A note of values of minors for Hadamard matrices

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Abstract. In this note we present a formulae for all possible values of $(n-j)\times(n-j)$ minors of an Hadamard matrix of order n

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1 Introduction

An Hadamard matrix is a type of square $(-1,1)$-matrix, whose rows are pairwise orthogonal.Two Hadamard matrices H and K are said to be said *Hadamard equivalent* or *H-equivalent* if there exist $(-1,0,1)$-monomial matrices A,B with $K=AHB$. Hadamard matrices of order n have determinant $n^{\frac{n}{2}}$. Whenever a determinant or minor is mentioned in this note, we just mean its absolute value. Sharpe [1] observed that all the $(n-1)\times(n-1)$ minors of an Hadamard matrix of order n are zero or $n^{\frac{n}{2}-1}$, and that all $(n-2)\times(n-2)$ minors are zero or $2n^{\frac{n}{2}-2}$, and that all $(n-3)\times(n-3)$ minors are zero or $4n^{\frac{n}{2}-3}$. The authors in [2,3] give an algorithm for computing $(n-j)\times(n-j)(j=1,2,\ldots)$ minors of Hadamard matrices of order n. The authors in [3] give the following open problem.

Conjecture 1. The $n-8$ minors of Hadamard matrics of order $n\geq 8$ can take the possible values

$$k\cdot 2^7\cdot n^{(n/2)-8}, k=1,2,\ldots,32.$$

Based upon the formulae of all possible values of $(n-j)\times(n-j)$ minors of an Hadamard matrices of order n presented in this note, the above conjecture is true.

2 Main results

We write

$$H_n=\begin{bmatrix} M_k & B \\ C & M_{n-k}\end{bmatrix}$$

for the Hadamard matrix of order n, and M_k is the k order leading principal sub-matrix of H_n.

Theorem 2. *Let* $H_n=\begin{bmatrix} M_k & B \\ C & M_{n-k}\end{bmatrix}$ *be an Hadamard matrix of order* n *where* M_k *is the* k *order leading principal sub-matrix of* H_n. *Then*

$$\det(M_{n-k})=n^{\frac{n}{2}-k}\det(M_k)$$

for $1 < k < n$.

Proof. Consider that

$$H_n \cdot (n^{-1} H_n) = I_n.$$

This implies that

$$\begin{bmatrix} M_k & B \\ C & M_{n-k} \end{bmatrix} \cdot \begin{bmatrix} n^{-1}M_k^T & n^{-1}C^T \\ n^{-1}B^T & n^{-1}M_{n-k}^T \end{bmatrix} = \begin{bmatrix} I_k & \\ & I_{n-k} \end{bmatrix}.$$

Thus, $M_k C^T + B M_{n-k}^T = \mathbf{0}$. Now consider that

$$\begin{bmatrix} M_k & B \\ & I_{n-k} \end{bmatrix} \cdot \begin{bmatrix} n^{-1}M_k^T & n^{-1}C^T \\ n^{-1}B^T & n^{-1}M_{n-k}^T \end{bmatrix} = \begin{bmatrix} I_k & \\ n^{-1}B^T & n^{-1}M_{n-k}^T \end{bmatrix}.$$

By taking determinants, we have

$$\det(M_k)\det(n^{-1}H_n) = \det(n^{-1}M_{n-k}).$$

Thus,

$$\det(M_{n-k}) = n^{\frac{n}{2}-k}\det(M_k).$$

$$\det(M_{n-k}) = n^{\frac{n}{2}}\det(\frac{1}{n}M_k).$$

□

The next following result is straightforward from linear algebra.

Lemma 3. *Let* A_k *be an integer* $(-1,1)$*-matrix of order* k. *Then there exists an integer* $(0,1)$*-matrix* $\overline{A}_{k-1}$ *of order* $k-1$

$$\det(A_k) = 2^{k-1}\det(\overline{A}_{k-1}).$$

For a $(0,1)$-matrix of order k, the largest possible determinants β_k for $k = 1, 2, \ldots$. Eric W.Weisstein of Wolfram Research, Inc., computed the sequence $\beta_1 = 1, \beta_2 = 1, \beta_3 = 2, \beta_4 = 3, \beta_5 = 5, \beta_6 = 9, \beta_7 = 32, \beta_8 = 56, \beta_9 = 144, \beta_{10} = 320, \beta_{11} = 1458, \beta_{12} = 3645, \beta_{13} = 9477$, see [4] for more.

The next result follows directly from Lemma 3.

Corollary 4. *Let* $H_n = \begin{bmatrix} M_k & B \\ C & M_{n-k} \end{bmatrix}$ *be an Hadamard matrix of order* n *where* M_k *is the* k *order leading principal sub-matrix of* H_n. *Then*

$$\det(M_{n-k}) = n^{\frac{n}{2}-k}\det(M_k) = n^{\frac{n}{2}-k}2^{k-1}\det(\overline{M}_{k-1})$$

where $\overline{M}_{k-1}$ *is a* $(0,1)$*-matrix of order* $k-1$.

The next result tries to answer Conjecture 1.

Theorem 5. *Let M_{n-k} be an $(n-k)\times(n-k)$ minor of an Hadamard matrix of order n with $n>k>1$. If β_k be the largest possible determinants of a $(0,1)$-matrix of order k, then*

$$\det(M_{n-k}) \in \{m \cdot 2^{k-1} \cdot n^{\frac{n}{2}-k} \,|\, m = 0, 1, 2, \ldots, \beta_{k-1}\}.$$

Proof. It follows directly from Corollary 4 and the definition of β_{k-1} that

$$\det(M_{n-k}) = n^{\frac{n}{2}-k}\det(M_k) = 2^{k-1} \cdot n^{\frac{n}{2}-k} \cdot \det(\overline{M}_{k-1}) \leq 2^{k-1} \cdot n^{\frac{n}{2}-k} \cdot \beta_{k-1}.$$

□

Example 6. Consider $\beta_7 = 32$. we know that the $n-8$ minors of Hadamard matrices of order $n \gg 8$ can take the possible values

$$m \cdot 2^7 \cdot n^{(n/2)-8}, m = 0, 1, 2, \ldots, 32.$$

Therefore, Conjecture 1 in [3] is true. The next table presents all possible $(n-k)\times(n-k)$ minors of an Hadamard matrix of order $n > 13$ for $k = 1, 2, \cdots, 13$.

order	Values of minors
$n-8$	$0, 1 \cdot 2^7 \cdot n^{(n/2)-8}, 2 \cdot 2^7 \cdot n^{(n/2)-8}, \cdots, 32 \cdot 2^7 \cdot n^{(n/2)-8}$
$n-9$	$0, 1 \cdot 2^8 \cdot n^{(n/2)-9}, 2 \cdot 2^8 \cdot n^{(n/2)-9}, \cdots, 56 \cdot 2^8 \cdot n^{(n/2)-9}$
$n-10$	$0, 1 \cdot 2^9 \cdot n^{(n/2)-10}, 2 \cdot 2^9 \cdot n^{(n/2)-10}, \cdots, 144 \cdot 2^9 \cdot n^{(n/2)-10}$
$n-11$	$0, 1 \cdot 2^{10} \cdot n^{(n/2)-11}, 2 \cdot 2^{10} \cdot n^{(n/2)-11}, \cdots, 320 \cdot 2^{10} \cdot n^{(n/2)-11}$
$n-12$	$0, 1 \cdot 2^{11} \cdot n^{(n/2)-12}, 2 \cdot 2^{11} \cdot n^{(n/2)-12}, \cdots, 1458 \cdot 2^{11} \cdot n^{(n/2)-12}$
$n-13$	$0, 1 \cdot 2^{12} \cdot n^{(n/2)-13}, 2 \cdot 2^{12} \cdot n^{(n/2)-13}, \cdots, 9477 \cdot 2^{12} \cdot n^{(n/2)-13}$

3 An application

A *D-optional design of order* n is a $(-1,1)$-matrix having maximum determinant. Throughout this note we write H_n for a Hadmard matrix of order n and D_j for D-optional design of order j. The notation $D_j \in H_n$ means that D_j is embedded in some H_n with $j < n$. The authors in [5] show that every Hadamard matrix of order ≥ 4 contains a sub-matrix equivalent to

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix}.$$

Theorem 7. *If $\frac{\det(D_k)}{\det(D_{n-k})} > n^{\frac{n}{2}-k}$ with $k > \frac{n}{2}$, then $D_k \notin H_n$.*

Proof. If $D_k \in H_n$, with without loss of generality we suppose that

$$H_n = \begin{bmatrix} A_{n-k} & B \\ C & D_k \end{bmatrix},$$

then

$$\det(D_k) = n^{\frac{n}{2}-k}\det(A_{n-k})$$

by Theorem 2. It implies that

$$\det(D_k) \leq n^{\frac{n}{2}-k}\det(D_{n-k}).$$

Thus,

$$\frac{\det(D_k)}{\det(D_{n-k})} \leq n^{\frac{n}{2}-k}.$$

□

Example 8. Consider

$$\begin{aligned} \det(D_5) &= 48,\ \det(D_3) = 4, \\ \det(D_6) &= 160,\ \det(D_2) = 2 \\ \det(D_7) &= 576,\ \det(D_1) = 1. \end{aligned}$$

We have

$$\begin{aligned} \frac{\det(D_5)}{\det(D_3)} = 12 &> 8^{4-3} = 8, \\ \frac{\det(D_6)}{\det(D_2)} = 80 &> 8^{4-2} = 64, \\ \frac{\det(D_7)}{\det(D_1)} = 576 &> 8^{4-1} = 512. \end{aligned}$$

It follows from Theorem 7 that $D_5, D_6, D_7 \notin H_8$ which are presented in Lemmas 1,2 and 3 in [6].

References

[1] F.R. Sharpe, The maximum value of a determinant, Bull. Amer. Math. Soc., 14,121-123,1907.

[2] C. Koukouvinos, M. Mitrouli, and J. Seberry. An algorithm to find formulae and values of minors for Hadamard matrices. *Linear Algebra Appl.,* 330:129-147,2001.

[3] C. Koukouvinos, M. Mitrouli, and J. Seberry. An algorithm to find formulae and values of minors for Hadamard matrices II. *Linear Algebra Appl.,* 371:111-124,2003.

[4] N.J.A. Sloane, The On-Line Encyclopedia of Integer sequences, published electronically at www.research.att.com/ njas/sequences/:1996-2011

[5] Day,J.,Peterson,B.,Growth in Gaussian elimination. Amer. Math. Monthly, 95:489-513,1988.

[6] Jennifer Seberry, Marilena Mitrouli, Some remarks on Hadamard matrices, Cryptogr. Commun., 2: 293-306,2010.