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The growth factor of a Hadamard matrix of order 16 is 16

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SUMMARY

In 1968 Cryer conjectured that the growth factor of an $n\times n$ Hadamard matrix is n. In 1988 Day and Peterson proved this only for the Hadamard–Sylvester class. In 1995 Edelman and Mascarenhas proved that the growth factor of a Hadamard matrix of order 12 is 12. In the present paper we demonstrate the pivot structures of a Hadamard matrix of order 16 and prove for the first time that its growth factor is 16. The study is divided in two parts: we calculate pivots from the beginning and pivots from the end of the pivot pattern. For the first part we develop counting techniques based on symbolic manipulation for specifying the existence or non-existence of specific submatrices inside the first rows of a Hadamard matrix, and so we can calculate values of principal minors. For the second part we exploit sophisticated numerical techniques that facilitate the computations of all possible $(n-j)\times(n-j)$ minors of Hadamard matrices for various values of j. The pivot patterns are obtained by utilizing appropriately the fact that the pivots appearing after the application of Gaussian elimination on a completely pivoted matrix are given as quotients of principal minors of the matrix. Copyright © 2009 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Traditionally, backward error analysis for Gaussian elimination (GE), see e.g. [1, 2], on a matrix $A=(a_{ij}^{(1)})$ is expressed in terms of the *growth factor*

$$g(n,A)=\frac{\max_{i,j,k}|a_{ij}^{(k)}|}{\max_{i,j}|a_{ij}^{(1)}|}$$

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which involves all the elements $a_{ij}^{(k)}$, $k=1,2,\ldots,n$, that occur during the elimination. Matrices with the property that no row and column interchanges are needed during GE with complete pivoting are called *completely pivoted* (CP) or feasible. In other words, at each step of the elimination the element of largest magnitude (the 'pivot') is located at the top left position of every submatrix, as it appears during the process. For a CP matrix A we have

$$g(n,A)=\frac{\max\{p_1,p_2,\ldots,p_n\}}{|a_{11}^{(1)}|}$$

where $p_1,p_2,\ldots,p_n$ are the pivots of A.

A *Hadamard matrix* H of order n is a matrix with elements ± 1 satisfying the orthogonality relation

$$HH^{\mathrm{T}}=H^{\mathrm{T}}H=nI_n$$

From this definition it clearly follows that way that every two distinct rows or columns of a Hadamard matrix are orthogonal, i.e. their inner product is zero. It can be proved [3] that if H is a Hadamard matrix of order n then $n=1,2$ or $n\equiv 0 \pmod 4$. However it is still an open conjecture whether Hadamard matrices exist for every n being a multiple of 4. For further information on Hadamard matrices, the reader can consult [4–7].

Two matrices are said to be *Hadamard equivalent* or *H-equivalent* if one can be obtained from the other by a sequence of the operations:

1. interchange any pair of rows and/or columns;
2. multiply any row and/or column through by -1.

In [8] Tornheim proved that $g(n,H)\geqslant n$ for a CP $n\times n$ Hadamard matrix H. In [9] Cryer conjectured that '$g(n,A)\leqslant n$, with equality iff A is a Hadamard matrix'. Day and Peterson [10] proved the equality only for the Hadamard–Sylvester class. They provided some experimental results for pivot patterns of 16×16 Hadamard matrices and conjectured that the fourth pivot from the end is $n/4$, which was shown to be false in [11] with an appropriate counterexample. In 1991 Gould discovered a 13×13 matrix for which the growth factor is 13.0205 [12], [2, p. 170]. Thus the first part of Cryer's conjecture was shown to be false. The second part of the conjecture concerning the growth factor of Hadamard matrices still remains open. In [13] Edelman and Mascarenhas proved that the growth factor of a Hadamard matrix of order 12 is 12 by demonstrating its unique pivot pattern.

As it is already stated in [13], it appears very difficult to prove that $g(16,H_{16})=16$. In this work we take a small step toward proving the Hadamard part of Cryer's conjecture by demonstrating all 34 possible pivot patterns for a Hadamard matrix of order 16 (denoted by H_{16}). A major difficulty arises in the study of this problem because H-equivalence operations do not preserve pivots, i.e. the pivot pattern is not invariant under H-equivalence, and many pivot patterns can be observed. So, H-equivalent matrices do not necessarily have the same pivot pattern. Although Hadamard matrix problems might sound seemingly easy, they are non-trivial, because, for example, for the case of proving the pivot structures of H_{16}, a naive computer exhaustive search performing all possible H-equivalence operations would require $(16!)^2(2^{16})^2\approx 10^{36}$ trials. Additionally, the pivot pattern of each one of these matrices should be computed with application of GE. Furthermore, the existence of relatively small Hadamard matrices ($n=668$) is still not known [6].

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It is known [14] that if GE with complete pivoting is applied to Hadamard matrices of order 16, over 30 different pivot patterns are attained, by contrast with Hadamard matrices of order 12, which yield only one pivot pattern [13]. Equivalent Hadamard matrices of order 16 can be classified with respect to the H-equivalence into five classes of equivalence I, ..., V, see [7, 15]. Classes IV and V are one another's transpose, and so they are identical for GE with complete pivoting, since a matrix is CP iff its transpose is CP, in which case both give the same pivot pattern, see [10]. Extensive experiments revealed 34 possible pivot patterns for Hadamard matrices of order 16, though not all patterns appeared for each equivalence class. The numerical experiments were designed as follows. 200 000 H-equivalent Hadamard matrices of order 16 have been randomly generated for each equivalence class, i.e. starting from one representative matrix of each of the five equivalence classes, random H-equivalent matrices to it were created by performing arbitrary sequences of H-equivalence operations and finally GE with complete pivoting was applied to them.

The challenge of our work lies in proving theoretically that every possible H_{16} can have one of these 34 pivot patterns only, in the sense of H-equivalence, which actually means to show that an *arbitrary* matrix H_{16} being H-equivalent to a representative of the five equivalence classes will have definitely one of these 34 pivot patterns. Then, since the maximum pivot appearing is 16, we are able to state that the growth factor for H_{16} is equal to 16 and the growth conjecture for H_{16} is now proved.

The paper is organized as follows. In Section 2 we present the algorithm Exist, which is used for proving the existence or non-existence of some matrices within the first few rows of a Hadamard matrix. This idea leads to calculating pivots from the beginning of the pivot structure of H_{16}. In Section 3 we describe the algorithm Minors, which is developed for calculating all possible $(n-j)\times(n-j)$ minors of Hadamard matrices, and furthermore it is useful for finding pivots just before the last. The ninth pivot is proved separately, and after the pivot patterns are specified, the growth factor is given. Finally, Section 4 outlines the results of this work and highlights further possibilities. The appendices at the end aim at offering a better insight and understanding on the algorithms and ideas used in this work.

Notations. Throughout this paper the elements of a $(1,-1)$ matrix will be denoted by $(+,-)$. I_n and J_n stand for the identity matrix of order n and the matrix with ones of order n, respectively, and if the indices are omitted, the general form of these matrices is meant. We assume, without loss of generality, that the first entry of a row and a column of a Hadamard matrix is always $+1$ (*normalized* form of a Hadamard matrix), because this can be achieved with the H-equivalence operation of multiplying by -1 and leaves unaffected the basic properties of the initial matrix. Wherever 'determinant' or 'minor' is mentioned in this work, we mean its magnitude, i.e. the absolute value. We write $A(j)$ for the absolute value of the determinant of the $j\times j$ principal submatrix in the upper left corner of a matrix A. An $m\times n$ matrix having all its entries equal to $x\in\mathbb{R}$ will be denoted by $x_{m\times n}$. The notation $(\kappa-\lambda)I+\lambda J$ for describing briefly a matrix of the form

$$\begin{bmatrix} \kappa & \lambda & \dots & \lambda \\ \lambda & \kappa & \dots & \lambda \\ \vdots & & \ddots & \\ \lambda & \lambda & \dots & \kappa \end{bmatrix}$$

will be frequently used. If it is necessary to specify the order n of such a matrix $X=(\kappa-\lambda)I+\lambda J$, then it will be denoted by $X_{n\times n}\equiv(\kappa-\lambda)I_n+\lambda J_n$. So, for instance, $X_{2\times 2}=\left[\begin{smallmatrix}\kappa & \lambda\\ \lambda & \kappa\end{smallmatrix}\right]$.

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Let $\underline{y}_{\beta+1}^{\mathrm{T}}$ be the vectors containing the binary representation of each integer $\beta+2^{j-1}$ for $\beta=0,\dots,2^{j-1}-1$. Replace all zero entries of $\underline{y}_{\beta+1}^{\mathrm{T}}$ by -1 and define the $j\times 1$ vectors $\underline{u}_k=\underline{y}_{2^{j-1}-k+1}$, $k=1,\dots,2^{j-1}$. We write U_j for all the matrices with j rows and the appropriate number of columns, in which $\underline{u}_k$ occurs u_k times. In other words, U_j is the matrix containing all possible 2^{j-1} columns of size j with elements ± 1 starting with $+1$. So,

$$U_j=\begin{array}{ccccc} \overbrace{+\cdots+}^{u_1} & \overbrace{+\cdots+}^{u_2} & \cdots & \overbrace{+\cdots+}^{u_{2^{j-1}-1}} & \overbrace{+\cdots+}^{u_{2^{j-1}}} \\ +\cdots+ & +\cdots+ & \cdots & -\cdots- & -\cdots- \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ +\cdots+ & +\cdots+ & \cdots & -\cdots- & -\cdots- \\ +\cdots+ & -\cdots- & \cdots & +\cdots+ & -\cdots- \end{array} = \begin{array}{ccccc} u_1 & u_2 & \dots & u_{2^{j-1}-1} & u_{2^{j-1}} \\ + & + & \cdots & + & + \\ + & + & \cdots & - & - \\ \vdots & \vdots & & \vdots & \vdots \\ + & + & \cdots & - & - \\ + & - & \cdots & + & - \end{array}$$

Example 1

$$U_3=\begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \\ + & + & + & + \\ + & + & - & - \\ + & - & + & - \end{array}, \quad U_4=\begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\ + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - \end{array}$$

The matrix U_j is important in this study because it depicts a general form for the first j rows of a normalized Hadamard matrix.

Preliminary results. The following lemma, which gives a useful relation between pivots and minors and a characteristic property for CP matrices, is essential for the ideas developed in this work.

Lemma 1 (*Cryer* [*9*], *Grantmacher* [*16, p.26*], *Kravvaritis et al.* [*17*])
Let A be a CP matrix.

(i) The magnitude of the pivots appearing after application of GE operations on A is given by

$$p_j=\frac{A(j)}{A(j-1)},\quad j=1,2,\dots,n,\quad A(0)=1 \tag{1}$$

(ii) The maximum $j\times j$ leading principal minor of A, when the first $j-1$ rows and columns are fixed, is $A(j)$.

Therefore it is clear that the calculation of minors is important in order to study pivot structures, and moreover the growth problem for CP Hadamard matrices.

Remark 1
In this work we deal only with CP Hadamard matrices of order 16. Hence, we can exploit (1) for calculating pivots. However it is important to emphasize that the results are valid for every H_{16} if

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GE with complete pivoting is applied. Indeed, if a matrix isn't initially CP, the row and column operations of GE with complete pivoting bring it always in CP form, hence we can deal, without loss of generality, only with CP H_{16} matrices. In other words, it is totally equivalent to apply GE with complete pivoting to a matrix and to apply GE operations on a CP matrix.

Remark 2
The second part of Lemma 1 assures that the maximum $j\times j$ minor appears in the upper left $j\times j$ corner of a CP matrix, when its upper left $(j-1)\times(j-1)$ corner is fixed. So, if the existence of a matrix with maximal determinant is proved for a CP H_{16}, we can indeed assume that it always appears in the upper left corner. It is important to stress that the maximum minor of a CP matrix A of order j appears as $A(j)$, i.e. in the upper left $j\times j$ corner of A, only if the first $j-1$ rows and columns of A are fixed. This is not necessarily the maximum $j\times j$ minor of A, but only the maximum $j\times j$ minor of A when its first $j-1$ rows and columns are fixed.

We give the following Lemma 2, which specifies the possible number of columns for the first j rows of a Hadamard matrix. This result is used in 2.1 and 3.1 in order to establish bounds for the parameters in the solutions of the systems, which represent columns of H_{16}. It is useful for obtaining constraints on the number of columns of a Hadamard matrix, and moreover for limiting the calculations in the algorithms. The proof of Lemma 2 is given in Appendix A.1.

Lemma 2
For the first j rows, $j\geqslant 3$, of a normalized Hadamard matrix H of order n, $n>3$, and for all the 2^{j-1} possible columns $\underline{u}_1,\ldots,\underline{u}_{2^{j-1}}$ of U_j, we have

$$0\leqslant u_i\leqslant \frac{n}{4} \quad \text{for } i=1,\ldots,2^{j-1}$$

We provide also some useful formulas for two matrices with special structure.

Lemma 3
Let $A=(k-\lambda)I_v+\lambda J_v$, where k,λ are integers. Then,

$$\det A=[k+(v-1)\lambda](k-\lambda)^{v-1} \tag{2}$$

and for $k\neq\lambda, -(v-1)\lambda$, A is non-singular with

$$A^{-1}=\frac{1}{k^2+(v-2)k\lambda-(v-1)\lambda^2}\{[k+(v-1)\lambda]I_v-\lambda J_v\} \tag{3}$$

Equation (2) can be proved straightforwardly by appropriate row operations. Equation (3) is a special case of the Sherman–Morrison formula [1, p. 239], which computes the inverse of a rank-one-correction of a non-singular matrix B as

$$(B-uv^{\mathrm{T}})^{-1}=B^{-1}+\frac{B^{-1}uv^{\mathrm{T}}B^{-1}}{1-v^{\mathrm{T}}B^{-1}u}$$

where u, v are vectors and $v^{\mathrm{T}}B^{-1}u\neq 1$. Indeed, (3) occurs for $B=(\kappa-\lambda)I_v$ and $u=-\lambda[1\ 1\ \ldots\ 1]^{\mathrm{T}}$ and $v=[1\ 1\ \ldots\ 1]^{\mathrm{T}}$.

Lemma 4 (*Schur determinant formula* [*18, p. 21*])
Let $B=\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$, B_1 non-singular. Then

$$\det B=\det B_1\cdot\det(B_4-B_3B_1^{-1}B_2) \tag{4}$$

2. PIVOTS FROM THE BEGINNING

In order to apply formula (1) for evaluating the pivots, we must determine the values of the principal minors $H_{16}(j)$ of a CP H_{16}. This can be achieved if for various values of j we specify all possible $j\times j$ matrices that can appear as upper left corner of the H_{16} and then directly compute the value of their determinant.

2.1. *An algorithm specifying the existence of submatrices in Hadamard matrices*

The following algorithm specifies whether a set of $k\times k$ matrices A_i always exists or not among all possible columns that can appear in the first k rows of a Hadamard matrix of order n, or equivalently in its upper left corner (this can be done with column interchanges leading to H-equivalent matrices). The main idea of the algorithm is to form the linear system that results (i) from summing all possible columns of length k with elements ± 1 starting with $+1$ (namely, the total number of columns of the matrix U_k must be equal to the order n) and (ii) from the property that every two distinct rows of a Hadamard matrix are mutually orthogonal. The system is solved (the parameters, if there are any, are bounded with the help of Lemma 2) and it is determined whether the columns, which constitute the matrices under investigation A_i, always appear among the solutions. If they appear, we state that the matrices A_i always exist within the first k rows of a Hadamard matrix of order n, otherwise not.

Algorithm Exist

Step 1: **Read** the $k\times k$ matrices A_i

Step 2: **Create** the matrix U_k

$$U_k=\begin{array}{ccccc} u_1 & u_2 & \cdots & u_{2^{k-1}-1} & u_{2^{k-1}} \\ + & + & \cdots & + & + \\ + & + & \cdots & - & - \\ \vdots & \vdots & & \vdots & \vdots \\ + & + & \cdots & - & - \\ + & - & \cdots & + & - \end{array}$$

Step 3:

Procedure Solve

Form the system of $1+\binom{k}{2}$ equations and 2^{k-1} variables, which results from counting the columns and the orthogonality of every two distinct rows of U_k

Solve the system for all u_i, $i=1,\ldots,2^{k-1}$

For $j=1,\ldots,2^{k-1}-1-\binom{k}{2}$

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For all the parameters $u_{p_j}=0,\ldots,n/4$
  If $u_1\geqslant 1$ and $u_i\geqslant 0$, $i=2,\ldots,2^{k-1}$ and u_i integers, $i=1,\ldots,2^{k-1}$
    If all columns of some A_i appear in every solution
      (the corresponding u_i are all $\geqslant 1$)
      the matrices A_i can always exist among all possible columns that can appear in the first k rows of a Hadamard matrix of order n
    End If
  Else there are no acceptable solutions
  End If
End
End
End{of Procedure Solve}
End{of Algorithm}

An implementation of the algorithm can be found in Appendix A.2. It is important to stress that algorithm Exist is used for three purposes in this work. First, it is used for proving the existence or non-existence of a set of CP $j\times j$ matrices among the first j rows of H_{16}, so that the $H_{16}(j)$ minors can be computed and relation (1) can be exploited for calculating pivots from the beginning. Furthermore, algorithm Exist is used for specifying the existence of $j\times j$ matrices among j rows of H_{16} (independently of the CP property), which will be used as input matrices M in algorithm Minors, developed later in Section 3, and finally, algorithm Exist is used for rejecting the occurrence of matrices within H_{16}, which have embedded inside them submatrices with specific determinants. This idea can be understood better in view of the argument for rejecting quotients in the proof of Proposition 2.

2.2. The first eight pivots

We aim at deriving the first eight pivots by specifying the possible matrices that always exist in an H_{16}. The matrices appearing occur as extensions of matrices of smaller orders that are proved to always exist. The idea with the extensions, which is used repeatedly in the proof of Proposition 1, is presented for first time in [19] and is described there in detail for skew and symmetric conference matrices. From all possible extended matrices, only the CP matrices come into question because it is known that the upper left principal submatrix of a CP matrix is always CP.

Proposition 1
If GE with complete pivoting is performed on an H_{16} the first eight pivots are

Pivot	Values
p_1	1
p_2	2
p_3	2
p_4	4
p_5	2,3
p_6	$4, \frac{8}{3}, \frac{10}{3}$
p_7	$2, 4, \frac{8}{10/3}, \frac{16}{5}, \frac{18}{5}$
p_8	$4, \frac{9}{2}, 5, 6, 8$

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Proof
It is already proved that the first four pivots of every Hadamard matrix are 1, 2, 2 and 4 [10]. In Appendix A.2 we see that

$$A=\begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

is the only CP matrix, which can always exist among all possible columns that can appear in the first four rows. A attains the maximum determinant for ± 1 matrices of order 4, which is 16. Hence, from Lemma 1(ii) we have

$$H_{16}(4)=\det A=16$$

From now on, the following procedure will be adopted throughout the rest of the proof in order to obtain the probable existing $j\times j$ submatrices inside an H_{16} occurring as extensions from $(j-1)\times(j-1)$ matrices that are already proved to exist always inside H_{16}.

Extension procedure

1. Extension of the $(j-1)\times(j-1)$ existing matrices to all possible normalized $\pm 1\, j\times j$ matrices;
2. From all possible extensions only the CP ones are kept;
3. The CP extensions are separated into classes according to the values of determinants appearing;
4. The set $\mathscr{M}_j$ with the matrices that can always exist among the first j rows of H_{16} is kept. In order to create $\mathscr{M}_j$, we examine the extensions with the maximum determinant. If they always exist, we denote them as $\mathscr{M}_j$; otherwise, we proceed with the second maximum determinant value appearing. If the matrices with the two largest determinant values always exist, we denote them as $\mathscr{M}_j$; otherwise, we proceed with the third maximum value appearing, and so on.

According to this procedure, we extend the matrix A to all possible 5×5 matrices of the form

$$\begin{bmatrix} + & + & + & + & + \\ + & - & + & - & * \\ + & + & - & - & * \\ + & - & - & + & * \\ + & * & * & * & * \end{bmatrix}$$

where the elements $*$ can be ± 1. From these $2^7=128$ possible 5×5 matrices, only the 52 CP matrices with determinants 48 and 32 given in Appendix A.2, denoted by $\mathscr{M}_5$, and tested with algorithm Exist can always exist among all possible columns that can appear in the first five

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rows, and furthermore in the upper left corner of a CP H_{16}. Since the 52 matrices $\mathscr{M}_5$ with determinants 48 and 32 always exist among the rows of H_{16}, from Lemma 1(ii) we derive that

$$H_{16}(5) = 32 \text{ or } 48$$

Applying formula (1) yields

$$p_5 = \frac{H_{16}(5)}{H_{16}(4)} = \frac{32}{16} \text{ or } \frac{48}{16} = 2 \text{ or } 3$$

In the next step we extended the 52 matrices $\mathscr{M}_5$ to all possible 6×6 matrices of the form

$$\begin{bmatrix} & & & & & + \\ & & & & & * \\ & & B & & & * \\ & & & & & * \\ & & & & & * \\ + & * & * & * & * & * \end{bmatrix}$$

where the elements $*$ can be ± 1 and B is a matrix from the set $\mathscr{M}_5$. For each of the 52 matrices B we obtain $2^9 = 512$ possible extensions. Again, by using algorithm Exist with a procedure similar to Appendix A.2, but for $k = 6$ and with input all the resulting $52\times 512 = 26\,624$ matrices this time, we conclude that only the 836 CP matrices of order 6 of the set $\mathscr{M}_6$ can always exist in the upper left corner of a CP H_{16}, where

$$\begin{aligned} \mathscr{M}_6 = \{ & 672 \text{ matrices with det} = 128 \text{ obtained from matrices of order 5 with det} = 32, \\ & 124 \text{ matrices with det} = 128 \text{ obtained from matrices of order 5 with det} = 48, \\ & 40 \text{ matrices with det} = 160 \text{ obtained from matrices of order 5 with det} = 48\} \end{aligned}$$

These results are presented briefly in Table I. So, from Lemma 1(ii) we have

$$H_{16}(6) = 128 \text{ or } 160$$

and from (1) we get

$$p_6 = \frac{H_{16}(6)}{H_{16}(5)} = \frac{128}{32} \text{ or } \frac{128}{48} \text{ or } \frac{160}{48} = 4 \text{ or } \frac{8}{3} \text{ or } \frac{10}{3}$$

Table I. Determinants of the possible existing 6×6 matrices (d_6) inside an H_{16} and of the corresponding embedded 5×5 submatrices (d_5).

d_6	d_5
128	32
128	48
160	48

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In the same manner we proceed with the calculation of the seventh pivot, which will be obviously more strenuous from a computational point of view.

We extended the 836 matrices of $\mathscr{M}_6$ to all possible 7×7 matrices of the form

$$\begin{bmatrix} & & & & & & + \\ & & & & & & * \\ & & & & & & * \\ & & & C & & & * \\ & & & & & & * \\ & & & & & & * \\ + & * & * & * & * & * & * \end{bmatrix}$$

where the elements $*$ can be ± 1 and C is a matrix from the set $\mathscr{M}_6$. For each of the 836 matrices C we obtain $2^{11}=2048$ possible extensions. As before, by using algorithm Exist with a procedure similar to Appendix A.2, but for $k=7$ and with input all the resulting $836\times 2048 = 1\,712\,128$ matrices this time, we conclude that only the 40 993 CP matrices of order 7 of the set $\mathscr{M}_7$ can exist in the upper left corner of a CP H_{16}, where

$$\begin{aligned}\mathscr{M}_7 = \{&9140 \text{ matrices with det} = 256 \text{ obtained from matrices of order 6 with det} = 128,\\ &10\,586 \text{ matrices with det} = 512 \text{ obtained from matrices of order 6 with det} = 128,\\ &7294 \text{ matrices with det} = 384 \text{ obtained from matrices of order 6 with det} = 160,\\ &8916 \text{ matrices with det} = 512 \text{ obtained from matrices of order 6 with det} = 160,\\ &4683 \text{ matrices with det} = 576 \text{ obtained from matrices of order 6 with det} = 160\}\end{aligned}$$

The results are summarized in Table II. Hence, Lemma 1(ii) implies

$$H_{16}(7) = 256 \text{ or } 384 \text{ or } 512 \text{ or } 576$$

and from (1) we get

$$\begin{aligned} p_7 = \frac{H_{16}(7)}{H_{16}(6)} &= \frac{256}{128} \text{ or } \frac{512}{128} \text{ or } \frac{384}{160} \text{ or } \frac{512}{160} \text{ or } \frac{576}{160}\\ &= 2 \text{ or } 4 \text{ or } \frac{8}{10/3} \text{ or } \frac{16}{5} \text{ or } \frac{18}{5}\end{aligned}$$

Table II. Determinants of the possible existing 7×7 matrices (d_7) inside an H_{16} and of the corresponding embedded 6×6 submatrices (d_6).

d_7	d_6
256	128
512	128
384	160
512	160
576	160

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Table III. Determinants of the possible existing 8×8 matrices (d_8) inside an H_{16} and of the corresponding embedded 7×7 submatrices (d_7).

d_8	d_7
1024	256
1536	256
2048	256
2048	512
2304	512
2560	512
3072	512
4096	512
1536	384
2304	384
3072	384
2304	576

Following the familiar extension procedure, the set $\mathscr{M}_8$ is created similarly (consisting of 10 074 231 CP matrices out of $40993\times 2^{13}=335814656$ possible extensions of the 7×7 matrices), and the results are summarized in Table III. We conclude that

$$H_{16}(8)=1024 \text{ or } 1536 \text{ or } 2048 \text{ or } 2304 \text{ or } 2560 \text{ or } 3072 \text{ or } 4096$$

As before, by considering the appropriate quotients, we obtain

$$p_8=4 \text{ or } \tfrac{9}{2} \text{ or } 5 \text{ or } 6 \text{ or } 8$$

□

Remark 3

We explained previously that the pivot pattern is not an invariant of the equivalence class; in other words, it is possible that H-equivalent matrices can yield different pivot patterns. If the pivot pattern was invariant under H-equivalence, then we could work only with one matrix representing H-equivalent matrices of the same order and determinant and we wouldn't have to examine all of them. If we used only one matrix representing H-equivalent matrices of same order and determinant, the set $\mathscr{M}_5$ would have 19 matrices, $\mathscr{M}_6$ would have 333 matrices, etc. and the searches needed would be significantly less, but in this manner we would obtain insufficient results, as we wouldn't be able to prove the existence of pivot patterns that already appeared in the experiments. Thus, it is absolutely necessary to deal with all the matrices given in the proof of Proposition 1.

3. PIVOTS FROM THE END

In this section we give an algorithm for calculating all possible $(n-j)\times(n-j)$ minors of Hadamard matrices of order n for various values of j. The results of the algorithm will be substituted appropriately in relationship (1) and give the last eight pivots.

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3.1. An algorithm computing $(n-j)\times(n-j)$ minors of Hadamard matrices

In general it is difficult to obtain analytical formulas for minors of various orders for a given arbitrary matrix. For Hadamard matrices such computations are possible due to their specific properties. The first known effort for calculating minors of Hadamard matrices was accomplished in [20] for the $n-1$, $n-2$ and $n-3$ minors in a totally different manner than the one presented here. In [21], a method for evaluating all possible $(n-j)\times(n-j)$ minors of Hadamard matrices was developed theoretically, which could be generalized as an algorithm. In the present paper this technique was appropriately modified in order to work more effectively and to deal better with the particular problem. For the sake of better understanding we present the proposed strategy not as a pseudocode, but in a theoretical descriptive context as follows. V_j stands for all possible columns of length j with entries ± 1, like U_j, but we choose another letter to show that the matrices U_j and V_j are not necessarily the same.

Algorithm Minors

Step 1. In order to calculate all possible $(n-j)\times(n-j)$ minors of a Hadamard matrix H of order n, we consider it in the form

$$\begin{bmatrix} M & U_j \\ V_j & D \end{bmatrix}$$

All possible matrices M are specified with algorithm Exist. Our intention is to write down for every possible $j\times j$ corner M the values appearing for the determinant of D, which is the required minor.

Step 2. For every M we solve a system of $1+\binom{j}{2}$ equations with 2^{j-1} unknowns the numbers of columns of U_j, resulting from the order of H and from the orthogonality of its first j rows.

Step 3. If $j>3$, there exist parameters in the solution. So, we use as upper bound for them the value $n/4$ (given by Lemma 1), which gives all possible values for the parameters, and lets them attain all the values $0,\ldots,n/4$.

Step 4. In the following, for all acceptable solutions (i.e. $u_1\geqslant 1$, $u_i\geqslant 0$, $i=2,\ldots,2^{j-1}$, and u_i integers, $i=1,\ldots,2^{j-1}$) we calculate $D^{\mathrm{T}}D$ taking into account $H^{\mathrm{T}}H=nI_n$ and write the result as

$$D^{\mathrm{T}}D\equiv\begin{bmatrix} E_1 & F_1 \\ F_1^{\mathrm{T}} & G_1 \end{bmatrix}$$

where $E_1=nI_{u_1}-jJ_{u_1}$. Owing to the clustering of same columns in U_j, $D^{\mathrm{T}}D$ and all the intermediate matrices $G_k-F_k^{\mathrm{T}}E_k^{-1}F_k$, $k=1,\ldots,2^{j-1}-1$ will appear in block form with diagonal blocks of known orders $u_i\times u_i$, $i=k+1,\ldots,2^{j-1}$, and due to the orthogonality of H, all blocks will be of the form $(a-b)I+bJ$, for various a,b. If some $u_i=0$, the respective matrices vanish, see also Appendix A.3, case $j=5$. We compute initially

$$G_1-F_1^{\mathrm{T}}E_1^{-1}F_1\equiv\begin{bmatrix} E_2 & F_2 \\ F_2^{\mathrm{T}} & G_2 \end{bmatrix}$$

where E_2 is of order $u_2\times u_2$. Then, according to (4), $\det D^{\mathrm{T}}D=\det E_1\cdot\det(G_1-F_1^{\mathrm{T}}E_1^{-1}F_1)=\det E_1\cdot\det E_2\cdot\det(G_2-F_2^{\mathrm{T}}E_2^{-1}F_2)$. From (2) we calculate $\det E_1$ and $\det E_2$ and we proceed with

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calculating $G_2 - F_2^{\mathrm{T}} E_2^{-1} F_2$ with the help of (3). So we aim at deriving $\det D^{\mathrm{T}} D$ by consecutive applications of formula (4), with the help of (2) and (3).

Step 5. For $k = 2, \ldots, 2^{j-1} - 1$ we compute the sequence of matrices

$$G_k - F_k^{\mathrm{T}} E_k^{-1} F_k \equiv \begin{bmatrix} E_{k+1} & F_{k+1} \\ F_{k+1}^{\mathrm{T}} & G_{k+1} \end{bmatrix}$$

where each E_k is of order $u_k \times u_k$. The determinants of E_k are stored. The last matrix of the sequence

$$G_{2^{j-1}-1} - F_{2^{j-1}-1}^{\mathrm{T}} E_{2^{j-1}-1}^{-1} F_{2^{j-1}-1} = E_{2^{j-1}}$$

consists of one block of dimension $u_{2^{j-1}} \times u_{2^{j-1}}$ and its determinant is evaluated directly with (2).

Step 6. The required absolute value of the determinant is computed from the formulas

$$\det D^{\mathrm{T}} D := \prod_{i=1}^{2^{j-1}} \det E_i, \qquad |\det D| = \sqrt{\det D^{\mathrm{T}} D}$$

Remarks on the algorithm. In order to standardize a technique for calculating all possible $(n-j) \times (n-j)$ minors of the H_{16}, $j = 1, \ldots, 7$, and to facilitate the computations we assume, without loss of generality, a pattern for the first j rows, in which the same columns are clustered in U_j. This assumption is done indeed without loss of generality because, if the first j rows don't appear in the suggested form, we can make this form appear by interchanging appropriately columns and by multiplying columns by -1. Then, for every possible upper left $j \times j$ corner, we calculate the determinant of the lower right $(n-j) \times (n-j)$ submatrix. The fact that we examine *all possible upper left* $j \times j$ *corners* guarantees that with this technique we calculate *every possible* $(n-j) \times (n-j)$ *minors of* H_{16} and that we do not miss out any values that appear.

Theoretically, the proposed algorithm can work for every value of n and j. Algorithm Minors is implemented symbolically and this guarantees its precision. We note that the algorithm occurred as an effort to standardize the algebraic calculations done initially by hand (with the help of (2), (4) and (3)) for computing $(n-j) \times (n-j)$ minors of Hadamard matrices, as it was easy to observe that they follow a predictable, standard procedure, which seemed challenging to develop from an algorithmic point of view. The algorithm is designed in such a way that the special structure and properties of every matrix appearing are taken into account. So, all necessary matrix multiplications and inversions and determinant evaluations are not performed explicitly but in an efficient manner and the total computational cost remains at relatively low levels. More information on the algorithm's theoretical background, implementation on the computer and complexity properties is available from the authors on request. Appendix A.3 is intended to throw more light on all the aspects discussed of the algorithm Minors.

For the needs of this study, we applied algorithm Minors for $n = 16$ and for $j = 1, \ldots, 7$. The results obtained from the numerical experiments performed with algorithm Minors are subject to the formulas presented in Table IV.

3.2. The last eight pivots

The last seven pivots of the H_{16} will be calculated with careful combination of the results of algorithm Minors and Equation (1), after the non-appearing quotients have been rejected with the

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Table IV. Values of minors of orders $n-1,\dots,n-7$ for Hadamard matrices of order $n=16$.

Order	Values of minors
$n-1$	$n^{n/2-1}$
$n-2$	$0,\ 2n^{n/2-2}$
$n-3$	$0,\ 4n^{n/2-3}$
$n-4$	$0,\ 8n^{n/2-4},\ 16n^{n/2-4}$
$n-5$	$0,\ 16n^{n/2-5},\ 32n^{n/2-5},\ 48n^{n/2-5}$
$n-6$	$0,\ 32n^{n/2-6},\ 64n^{n/2-6},\ 96n^{n/2-6},\ 128n^{n/2-6},\ 160n^{n/2-6}$
$n-7$	$0,\ 64n^{n/2-7},\ 128n^{n/2-7},\ 192n^{n/2-7},\ 256n^{n/2-7},\ 320n^{n/2-7}$ $384n^{n/2-7},\ 448n^{n/2-7},\ 512n^{n/2-7},\ 576n^{n/2-7}$

help of the algorithm Exist. The ninth pivot will be computed separately by the property that the product of the pivots is equal to the determinant of the initial matrix.

Proposition 2
If GE with complete pivoting is performed on an H_{16}, the last seven pivots are

Pivot	Values
p_{10}	$4,\ \frac{16}{8/5}, 5, 8$
p_{11}	$4,\ \frac{16}{10/3}, 6, 8$
p_{12}	$8,\ \frac{16}{3}$
p_{13}	4, 8
p_{14}	8
p_{15}	8
p_{16}	16

Proof
It is known [22] that when GE is applied on a CP Hadamard matrix of order n, the last four pivots in backward order are n, $n/2$, $n/2$, $n/4$ or $n/2$. Particularly, for $n=16$ we have the values $p_{16}=16$, $p_{15}=8$, $p_{14}=8$ and $p_{13}=4$ or 8.

With application of the algorithm Minors, we succeeded in computing the $(16-j)\times(16-j)$ minors, $j=4,\dots,7$, of the H_{16}, which are already presented in Table IV. If we substitute these results appropriately in formula (1), we obtain the values of the pivots p_{12}, p_{11} and p_{10}. Thus, we have

$$p_{12}=\frac{H_{16}(12)}{H_{16}(11)}=\frac{16\cdot 16^4}{32\cdot 16^3} \text{ or } \frac{16\cdot 16^4}{48\cdot 16^3} \text{ or } \frac{8\cdot 16^4}{16\cdot 16^3} \text{ or } \frac{16\cdot 16^4}{16\cdot 16^3} \text{ or } \frac{8\cdot 16^4}{32\cdot 16^3} \text{ or } \frac{8\cdot 16^4}{48\cdot 16^3}$$

The three last quotients will be excluded using the following argument. First, we observe that, for instance, for the fifth quotient, the value of the $n-4$ minor $8\cdot 16^4$ appears if we use a 4×4 matrix M with determinant 8 in the algorithm Minors and the value $32\cdot 16^3$ for the $n-5$ minor appears if we use a 5×5 M with determinant 32. With this logic we create the set $\mathscr{M}'_5$ of 5×5

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matrices, where we note

$$\begin{aligned}\mathscr{M}'_5 = \{&1777 \text{ matrices with det}=16 \text{ obtained from matrices of order 4 with det}=16,\\ &256 \text{ matrices with det}=32 \text{ obtained from matrices of order 4 with det}=8,\\ &16 \text{ matrices with det}=48 \text{ obtained from matrices of order 4 with det}=8\}\end{aligned}$$

We find with the algorithm Exist that the matrices of $\mathscr{M}'_5$ *do not* always exist in an H_{16} and we can reject the quotients $(16\cdot 16^4)/(16\cdot 16^3)$, $(8\cdot 16^4)/(32\cdot 16^3)$ and $(8\cdot 16^4)/(48\cdot 16^3)$. So,

$$p_{12}=8 \text{ or } \tfrac{16}{3}$$

After rejecting the non-feasible quotients for p_{11} with the use of the same argument, we have

$$p_{11}=\frac{H_{16}(11)}{H_{16}(10)}=\frac{16\cdot 16^3}{64\cdot 16^2} \text{ or } \frac{32\cdot 16^3}{128\cdot 16^2} \text{ or } \frac{48\cdot 16^3}{128\cdot 16^2} \text{ or } \frac{48\cdot 16^3}{160\cdot 16^2}=4 \text{ or } 6 \text{ or } \frac{16}{10/3}$$

According to the above-described argument for excluding quotients for p_{12} and p_{11}, we obtain

$$\begin{aligned}p_{10}&=\frac{H_{16}(10)}{H_{16}(9)}=\frac{64\cdot 16^2}{128\cdot 16} \text{ or } \frac{64\cdot 16^2}{256\cdot 16} \text{ or } \frac{128\cdot 16^2}{256\cdot 16} \text{ or } \frac{128\cdot 16^2}{512\cdot 16} \text{ or } \frac{160\cdot 16^2}{512\cdot 16} \text{ or } \frac{160\cdot 16^2}{576\cdot 16}\\ &=4 \text{ or } \frac{16}{18/5} \text{ or } 5 \text{ or } 8\end{aligned}$$

So, we have proved all the values appearing of the pivots of the H_{16}, except for the ninth pivot. It remains to calculate this value and to show how the concrete pivot patterns of the H_{16} are obtained. For this purpose, we need to observe the origin of each of the minor values that appear.

Proposition 3
If GE with complete pivoting is performed on an H_{16}, the ninth pivot can have the values 2, $\frac{8}{3}$, $\frac{16}{5}$, 4, $\frac{9}{2}$ or $\frac{16}{3}$.

Proof
We show how one value is obtained, since the rest can be handled similarly. For this purpose we need to derive first a particular pivot structure. The first four pivots are always 1, 2, 2 and 4. For the fifth pivot there were proved two values, 2 and 3. We consider one of them as the fifth pivot for this example, e.g. $p_5=2$. In this case, it can be stated that the 5×5 matrix, which gave this value, has determinant equal to 32, as it can be seen in the procedure for calculating p_5 in the proof of Proposition 1. In a similar manner we observe that the corresponding 6×6 minor will be 128 and the resulting value for the sixth pivot is $p_6=4$. Similarly, the corresponding 7×7 determinant will attain two values in this case, 256 and 512. For this example, we carry out the demonstration by choosing the value 512, for which the resulting seventh pivot is $p_7=4$. The corresponding 8×8 determinant will attain five values in this case, 2048, 2304, 2560, 3072 and 4096. We carry out the demonstration for the value 4096 and the resulting pivot value is $p_8=8$.

Next we will demonstrate the seven last pivots for these specific eight first pivots, and eventually the ninth pivot. The three last pivots can only be 16, 8 and 8. The fourth pivot from the end can be 4 or 8. These values are obtained with $H_{16}(12)=16\cdot 16^4$ or $8\cdot 16^4$, respectively. We continue this example by choosing the value $p_{13}=4$. As it can be seen from the computations for p_{12} in

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Proposition 2, we might have as fifth pivot from the end the values 8 and $\frac{16}{3}$. For this example we choose to go on with $p_{12}=8$. Similarly, we can have as sixth and seventh pivots from the end the values 4 and 4, respectively. The ninth pivot will be calculated as follows:

$$\det H_{16}=\prod_{i=1}^{16} p_i \Rightarrow p_9=\frac{\det H_{16}}{\prod_{i=1,i\neq 9}^{16} p_i}=\frac{16^8}{1\cdot 2\cdot 2\cdot 4\cdot 2\cdot 4\cdot 4\cdot 8\cdot 4\cdot 4\cdot 8\cdot 4\cdot 8\cdot 8\cdot 16}=2$$

So, we have demonstrated the tenth possible pivot pattern given in Appendix A.4. Similarly we can derive the rest of the values of the Proposition and furthermore the 34 different pivot patterns of Appendix A.4. □

Remark 4

It becomes clear that in order to calculate the ninth pivot with the determinant property, one has to derive a complete pivot sequence of H_{16}. So, as a collateral result of the proof of Proposition 3 we have the exact pivot structures of the H_{16}, which are given in Appendix A.4.

Theorem 1

If GE with complete pivoting is performed on an H_{16}, the growth factor of H_{16} is 16.

Proof

The result follows easily from Propositions 1–3 and from the definition of the growth factor for CP matrices given in the introduction. □

The results of this work can be also summarized in Table V, where the last eight pivots are given as functions of n.

Table V. The appearing pivots for Hadamard matrices of order $n=16$.

Pivot	Values
p_1	1
p_2	2
p_3	2
p_4	4
p_5	2,3
p_6	$4,\frac{8}{3},\frac{10}{3}$
p_7	$2,4,\frac{8}{10/3},\frac{16}{5},\frac{18}{5}$
p_8	$4,\frac{9}{2},5,6,8$
p_{n-7}	$\frac{n}{8},\frac{n}{6},\frac{n}{5},\frac{n}{4},\frac{n}{32/9},\frac{n}{3}$
p_{n-6}	$\frac{n}{4},\frac{n}{8/5},\frac{n}{16/5},\frac{n}{2}$
p_{n-5}	$\frac{n}{4},\frac{n}{8/3},\frac{n}{2},\frac{n}{10/3}$
p_{n-4}	$\frac{n}{3},\frac{n}{2}$
p_{n-3}	$\frac{n}{4},\frac{n}{2}$
p_{n-2}	$\frac{n}{2}$
p_{n-1}	$\frac{n}{2}$
p_n	n

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4. CONCLUSIONS AND FURTHER WORK

With the use of sophisticated numerical techniques we demonstrated the pivot patterns of H_{16}, which until now were observed only experimentally. In this manner we proved that the growth factor of any H_{16} is equal to 16, and thus an open problem is solved. For this purpose, we split the work into two tasks: specification of the existence of $k\times k$ matrices inside H_{16}, which led to the computation of pivots from the beginning of the pivot pattern, and calculation of $(n-j)\times(n-j)$ minors, which led to the computation of pivots from the end.

The methods presented in this work can be used as the basis for calculating the pivot pattern of Hadamard matrices of higher orders, such as H_{20}, H_{24}, etc. The complexity of such problems points out the need for developing algorithms that can implement very effectively the ideas introduced in this work, or other, more elaborate ideas. For instance, a question toward this direction is: Is there a reliable criterion for reducing the total amount of matrices M used as input for the algorithm Minors, so that no values, which should appear, are skipped? The reduction of the number of matrices occurring with the extension procedure in the proof of Proposition 1 is a matter of concern, too. Another observation, which could lead to a computational improvement, is the fact that all acceptable solutions of the linear systems of algorithm Exist, $k\geqslant 5$, are always obtained for the parameter values $0,\dots,n/8$, as in Appendix A.2, case $k=5$. Hence, there arises the obvious question whether there is a more precise upper bound for the possible columns in the first rows of a Hadamard matrix than the one given in Lemma 2.

Furthermore, it would be interesting to classify theoretically the pivot patterns of the H_{16} with respect to the five equivalence classes, as it is discussed in Appendix A.4, and also to explain the occurrence of the value 8 as fourth pivot from the end only for matrices from the Hadamard–Sylvester class. Finally, the parallel implementation of the two independent tasks for calculating pivots simultaneously from the beginning and from the end, which would limit significantly the computational time needed by the algorithms, is under investigation, too.

APPENDIX A

A.1. Proof of Lemma 2

Proof
If we consider separately the first three rows from the first j rows of the enunciation, we observe for the 2^{j-1} possible columns $\underline{u}_i, i=1,\dots,2^{j-1}$, that

$$\underline{u}_1(1:3)=\cdots=\underline{u}_{\frac{1}{4}2^{j-1}}(1:3)=(+,+,+)^{\mathrm{T}}$$
$$\underline{u}_{\frac{1}{4}2^{j-1}+1}(1:3)=\cdots=\underline{u}_{\frac{2}{4}2^{j-1}}(1:3)=(+,+,-)^{\mathrm{T}}$$
$$\underline{u}_{\frac{2}{4}2^{j-1}+1}(1:3)=\cdots=\underline{u}_{\frac{3}{4}2^{j-1}}(1:3)=(+,-,+)^{\mathrm{T}}$$
$$\underline{u}_{\frac{3}{4}2^{j-1}+1}(1:3)=\cdots=\underline{u}_{2^{j-1}}(1:3)=(+,-,-)^{\mathrm{T}}$$

where $\underline{u}_i(1:3)$ denotes the first three entries (as in Matlab notation) of the column $\underline{u}_i$ of U_j. This observation accrues easily from a combinatorial counting and can be verified with the matrices U_3, U_4 and U_5 given in this paper.

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It is known [21, Lemma 1] that for every triple of rows of any Hadamard matrix of order $n>3$, there are precisely (under H-equivalence) $n/4$ columns, which are $(+,+,+)^{\mathrm{T}}$, $(+,+,-)^{\mathrm{T}}$, $(+,-,+)^{\mathrm{T}}$ and $(+,-,-)^{\mathrm{T}}$.

Hence, we obtain

$$\begin{aligned} u_1+\cdots+u_{(1/4)2^{j-1}} &= \frac{n}{4} \\ u_{(1/4)2^{j-1}+1}+\cdots+u_{(2/4)2^{j-1}} &= \frac{n}{4} \\ u_{(2/4)2^{j-1}+1}+\cdots+u_{(3/4)2^{j-1}} &= \frac{n}{4} \\ u_{(3/4)2^{j-1}+1}+\cdots+u_{2^{j-1}} &= \frac{n}{4} \end{aligned}$$

The result follows straightforwardly from these relations by taking into account that $u_i\geqslant 0$, since u_i denote number of columns. □

A.2. Implementation of algorithm Exist

We would like to note that algorithm Exist is mainly designed for working with a set of matrices with a special property as input and is executed in order to decide whether all the matrices treated as a set can exist inside a Hadamard matrix or not. We developed this idea with the set of matrices because, with the exception of the matrix A in the following application for $k=4$, it is never possible that a matrix exists among all solutions for all feasible values of parameters. On the contrary, it seems more sensible to apply this idea for a group of matrices with some property. For instance, the matrices $\mathscr{M}_5$ of the application for $k=5$ are extensions of the 4×4 matrix A to CP 5×5 matrices with absolute determinant 32 or 48. For the purpose of this work it is not reasonable to examine the existence properties of each one of them, but to handle them as a set and to ensure the existence of one such matrix always inside H_{16}. This idea is further justified by the technique used in the proof of Proposition 1. So, in order to prove that a set of matrices always exists, we must show that at least one of the matrices of the set appears in every acceptable solution of the linear system.

We demonstrate the application of algorithm Exist for $k=4,5$ and $n=16$.

k=4:

Step 1: We want to establish whether the matrix

$$A=\begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}=[\underline{u}_1\ \ \underline{u}_6\ \ \underline{u}_4\ \ \underline{u}_7]$$

always exists in the upper left 4×4 corner of a Hadamard matrix of order 16.

Step 2:

$$U_4=\begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\ + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - \end{array}$$

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Step 3: The system of seven equations and eight variables, which results from counting of columns and the orthogonality of every two distinct rows of U_4, is

$$
\begin{aligned}
u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8&=n\\
u_1+u_2+u_3+u_4-u_5-u_6-u_7-u_8&=0\\
u_1+u_2-u_3-u_4+u_5+u_6-u_7-u_8&=0\\
u_1+u_2-u_3-u_4-u_5-u_6+u_7+u_8&=0\\
u_1-u_2+u_3-u_4+u_5-u_6+u_7-u_8&=0\\
u_1-u_2+u_3-u_4-u_5+u_6-u_7+u_8&=0\\
u_1-u_2-u_3+u_4+u_5-u_6-u_7+u_8&=0
\end{aligned}
$$

The solution is

$$
\begin{aligned}
u_1&=4-u_8\\
u_2&=u_8\\
u_3&=u_8\\
u_4&=4-u_8\\
u_5&=u_8\\
u_6&=4-u_8\\
u_7&=4-u_8\\
u_8&=u_8
\end{aligned}
$$

According to Lemma 2, u_8 is allowed to take the values 0, 1, 2, 3, 4.

For $u_8=0$ we have $(u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8)=(4,0,0,4,0,4,4,0)$.
For $u_8=1$ we have $(u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8)=(3,1,1,3,1,3,3,1)$.
For $u_8=2$ we have $(u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8)=(2,2,2,2,2,2,2,2)$.
For $u_8=3$ we have $(u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8)=(1,3,3,1,3,1,1,3)$.
For $u_8=4$ we have $(u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8)=(0,4,4,0,4,0,0,4)$.

The solution for $u_8=4$ is not accepted because we assume without loss of generality that the matrix is normalized, so that we always have $u_1\geqslant 1$. In the remainder of the solutions we always see that $u_1,u_4,u_6,u_7\geqslant 1$; hence, we can conclude that A always exists in the upper left corner of a Hadamard matrix of order 16. Precisely, it is the only CP 4×4 matrix that can exist there.
$k=5$:
The matrices of this example occurred as extensions of the 4×4 matrix A to 5×5 ± 1 matrices; they are CP and have determinants 32 and 48; according to the idea described in more detail in Section 2.2 in the proof of Proposition 1.
Step 1: We want to establish whether the matrices

$$
\begin{aligned}
\mathscr{M}_5=\{&[\underline{u}_1\ \underline{u}_{15}\ \underline{u}_8\ \underline{u}_{14}\ \underline{u}_{13}],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_8\ \underline{u}_{12}\ \underline{u}_{13}],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_8\ \underline{u}_{14}\ \underline{u}_{12}],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_8\ \underline{u}_{12}\ \underline{u}_{14}],\\
&[\underline{u}_1\ \underline{u}_{15}\ \underline{u}_7\ \underline{u}_{14}\ \underline{u}_{13}],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_7\ \underline{u}_{14}\ \underline{u}_{12}],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_7\ \underline{u}_{12}\ \underline{u}_{14}],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_8\ \underline{u}_{14}\ \underline{u}_{11}],
\end{aligned}
$$

$$[\underline{u}_1\ \underline{u}_{15}\ \underline{u}_8\ \underline{u}_{12}\ \underline{u}_{11}],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_8\ \underline{u}_{12}\ \underline{u}_{10}],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_7\ \underline{u}_{14}\ \underline{u}_{11}],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_7\ \underline{u}_{12}\ \underline{u}_{11}],$$

$$[\underline{u}_1\ \underline{u}_{15}\ \underline{u}_7\ \underline{u}_{14}\ \underline{u}_9],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_8\ \underline{u}_{14}\ \underline{u}_{13}],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_8\ \underline{u}_{14}\ \underline{u}_{12}],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_8\ \underline{u}_{12}\ \underline{u}_{14}],$$

$$[\underline{u}_1\ \underline{u}_{11}\ \underline{u}_7\ \underline{u}_{12}\ \underline{u}_{16}],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_7\ \underline{u}_{14}\ \underline{u}_{12}],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_7\ \underline{u}_{12}\ \underline{u}_{14}],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_8\ \underline{u}_{14}\ \underline{u}_{15}],$$

$$[\underline{u}_1\ \underline{u}_{11}\ \underline{u}_8\ \underline{u}_{12}\ \underline{u}_{15}],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_8\ \underline{u}_{14}\ \underline{u}_{10}],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_8\ \underline{u}_{12}\ \underline{u}_{10}],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_7\ \underline{u}_{14}\ \underline{u}_{15}],$$

$$[\underline{u}_1\ \underline{u}_{11}\ \underline{u}_7\ \underline{u}_{12}\ \underline{u}_{15}],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_7\ \underline{u}_{12}\ \underline{u}_{10}],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_8\ \underline{u}_{14}\ \underline{u}_7],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_8\ \underline{u}_{12}\ \underline{u}_7],$$

$$[\underline{u}_1\ \underline{u}_{15}\ \underline{u}_8\ \underline{u}_{12}\ \underline{u}_6],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_7\ \underline{u}_{14}\ \underline{u}_8],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_7\ \underline{u}_{12}\ \underline{u}_8],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_7\ \underline{u}_{14}\ \underline{u}_6],$$

$$[\underline{u}_1\ \underline{u}_{15}\ \underline{u}_7\ \underline{u}_{12}\ \underline{u}_6],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_8\ \underline{u}_{12}\ \underline{u}_3],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_8\ \underline{u}_{14}\ \underline{u}_2],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_8\ \underline{u}_{12}\ \underline{u}_2],$$

$$[\underline{u}_1\ \underline{u}_{15}\ \underline{u}_7\ \underline{u}_{14}\ \underline{u}_4],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_7\ \underline{u}_{14}\ \underline{u}_2],\ [\underline{u}_1\ \underline{u}_{15}\ \underline{u}_7\ \underline{u}_{12}\ \underline{u}_2],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_8\ \underline{u}_{14}\ \underline{u}_7],$$

$$[\underline{u}_1\ \underline{u}_{11}\ \underline{u}_8\ \underline{u}_{12}\ \underline{u}_7],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_8\ \underline{u}_{14}\ \underline{u}_5],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_7\ \underline{u}_{14}\ \underline{u}_8],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_7\ \underline{u}_{12}\ \underline{u}_8],$$

$$[\underline{u}_1\ \underline{u}_{11}\ \underline{u}_7\ \underline{u}_{12}\ \underline{u}_6],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_8\ \underline{u}_{14}\ \underline{u}_4],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_8\ \underline{u}_{14}\ \underline{u}_2],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_8\ \underline{u}_{12}\ \underline{u}_2],$$

$$[\underline{u}_1\ \underline{u}_{11}\ \underline{u}_7\ \underline{u}_{14}\ \underline{u}_4],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_7\ \underline{u}_{12}\ \underline{u}_4],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_7\ \underline{u}_{14}\ \underline{u}_2],\ [\underline{u}_1\ \underline{u}_{11}\ \underline{u}_7\ \underline{u}_{12}\ \underline{u}_2]\}$$

always exist in the upper left 5×5 corner of a Hadamard matrix of order 16, where the vector $\underline{u}_i$ denotes the ith column of U_5 in this example.

Step 2

$$U_5 = \begin{array}{cccccccccccccccc}
u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} \\
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\
+ & + & + & + & - & - & - & - & + & + & + & - & - & - & + & - \\
+ & + & - & - & + & + & - & - & + & + & - & + & - & + & - & - \\
+ & - & + & - & + & - & + & - & + & - & + & + & + & - & - & -
\end{array}$$

Step 3

The system of 11 equations and 16 variables, which results from counting of columns and the orthogonality of every two distinct rows, is

$$u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9+u_{10}+u_{11}+u_{12}+u_{13}+u_{14}+u_{15}+u_{16}=16$$

$$u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8-u_9-u_{10}-u_{11}-u_{12}-u_{13}-u_{14}-u_{15}-u_{16}=0$$

$$u_1+u_2+u_3+u_4-u_5-u_6-u_7-u_8+u_9+u_{10}+u_{11}-u_{12}-u_{13}-u_{14}+u_{15}-u_{16}=0$$

$$u_1+u_2-u_3-u_4+u_5+u_6-u_7-u_8+u_9+u_{10}-u_{11}+u_{12}-u_{13}+u_{14}-u_{15}-u_{16}=0$$

$$u_1-u_2+u_3-u_4+u_5-u_6+u_7-u_8+u_9-u_{10}+u_{11}+u_{12}+u_{13}-u_{14}-u_{15}-u_{16}=0$$

$$u_1+u_2+u_3+u_4-u_5-u_6-u_7-u_8-u_9-u_{10}-u_{11}+u_{12}+u_{13}+u_{14}-u_{15}+u_{16}=0$$

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$$u_1+u_2-u_3-u_4+u_5+u_6-u_7-u_8-u_9-u_{10}+u_{11}-u_{12}+u_{13}-u_{14}+u_{15}+u_{16}=0$$
$$u_1-u_2+u_3-u_4+u_5-u_6+u_7-u_8-u_9+u_{10}-u_{11}-u_{12}-u_{13}+u_{14}+u_{15}+u_{16}=0$$
$$u_1+u_2-u_3-u_4-u_5-u_6+u_7+u_8+u_9+u_{10}-u_{11}-u_{12}+u_{13}-u_{14}-u_{15}+u_{16}=0$$
$$u_1-u_2+u_3-u_4-u_5+u_6-u_7+u_8+u_9-u_{10}+u_{11}-u_{12}-u_{13}+u_{14}-u_{15}+u_{16}=0$$
$$u_1-u_2-u_3+u_4+u_5-u_6-u_7+u_8+u_9-u_{10}-u_{11}+u_{12}-u_{13}-u_{14}+u_{15}+u_{16}=0$$

The solution is

$$u_1=8-u_8-u_{12}-u_{14}-u_{15}-3u_{16}$$
$$u_2=u_8-4+2u_{16}+u_{14}+u_{12}$$
$$u_3=u_8+u_{12}-4+2u_{16}+u_{15}$$
$$u_4=-u_8-u_{12}+4-u_{16}$$
$$u_5=u_8-4+2u_{16}+u_{14}+u_{15}$$
$$u_6=-u_8+4-u_{16}-u_{14}$$
$$u_7=-u_8+4-u_{16}-u_{15}$$
$$u_8=u_8$$
$$u_9=u_{12}-4+2u_{16}+u_{15}+u_{14}$$
$$u_{10}=-u_{12}+4-u_{16}-u_{14}$$
$$u_{11}=-u_{12}+4-u_{16}-u_{15}$$
$$u_{12}=u_{12}$$
$$u_{13}=4-u_{16}-u_{14}-u_{15}$$
$$u_{14}=u_{14}$$
$$u_{15}=u_{15}$$
$$u_{16}=u_{16}$$

The parameters can have the values $0,\ldots,4$. For all acceptable solutions we give

$$(u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8,u_9,u_{10},u_{11},u_{12},u_{13},u_{14},u_{15},u_{16})$$

and the respective values for the parameters of the solution.

$$(2,0,0,2,0,2,2,0,0,2,2,0,2,0,0,2),\ u_8=0,\ u_{12}=0,\ u_{14}=0,\ u_{15}=0,\ u_{16}=2$$
$$(1,0,1,2,1,2,1,0,1,2,1,0,1,0,1,2),\ u_8=0,\ u_{12}=0,\ u_{14}=0,\ u_{15}=1,\ u_{16}=2$$
$$(1,1,0,2,1,1,2,0,1,1,2,0,1,1,0,2),\ u_8=0,\ u_{12}=0,\ u_{14}=1,\ u_{15}=0,\ u_{16}=2$$
$$(1,1,1,1,0,2,2,0,1,1,1,1,2,0,0,2),\ u_8=0,\ u_{12}=1,\ u_{14}=0,\ u_{15}=0,\ u_{16}=2$$

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$(2,0,0,2,0,2,2,0,1,1,1,1,1,1,1,1),\ u_8=0,\ u_{12}=1,\ u_{14}=1,\ u_{15}=1,\ u_{16}=1$

$(1,0,1,2,1,2,1,0,2,1,0,1,0,1,2,1),\ u_8=0,\ u_{12}=1,\ u_{14}=1,\ u_{15}=2,\ u_{16}=1$

$(1,1,0,2,1,1,2,0,2,0,1,1,0,2,1,1),\ u_8=0,\ u_{12}=1,\ u_{14}=2,\ u_{15}=1,\ u_{16}=1$

$(1,1,1,1,0,2,2,0,2,0,0,2,1,1,1,1),\ u_8=0,\ u_{12}=2,\ u_{14}=1,\ u_{15}=1,\ u_{16}=1$

$(2,0,0,2,0,2,2,0,2,0,0,2,0,2,2,0),\ u_8=0,\ u_{12}=2,\ u_{14}=2,\ u_{15}=2,\ u_{16}=0$

$(1,1,1,1,1,1,1,1,0,2,2,0,2,0,0,2),\ u_8=1,\ u_{12}=0,\ u_{14}=0,\ u_{15}=0,\ u_{16}=2$

$(2,0,0,2,1,1,1,1,0,2,2,0,1,1,1,1),\ u_8=1,\ u_{12}=0,\ u_{14}=1,\ u_{15}=1,\ u_{16}=1$

$(1,0,1,2,2,1,0,1,1,2,1,0,0,1,2,1),\ u_8=1,\ u_{12}=0,\ u_{14}=1,\ u_{15}=2,\ u_{16}=1$

$(1,1,0,2,2,0,1,1,1,1,2,0,0,2,1,1),\ u_8=1,\ u_{12}=0,\ u_{14}=2,\ u_{15}=1,\ u_{16}=1$

$(2,0,1,1,0,2,1,1,0,2,1,1,2,0,1,1),\ u_8=1,\ u_{12}=1,\ u_{14}=0,\ u_{15}=1,\ u_{16}=1$

$(1,0,2,1,1,2,0,1,1,2,0,1,1,0,2,1),\ u_8=1,\ u_{12}=1,\ u_{14}=0,\ u_{15}=2,\ u_{16}=1$

$(2,1,0,1,0,1,2,1,0,1,2,1,2,1,0,1),\ u_8=1,\ u_{12}=1,\ u_{14}=1,\ u_{15}=0,\ u_{16}=1$

$(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1),\ u_8=1,\ u_{12}=1,\ u_{14}=1,\ u_{15}=1,\ u_{16}=1$

$(1,2,0,1,1,0,2,1,1,0,2,1,1,2,0,1),\ u_8=1,\ u_{12}=1,\ u_{14}=2,\ u_{15}=0,\ u_{16}=1$

$(2,0,0,2,1,1,1,1,1,1,1,1,0,2,2,0),\ u_8=1,\ u_{12}=1,\ u_{14}=2,\ u_{15}=2,\ u_{16}=0$

$(1,1,2,0,0,2,1,1,1,1,0,2,2,0,1,1),\ u_8=1,\ u_{12}=2,\ u_{14}=0,\ u_{15}=1,\ u_{16}=1$

$(1,2,1,0,0,1,2,1,1,0,1,2,2,1,0,1),\ u_8=1,\ u_{12}=2,\ u_{14}=1,\ u_{15}=0,\ u_{16}=1$

$(2,0,1,1,0,2,1,1,1,1,0,2,1,1,2,0),\ u_8=1,\ u_{12}=2,\ u_{14}=1,\ u_{15}=2,\ u_{16}=0$

$(2,1,0,1,0,1,2,1,1,0,1,2,1,2,1,0),\ u_8=1,\ u_{12}=2,\ u_{14}=2,\ u_{15}=1,\ u_{16}=0$

$(1,1,1,1,1,1,1,1,2,0,0,2,0,2,2,0),\ u_8=1,\ u_{12}=2,\ u_{14}=2,\ u_{15}=2,\ u_{16}=0$

$(1,1,1,1,2,0,0,2,0,2,2,0,1,1,1,1),\ u_8=2,\ u_{12}=0,\ u_{14}=1,\ u_{15}=1,\ u_{16}=1$

$(2,0,0,2,2,0,0,2,0,2,2,0,0,2,2,0),\ u_8=2,\ u_{12}=0,\ u_{14}=2,\ u_{15}=2,\ u_{16}=0$

$(1,1,2,0,1,1,0,2,0,2,1,1,2,0,1,1),\ u_8=2,\ u_{12}=1,\ u_{14}=0,\ u_{15}=1,\ u_{16}=1$

$(1,2,1,0,1,0,1,2,0,1,2,1,2,1,0,1),\ u_8=2,\ u_{12}=1,\ u_{14}=1,\ u_{15}=0,\ u_{16}=1$

$(2,0,1,1,1,1,0,2,0,2,1,1,1,1,2,0),\ u_8=2,\ u_{12}=1,\ u_{14}=1,\ u_{15}=2,\ u_{16}=0$

$(2,1,0,1,1,0,1,2,0,1,2,1,1,2,1,0),\ u_8=2,\ u_{12}=1,\ u_{14}=2,\ u_{15}=1,\ u_{16}=0$

$(1,1,1,1,2,0,0,2,1,1,1,1,0,2,2,0),\ u_8=2,\ u_{12}=1,\ u_{14}=2,\ u_{15}=2,\ u_{16}=0$

$(2,0,2,0,0,2,0,2,0,2,0,2,2,0,2,0),\ u_8=2,\ u_{12}=2,\ u_{14}=0,\ u_{15}=2,\ u_{16}=0$

$(2,1,1,0,0,1,1,2,0,1,1,2,2,1,1,0),\ u_8=2,\ u_{12}=2,\ u_{14}=1,\ u_{15}=1,\ u_{16}=0$

$(1,1,2,0,1,1,0,2,1,1,0,2,1,1,2,0),\ u_8=2,\ u_{12}=2,\ u_{14}=1,\ u_{15}=2,\ u_{16}=0$

$(2,2,0,0,0,0,2,2,0,0,2,2,2,2,0,0),\ u_8=2,\ u_{12}=2,\ u_{14}=2,\ u_{15}=0,\ u_{16}=0$

$(1,2,1,0,1,0,1,2,1,0,1,2,1,2,1,0),\ u_8=2,\ u_{12}=2,\ u_{14}=2,\ u_{15}=1,\ u_{16}=0$

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In each of the solutions that arise we see that the columns of at least one of the required matrices of the set $\mathscr{M}_5$ are $\geqslant 1$; hence, we can conclude that these matrices can always exist in the upper left corner of a Hadamard matrix of order 16.

A.3. Implementation of algorithm Minors

We present an application of algorithm Minors for $j=3,5$ and $n=16$.
$j=3$:
We suppose initially that the following matrix (or an H-equivalent to it) appears in the upper left 3×3 corner of an H_{16}, which can be easily proved with algorithm Exist:

$$M=\begin{bmatrix} + & + & + \\ + & - & - \\ + & + & - \end{bmatrix}$$

The system of four equations and four unknowns, which results from counting of columns and the orthogonality of every two distinct rows of the matrix $[MU_3]$, is

$$u+v+x+y=n-3$$
$$u+v-x-y=1$$
$$u-v+x-y=-1$$
$$u-v-x+y=-1$$

The solution is

$$u=x=y=\frac{n}{4}-1$$
$$v=\frac{n}{4}$$

By denoting with D the remaining matrix after deleting the first three rows and columns of the initial Hadamard matrix, we have

$$D^{\mathrm{T}}D\equiv\begin{bmatrix} E_{1_{u\times u}} & F_1 \\ F_1^{\mathrm{T}} & G_1 \end{bmatrix}$$

where

$$E_1=nI-3J,\quad F_1=[-1_{u\times v}\ \ -1_{u\times x}\ \ 1_{u\times y}]\quad \text{and}\quad G_1=\begin{bmatrix} E_{1_{v\times v}} & 1_{v\times x} & -1_{v\times y} \\ 1_{x\times v} & E_{1_{x\times x}} & -1_{x\times y} \\ -1_{y\times v} & -1_{y\times x} & E_{1_{y\times y}} \end{bmatrix}$$

From now on we give the successive intermediate results provided by the algorithm. We have $\det E_{1_{u\times u}}=\frac{1}{4}\cdot(n+12)n^{(n-8)/4}$ and the calculations described previously in Section 3.1 give

$$G_1-F_1^{\mathrm{T}}E_{1_{u\times u}}^{-1}F_1\equiv\begin{bmatrix} E_{2_{v\times v}} & F_2 \\ F_2^{\mathrm{T}} & G_2 \end{bmatrix}$$

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where

$$E_2=\frac{n^2+8n-32}{n+12}I-\frac{4(n+8)}{n+12}J,\quad F_2=\frac{16}{n+12}[1_{v\times x}\quad -1_{v\times y}]\quad\text{and}$$

$$G_2=\begin{bmatrix} E_{2_{x\times x}} & -\dfrac{16}{n+12}_{x\times y}\\ -\dfrac{16}{n+12}_{y\times x} & E_{2_{y\times y}}\end{bmatrix},\quad \det E_{2_{v\times v}}=\frac{4n^{n/4}}{n+12}$$

$$G_2-F_2^{\mathrm{T}}E_{2_{v\times v}}^{-1}F_2\equiv\begin{bmatrix} E_{3_{x\times x}} & F_3\\ F_3^{\mathrm{T}} & G_3\end{bmatrix}$$

where $E_3=(n-4)I-4J$, $F_3=[0_{y\times y}]$ and $G_3=E_{3_{y\times y}}$, $\det E_{3_{x\times x}}=4n^{(n-8)/4}$,

$$G_3-F_3^{\mathrm{T}}E_{3_{x\times x}}^{-1}F_3\equiv E_{4_{y\times y}}=nI_y-4J_y$$

$\det E_{4_{y\times y}}=4n^{(n-8)/4}$. Finally, $\det D^{\mathrm{T}}D=\det E_{1_{u\times u}}\cdot\det E_{2_{v\times v}}\cdot\det E_{3_{x\times x}}\cdot\det E_{4_{y\times y}}=16n^{n-6}$. So $|\det D|=\sqrt{\det D^{\mathrm{T}}D}=4n^{(n/2)-3}=4\,194\,304$, which corresponds to the non-zero value of the $n-3$ minors for a Hadamard matrix of order 16.

$j=5$:

Next we provide the application of algorithm Minors for $j=5$ and $n=16$. For brevity, we will show the algorithm's implementation only for one 5×5 matrix. Let

$$M=\begin{bmatrix} + & + & + & + & +\\ + & - & + & - & -\\ + & + & - & - & -\\ + & - & - & + & +\\ + & - & - & + & -\end{bmatrix}$$

The system of 11 equations and 16 variables, which results from counting of columns and the orthogonality of every two distinct rows of the matrix $[M\quad U_5]$, is the same with the one in Appendix A.2, case k=4, Step 3, but with the right-hand side $[11,1,1,-1,1,-1,1,-1,1,-1,-3]^{\mathrm{T}}$.

The solution is

$$u_1=4-u_8-u_{12}-u_{14}-u_{15}-3u_{16}$$

$$u_2=u_8-1+2u_{16}+u_{14}+u_{12}$$

$$u_3=u_8+u_{12}-2+2u_{16}+u_{15}$$

$$u_4=-u_8-u_{12}+2-u_{16}$$

$$u_5=u_8-2+2u_{16}+u_{14}+u_{15}$$

$$u_6=-u_8+2-u_{16}-u_{14}$$

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$$
\begin{aligned}
u_7 &= -u_8+3-u_{16}-u_{15}\\
u_8 &= u_8\\
u_9 &= u_{12}-2+2u_{16}+u_{15}+u_{14}\\
u_{10} &= -u_{12}+2-u_{16}-u_{14}\\
u_{11} &= -u_{12}+3-u_{16}-u_{15}\\
u_{12} &= u_{12}\\
u_{13} &= 2-u_{16}-u_{14}-u_{15}\\
u_{14} &= u_{14}\\
u_{15} &= u_{15}\\
u_{16} &= u_{16}
\end{aligned}
$$

According to Lemma 2, the parameters u_8, u_{12}, u_{14}, u_{15}, u_{16} can take the values 0, 1, 2, 3, 4. We present as an example only the case $(u_8, u_{12}, u_{14}, u_{15}, u_{16}) = (0, 0, 0, 0, 1)$, since the rest of them can be handled absolutely similarly. In this case

$$
\begin{aligned}
&(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16})\\
&\quad = (1, 1, 0, 1, 0, 1, 2, 0, 0, 1, 2, 0, 1, 0, 0, 1)
\end{aligned}
$$

By denoting with D the remaining matrix after deleting the first five rows and columns of the initial Hadamard matrix, we have

$$
D^{\mathrm{T}}D \equiv \begin{bmatrix} E_{1_{u_1\times u_1}} & F_1 \\ F_1^{\mathrm{T}} & G_1 \end{bmatrix}
$$

where $E_1 = nI-5J$, $F_1 = [-3_{u_2} - 1_{u_4} - 1_{u_6} - 1_{u_7} - 1_{u_{10}} - 1_{u_{11}} - 1_{u_{13}} 3_{u_{16}}]$ and

$$
G_1 = \begin{bmatrix}
E_1 & \overline{3} & \overline{3} & 1 & \overline{3} & 1 & 1 & 1\\
 & E_1 & \overline{1} & \overline{1} & \overline{1} & \overline{1} & 3 & \overline{1}\\
 & & E_1 & \overline{1} & \overline{1} & 3 & \overline{1} & \overline{1}\\
 & & & E_1 & 3 & \overline{1} & \overline{1} & \overline{1}\\
 & & & & E_1 & \overline{1} & \overline{1} & \overline{1}\\
 & & & & & E_1 & \overline{1} & \overline{1}\\
 & & & & & & E_1 & \overline{1}\\
 & & & & & & & E_1
\end{bmatrix}
$$

For the sake of better presentation we introduce $\overline{d}$ standing for $-d$, $d = 1, 3$, and we omit the subscripts of $\pm 1, \pm 3$ in G_1, which actually represent blocks with these elements of appropriate

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size, and of the diagonal blocks E_1, which are of orders $u_2\times u_2, u_4\times u_4, \ldots, u_{16}\times u_{16}$, respectively. The same holds for the next matrices $E_2,\ldots,E_{16}$. The lower triangular part of G_1 is symmetric to the upper triangular.

We have

$$\det E_{1_{u_1\times u_1}} = n-5 \quad \text{and} \quad G_1 - F_1^{\mathrm{T}} E_{1_{u_1\times u_1}}^{-1} F_1 \equiv \begin{bmatrix} E_2 & F_2 \\ F_2^{\mathrm{T}} & G_2 \end{bmatrix}$$

From now on, all the intermediate matrices appearing are obtained according to the method of Section 3.1 and it is sensible not to give all of them analytically. For all the remaining steps, the algorithm gives

$$\det E_2 = \frac{(n-2)(n-8)}{n-5}, \quad \det E_4 = \frac{(n-4)(n^2-11n+12)}{(n-8)(n-2)}, \quad \det E_6 = \frac{(n-4)(n^2-12n+8)}{n^2-11n+12}$$

$$\det E_7 = \frac{n^2(n-12)(n-10)}{n^2-12n+8}, \quad \det E_{10} = \frac{n^2-15n+20}{n-10}, \quad \det E_{11} = \frac{(n-12)(n^2-17n+36)n^2}{(n^2-15n+20)(n-4)}$$

$$\det E_{13} = \frac{(n-8)(n^2-18n+48)n}{(n-4)(n^2-17+36)} \quad \text{and} \quad \det E_{16} = \frac{n(n-8)(n-15)}{(n^2-18n+48)}$$

Finally,

$$\det D^{\mathrm{T}} D = \prod_{i=1, u_i\neq 0}^{16} \det E_i = n^6(n-12)^2(n-8)^2(n-15) = 17\,179\,869\,184$$

So $|\det D| = \sqrt{\det D^{\mathrm{T}} D} = 131\,072$ is the required value for the minor of order $n-5$ for a Hadamard matrix of order 16, if the matrix M was used as upper left corner and for the specific values of parameters. We mention that this resulting value is subject to the formula $32n^{(n/2)-5}$. The algorithm proceeds in an absolutely similar manner with the other parameter values, and also with the other possible matrices M.

A.4. The pivot patterns of the H_{16}

If GE with complete pivoting is applied to Hadamard matrices of order 16, the 34 different pivot patterns of Table AI are obtained, as is proved with Propositions 1–3.

According to the experiments performed, the pivot patterns can be classified with respect to the five equivalence classes as they are given in [22]. It is important to stress again that H-equivalent matrices do not have necessarily the same pivot pattern and non-H-equivalent matrices can have the same pivot pattern. For instance, the last pivot pattern of Table AI corresponds to matrices from the equivalence classes III and IV/V.

We observe that the value 8 as fourth pivot from the end occurs only in one pivot pattern, and particularly for matrices from the I-Class, according to the experiments. It is interesting to study this issue and to find out whether the construction properties of the I-Class can explain this exceptional case.

Another open problem is to classify theoretically the pivot patterns appearing in the appropriate equivalence classes. This research would require more information on the properties of the classes,

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Table AI. The 34 pivot patterns of H_{16}.

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}
1	2	2	4	2	4	4	4	4	4	4	8	4	8	8	16
1	2	2	4	2	4	4	4	4	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	4	4	5	16/(10/3)	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	4	$\frac{9}{2}$	16/(18/5)	16/(10/3)	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	$\frac{9}{2}$	4	16/(18/5)	16/(10/3)	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	5	$\frac{16}{5}$	4	4	8	4	8	8	16
1	2	2	4	2	4	4	5	$\frac{16}{5}$	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	5	$\frac{16}{5}$	5	16/(10/3)	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	6	$\frac{8}{3}$	4	4	8	4	8	8	16
1	2	2	4	2	4	4	6	$\frac{8}{3}$	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	8	2	4	4	8	4	8	8	16
1	2	2	4	3	$\frac{8}{3}$	2	4	4	4	4	8	8	8	8	16
1	2	2	4	3	$\frac{8}{3}$	2	4	4	4	8	8	4	8	8	16
1	2	2	4	3	$\frac{8}{3}$	2	4	4	8	4	8	4	8	8	16
1	2	2	4	3	$\frac{8}{3}$	2	4	4	8	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{8}{3}$	4	4	4	4	4	8	4	8	8	16
1	2	2	4	3	$\frac{8}{3}$	4	4	4	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{8}{3}$	4	4	4	5	16/(10/3)	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{8}{3}$	4	4	$\frac{9}{2}$	16/(18/5)	16/(10/3)	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{8}{3}$	4	5	$\frac{16}{5}$	4	4	8	4	8	8	16
1	2	2	4	3	$\frac{8}{3}$	4	5	$\frac{16}{5}$	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{8}{3}$	4	5	$\frac{16}{5}$	5	16/(10/3)	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{8}{3}$	4	6	$\frac{8}{3}$	4	4	8	4	8	8	16
1	2	2	4	3	$\frac{8}{3}$	4	6	$\frac{8}{3}$	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	8/(10/3)	4	$\frac{16}{3}$	4	4	8	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	8/(10/3)	4	$\frac{16}{3}$	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	8/(10/3)	4	$\frac{16}{3}$	5	16/(10/3)	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	4	4	4	4	8	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	4	4	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	4	4	5	16/(10/3)	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	5	$\frac{16}{5}$	4	4	8	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	5	$\frac{16}{5}$	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	5	$\frac{16}{5}$	5	16/(10/3)	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	16/(18/5)	16/(10/3)	$\frac{16}{3}$	4	8	8	16

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see [7, 15]. For example, the following reasoning could be a motivation for such an effort. We observe [22] that the value $\frac{10}{3}$ for the sixth pivot does not appear in the I-Class of equivalence. Looking closely at the proof of Proposition 1, we see that this value corresponds to $H_{16}(6)=160$. Hence, in order to prove the non-occurrence of $p_6=\frac{10}{3}$ in I-Class, it is equivalent to show that a 6×6 submatrix with determinant 160 cannot exist inside a matrix of this class. The 6×6 ± 1 matrix attaining the maximum determinant value 160 is called *the D-optimal design of order 6* (D_6).

We know that a representative H_{16} matrix of the I-Class, which is also called *Sylvester equivalence class* (H_{16}^{S}), is created according to the construction

$$H_{16}=\begin{bmatrix} H_8 & H_8 \\ H_8 & -H_8 \end{bmatrix}$$

where H_8 is the Hadamard matrix of order 8. It must be shown that $D_6\notin H_{16}^{\mathrm{S}}$. Hence, the result would be that pivot patterns with $p_6=\frac{10}{3}$ are not obtained from H_{16}^{S}.

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