \\
d_{17} &= 327680 \cdot 2^{16} \\
d_{18} &= 1114212 \cdot 2^{17} \\
d_{19} &= 3411968 \cdot 2^{18}
\end{aligned}
$$

Thank you

for your attention