Vector Balancing in Lebesgue Spaces

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Abstract

A tantalizing conjecture in discrete mathematics is the one of *Komlós*, suggesting that for any vectors $\boldsymbol{a}_1,\dots,\boldsymbol{a}_n \in B_2^m$ there exist signs $x_1,\dots,x_n \in \{-1,1\}$ so that $\|\sum_{i=1}^n x_i \boldsymbol{a}_i\|_\infty \leq O(1)$. It is a natural extension to ask what ℓ_q-norm bound to expect for $\boldsymbol{a}_1,\dots,\boldsymbol{a}_n \in B_p^m$. We prove that, for $2 \leq p \leq q \leq \infty$, such vectors admit fractional colorings $x_1,\dots,x_n \in [-1,1]$ with a linear number of ± 1 coordinates so that $\|\sum_{i=1}^n x_i \boldsymbol{a}_i\|_q \leq O(\sqrt{\min(p,\log(2m/n))}) \cdot n^{1/2-1/p+1/q}$, and that one can obtain a full coloring at the expense of another factor of $\frac{1}{1/2-1/p+1/q}$. In particular, for $p \in (2,3]$ we can indeed find signs $\boldsymbol{x} \in \{-1,1\}^n$ with $\|\sum_{i=1}^n x_i \boldsymbol{a}_i\|_\infty \leq O(n^{1/2-1/p} \cdot \frac{1}{p-2})$. Our result generalizes Spencer's theorem, for which $p = q = \infty$, and is tight for $m = n$.

Additionally, we prove that for *any* fixed constant $\delta > 0$, in a centrally symmetric body $K \subseteq \mathbb{R}^n$ with measure at least $e^{-\delta n}$ one can find such a fractional coloring in polynomial time. Previously this was known only for a *small* enough constant — indeed in this regime classical nonconstructive arguments do not apply and partial colorings of the form $\boldsymbol{x} \in \{-1,0,1\}^n$ do not necessarily exist.

1 Introduction

The celebrated *Spencer's Theorem* in discrepancy theory [Spe85] shows that "six standard deviations suffice" for balancing vectors in the ℓ_∞-norm: for any $\boldsymbol{a}_1,\dots,\boldsymbol{a}_n \in [-1,1]^n$, there exist signs $\boldsymbol{x} \in \{-1,1\}^n$ such that $\|\sum_{i=1}^n x_i \boldsymbol{a}_i\|_\infty \leq 6\sqrt{n}$. More generally, Spencer showed that for vectors in $[-1,1]^m$ with $n \leq m$ one can achieve a bound of $O(\sqrt{n\log(2m/n)})$. While his proof used a nonconstructive form of the *partial coloring lemma* based on the pigeonhole principle, in the past decade several approaches starting with the breakthrough work of Bansal [Ban10] did succeed in computing such signs in polynomial time [LM12, Rot14, LRR16, ES18].

As for balancing vectors of bounded ℓ_2-norm, the situation has been more delicate. In the same paper, Spencer [Spe85] showed a nonconstructive bound of $O(\log n)$ for the ℓ_∞ discrepancy of vectors $\boldsymbol{a}_1,\dots,\boldsymbol{a}_n \in B_2^m$ and also stated a conjecture of Komlós that this may be improved to $O(1)$. This was improved to $O(\sqrt{\log n})$ by Banaszczyk [Ban98] who showed that in fact for any set of n vectors of ℓ_2-norm at most 1 and any convex body $K \subseteq \mathbb{R}^m$

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of Gaussian measure at least $1/2$, some ± 1 combination of such vectors lies in $5 \cdot K$. For the more general setting of ℓ_q discrepancy, the work of Barthe, Guédon, Mendelson and Naor [BGMN05] shows that, for $q \geq 2$, a $O(\sqrt{q} \cdot n^{1/q})$ scaling of n-dimensional slices of the ℓ_q ball in $\mathbb{R}^m$ does have Gaussian measure at least $1/2$, thus implying a corresponding $O(\sqrt{q} \cdot n^{1/q})$ upper bound for balancing vectors from ℓ_2 to ℓ_q. For $q = \log n$, this matches the ℓ_2 to ℓ_∞ bound of $O(\sqrt{\log n})$. Banaszczyk's proof was nonconstructive and the first polynomial time algorithm in the general convex body setting was found only recently by Bansal, Dadush, Garg and Lovett [BDGL18], while the Komlós conjecture remains an open problem. The work of [BDGL18] actually shows that for any vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in B_2^m$ there exists an efficiently computable distribution over signs $\boldsymbol{x} \in \{-1,1\}^n$ so that the sum $\sum_{i=1}^n x_i \boldsymbol{a}_i$ is $O(1)$-subgaussian and will be in $O(1) \cdot K$ with good probability. Interestingly, this means their algorithm is *oblivious* to the body K, which is a striking difference to the regime of $\gamma_n(K) = e^{-\Theta(n)}$ where any algorithm needs to be dependent on K. The connection between Banaszczyk's theorem and subgaussianity is due to Dadush et al. [DGLN16].

For the general setting of balancing vectors from ℓ_p to ℓ_q norms, not much was known beyond Spencer's theorem ($p = \infty$) or what can be deduced from Banaszczyk's theorem as above: any vector in B_p^m also belongs to $m^{\max(0,1/2-1/p)} \cdot B_2^m$, thus implying a discrepancy bound of $O(\sqrt{q}) \cdot m^{\max(0,1/2-1/p)} \cdot n^{1/q}$. Even in the square case $m = n$, it has been an open problem to remove the dependency on $\sqrt{q}$ [DNTT18]. The goal of this paper is to provide a unified approach for balancing from ℓ_p to ℓ_q via optimal constructive fractional partial colorings, which yield optimal bounds for most of the range $1 \leq p \leq q \leq \infty$. We obtain such fractional partial colorings by proving a new measure lower bound on the relevant linear preimages of ℓ_q balls (Section 3) and an improved algorithm which works for sets of Gaussian measure $e^{-\delta n}$ for any $\delta > 0$ (Section 4), as opposed to previous work ([Rot14, ES18]) which required measure $e^{-\delta n}$ for *sufficiently small* $\delta > 0$.

As an application of our results, we show a slight improvement to the bounds for the well-known Beck-Fiala conjecture [BF81], a discrete version of Komlós. It asks for a $O(\sqrt{t})$ bound on the ℓ_∞ discrepancy of any $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \{0,1\}^m$, each with at most t ones. We establish the conjecture for $t \geq n$ and show slightly improved bounds when t is close to n (Corollary 4).

Notation. Let $B_p^m := \{\boldsymbol{x} \in \mathbb{R}^m : \|\boldsymbol{x}\|_p \leq 1\}$ denote the unit ball in the ℓ_p-norm. The *Gaussian measure* of a measurable set $K \subseteq \mathbb{R}^n$ is given by $\gamma_n(K) := \Pr_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)}[\boldsymbol{x} \in K]$. We denote the *mean width* of a convex set as $w(K) := \mathbb{E}_{\boldsymbol{\theta} \in S^{n-1}}[\sup_{\boldsymbol{x} \in K} \langle \boldsymbol{\theta}, \boldsymbol{x} \rangle]$. The Euclidean distance to a set $S \subseteq \mathbb{R}^n$ is denoted by $d(\boldsymbol{x}, S) := \min\{\|\boldsymbol{x} - \boldsymbol{y}\|_2 : \boldsymbol{y} \in S\}$. If $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is a matrix, we denote its rows by $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathbb{R}^n$ and its columns by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{R}^m$. Naturally, a matrix can also be interpreted as a (not necessarily invertible) linear map. Then for any set $K \subseteq \mathbb{R}^m$, we use the notation $\boldsymbol{A}^{-1}(K) := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \in K\}$.

1.1 Our contribution

Our main contribution is a tight bound on partial colorings for balancing from ℓ_p to ℓ_q:

Theorem 1. *Let $n \le m$ and $1 \le p \le q \le \infty$. Then for any $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in B_p^m$, there exists a polynomial-time computable partial coloring $\boldsymbol{x} \in [-1,1]^n$ with $|\{i : x_i^2 = 1\}| \ge n/2$ so that*

$$\Big\| \sum_{i=1}^{n} x_i \boldsymbol{a}_i \Big\|_q \le C \sqrt{\min\Big(p, \log\Big(\frac{2m}{n}\Big)\Big)} \cdot n^{\max(0,1/2-1/p)+1/q},$$

for some universal constant $C > 0$.

We would like to mention that, as noted by Banaszczyk [Ban93], the condition $n \le m$ does not weaken the theorem: in fact for $n > m$ the upper bound can only be larger than that of $n = m$ by a factor of two. On the other hand, the condition $p \le q$ is natural, for otherwise if $p > q$ we would need a polynomial dependence on the dimension m, even for $n = 1$. By iteratively applying Theorem 1 we can obtain a full coloring at the expense of another factor of $\frac{1}{\max(0,1/2-1/p)+1/q}$, with the caveat that $p > 2$ whenever $q = \infty$:

Theorem 2. *Let $n \le m$ and $1 \le p \le q \le \infty$ with $\max(0, 1/2 - 1/p) + 1/q > 0$. Then for any $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in B_p^m$, there exist polynomial-time computable signs $\boldsymbol{x} \in \{-1,1\}^n$ so that*

$$\Big\| \sum_{i=1}^{n} x_i \boldsymbol{a}_i \Big\|_q \le \frac{C\sqrt{\min\Big(p, \log\Big(\frac{2m}{n}\Big)\Big)}}{\max(0, 1/2 - 1/p) + 1/q} \cdot n^{\max(0,1/2-1/p)+1/q},$$

for some universal constant $C > 0$.

This significantly improves upon the general $\sqrt{q} \cdot m^{\max(0,1/2-1/p)} \cdot n^{1/q}$ bound from Banaszczyk's theorem in [DNTT18] when $p = 2 + \varepsilon$ for (not too small) $\varepsilon > 0$ and $q \gg 1$.

When $p = q$ and $m = n$, we get the following corollary which matches, up to a constant, the lower bound $\Omega(\sqrt{n})$ of [Ban93] known to hold for any norm:

Corollary 3 (ℓ_p version of Spencer's theorem)**.** *Let $2 \le p \le \infty$ and $n \in \mathbb{N}$. Then for any $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in B_p^n$, there exist polynomial-time computable signs $\boldsymbol{x} \in \{-1,1\}^n$ so that*

$$\Big\| \sum_{i=1}^{n} x_i \boldsymbol{a}_i \Big\|_p \le C\sqrt{n},$$

for some universal constant $C > 0$.

The following corollary shows the Beck-Fiala conjecture holds for $t \ge n$ and slightly improves upon the best known bound of $O(\sqrt{t \log n})$ [Ban98] when t is close to n:

Corollary 4 (Bound for Beck-Fiala)**.** *Let $n \le m$ and $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in \{0,1\}^m$, each with at most $t \in [m]$ ones. Then there exist polynomial-time computable signs $\boldsymbol{x} \in \{-1,1\}^n$ so that*

$$\Big\| \sum_{i=1}^n x_i \boldsymbol{a}_i \Big\|_\infty \le C\sqrt{t}\log\Big(\frac{2\max(n,t)}{t}\Big),$$

for some universal constant $C > 0$.

Finally, we show the partial coloring bound in Theorem 1 is tight at least when $m = n$:

Theorem 5. *Let $1 \le p \le q \le \infty$. There exist infinitely many positive integers n for which we can find $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in B_p^n$ such that for any $\boldsymbol{x} \in [-1,1]^n$ with $|\{i : x_i^2 = 1\}| \ge n/2$ one has*

$$\Big\| \sum_{i=1}^n x_i \boldsymbol{a}_i \Big\|_q \ge C \cdot n^{\max(0,1/2-1/p)+1/q},$$

for some universal constant $C > 0$.

As we mentioned earlier, the result of Gluskin [Glu89] and Giannopoulos [Gia97] shows that for a *small enough* constant, a symmetric convex body K with $\gamma_n(K) \ge e^{-\alpha n}$ contains a partial coloring $\boldsymbol{x} \in \{-1,0,1\}^n \setminus \{\boldsymbol{0}\}$ with a linear number of entries in ± 1. We can prove that for fractional colorings *any* constant $\alpha > 0$ suffices. Our argument even works for intersections with a large enough subspace.

Theorem 6. *For all $\alpha, \beta, \gamma > 0$, there is a constant $C := C(\alpha,\beta,\gamma) > 0$ so that the following holds: There is a randomized polynomial time algorithm which for a symmetric convex set $K \subseteq \mathbb{R}^n$ with $\gamma_n(K) \ge e^{-\alpha n}$, a shift $\boldsymbol{y} \in [-1,1]^n$ and a subspace $H \subseteq \mathbb{R}^n$ with $\dim(H) \ge \beta n$, finds an $\boldsymbol{x} \in (C \cdot K \cap H)$ with $\boldsymbol{x} + \boldsymbol{y} \in [-1,1]^n$ and $|\{i \in [n] : (\boldsymbol{x}+\boldsymbol{y})_i \in \{\pm 1\}\}| \ge (\beta - \gamma)n$.*

2 Preliminaries

We will use two elementary inequalities dealing with ℓ_p-norms. The first one estimates the ratio between different norms:

Lemma 7. *For any $\boldsymbol{z} \in \mathbb{R}^m$ and $1 \le p \le q \le \infty$, we have $\|\boldsymbol{z}\|_q \le \|\boldsymbol{z}\|_p \le m^{1/p-1/q}\|\boldsymbol{z}\|_q$.*

It is instructive to note that this bound implies $\|\boldsymbol{z}\|_\infty \le \|\boldsymbol{z}\|_{\log_2(m)} \le 2\|\boldsymbol{z}\|_\infty$. If one has an upper bound on the largest entry in a vector — say $\|\boldsymbol{z}\|_\infty \le 1$ — then one can strengthen the first inequality to $\|\boldsymbol{z}\|_q^q \le \|\boldsymbol{z}\|_p^p$. More generally:

Lemma 8. *For any $\boldsymbol{z} \in \mathbb{R}^m$ and $1 \le p \le q \le \infty$, we have $\|\boldsymbol{z}\|_q^q \le \|\boldsymbol{z}\|_p^p \cdot \|\boldsymbol{z}\|_\infty^{q-p}$.*

We will also need the following version of *Khintchine's inequality*, see e.g. the excellent textbook of Artstein-Avidan, Giannopoulos and Milman [AAGM15].

Lemma 9 (Khintchine's inequality)**.** *Given $p > 0$, $a_1, \dots, a_n \in \mathbb{R}$ and $\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)$, we have*

$$\mathbb{E}\Big[\Big|\sum_{i=1}^n x_i a_i\Big|^p\Big] \le C\sqrt{p} \cdot \Big(\sum_{i=1}^n a_i^2\Big)^{p/2}$$

where $C > 0$ is a universal constant.

This fact can be derived from a standard Chernov bound which guarantees that for a vector with $\|\boldsymbol{a}\|_2 = 1$ one has $\Pr[|\langle \boldsymbol{a}, \boldsymbol{x}\rangle| > \lambda] \le 2e^{-\lambda^2/2}$; then one can analyze that the regime of $\lambda = \Theta(\sqrt{p})$ dominates the contribution to $\mathbb{E}[|\langle \boldsymbol{a}, \boldsymbol{x}\rangle|^p]$. We use it to show the following:

Lemma 10. *Given $p \ge 1$ and $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in B_p^m$ and $\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)$, we have*

$$\mathbb{E}\Big[\Big\|\sum_{i=1}^n x_i \boldsymbol{a}_i\Big\|_p\Big] \le O(\sqrt{p} \cdot n^{\max(1/2, 1/p)}).$$

Proof. By convexity of $z \mapsto |z|^p$, Jensen's inequality in $(*)$ and Khintchine's inequality in $(**)$ (Lemma 9) we have

$$\begin{aligned}
\mathbb{E}\Big[\Big\|\sum_{i=1}^n x_i \boldsymbol{a}_i\Big\|_p\Big] &\overset{(*)}{\le} \mathbb{E}\Big[\Big\|\sum_{i=1}^n x_i \boldsymbol{a}_i\Big\|_p^p\Big]^{1/p} \\
&= \Big(\sum_{j\in[m]} \mathbb{E}\Big[\Big|\sum_{i\in[n]} x_i a_{ij}\Big|^p\Big]\Big)^{1/p} \\
&\overset{(**)}{\le} C\sqrt{p} \cdot \Big(\sum_{j\in[m]} \Big(\sum_{i\in[n]} a_{ij}^2\Big)^{p/2}\Big)^{1/p}.
\end{aligned}$$

If $p \in [1,2]$, write $\boldsymbol{A}_j \in \mathbb{R}^n$ as $(\boldsymbol{A}_j)_i := a_{ij}$. Then by Lemma 7,

$$\Big(\sum_{j\in[m]} \Big(\sum_{i\in[n]} a_{ij}^2\Big)^{p/2}\Big)^{1/p} = \Big(\sum_{j\in[m]} \|\boldsymbol{A}_j\|_2^p\Big)^{1/p} \le \Big(\sum_{j\in[m]} \|\boldsymbol{A}_j\|_p^p\Big)^{1/p} = \Big(\sum_{i\in[n]} \|\boldsymbol{a}_i\|_p^p\Big)^{1/p} \le n^{1/p}.$$

Now suppose that $p \ge 2$. Define $(\boldsymbol{a}_i)^2 \in \mathbb{R}^m$ to be the vector with jth coordinate a_{ij}^2. Since $\|\cdot\|_{p/2}$ is a norm, we can use the triangle inequality to get

$$\Big(\sum_{j\in[m]} \Big(\sum_{i\in[n]} a_{ij}^2\Big)^{p/2}\Big)^{1/p} = \Big\|\sum_{i\in[n]} (\boldsymbol{a}_i)^2\Big\|_{p/2}^{1/2} \le \Big(\sum_{i\in[n]} \|(\boldsymbol{a}_i)^2\|_{p/2}\Big)^{1/2} = \Big(\sum_{i\in[n]} \|\boldsymbol{a}_i\|_p^2\Big)^{1/2} \le n^{1/2}.$$

Either way, we conclude that $\mathbb{E}[\|\sum_{i=1}^n x_i \boldsymbol{a}_i\|_p] \le O(\sqrt{p} \cdot n^{\max(1/2,1/p)})$, as desired. □

A well-known correlation inequality for Gaussian measure is the following:

Lemma 11 (Šidak [Šid67] and Kathri [Kha67])**.** *For any symmetric convex set $K \subseteq \mathbb{R}^n$ and strip $S = \{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{a}, \boldsymbol{x}\rangle| \le 1\}$, one has $\gamma_n(K \cap S) \ge \gamma_n(K) \cdot \gamma_n(S)$.*

It is worth noting that a recent result of Royen [Roy14] extends this to any two arbitrary symmetric sets, though its full power will not be needed. We refer to the exposition of Latała and Matlak [LM17]. We also need a one-dimensional estimate:

Lemma 12. *For a strip $S = \{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{a}, \boldsymbol{x}\rangle| \le 1\}$, one has*

$$\gamma_n(S) = \gamma_1(\{x \in \mathbb{R} : |x| \le \|\boldsymbol{a}\|_2^{-1}\}) \ge 1 - \exp(-\|\boldsymbol{a}\|_2^{-2}/2).$$

We use the following scaling lemma to deal with constant factors:

Lemma 13. *Let $K \subset \mathbb{R}^n$ be a measurable set and B be a closed Euclidean ball such that $\gamma_n(K) = \gamma_n(B)$. Then $\gamma_n(tK) \geq \gamma_n(tB)$ for all $t \in [0,1]$. In particular, if $\gamma_n(C \cdot K) \geq 2^{-O(n)}$ for some constant $C > 1$ then also $\gamma_n(K) \geq 2^{-O(n)}$.*

For Section 4 we also need two helpful results. For the first one, see [vH14].

Theorem 14. *If $F : \mathbb{R}^m \to \mathbb{R}$ is 1-Lipschitz, then for $t \geq 0$ one has*

$$\Pr_{\boldsymbol{y} \sim N(\boldsymbol{0}, \boldsymbol{I}_m)} \left[F(\boldsymbol{y}) > \mathbb{E}[F(\boldsymbol{y})] + t \right] \leq e^{-t^2/2}.$$

The classical *Urysohn Inequality* states that among all convex bodies of identical volume, the Euclidean ball minimizes the width. We will need a variant that is phrased in terms of the Gaussian measure rather than volume. For a proof, see Eldan and Singh [ES18].

Theorem 15 (Gaussian Variant of Urysohn's Inequality)**.** *Let $K \subseteq \mathbb{R}^n$ be a convex body and let $r > 0$ be so that $\gamma_n(K) = \gamma_n(rB_2^n)$. Then $w(K) \geq w(rB_2^n) = r$.*

3 Main technical result

In this section we show our measure lower bound for balancing vectors from ℓ_p to ℓ_q:

Theorem 16. *Let $n \leq m$ and $1 \leq p \leq q \leq \infty$. Then for any $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in B_p^m$,*

$$\gamma_n\Big(\Big\{ \boldsymbol{x} \in \mathbb{R}^n : \Big\| \sum_{i=1}^n x_i \boldsymbol{a}_i \Big\|_q \leq \sqrt{\min\Big(p, \log\Big(\frac{2m}{n}\Big)\Big)} \cdot n^{\max(0,1/2-1/p)+1/q} \Big\} \Big) \geq 2^{-O(n)}.$$

In order to show Theorem 16, roughly speaking it will suffice to show the corresponding bounds for the two special cases of $q \in \{p, \infty\}$, which can be bootstrapped into a general bound. First we address the simpler case $p = q$ which at heart is based on Khintchine's inequality:

Lemma 17. *Let $n \leq m$ and $p \geq 1$. Then for any $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in B_p^m$,*

$$\gamma_n\Big(\Big\{ \boldsymbol{x} \in \mathbb{R}^n : \Big\| \sum_{i=1}^n x_i \boldsymbol{a}_i \Big\|_p \leq \sqrt{p} \cdot n^{\max(1/2,1/p)} \Big\} \Big) \geq 2^{-O(n)}.$$

Proof. By Lemma 10 we know that, for some constant $C > 0$,

$$\mathop{\mathbb{E}}_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)} \Big[\Big\| \sum_{i=1}^n x_i \boldsymbol{a}_i \Big\|_p \Big] \leq C\sqrt{p} \cdot n^{\max(1/2,1/p)}.$$

By Markov's inequality it follows that

$$\gamma_n\Big(\Big\{ \boldsymbol{x} \in \mathbb{R}^n : \Big\| \sum_{i=1}^n x_i \boldsymbol{a}_i \Big\|_p \leq 2C\sqrt{p} \cdot n^{\max(1/2,1/p)} \Big\} \Big) \geq 1/2,$$

so that the result follows by Lemma 13. $\square$

Next, we deal with the crucial case $q = \infty$:

Lemma 18. *Let $n \leq m$ and $p \geq 1$. Then for any $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with columns $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in B_p^m$ and rows $\boldsymbol{A}_1, \dots, \boldsymbol{A}_m \in \mathbb{R}^n$, the body $K := \{\boldsymbol{x} \in \mathbb{R}^n : \|\sum_{i=1}^n x_i \boldsymbol{a}_i\|_\infty \leq \sqrt{p}\cdot n^{\max(0,1/2-1/p)}\}$ satisfies*

$$\gamma_n(K) \geq \prod_{j\in[m]} \gamma_n(\{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{x}, \boldsymbol{A}_j\rangle| \leq \sqrt{p} n^{\max(0,1/2-1/p)}\}) \geq 2^{-O(n)}.$$

Proof. The main idea in the proof is that we can convert the bound on the ℓ_p-norm of the columns $\boldsymbol{a}_i$ into information about the ℓ_2-norm of the rows $\boldsymbol{A}_j$. Namely,

$$\Big(\frac{1}{n}\sum_{j\in[m]} \|\boldsymbol{A}_j\|_2^p\Big)^{1/p} \overset{\text{Lem 7}}{\leq} n^{\max(0,1/2-1/p)} \cdot \Big(\frac{1}{n}\underbrace{\sum_{j\in[m]} \|\boldsymbol{A}_j\|_p^p}_{\leq n}\Big)^{1/p} \leq n^{\max(0,1/2-1/p)}. \tag{1}$$

We rescale the row vectors to $\boldsymbol{V}_j := (\sqrt{p} n^{\max(0,1/2-1/p)})^{-1}\boldsymbol{A}_j$ and abbreviate $y_j := \|\boldsymbol{V}_j\|_2^2$, so that Eq. (1) simplifies to $\sum_{j=1}^m y_j^{p/2} \leq n \cdot p^{-p/2}$. We may then apply Šidak's Lemma 11 and bound the one-dimensional measure:

$$\begin{aligned}
\gamma_n(K) &= \gamma_n\big(\{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{x}, \boldsymbol{V}_j\rangle| \leq 1 \;\; \forall j \in [m]\}\big) \\
&\overset{\text{Lem 11}}{\geq} \prod_{j\in[m]} \gamma_n\big(\{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{x}, \boldsymbol{V}_j\rangle| \leq 1\}\big) \\
&\overset{\text{Lem 12}}{\geq} \prod_{j\in[m]} \big(1 - \exp(-y_j^{-1}/2)\big) \\
&\overset{\text{Claim I}}{\geq} \prod_{j\in[m]} \exp\Big(-C' p^{p/2} y_j^{p/2}\Big) = \exp\Big(-C' p^{p/2} \sum_{j\in[m]} y_j^{p/2}\Big) \geq \exp(-C' n)
\end{aligned}$$

Here we have used an estimate that remains to be proven:
Claim I. *For any $p \geq 1$ and $y > 0$ one has $1 - \exp(-\frac{1}{2y}) \geq \exp(-C' p^{p/2} y^{p/2})$ where $C' > 0$ is a universal constant.*
Proof of Claim I. It will suffice to show for any $y > 0$:

$$-\log(1 - \exp(-y^{-1}/2)) \leq O(p^{p/2} y^{p/2}).$$

To see this, let $z = y^{-1}/2$ and note that it suffices to show

$$-\log(1 - \exp(-z)) \cdot z^{p/2} \leq O((p/2)^{p/2}).$$

For $z \leq 1$ we can use the inequality $-\log(1 - \exp(-z)) \leq z^{-1/2}$ to see that the left side is at most 1. For $z > 1$ we use instead $-\log(1 - \exp(-z)) \leq \exp(-z/2)$ to get

$$\begin{aligned}
-\log(1 - \exp(-z)) \cdot z^{p/2} &\leq z^{p/2} \cdot \exp(-z/2) \\
&\leq z^{p/2} \cdot \lceil p/2 \rceil ! / ((z/2)^{p/2}) \\
&= 2^{p/2} \cdot \lceil p/2 \rceil ! \leq O((p/2)^{p/2}),
\end{aligned}$$

where in the last step we use the Stirling bound $a! \leq O(\sqrt{a} \cdot (a/e)^a)$ for $a := \lceil p/2 \rceil$. $\square$

Remark 1. This argument is largely motivated by the result of Ball and Pajor [BP90] which bounds volume instead of Gaussian measure. More specifically, [BP90] prove that for $1 \le p \le \infty$ and any matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, the set

$$K = \left\{ \boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{A}_j, \boldsymbol{x}\rangle| \le \sqrt{p} \cdot \Big(\frac{1}{n}\sum_{j=1}^m \|\boldsymbol{A}_j\|_2^p\Big)^{1/p} \ \forall j \in [m] \right\}$$

satisfies $\mathrm{vol}_n(K) \ge 1$. In contrast, our Lemma 18 provides a simpler proof of a stronger result (up to a constant scaling), since the volume of a convex body is always at least its Gaussian measure.

We are now ready to show Theorem 16:

Proof of Theorem 16. Let $1 \le p \le q \le \infty$ and let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ denote the matrix with columns $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in B_p^m$. By Lemma 8 we know that for any $\boldsymbol{z} \in \mathbb{R}^m$ with $\|\boldsymbol{z}\|_p \le n^{1/p}$ and $\|\boldsymbol{z}\|_\infty \le 1$ one has $\|\boldsymbol{z}\|_q \le (\|\boldsymbol{z}\|_p^p \cdot \|\boldsymbol{z}\|_\infty^{q-p})^{1/q} \le n^{1/q}$. Phrased in geometric terms this means $n^{1/q}B_q^m \supseteq n^{1/p}B_p^m \cap B_\infty^m$. We would like to point out that this is a crucial point to obtain a dependence solely on n rather than the larger parameter m. Next, note the fact that $\boldsymbol{A}^{-1}(S\cap T) = \boldsymbol{A}^{-1}(S) \cap \boldsymbol{A}^{-1}(T)$ for any sets S and T which we use together with the inequality of Šidak and Kathri (Lemma 11) to obtain the estimate

$$\begin{aligned}
& \gamma_n\Big(\boldsymbol{A}^{-1}\big(\sqrt{p}\cdot n^{\max(0,1/2-1/p)+1/q} B_q^m\big)\Big) \\
\ge\ & \gamma_n\Big(\boldsymbol{A}^{-1}\big(\sqrt{p}\cdot n^{\max(0,1/2-1/p)} (n^{1/p}B_p^m \cap B_\infty^m)\big)\Big) \\
\ge\ & \gamma_n\Big(\boldsymbol{A}^{-1}\big(\sqrt{p}\cdot n^{\max(1/2,1/p)} B_p^m\big)\Big) \cdot \prod_{j\in[m]} \gamma_n\big(\{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{x}, \boldsymbol{A}_j\rangle| \le \sqrt{p}\, n^{\max(0,1/2-1/p)}\}\big) \\
\ge\ & 2^{-O(n)} \cdot 2^{-O(n)} = 2^{-O(n)},
\end{aligned}$$

where we have used the measure lower bounds from Lemmas 17 and 18. This shows the claimed bound whenever $p \le O(\log(\frac{2m}{n}))$, where the hidden constant can be removed by scaling the corresponding convex body, see Lemma 13.

It remains to prove that we can bootstrap the existing bound for the regime of large p. So let us assume that $p \ge 2\cdot\max\{1, \log(m/n)\}$. Let $p_0 \in [2,p]$ be a parameter to be determined and remark that Lemma 7 gives $\|\boldsymbol{a}_i\|_{p_0} \le m^{1/p_0-1/p}\cdot\|\boldsymbol{a}_i\|_p \le m^{1/p_0-1/p}$. Applying the above measure lower bound for p_0 implies

$$\gamma_n\left(\left\{\boldsymbol{x} \in \mathbb{R}^n : \Big\|\sum_{i=1}^n x_i \boldsymbol{a}_i\Big\|_q \le \sqrt{p_0}\cdot n^{1/2-1/p_0+1/q}\cdot m^{1/p_0-1/p}\right\}\right) \ge 2^{-O(n)}.$$

We can rewrite the above upper bound on ℓ_q-norm as

$$\sqrt{p_0}\cdot n^{1/2-1/p_0+1/q}\cdot m^{1/p_0-1/p} = n^{1/2-1/p+1/q}\cdot \underbrace{\Big(\frac{m}{n}\Big)^{-1/p}}_{\le 1} \cdot \sqrt{p_0}\cdot\Big(\frac{m}{n}\Big)^{1/p_0}.$$

Taking $p_0 := 2\cdot\max\{1,\log(m/n)\}$ gives the desired result as then $(m/n)^{1/p_0} \le \sqrt{e}$ and Lemma 13 can again deal with such constant scaling. □

Now our main result on existence of partial colorings easily follows:

Proof of Theorem 1. Apply Theorem 6 to the set

$$K := \Big\{ \boldsymbol{x} \in \mathbb{R}^n : \Big\| \sum_{i=1}^n x_i \boldsymbol{a}_i \Big\|_q \leq \sqrt{\min\Big(p, \log\Big(\frac{2m}{n}\Big)\Big)} \cdot n^{\max(0,1/2-1/p)+1/q} \Big\},$$

which by Theorem 16 indeed has a Gaussian measure of $\gamma_n(K) \geq 2^{-O(n)}$. □

Next, we show how to obtain a full coloring by iteratively finding partial colorings.

Proof of Theorem 2. Let again $1 \leq p \leq q \leq \infty$ and let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in B_p^m$. We begin with $\boldsymbol{x}^{(0)} := \mathbf{0}$ and given $\boldsymbol{x}^{(0)}, \ldots, \boldsymbol{x}^{(t)}$ we set $S^{(t)} := \{i \in [n] : -1 < x_i^{(t)} < 1\}$ as the *active variables*. Then combining Theorem 6 and Theorem 16 we can find a partial coloring $\boldsymbol{x}^{(t+1)} \in [-1,1]^n$ in polynomial time so that $|S^{(t+1)}| \leq |S^{(t)}|/2$ and $\| \sum_{i=1}^n (x_i^{(t+1)} - x_i^{(t)}) \boldsymbol{a}_i \|_q \leq C_1 \sqrt{\min(p, \log(\frac{2m}{|S^{(t)}|}))} \cdot |S^{(t)}|^{\max(0,1/2-1/p)+1/q}$. Let $\boldsymbol{x}^{(T)}$ be the first iterate with $\boldsymbol{x}^{(T)} \in \{-1,1\}^n$. Clearly $|S^{(t)}| \leq n 2^{-t}$ and $T \leq \log_2(n)$. Using the triangle inequality we get

$$\begin{aligned}
\Big\| \sum_{i=1}^n x_i^{(T)} \boldsymbol{a}_i \Big\|_q &\leq \sum_{t=0}^{T-1} \Big\| \sum_{i=1}^n (x_i^{(t+1)} - x_i^{(t)}) \boldsymbol{a}_i \Big\|_q \\
&\leq C_1 \sum_{t=0}^{T-1} \sqrt{\min\Big(p, \log\Big(\frac{2m}{2^{-t} \cdot n}\Big)\Big)} \cdot (2^{-t} \cdot n)^{\max(0,1/2-1/p)+1/q} \\
&\leq \frac{C_1 C_2 \sqrt{\min\Big(p, \log\Big(\frac{2m}{n}\Big)\Big)}}{\max(0, 1/2 - 1/p) + 1/q} \cdot n^{\max(0,1/2-1/p)+1/q}. \quad \square
\end{aligned}$$

The intuition behind the extra factor for obtaining a full coloring is as follows: abbreviate the exponent as $\beta := \max(0, 1/2 - 1/p) + 1/q$. Then it takes $\frac{1}{\beta}$ iterations until the term $|S^{(t)}|^{\beta}$ decreases by a factor of 1/2 which dominates the miniscule growth of the logarithmic term. Then indeed the overall discrepancy is dominated by the discrepancy from the first $\frac{1}{\beta}$ iterations.

We can now demonstrate how a nontrivial choice of ℓ_p-norms can be beneficial in classical discrepancy settings:

Proof of Corollary 4. Consider column vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \{0,1\}^m$ with at most t nonzero entries per $\boldsymbol{a}_i$. First let us study the case $t \geq n/10$. Since for each column $\|\boldsymbol{a}_i\|_4 \leq t^{1/4}$, Theorem 2 provides a coloring $\boldsymbol{x} \in \{-1,1\}^n$ with $\| \sum_{i=1}^n x_i \boldsymbol{a}_i \|_\infty \leq O(n^{1/4} \cdot t^{1/4}) = O(\sqrt{t})$. [1]

Now if $t < n/10$, we take $p \in [2, 16)$ with $1/2 - 1/p = 1/\log(n/t)$. Then $\|\boldsymbol{a}_i\|_p \leq t^{1/p}$ and Theorem 2 gives $\boldsymbol{x} \in \{-1,1\}^n$ with

$$\Big\| \sum_{i=1}^n x_i \boldsymbol{a}_i \Big\|_\infty \leq \frac{C \cdot n^{1/2-1/p} \cdot t^{1/p}}{1/2 - 1/p} = C\sqrt{t} \log(n/t) \cdot \underbrace{(n/t)^{1/\log(n/t)}}_{=e}. \quad \square$$

[1] In fact for $t \geq n$ a more careful choice of $p = \log(2t/n)$ gives a better ℓ_∞ discrepancy bound of $O(\sqrt{n \log(2t/n)})$, even though the Beck-Fiala conjecture asks only for $O(\sqrt{t})$.

We conclude this section by showing that the term $n^{\max(0,1/2-1/p)+1/q}$ in our bounds is necessary:

Proof of Theorem 5. Consider the case $p \geq 2$. Consider an $n \times n$ *Hadamard matrix*, which is a matrix $\boldsymbol{H} \in \{-1,1\}^{n\times n}$ so that all rows and columns are orthogonal. Such matrices are known to exist at least whenever n is a power of 2. The columns satisfy $\|\boldsymbol{h}_i\|_p = n^{1/p}$ and for any $\boldsymbol{x} \in [-1,1]^n$ with $|\{i : x_i^2 = 1\}| \geq n/2$ we know that $\|\boldsymbol{x}\|_2 \geq \Omega(\sqrt{n})$ and $\|\boldsymbol{H}\boldsymbol{x}\|_2 \geq \Omega(n)$, so that by Lemma 7 we have

$$\|\boldsymbol{H}\boldsymbol{x}\|_q \geq \|\boldsymbol{H}\boldsymbol{x}\|_2 \cdot n^{1/q-1/2} = \Omega(n^{1/2+1/q}).$$

For $p \in [1,2]$, take an identity matrix $\boldsymbol{I}_n$. For every $\boldsymbol{x} \in [-1,1]^n$ with $|\{i : x_i^2 = 1\}| \geq n/2$ we have $\|\boldsymbol{I}_n\boldsymbol{x}\|_q = \|\boldsymbol{x}\|_q \geq \Omega(n^{1/q})$, and the columns of $\boldsymbol{I}_n$ are certainly in B_p^m. □

4 Partial coloring via measure lower bound

In this chapter, we want to show the existence of partial fractional colorings for bodies K with $\gamma_n(K) \geq e^{-\alpha n}$ as promised in Theorem 6. The main innovation of this work compared to e.g. [Rot14] is to handle an arbitrarily small constant $\alpha > 0$. For the sake of a simpler exposition we first prove such a result without the shift $\boldsymbol{y}$, without the hyperplane H and with a small fraction δ of colored elements.

Theorem 19. *For any $\alpha > 0$, there are constants $\varepsilon := \varepsilon(\alpha), \delta := \delta(\alpha) > 0$ so that the following holds: There is a polynomial time algorithm that for a symmetric convex set $K \subseteq \mathbb{R}^n$ with $\gamma_n(K) \geq e^{-\alpha n}$ finds an $\boldsymbol{x} \in (\frac{1}{\varepsilon}K) \cap [-1,1]^n$ so that $|\{i \in [n] : x_i \in \{-1,1\}\}| \geq \delta n$.*

Note that the standard nonconstructive proof by Gluskin [Glu89] and Giannopoulos [Gia97] requires a *small enough* constant $\alpha > 0$ to guarantee a partial coloring $\boldsymbol{x} \in \{-1,0,1\}^n$ with support $\Omega(n)$. Moreover, the statement of Theorem 19 does not hold if $\boldsymbol{x} \in [-1,1]^n$ is replaced by $\boldsymbol{x} \in \{-1,0,1\}^n$. In fact, it is not hard to construct a thin strip K with $\gamma_n(K) \geq e^{-\Omega(n)}$ so that K does not intersect $\{-1,0,1\}^n \setminus \{\boldsymbol{0}\}$ (even after a subexponential scaling). We show the construction in Appendix B.

For our proof we make use of the mean width $w(Q) := \mathbb{E}_{\boldsymbol{\theta}\in S^{n-1}}[\sup_{\boldsymbol{x}\in Q}\langle\boldsymbol{\theta},\boldsymbol{x}\rangle]$ of a body. We should point out that the connection between partial coloring arguments and mean width is due to Eldan and Singh [ES18]. Several of the claims require that n is chosen large enough.

Lemma 20. *Let $Q \subseteq \mathbb{R}^n$ be a symmetric convex body with $\gamma_n(Q) \geq e^{-\alpha n}$ for $\alpha > 0$. Then $w(Q) \geq \frac{1}{2}e^{-\alpha}\sqrt{n}$.*

Proof. Let $r > 0$ be the radius $\gamma_n(rB_2^n) = \gamma_n(Q)$. By *Urysohn's Inequality* (Theorem 15) one has $w(Q) \geq w(rB_2^n) = r$ so it suffices to give a lower bound on the radius r. A simple but useful estimate is that $2^n \leq \mathrm{Vol}_n(\sqrt{n}B_2^n) \leq 5^n$ for any $n \geq 1$. Moreover, the Gaussian density is maximized at $\gamma_n(\mathbf{0}) = \frac{1}{(\sqrt{2\pi})^n}$. Then for $\beta := 2e^\alpha \geq 2$ we have

$$\gamma_n\Big(\frac{\sqrt{n}}{\beta}B_2^n\Big) \leq \mathrm{Vol}_n\Big(\frac{\sqrt{n}}{\beta}B_2^n\Big)\cdot\gamma_n(\mathbf{0}) \leq \Big(\frac{5}{\beta}\Big)^n \cdot \frac{1}{(\sqrt{2\pi})^n} \leq \Big(\frac{2}{\beta}\Big)^n \overset{\beta=2e^\alpha}{\leq} e^{-\alpha n}$$

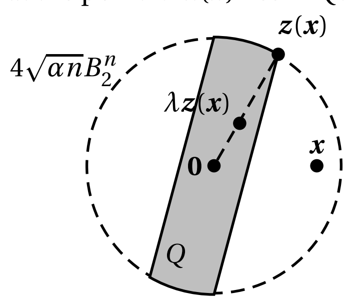
and so $r \geq \frac{\sqrt{n}}{\beta} = \frac{\sqrt{n}}{2e^\alpha}$. □

The key modification of our work in contrast to [Rot14] is a finer upper bound on the distance of a Gaussian to K:

Lemma 21. *Let $K \subseteq \mathbb{R}^n$ be a symmetric convex set with $\gamma_n(K) \geq e^{-\alpha n}$ where $\alpha \geq 1$ and n is large enough. Then*

$$\mathop{\mathbb{E}}_{\boldsymbol{x}\sim N(\mathbf{0},\boldsymbol{I}_n)}[d(\boldsymbol{x},K)] \leq \sqrt{n}\cdot\Big(1-\frac{1}{512\alpha e^{4\alpha}}\Big)$$

Proof. Note that by Theorem 14 we have $\Pr_{\boldsymbol{x}\sim N(\mathbf{0},\boldsymbol{I}_n)}[\|\boldsymbol{x}\|_2 \geq 4\sqrt{\alpha n}] \leq e^{-2\alpha n}$, hence the restriction $Q := K \cap 4\sqrt{\alpha n}B_2^n$ still has $\gamma_n(Q) \geq \gamma_n(K) - e^{-2\alpha n} \geq e^{-2\alpha n}$ for n large enough. Then by the previous Lemma we know that $w(Q) \geq \frac{\sqrt{n}}{2e^{2\alpha}}$. For a vector $\boldsymbol{x}$, let $\boldsymbol{z}(\boldsymbol{x}) := \mathrm{argmax}\{\langle \boldsymbol{z},\boldsymbol{x}\rangle : \boldsymbol{z} \in Q\}$. As we just showed, $\mathbb{E}_{\boldsymbol{x}\sim N(\mathbf{0},\boldsymbol{I}_n)}[\langle \boldsymbol{z}(\boldsymbol{x}), \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}\rangle] \geq \frac{\sqrt{n}}{2e^{2\alpha}}$. Let $\lambda \in [0,1]$ be a parameter that we determine later. Note that the point $\lambda\cdot\boldsymbol{z}(\boldsymbol{x})$ lies in Q.



This point can be used to bound

$$\begin{aligned}
\mathop{\mathbb{E}}_{\boldsymbol{x}\sim N(\mathbf{0},\boldsymbol{I}_n)}[\|\boldsymbol{x}-\lambda\boldsymbol{z}(\boldsymbol{x})\|_2^2] &= \mathbb{E}[\|\boldsymbol{x}\|_2^2] - 2\lambda\,\mathbb{E}[\langle \boldsymbol{x},\boldsymbol{z}\rangle] + \mathbb{E}[\lambda^2\|\boldsymbol{z}\|_2^2] \\
&= \underbrace{\mathbb{E}[\|\boldsymbol{x}\|_2^2]}_{=n} - 2\lambda\underbrace{\mathbb{E}[\|\boldsymbol{x}\|_2]}_{\geq\frac{1}{2}\sqrt{n}}\cdot\underbrace{\mathop{\mathbb{E}}_{\boldsymbol{\theta}\in S^{n-1}}[\langle\boldsymbol{\theta},\boldsymbol{z}(\boldsymbol{\theta})\rangle]}_{\geq\sqrt{n}/(2e^{2\alpha})} + \mathbb{E}[\lambda^2\underbrace{\|\boldsymbol{z}\|_2^2}_{\leq 16\alpha n}] \\
&\leq n - \frac{1}{2}e^{-2\alpha}\lambda n + \lambda^2\cdot 16\alpha n \overset{\lambda:=\frac{1}{64\alpha e^{2\alpha}}}{=} n\cdot\Big(1-\frac{1}{256\alpha e^{4\alpha}}\Big)
\end{aligned}$$

Then

$$\mathbb{E}[d(\boldsymbol{x},Q)] \overset{\lambda\boldsymbol{z}\in Q}{\leq} \mathbb{E}[\|\boldsymbol{x}-\lambda\boldsymbol{z}\|_2] \overset{\text{Jensen}}{\leq} \mathbb{E}[\|\boldsymbol{x}-\lambda\boldsymbol{z}\|_2^2]^{1/2} \leq \sqrt{n}\cdot\sqrt{1-\frac{1}{256\alpha e^{4\alpha}}} \leq \sqrt{n}\cdot\Big(1-\frac{1}{512\alpha e^{4\alpha}}\Big)$$

using $\sqrt{1-x} \leq 1-\frac{x}{2}$ for $0 \leq x \leq 1$. □

Next, we show the average distance of a Gaussian to the cube $[-\varepsilon,\varepsilon]^n$ is $\sqrt{n}\cdot(1-\Theta(\varepsilon))$.

Lemma 22. *Let $\varepsilon > 0$. Then for n large enough one has*

$$\Pr_{\boldsymbol{x}\sim N(\boldsymbol{0},\boldsymbol{I}_n)}\Big[d(\boldsymbol{x},[-\varepsilon,\varepsilon]^n) \geq (1-5\varepsilon)\sqrt{n}\Big] \geq 1-\exp\Big(-\frac{\varepsilon^2}{2}n\Big)$$

Proof. Let $\boldsymbol{y} := \boldsymbol{y}(\boldsymbol{x}) := \operatorname{argmin}\{\|\boldsymbol{x}-\boldsymbol{y}\|_2 : \boldsymbol{y} \in [-\varepsilon,\varepsilon]^n\}$ be the closest point in the cube to $\boldsymbol{x}$. For an individual coordinate $i \in [n]$ the expected contribution to the distance is

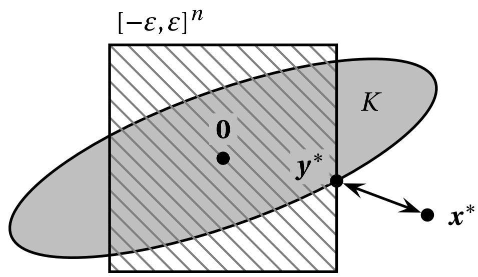
$$\mathbb{E}\left[d(x_i,[-\varepsilon,\varepsilon])^2\right] = \mathbb{E}\left[|x_i-y_i|^2\right] = \underbrace{\mathbb{E}[x_i^2]}_{=1} - 2\underbrace{\mathbb{E}[x_iy_i]}_{\leq\varepsilon\mathbb{E}[|x_i|]} + \underbrace{\mathbb{E}[y_i^2]}_{\geq 0} \geq 1-2\sqrt{\frac{2}{\pi}}\cdot\varepsilon \geq 1-2\varepsilon.$$

Then by linearity $\mathbb{E}[d(\boldsymbol{x},[-\varepsilon,\varepsilon]^n)^2]^{1/2} \geq \sqrt{n\cdot(1-2\varepsilon)} \geq \sqrt{n}\cdot(1-2\varepsilon)$. Recall that the distance function $F(\boldsymbol{x}) := d(\boldsymbol{x},[-\varepsilon,\varepsilon]^n)$ is 1-Lipschitz and for such functions the difference $|\mathbb{E}[F(\boldsymbol{x})]-\mathbb{E}[F(\boldsymbol{x})^2]^{1/2}|$ is bounded by an absolute constant. Then $\mathbb{E}[F(\boldsymbol{x})] \geq \sqrt{n}\cdot(1-4\varepsilon)$ for n large enough. Finally by Theorem 14 one has $\Pr[F(\boldsymbol{x}) < \mathbb{E}[F(\boldsymbol{x})]-\varepsilon\sqrt{n}] \leq e^{-\varepsilon^2 n/2}$ for $\boldsymbol{x} \sim N(\boldsymbol{0},\boldsymbol{I}_n)$ which then gives the claim as $\mathbb{E}[F(\boldsymbol{x})]-\varepsilon\sqrt{n} \geq (1-5\varepsilon)\sqrt{n}$. ☐

We will now prove Theorem 19. Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body with $\gamma_n(K) \geq e^{-\alpha n}$. Instead of providing a vector $\boldsymbol{x} \in (\frac{1}{\varepsilon}K)\cap[-1,1]^n$ directly, we will instead find an $\boldsymbol{x} \in K\cap[-\varepsilon,\varepsilon]^n$ with $|\{i \in [n] : x_i \in \{-\varepsilon,\varepsilon\}\}| \geq \delta n$ where $\varepsilon,\delta > 0$ will be chosen small enough, depending on α — the result in Theorem 19 then follows by scaling $\boldsymbol{x}$ by $\frac{1}{\varepsilon}$. We will use the following algorithm:

(1) Pick $\boldsymbol{x}^* \sim N(\boldsymbol{0},\boldsymbol{I}_n)$ at random.

(2) Compute $\boldsymbol{y}^* := \operatorname{argmin}\{\|\boldsymbol{x}^*-\boldsymbol{y}\|_2 : \boldsymbol{y} \in K\cap[-\varepsilon,\varepsilon]^n\}$.



Note that the step (2) is a convex program which can be solved in polynomial time, see [GLS88]. Now we can finish the proof of Theorem 19.

Lemma 23. *If $\varepsilon,\delta > 0$ are chosen small enough (depending on α), then with probability $1-e^{-\Omega_{\varepsilon,\delta}(n)}$ one has $|\{i \in [n] : y_i^* \in \{-\varepsilon,\varepsilon\}\}| \geq \delta n$.*

Proof. For a set of indices $I \subseteq [n]$ we abbreviate $K(I) := \{\boldsymbol{x} \in K : |x_i| \leq \varepsilon\ \forall i \in I\}$ as the intersection of K with the slabs corresponding to coordinates in I. Consider the two events

$$\mathcal{E}_1 := \text{“}d(\boldsymbol{x}^*,K\cap[-\varepsilon,\varepsilon]^n) \geq (1-5\varepsilon)\cdot\sqrt{n}\text{”}$$

$$\mathcal{E}_2 := \text{“for all } I \subseteq [n] \text{ with } |I| \leq \delta n \text{ one has } d(\boldsymbol{x}^*,K(I)) \leq (1-10\varepsilon)\sqrt{n}\text{”}$$

We will see that both events $\mathcal{E}_1$ and $\mathcal{E}_2$ happen with overwhelming probability.
Claim I. *One has* $\Pr[\mathcal{E}_1] \geq 1 - \exp(-\frac{\varepsilon^2}{2} n)$.
Proof of Claim I. Follows from Lemma 22 as $d(\boldsymbol{x}^*, K \cap [-\varepsilon,\varepsilon]^n) \geq d(\boldsymbol{x}^*, [-\varepsilon,\varepsilon]^n)$.
Claim II. *If* $\varepsilon := \varepsilon(\alpha), \delta := \delta(\alpha) > 0$ *are small enough, then* $\Pr[\mathcal{E}_2] \geq 1 - e^{-\Theta_\varepsilon(n)}$.
Proof of Claim II. For an index set I with $|I| \leq \delta n$ one can lower bound the measure as

$$\gamma(K(I)) \overset{\text{Šidak-Kathri (Lem 11)}}{\geq} \gamma_n(K) \cdot \gamma_1([-\varepsilon,\varepsilon])^{|I|} \geq e^{-\alpha n} \cdot (\varepsilon/2)^{|I|} \geq e^{-\alpha n - \ln(\frac{2}{\varepsilon}) \cdot \delta n} \geq e^{-2\alpha n},$$

assuming $\delta > 0$ is chosen small enough so that $\ln(\frac{2}{\varepsilon}) \cdot \delta \leq \alpha$. Here we use that $\gamma_1([-\varepsilon,\varepsilon]) \geq 2\varepsilon \cdot \gamma_1(1/2) \geq 2\varepsilon \frac{1}{\sqrt{2\pi}} e^{-(1/2)^2/2} \geq \frac{\varepsilon}{2}$ for $0 < \varepsilon \leq \frac{1}{2}$. Let us abbreviate $\mathcal{I} := \{I \subseteq [n] : |I| \leq \delta n\}$ as the family of small index sets. Then by Lemma 21 we know that a fixed $I \in \mathcal{I}$ has $\mathbb{E}_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)}[d(\boldsymbol{x}, K(I))] \leq \sqrt{n} \cdot \left(1 - \frac{1}{512 \cdot (2\alpha) e^{8\alpha}}\right) \leq (1 - 20\varepsilon)\sqrt{n}$, if we choose $\varepsilon \leq \frac{1}{20 \cdot 512 \alpha e^{8\alpha}}$. Then by concentration one has $\Pr[d(\boldsymbol{x}, K(I)) > (1 - 10\varepsilon)\sqrt{n}] \leq \exp(-50\varepsilon^2 n)$, see Theorem 14. A useful bound is $|\mathcal{I}| \leq e^{2\delta \log_2(\frac{1}{\delta}) n} \leq e^{\varepsilon^2 n}$ if we choose δ small enough compared to ε. Then

$$\Pr[\mathcal{E}_2] \overset{\text{union bound}}{\leq} \sum_{I \in \mathcal{I}} \Pr\left[d(\boldsymbol{x}^*, K(I)) > (1-10\varepsilon)\sqrt{n}\right] \leq e^{\varepsilon^2 n} \cdot \exp(-50\varepsilon^2 n) \leq \exp\left(-40\varepsilon^2 n\right). \quad \square$$

Now we have everything to finish the proof. Fix an outcome of the vector $\boldsymbol{x}^*$ so that the events $\mathcal{E}_1$ and $\mathcal{E}_2$ are both true, and abbreviate $I^* := \{i \in [n] : y_i^* \in \{-\varepsilon, \varepsilon\}\}$. Suppose for the sake of contradiction that $|I^*| < \delta n$. Then

$$(1 - 10\varepsilon)\sqrt{n} \overset{\mathcal{E}_2 \text{ true \& } I^* \in \mathcal{I}}{\geq} d(\boldsymbol{x}^*, K(I^*)) \overset{(*)}{=} d(\boldsymbol{x}^*, K \cap [-\varepsilon,\varepsilon]^n) \overset{\mathcal{E}_1 \text{ true}}{\geq} (1 - 5\varepsilon)\sqrt{n}$$

which is a contradiction. Here the crucial argument for $(*)$ is that $d(\boldsymbol{x}^*, K \cap [-\varepsilon,\varepsilon]^n) = \min\{\|\boldsymbol{x}^* - \boldsymbol{y}\|_2 : \boldsymbol{y} \in K \text{ and } |y_i| \leq \varepsilon \ \forall i \in [n]\}$ is a *convex minimization* problem and the optimum value will not change if linear constraints are discarded that are not tight for the optimum $\boldsymbol{y}^*$, and the cube constraints for coordinates $I^* \setminus [n]$ are indeed not tight. $\square$

In order to obtain a full coloring $\boldsymbol{x} \in \{-1,1\}^n$ one typically applies the partial coloring lemma $O(\log n)$ times. This requires a slight variant of Theorem 19 where the set K is shifted (and the shift corresponds to the sum of vectors from previous iterations). It can also be convenient for applications to allow the intersection of K with a subspace, so we incorporate that feature as well:

Theorem 24. *For all* $\alpha, \beta > 0$*, there are constants* $\varepsilon := \varepsilon(\alpha,\beta)$ *and* $\delta := \delta(\alpha,\beta) > 0$ *so that the following holds: There is a randomized polynomial time algorithm which for a symmetric convex set* $K \subseteq \mathbb{R}^n$ *with* $\gamma_n(K) \geq e^{-\alpha n}$*, a shift* $\boldsymbol{y} \in [-1,1]^n$ *and a subspace* $H \subseteq \mathbb{R}^n$ *with* $\dim(H) \geq \beta n$*, finds an* $\boldsymbol{x} \in (\frac{1}{\varepsilon} K \cap H)$ *with* $\boldsymbol{x} + \boldsymbol{y} \in [-1,1]^n$ *and* $|\{i \in [n] : (\boldsymbol{x}+\boldsymbol{y})_i \in \{\pm 1\}\}| \geq \delta n$.

The proof is very similar to the arguments presented above; see Appendix A for details. Another extension which can be often convenient in applications is to color close to $(1-\beta)n$ many elements rather than δn for some small constant δ. We stated such a result earlier in Theorem 6. Now we are ready to prove it:

Proof of Theorem 6. The idea is to simply apply Theorem 24 a constant number of times until the desired number of elements is colored. We assume $\beta > \gamma$ since otherwise there is nothing to prove. We set $\boldsymbol{y}^{(0)} := \boldsymbol{y}$ and for $t \geq 0$ we set $S^{(t)} := \{i \in [n] : -1 < y_i^{(t)} < 1\}$. Suppose for some t we have constructed a sequence $\boldsymbol{y}^{(0)}, \ldots, \boldsymbol{y}^{(t)}$ and still $|S^{(t)}| \geq (1 - \beta + \gamma)n$. Let $K_{S^{(t)}} := \{\bar{\boldsymbol{x}} \in \mathbb{R}^{S^{(t)}} : (\bar{\boldsymbol{x}}, \boldsymbol{0}) \in K\}$ and note that $\gamma_{|S^{(t)}|}(K_{S^{(t)}}) \geq \gamma_n(K) \geq e^{-\alpha n} \geq \exp(-\frac{\alpha}{1-\beta+\gamma}|S^{(t)}|)$. Moreover $\dim(H_{S^{(t)}}) \geq \dim(H) - (n - |S^{(t)}|) \geq \beta n - (\beta - \gamma)n = \gamma n$. Hence by Theorem 24 there exists a $\boldsymbol{x}^{(t)}$ so that $\boldsymbol{y}^{(t+1)} := \boldsymbol{y}^{(t)} + \boldsymbol{x}^{(t)} \in [-1,1]^n$ with $\boldsymbol{x}^{(t)} \in (C' \cdot K \cap H)$ and $|S^{(t+1)}| \leq (1-\delta)|S^{(t)}|$ for some constants $C', \delta > 0$. As soon as we reach an iteration t with $|S^{(t)}| < (1 - \beta + \gamma)n$ we stop and return the desired vector $\boldsymbol{x} := \boldsymbol{x}^{(0)} + \ldots + \boldsymbol{x}^{(t-1)}$. ☐

5 Open problems

We conjecture that Theorem 2 can be improved to match Theorem 1:

Conjecture 1 ($\ell_p \to \ell_q$ version of Komlós conjecture)**.** *Given $n \leq m$, $1 \leq p \leq q \leq \infty$ and $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in B_p^m$, do there always exist signs $\boldsymbol{x} \in \{-1,1\}^n$ so that*

$$\Big\| \sum_{i=1}^n x_i \boldsymbol{a}_i \Big\|_q \leq C \sqrt{\min\Big(p, \log\Big(\frac{2m}{n}\Big)\Big)} \cdot n^{\max(0, 1/2 - 1/p) + 1/q},$$

for some universal constant $C > 0$?

Since Conjecture 1 is at least as hard as the Komlós conjecture, a more realistic goal would be to improve the full coloring of Theorem 2 by a factor of $(1/2 - 1/p + 1/q)^{-1/2}$ so as to match the best known bound of $O(\sqrt{\log n})$ for Komlós.

Recall that for a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $1 \leq p \leq \infty$, the *Schatten-p norm* is defined as $\|\boldsymbol{A}\|_{S(p)} := (\sum_{i=1}^n \sigma_i(\boldsymbol{A})^p)^{1/p}$ where $\sigma_i(\boldsymbol{A}) \geq 0$ is the ith *singular value* of the matrix. In particular $\|\boldsymbol{A}\|_{S(\infty)}$ is the maximum singular value and $\|\boldsymbol{A}\|_{S(1)}$ is known as *Trace norm* or *Nuclear norm*. One might wonder whether Theorem 1 could be extended for *matrices* instead of vectors in the corresponding Schatten norms. In fact this is not possible: even for $p = 2$ and $q = \infty$, there exist n rank-one matrices $\boldsymbol{A}_i := \boldsymbol{v}_i \boldsymbol{v}_i^\top \in \mathbb{R}^{n \times n}$ with unit $\boldsymbol{v}_i$ for which any fractional coloring has discrepancy $\Omega(\sqrt{n})$ in the operator norm ([Wea02], Section 3). It is still possible nevertheless that Corollary 3 extends in the following way:

Conjecture 2 (ℓ_p version of Matrix Spencer)**.** *Given $2 \leq p \leq \infty$ and symmetric $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n \in \mathbb{R}^{n \times n}$ with Schatten-p norm at most 1, can we always find signs $\boldsymbol{x} \in \{-1,1\}^n$ so that*

$$\Big\| \sum_{i=1}^n x_i \boldsymbol{A}_i \Big\|_{S(p)} \leq C\sqrt{n}$$

for some universal constant $C > 0$?

This is a more general form of the Matrix Spencer conjecture [Zou12], and one can show a weaker bound of $O(\sqrt{pn})$ with random signs similar to Lemma 10 using matrix concentration. In fact, it is an open problem to show even a partial coloring for Conjecture 2. This would be implied by the following measure lower bound:

Conjecture 3. *Given $1 \le p \le \infty$ and symmetric $\boldsymbol{A}_1, \dots, \boldsymbol{A}_n \in \mathbb{R}^{n \times n}$, can we show that*

$$K := \left\{ \boldsymbol{x} \in \mathbb{R}^n : \left\| \sum_{i=1}^n x_i \boldsymbol{A}_i \right\|_{S(p)} \le \left\| \left(\sum_{i=1}^n \boldsymbol{A}_i^2 \right)^{1/2} \right\|_{S(p)} \right\}$$

satisfies $\gamma_n(K) \ge 2^{-O(n)}$?

The reader may notice our techniques establish Conjecture 3 in the case where the matrices $\boldsymbol{A}_i$ are all diagonal.

References

[AAGM15] S. Artstein-Avidan, A. Giannopoulos, and V. Milman. *Asymptotic Geometric Analysis. Part I.* 2015.

[Ban93] W. Banaszczyk. Balancing vectors and convex bodies. *Studia Mathematica*, 106(1):93–100, 1993.

[Ban98] W. Banaszczyk. Balancing vectors and Gaussian measures of n-dimensional convex bodies. *Random Structures Algorithms*, 12(4):351–360, 1998.

[Ban10] N. Bansal. Constructive algorithms for discrepancy minimization. In *FOCS*, pages 3–10, 2010.

[BDGL18] N. Bansal, D. Dadush, S. Garg, and S. Lovett. The gram-schmidt walk: a cure for the banaszczyk blues. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2018, Los Angeles, CA, USA, June 25-29, 2018*, pages 587–597, 2018.

[BF81] J. Beck and T. Fiala. Integer-making theorems. *Discrete Applied Mathematics*, 3(1):1 – 8, 1981.

[BGMN05] F. Barthe, O. Guédon, S. Mendelson, and A. Naor. A probabilistic approach to the geometry of the ℓ_p^n -ball. *Ann. Probab.*, 33(2):480–513, 03 2005.

[BP90] K. Ball and A. Pajor. Convex bodies with few faces. *Proceedings of the American Mathematical Society*, 110(1):225–231, 1990.

[DGLN16] D. Dadush, S. Garg, S. Lovett, and A. Nikolov. Towards a constructive version of banaszczyk's vector balancing theorem. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2016, September 7-9, 2016, Paris, France*, pages 28:1–28:12, 2016.

[DNTT18] D. Dadush, A. Nikolov, K. Talwar, and N. Tomczak-Jaegermann. Balancing vectors in any norm. In *2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 1–10, 2018.

[ES18] R. Eldan and M. Singh. Efficient algorithms for discrepancy minimization in convex sets. *Random Struct. Algorithms*, 53(2):289–307, 2018.

[Gia97] A. Giannopoulos. On some vector balancing problems. *Studia Mathematica*, 122(3):225–234, 1997.

[GLS88] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*, volume 2 of *Algorithms and Combinatorics*. Springer, 1988.

[Glu89] E. D. Gluskin. Extremal properties of orthogonal parallelepipeds and their applications to the geometry of banach spaces. *Mathematics of the USSR-Sbornik*, 64(1):85, 1989.

[Kha67] C. G. Khatri. On certain inequalities for normal distributions and their applications to simultaneous confidence bounds. *Ann. Math. Statist.*, 38:1853–1867, 1967.

[LM12] S. Lovett and R. Meka. Constructive discrepancy minimization by walking on the edges. In *FOCS*, pages 61–67, 2012.

[LM17] R. Latala and D. Matlak. Royen's proof of the gaussian correlation inequality. *Geometric Aspects of Functional Analysis*, page 265âĂŞ275, 2017.

[LRR16] A. Levy, H. Ramadas, and T. Rothvoss. Deterministic discrepancy minimization via the multiplicative weight update method. *CoRR*, abs/1611.08752, 2016.

[Rot14] T. Rothvoß. Constructive discrepancy minimization for convex sets. In *55th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2014, Philadelphia, PA, USA, October 18-21, 2014*, pages 140–145, 2014.

[Roy14] T. Royen. A simple proof of the gaussian correlation conjecture extended to multivariate gamma distributions, 2014.

[Šid67] Z. Šidák. Rectangular confidence regions for the means of multivariate normal distributions. *J. Amer. Statist. Assoc.*, 62:626–633, 1967.

[Spe85] J. Spencer. Six standard deviations suffice. 1985.

[vH14] R. van Handel. Probability in high dimension. 2014.

[Wea02] N. Weaver. The kadison-singer problem in discrepancy theory, 2002.

[Zou12] A. Zouzias. A matrix hyperbolic cosine algorithm and applications. In *Automata, Languages, and Programming - 39th International Colloquium, ICALP 2012, Warwick, UK, July 9-13, 2012, Proceedings, Part I*, pages 846–858, 2012.

A Partial colorings in shifted sets

In this section we show the postponed proof of Theorem 24. It turns out that we need only an extension that can handle the intersection with a subspace — the shift can be obtained by a scaling argument. Hence in this section we will prove the following main technical theorem:

Theorem 25. *For any $\alpha, \beta > 0$, there are constants $\varepsilon := \varepsilon(\alpha, \beta) > 0$ and $\delta := \delta(\alpha, \beta) > 0$ so that the following holds: Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body with $\gamma_n(K) \geq e^{-\alpha n}$ and let $H \subseteq \mathbb{R}^n$ be a subspace with $\dim(H) \geq \beta n$. Then there is a randomized polynomial time algorithm that finds an $\boldsymbol{x} \in K \cap H$ so that $|\{i \in [n] : x_i \in \{-\varepsilon, \varepsilon\}\}| \geq \delta n$ with probability $1 - e^{-\Omega_{\varepsilon,\delta}(n)}$.*

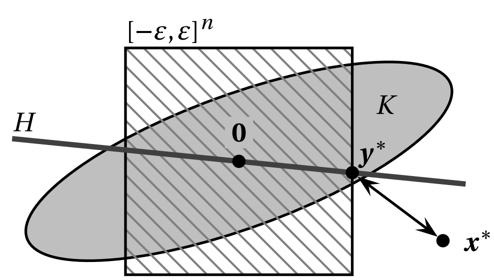
Before we prove Theorem 25 we argue how it implies the desired Theorem 24.

Proof of Theorem 24. Consider the input of Theorem 24 which is a set $K \subseteq \mathbb{R}^n$ with $\gamma_n(K) \geq e^{-\alpha n}$ and a subspace $H \subseteq \mathbb{R}^n$ with $\dim(H) \geq \beta n$. Instead of working with a translate $\boldsymbol{y}$, we allow asymmetric bounds $-2 \leq L_i < 0 < R_i \leq 2$ with $|R_i - L_i| \leq 2$ and the goal will be to find a vector $\boldsymbol{x} \in \frac{1}{\varepsilon'} K \cap H$ with a linear number of coordinates i satisfying $x_i \in \{L_i, R_i\}$. For symmetry reasons we may assume that $|R_i| \leq |L_i|$ for all $i \in [n]$, meaning that the upper boundary is the closer one for every coordinate. Note that then $0 < R_i \leq 1$. Now consider the linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ with $T(\boldsymbol{x}) := (\frac{x_1}{R_1}, \ldots, \frac{x_n}{R_n})$. Intuitively, this map stretches the ith coordinate axis by a factor of $\frac{1}{R_i} \geq 1$, which implies that $\gamma_n(T(K)) \geq \gamma_n(K)$. Now we apply Theorem 25 to the body $T(K)$ and the subspace $T(H)$. Let us suppose that the randomized algorithm is successful and delivers a vector $\boldsymbol{x} \in T(K) \cap T(H) \cap [-\varepsilon, \varepsilon]^n$ with $|x_i| = \varepsilon$ for at least δn many coordinates, where $\varepsilon, \delta > 0$ are the constants depending on α and β that make Theorem 25 work. Transforming this vector back to $\boldsymbol{y} := \frac{1}{\varepsilon} T^{-1}(\boldsymbol{x})$, we see that $\boldsymbol{y} \in \frac{1}{\varepsilon}(K \cap H)$ with $-R_i \leq y_i \leq R_i$ and $|y_i| = R_i$ for at least δn many coordinates $i \in [n]$. Then for at least one choice $\boldsymbol{z} \in \{-\boldsymbol{y}, \boldsymbol{y}\}$ one has $z_i = R_i$ for at least $\frac{\delta n}{2}$ many coordinates $i \in [n]$, while still $L_i \leq z_i \leq R_i$ for all $i \in [n]$. This concludes the claim. ☐

The algorithm for Theorem 25 is simply the previous one where K is replaced by $K \cap H$. We restate it for the sake of readability:

(1) Pick $\boldsymbol{x}^* \sim N(\boldsymbol{0}, \boldsymbol{I}_n)$ at random.

(2) Compute $\boldsymbol{y}^* := \operatorname{argmin}\{\|\boldsymbol{x}^* - \boldsymbol{y}\|_2 : \boldsymbol{y} \in K \cap H \cap [-\varepsilon, \varepsilon]^n\}$.



Luckily it suffices to prove one additional lemma to guarantee that a random Gaussian is close to the intersection $K \cap H$.

Lemma 26. *For any $\alpha, \beta > 0$ there are small enough constants $\varepsilon, \delta > 0$ so that the following holds for a convex symmetric body $K \subseteq \mathbb{R}^n$ with $\gamma_n(K) \geq e^{-\alpha n}$ and any subspace $H \subseteq \mathbb{R}^n$ with $\dim(H) \geq \beta n$. Let $I \subseteq [n]$ with $|I| \leq \delta n$ and abbreviate $K(I) := \{\boldsymbol{x} \in K : -\varepsilon \leq x_i \leq \varepsilon \ \forall i \in I\}$. Then*

$$\mathop{\mathbb{E}}_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)} \left[d(\boldsymbol{x}, K(I) \cap H) \right] \leq (1 - 20\varepsilon)\sqrt{n}.$$

Proof. We denote γ_H as the Gaussian measure restricted to a subspace H. Moreover, let $N(H)$ be the standard Gaussian in that same subspace. We can again lower bound the Gaussian measure of $K(I)$. We abbreviate $S_i := \{\boldsymbol{x} \in \mathbb{R}^n : |x_i| \leq \varepsilon\}$ as the strip in ith coordinate direction.

$$\begin{aligned}
\gamma_H(K(I) \cap H) &\overset{\text{Šidak-Kathri (Lem 11)}}{\geq} \gamma_H(K \cap H) \cdot \prod_{i \in I} \gamma_H(S_i \cap H) \\
&\overset{(*)}{\geq} \gamma_n(K) \cdot \prod_{i \in I} \gamma_n(S_i) \\
&\geq \gamma_n(K) \cdot (\varepsilon/2)^{|I|} \geq e^{-2\alpha n} \geq e^{-\frac{2\alpha}{\beta} \cdot \dim(H)}
\end{aligned}$$

assuming we choose ε, δ small enough. In $(*)$ we have used that $\gamma_H(K \cap H) \geq \gamma_n(K)$, as slices through the origin of a symmetric convex body maximize Gaussian measure.

Next, we use that by orthogonality one has

$$\begin{aligned}
\mathop{\mathbb{E}}_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)} \left[d(\boldsymbol{x}, K(I) \cap H)^2 \right] &= \underbrace{\mathop{\mathbb{E}}_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)} \left[d(\boldsymbol{x}, H)^2 \right]}_{= n - \dim(H)} + \underbrace{\mathop{\mathbb{E}}_{\boldsymbol{x} \sim N(H)} \left[d(\boldsymbol{x}, K(I) \cap H)^2 \right]}_{\leq \dim(H) \cdot (1 - \frac{1}{256 \cdot \frac{2\alpha}{\beta} \cdot \exp(4 \frac{2\alpha}{\beta})})} \\
&\overset{(**)}{\leq} n - \frac{1}{256 \cdot \frac{2\alpha}{\beta} \exp(4 \cdot \frac{2\alpha}{\beta})} \cdot \dim(H) \overset{(***)}{\leq} n \cdot (1 - 40\varepsilon)
\end{aligned}$$

where we had proven the inequality for $(**)$ already in Lemma 21. Morever $(***)$ follows from $\dim(H) \geq \beta n$ and choosing ε small enough. Consequently $\mathbb{E}_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)}[d(\boldsymbol{x}, K(I) \cap H)] \leq \mathbb{E}_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)}[d(\boldsymbol{x}, K(I) \cap H)^2]^{1/2} \leq \sqrt{n \cdot (1 - 40\varepsilon)} \leq \sqrt{n} \cdot (1 - 20\varepsilon)$ by Jensen's Inequality. $\square$

Now, let us revisit the proof of Lemma 23 and observe that the only properties for a body K that are needed for the projection algorithm to work are: (i) K is convex; (ii) one has $\mathbb{E}_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)}[d(\boldsymbol{x}, K(I))] \leq (1 - 20\varepsilon)\sqrt{n}$ for all $I \subseteq [n]$ with $|I| \leq \delta n$ for some constants $\varepsilon, \delta > 0$. But as we have just proven in Lemma 26, those same properties holds for $\tilde{K} := K \cap H$. That concludes the proof of Theorem 25 and hence the proof of Theorem 24.

B Large convex sets without partial colorings

We have mentioned earlier that a symmetric convex set K with measure $\gamma_n(K) \geq e^{-\delta n}$ contains a partial coloring $\boldsymbol{x} \in \{-1, 0, 1\}^n \setminus \{\boldsymbol{0}\}$ if the constant δ is small enough — but we claimed that this is false for constants beyond a certain threshold, even if one is allowed to rescale the body by some parameter dependent on δ. The construction for such a set is a very thin strip that avoids any point in $\{-1,0,1\}^n \setminus \{\mathbf{0}\}$.

Lemma 27. *For any $C \geq 1$, there exists a $\delta > 0$ so that the following holds: for any $n \in \mathbb{N}$ large enough there is a symmetric convex body $K \subseteq \mathbb{R}^n$ so that (i) $(C^n K) \cap (\{-1,0,1\}^n \setminus \{\mathbf{0}\}) = \emptyset$ and (ii) $\gamma_n(K) \geq e^{-\delta n}$.*

Proof. The construction is probabilistic. We sample a Gaussian $\boldsymbol{g} \sim N(\mathbf{0}, \boldsymbol{I}_n)$ and for a tiny parameter $s > 0$ that we determine later, we consider the strip $K := \{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{g}, \boldsymbol{x}\rangle| \leq s\}$. Consider the set of nontrivial partial colorings $X := \{-1,0,1\}^n \setminus \{\mathbf{0}\}$ and recall that $|X| \leq 3^n$. For any $\boldsymbol{x} \in X$, the distribution of $\langle \boldsymbol{g}, \boldsymbol{x}\rangle$ is Gaussian with variance $\|\boldsymbol{x}\|_2^2 \geq 1$ and hence the density of this 1-dimensional Gaussian is at most $\frac{1}{\sqrt{2\pi}} e^0 \leq \frac{1}{2}$ everywhere. In particular for a fixed $\boldsymbol{x} \in X$, one can obtain the simple estimate of $\Pr[|\langle \boldsymbol{g}, \boldsymbol{x}\rangle| \leq t] \leq 4t$ for any $t > 0$. Then choosing $s := \frac{1}{16} \cdot C^{-n} 3^{-n}$ we obtain

$$\Pr_{\boldsymbol{g}}\left[(C^n K) \cap X \neq \emptyset\right] \leq \sum_{\boldsymbol{x} \in X} \Pr_{\boldsymbol{g}}\left[|\langle \boldsymbol{g}, \boldsymbol{x}\rangle| > C^n s\right] \leq \frac{1}{4} \cdot |X| \cdot 3^{-n} \leq \frac{1}{4} \qquad (*)$$

Moreover using Markov's Inequality we obtain the (rather weak) estimate

$$\Pr\left[\|\boldsymbol{g}\|_2^2 > 4n\right] \leq \frac{1}{4} \qquad (**)$$

Then with probability at least $1/2$ none of the events $(*)$ and $(**)$ happen. We fix such an outcome of $\boldsymbol{g}$ and estimate that the measure of our strip is

$$\gamma_n(K) = \int_{-s/\|\boldsymbol{g}\|_2}^{s/\|\boldsymbol{g}\|_2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \geq \frac{1}{\sqrt{2\pi}} e^{-1/2} \frac{2s}{\sqrt{n}} \geq e^{-\delta n}$$

for a suitable choice of δ using $\frac{s}{\|\boldsymbol{g}\|_2} \leq 1$. ∎