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Thrifty Approximations of Convex Bodies by Polytopes

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Given a convex body $C \subset \mathbb{R}^d$ containing the origin in its interior and a real number $\tau > 1$, we seek to construct a polytope $P \subset C$ with as few vertices as possible such that $C \subset \tau P$. Our construction is nearly optimal for a wide range of d and τ. In particular, we prove that if $C = -C$, then for any $1 > \epsilon > 0$ and $\tau = 1 + \epsilon$ one can choose P having roughly $\epsilon^{-d/2}$ vertices and for $\tau = \sqrt{\epsilon d}$ one can choose P having roughly $d^{1/\epsilon}$ vertices. Similarly, we prove that if $C \subset \mathbb{R}^d$ is a convex body such that $-C \subset \mu C$ for some $\mu \geq 1$, then one can choose P having roughly $((\mu+1)/(\tau-1))^{d/2}$ vertices provided $(\tau-1)/(\mu+1) \ll 1$.

1 Introduction and Main Results

We discuss how well convex bodies (compact convex sets with nonempty interior) can be approximated by polytopes (convex hulls of finite sets of points). There is, of course, vast literature on the topic, as there are many different notions of approximation, see surveys [5, 9]. Our setup is as follows. Let $C \subset \mathbb{R}^d$ be a convex body containing the origin in its interior. We seek to construct a polytope $P \subset \mathbb{R}^d$ with as few vertices as possible, so that

$$P \subset C \subset \tau P,$$

for some given $\tau > 1$.

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Our first main result concerns *symmetric* convex bodies C for which $C=-C$ and τ measures the Banach–Mazur distance between P and C.

Theorem 1.1. Let k and d be positive integers and let $\tau > 1$ be a real number such that

$$(\tau - \sqrt{\tau^2 - 1})^k + (\tau + \sqrt{\tau^2 - 1})^k \geq 6\binom{d+k}{k}^{1/2}.$$

Then for any symmetric convex body $C \subset \mathbb{R}^d$, there is a symmetric polytope $P \subset \mathbb{R}^d$ with N vertices such that

$$N \leq 8\binom{d+k}{k}$$

and

$$P \subset C \subset \tau P. \qquad \square$$

In fact (see Remark 3.1), we can replace $\binom{d+k}{k}$ throughout the statement of Theorem 1.1 by a slightly smaller number

$$D(d,k) = \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{d+k-1-2m}{k-2m}. \tag{1}$$

For example, taking $d=20$ and $k=3$, we conclude that any 20-dimensional symmetric convex body can be approximated within a factor of $\tau = 3.18$ by a symmetric polytope with at most $12,480$ vertices.

Tuning up the parameter k in Theorem 1.1, we obtain different asymptotic regimes relating the dimension d, the number N of vertices of the approximating polytope, and the quality τ of the best approximation. Since the first version of this paper appeared as a preprint, in [11], the following compact formula describing the relation between τ, d, and N was deduced from Theorem 1.1:

$$\tau \leq \gamma \max\left\{1, \sqrt{\frac{d}{\ln N} \cdot \ln \frac{d}{\ln N}}\right\},$$

for an absolute constant $\gamma > 0$. If τ is close to 1, the above bound can be essentially sharpened.

Corollary 1.2. For any

$$\gamma > \frac{e}{4\sqrt{2}} \approx 0.48,$$

there exists $\epsilon_0 = \epsilon_0(\gamma) > 0$ such that for any $0 < \epsilon < \epsilon_0$ and for any symmetric convex body $C \subset \mathbb{R}^d$, there is a symmetric polytope $P \subset \mathbb{R}^d$ with N vertices such that

$$N \leq \left(\frac{\gamma}{\sqrt{\epsilon}} \ln \frac{1}{\epsilon} \right)^d$$

and

$$P \subset C \subset (1+\epsilon)P. \qquad \square$$

The well-known volumetric argument (see, e.g., [13, Lemma 4.10]) produces polytopes with roughly $(3/\epsilon)^d$ vertices which approximate a given symmetric d-dimensional convex body within a factor of $1+\epsilon$. Hence, for small $\epsilon > 0$, the estimate of Corollary 1.2 gives us roughly the square root of the number of vertices required by the volumetric bound.

Dudley [7] and, independently, Bronshtein and Ivanov [6] considered how well a given convex body can be approximated *in the Hausdorff metric*. They proved that for any convex body C contained in the unit ball in $\mathbb{R}^d$ and for any $\epsilon > 0$, there is a polytope P with $N \leq (\gamma/\epsilon)^{(d-1)/2}$ vertices which approximates C within distance ϵ in the Hausdorff metric, where $\gamma > 0$ is an absolute constant. To convert such an approximation into a $(1+\epsilon)$-approximation in the Banach–Mazur distance considered in Corollary 1.2, one would need, generally speaking, to increase N roughly by a factor of $d^{d/4}$ (since a symmetric convex body C inscribed in the unit ball can be rather thin, with the radius of the largest inscribed ball no bigger than $d^{-1/2}$).

For a given convex body C with a $\mathcal{C}^2$-smooth boundary, the asymptotic of the number N of vertices of the best approximating polytope (both in the Hausdorff and in the Banach–Mazur metrics) as $\epsilon \longrightarrow 0$ was obtained by Gruber [10], see also [3] for sharpening. In this case, for any $0 < \epsilon < \epsilon_0(C)$, we have $N \leq (\gamma/\epsilon)^{(d-1)/2}$ for some absolute constant $\gamma > 0$ and the bound is attained when C is the Euclidean ball (note that the upper bound for ϵ depends on the convex body C). No such results appear to be known for nonsmooth bodies C.

Summarizing, the estimate of Corollary 1.2 is the first bound improving the volumetric bound uniformly over all symmetric convex bodies C of all dimensions d.

Next, we consider approximations for which we want to keep the number of vertices of the polytope polynomial in the dimension of the ambient space.

Corollary 1.3. For any

$$\gamma > \frac{\sqrt{e}}{2} \approx 0.82,$$

there is a positive integer $k_0 = k_0(\gamma)$ such that for any $k > k_0$ and for any symmetric convex body $C \subset \mathbb{R}^d$ of a sufficiently large dimension $d > d_0(k)$ there is a symmetric polytope $P \subset \mathbb{R}^d$ with N vertices such that

$$N \le 8\binom{d+k}{k}$$

and

$$P \subset C \subset \gamma\sqrt{\frac{d}{k}}P. \qquad \square$$

In other words, for any fixed $0 < \epsilon < 1$, any d-dimensional symmetric convex body C can be approximated within a factor of $\tau = \sqrt{\epsilon d}$ by a polytope P with roughly $d^{1/\epsilon}$ vertices. A simple computation shows that if C is the d-dimensional Euclidean ball and P has at most d^k vertices for some fixed k, then P cannot approximate C better than within a factor of $\tau = \gamma\sqrt{\frac{d}{k \ln d}}$ as d grows, where $\gamma > 0$ is an absolute constant, see [1].

Finally, we consider approximations of not necessarily symmetric convex bodies. We prove the following main result, generalizing Theorem 1.1. The quality of approximation depends on the *symmetry coefficient* of the convex body C, that is, on the smallest $\mu \ge 1$ such that $-C \subset \mu C$ (recall that the convex bodies we consider contain the origin in their interior).

Theorem 1.4. Let d and k be positive integers. For $\tau, \mu \ge 1$, let us define

$$\lambda = \lambda(\tau, \mu) = \frac{2}{\mu+1}\tau + \frac{\mu-1}{\mu+1} \ge 1.$$

If

$$(\lambda - \sqrt{\lambda^2 - 1})^k + (\lambda + \sqrt{\lambda^2 - 1})^k \ge 6\binom{d+k}{k}^{1/2},$$

then for any convex body $C \subset \mathbb{R}^d$ containing the origin in its interior and such that $-C \subset \mu C$, there is a polytope $P \subset \mathbb{R}^d$ with N vertices such that

$$N \le 8\binom{d+k}{k}$$

and

$$P \subset C \subset \tau P. \qquad \square$$

We also obtain the following extension of Corollary 1.2.

Corollary 1.5. (1) For $\tau, \mu \geq 1$, let us define

$$\delta = \delta(\tau, \mu) = \frac{2(\tau - 1)}{\mu + 1}.$$

For any

$$\gamma > \frac{e}{4\sqrt{2}} \approx 0.48,$$

there exists $\delta_0 = \delta_0(\gamma) > 0$ such that as long as $\delta(\tau, \mu) < \delta_0$, for any convex body $C \subset \mathbb{R}^d$ such that $-C \subset \mu C$ there exists a polytope with N vertices such that

$$N \leq \left(\frac{\gamma}{\sqrt{\delta}} \ln \frac{1}{\delta}\right)^d$$

and

$$P \subset C \subset \tau P.$$

(2) For any

$$\gamma > \frac{e}{8} \approx 0.34,$$

there exists $\epsilon_0 = \epsilon(\gamma) > 0$ such that for any $0 < \epsilon < \epsilon_0$ and for any convex body $C \subset \mathbb{R}^d$ such that $-C \subset \mu C$ for some $\mu \geq 1$ there exists a polytope $P \subset \mathbb{R}^d$ with N vertices such that

$$N \leq \left(\gamma \sqrt{\frac{\mu + 1}{\epsilon}} \ln \frac{1}{\epsilon}\right)^d$$

and

$$P \subset C \subset (1 + \epsilon) P.$$ □

As a function of the symmetry coefficient μ, the number of vertices of P grows roughly as $\mu^{d/2}$ as long as the ratio τ/μ is small enough. One can deduce from results of Gruber [10] that if the boundary of C is $\mathcal{C}^2$-smooth, then for all sufficiently small $0 < \epsilon < \epsilon_0(C)$ one can construct a polytope P with not more than $\mu^{d/2}(\gamma/\epsilon)^{(d-1)/2}$ vertices for some absolute constant γ which approximates C within a factor of $1 + \epsilon$. The estimates of Corollary 1.5 are uniform over all convex bodies C of all dimensions d.

The plan of the paper is as follows. In Section 2, we collect some facts needed for the proofs of Theorems 1.1 and 1.4. We recall the classical result on the John decomposition of the identity operator and the minimum volume ellipsoid of a convex body, a recent result of Batson et al. [2] which allows one to obtain certain "sparsification" of the John decomposition, the standard construction of tensor product from multilinear algebra which allows us to translate polynomial relations among vectors into linear identities among tensors, and the classical construction of the Chebyshev polynomials which solve a relevant extremal problem. As it turns out, the vertices of the approximating polytopes P are picked up by certain algebraic conditions.

We complete the proofs in Section 3.

2 Preliminaries

2.1 Chebyshev polynomials

For a positive integer k, let $T_k(t)$ be the Chebyshev polynomial of degree k, see, for example, [4, Section 2.1]. Thus, for real t the polynomial $T_k(t)$ can be defined by

$$T_k(t) = \cos(k \arccos t) \quad \text{provided } -1 \le t \le 1$$

and

$$T_k(t) = \tfrac{1}{2}(t - \sqrt{t^2-1})^k + \tfrac{1}{2}(t + \sqrt{t^2-1})^k \quad \text{provided } |t| \ge 1.$$

In particular,

$$|T_k(t)| \le 1 \quad \text{provided } |t| \le 1. \tag{2}$$

Writing $T_k(t)$ in the standard monomial basis, we obtain

$$T_k(t) = \frac{k}{2} \sum_{m=0}^{\lfloor k/2 \rfloor} (-1)^m \frac{(k-m-1)!}{m!(k-2m)!} (2t)^{k-2m}.$$

In particular,

$$T_1(t) = t, \quad T_2(t) = 2t^2 - 1, \quad T_3(t) = 4t^3 - 3t, \quad T_4(t) = 8t^4 - 8t^2 + 1.$$

We note that $T_k(-t) = T_k(t)$ if k is even and $T_k(-t) = -T_k(t)$ if k is odd. We also note that the polynomial $T_k(t)$ is strictly increasing for $t \ge 1$.

In particular,

$$|T_k(t)| > \frac{(\tau - \sqrt{\tau^2-1})^k + (\tau + \sqrt{\tau^2-1})^k}{2} \quad \text{provided } |t| > \tau \ge 1. \tag{3}$$

The polynomial $T_k(t)$ has the following extremal property relevant to us: for any $t_0 \notin [-1, 1]$, the maximum value of $|p(t_0)|$, where p is a polynomial of $\deg p \le k$ such that $|p(t)| \le 1$ for all $t \in [-1, 1]$, is attained for $p = T_k$, see, for example, [4, Section 5.1].

2.2 Tensor power

Let V be Euclidean space with scalar product $\langle \cdot, \cdot \rangle$. For a positive integer k, let

$$V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}$$

be the kth tensor power of V. We consider $V^{\otimes k}$ as Euclidean space endowed with scalar product $\langle \cdot, \cdot \rangle$ such that

$$\langle x_1 \otimes \cdots \otimes x_k,\ y_1 \otimes \cdots \otimes y_k \rangle = \prod_{i=1}^{k} \langle x_i, y_i \rangle,$$

for all $x_1, \ldots, x_k; y_1, \ldots, y_k \in V$. The space $V^{\otimes 2}$ is naturally identified with the space of all linear operators on V.

The symmetric part $\mathrm{Sym}(V^{\otimes k})$ of $V^{\otimes k}$ is the subspace spanned by the tensors

$$x^{\otimes k} = \underbrace{x \otimes \cdots \otimes x}_{k \text{ times}},$$

for $x \in V$. The space $\mathrm{Sym}(V^{\otimes k})$ is naturally identified with the space of all homogeneous polynomials $p: V \longrightarrow \mathbb{R}$ of degree k. In particular, $\mathrm{Sym}(V^{\otimes 2})$ can be identified with the space of quadratic forms on V and also with the space of all symmetric operators on V. We have

$$\dim \mathrm{Sym}(V^{\otimes k}) = \binom{\dim V + k - 1}{k}.$$

Let us consider the direct sum

$$W = \mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes k},$$

as Euclidean space with the standard scalar product, which we also denote by $\langle \cdot, \cdot \rangle$. For a real univariate polynomial $a(t)$ and a vector $x \in V$, we denote by $a^{\otimes}(x) \in W$ the vector

$$a^{\otimes}(x) = \alpha_0 \oplus \alpha_1 x \oplus \alpha_2 x^{\otimes 2} \oplus \cdots \oplus \alpha_k x^{\otimes k} \quad \text{where } a(t) = \sum_{m=0}^{k} \alpha_m t^m. \tag{4}$$

It is then easy to check that for any $x, y \in V$ and any polynomials $a(t)$ and $b(t)$, we have

$$\langle a^{\otimes}(x),\ b^{\otimes}(y)\rangle = c(\langle x, y\rangle),$$

$$\text{provided } a(t)=\sum_{m=0}^{k}\alpha_m t^m,\ b(t)=\sum_{m=0}^{k}\beta_m t^m, \quad \text{and} \quad c(t)=\sum_{m=0}^{k}(\alpha_m\beta_m)t^m. \tag{5}$$

2.3 The ellipsoid of the minimum volume

As is known, for any compact set $C \subset \mathbb{R}^d$, there is a unique ellipsoid of the minimum volume among all ellipsoids centered at the origin and containing C. If the minimum-volume ellipsoid is the unit ball

$$B = \{x \in \mathbb{R}^d : \ \|x\| \le 1\},$$

where $\|\cdot\|$ is the Euclidean norm, the contact points $x_i \in C \cap \partial B$ provide a certain decomposition of the identity operator I, called the John decomposition (recall that $x \otimes x$ for $x \in \mathbb{R}^d$ is interpreted as a $d \times d$ symmetric matrix). We need the following result, see, for example, [1].

Theorem 2.1. Let $C \subset \mathbb{R}^d$ be a compact set that spans $\mathbb{R}^d$ and let $B \subset \mathbb{R}^d$ be the unit ball. Suppose that $C \subset B$ and that B has the smallest volume among all ellipsoids centered at the origin and containing C. Then there exist points $x_1, \ldots, x_n \in C \cap \partial B$ and nonnegative real $\alpha_1, \ldots, \alpha_n$ such that

$$\sum_{i=1}^{n} \alpha_i (x_i \otimes x_i) = I,$$

where I is the identity operator on $\mathbb{R}^d$. Equivalently,

$$\sum_{i=1}^{n} \alpha_i \langle x_i, y\rangle^2 = \|y\|^2,$$

for every $y \in \mathbb{R}^d$. □

2.4 Sparsification

We need a recent result of Batson, Spielman, and Srivastava on a certain "sparsification" of the conclusion of Theorem 2.1, see also [12]. Namely, we want to be able to choose the number n of points in Theorem 2.1 linear in the dimension d at the cost of a controlled corruption of the identity operator I.

If A and B are $d \times d$ symmetric matrices, we say that $A \preceq B$ if $B - A$ is positive semidefinite. The following result is from [2].

Theorem 2.2. Let $\gamma > 1$ be a number and let $x_1, \ldots, x_n$ be vectors in $\mathbb{R}^d$ such that

$$\sum_{i=1}^{n} x_i \otimes x_i = I,$$

or equivalently,

$$\sum_{i=1}^{n} \langle x_i, y \rangle^2 = \|y\|^2,$$

for all $y \in \mathbb{R}^d$. Then there is a subset $J \subset \{1, \ldots, n\}$ with $|J| \leq \gamma d$ and $\beta_j > 0$ for $j \in J$ such that

$$I \preceq \sum_{j \in J} \beta_j (x_j \otimes x_j) \preceq \left(\frac{\gamma + 1 + 2\sqrt{\gamma}}{\gamma + 1 - 2\sqrt{\gamma}} \right) I,$$

or equivalently,

$$\|y\|^2 \leq \sum_{j \in J} \beta_j \langle x_j, y \rangle^2 \leq \left(\frac{\gamma + 1 + 2\sqrt{\gamma}}{\gamma + 1 - 2\sqrt{\gamma}} \right) \|y\|^2,$$

for all $y \in \mathbb{R}^d$. □

3 Proofs

We start with a lemma (a similar result was recently obtained by similar methods by Gluskin and Litvak, see [8, Lemma 4.2]).

Lemma 3.1. Let $C \subset \mathbb{R}^d$ be a compact set. Then there is a subset $X \subset C$ of

$$|X| \leq 4d,$$

points such that for any linear function $\ell : \mathbb{R}^d \longrightarrow \mathbb{R}$ we have

$$\max_{x \in X} |\ell(x)| \leq \max_{x \in C} |\ell(x)| \leq 3\sqrt{d} \max_{x \in X} |\ell(x)|.$$ □

Proof. Without loss of generality, we assume that C spans $\mathbb{R}^d$. Applying a linear transformation, if necessary, we may assume that C is contained in the unit ball B and that B is the minimum-volume ellipsoid among all ellipsoids centered at the origin and containing C. By Theorem 2.1, there exist vectors $x_1, \dots, x_n \in C \cap \partial B$ and numbers $\alpha_1, \dots, \alpha_n \geq 0$ such that

$$\sum_{i=1}^{n} \alpha_i (x_i \otimes x_i) = I.$$

Applying Theorem 2.2 with $\gamma = 4$ to vectors $\sqrt{\alpha_i} x_i$, we conclude that for some $J \subset \{1, \dots, n\}$ and $\beta_j > 0$ for $j \in J$ we have

$$I \preceq \sum_{j \in J} \alpha_j \beta_j (x_j \otimes x_j) \preceq 9I, \tag{6}$$

and $|J| \leq 4d$. We let

$$X = \{x_j : \ j \in J\}.$$

In particular, $x_j \in C$ and $\|x_j\| = 1$ for all $j \in J$. Comparing the traces of the operators in (6), we get

$$d \leq \sum_{j \in J} \alpha_j \beta_j \leq 9d. \tag{7}$$

A linear function $\ell : \mathbb{R}^d \longrightarrow \mathbb{R}$ can be written as $\ell(x) = \langle y, x \rangle$ for some $y \in \mathbb{R}^d$. It follows by (6) that

$$\sum_{j \in J} (\alpha_j \beta_j) \langle y, x_j \rangle^2 \geq \|y\|^2,$$

and then by (7) it follows that

$$|\langle y, x_j \rangle| \geq \frac{1}{3\sqrt{d}} \|y\| \quad \text{for some } j \in J.$$

Since $C \subset B$, we have

$$\max_{x \in C} |\langle y, x \rangle| \leq \|y\|,$$

and the proof follows. ■

We now prove Theorem 1.1.

Proof of Theorem 1.1. Let us denote $V = \mathbb{R}^d$ and let us consider the space

$$W = \mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes k},$$

see Section 2.2. Let us define a continuous map $\phi : V \longrightarrow W$ by

$$\phi(x) = 1 \oplus x \oplus x^{\otimes 2} \oplus \cdots \oplus x^{\otimes k} \quad \text{for } x \in V.$$

We consider the compact set

$$\hat{C} = \{\phi(x) : \; x \in C\}, \quad \hat{C} \subset W.$$

We note that $\hat{C}$ lies in the subspace

$$\mathbb{R} \oplus V \oplus \mathrm{Sym}(V^{\otimes 2}) \oplus \cdots \oplus \mathrm{Sym}(V^{\otimes k}).$$

In particular,

$$\dim \mathrm{span}(\hat{C}) \leq 1 + d + \binom{d+1}{2} + \cdots + \binom{d+k-1}{k} = \binom{d+k}{k}.$$

Applying Lemma 3.1 to $\hat{C}$, we conclude that there is a set $X \subset C$ such that

$$|X| \leq 4\binom{d+k}{k},$$

such that for any linear function $\mathcal{L} : W \longrightarrow \mathbb{R}$ we have

$$\max_{x\in X} |\mathcal{L}(\phi(x))| \leq \max_{x\in C} |\mathcal{L}(\phi(x))| \leq 3\binom{d+k}{k}^{1/2} \max_{x\in X} |\mathcal{L}(\phi(x))|. \tag{8}$$

We define P as the convex hull

$$P = \mathrm{conv}(X \cup -X).$$

Clearly, $P \subset C$ and P has at most $8\binom{d+k}{k}$ vertices. To conclude that P approximates C with the desired accuracy, we compare the maxima of linear functions $\ell : \mathbb{R}^d \longrightarrow \mathbb{R}$ on C and on P.

Suppose that

$$\ell(x) = \langle y, x\rangle \quad \text{for some } y \in V.$$

Let us define a linear function $\mathcal{L} : W \longrightarrow \mathbb{R}$ by

$$\mathcal{L}(w) = \langle T_k^{\otimes}(y), w\rangle \quad \text{for all } w \in W,$$

where T_k is the Chebyshev polynomial of degree k, see Section 2.1 and (4). Then by (5), we have

$$\mathcal{L}(\phi(x)) = T_k(\langle y, x\rangle).$$

Hence, from (8) we obtain

$$\max_{x\in X}|T_k(\ell(x))| \le \max_{x\in C}|T_k(\ell(x))| \le 3\binom{d+k}{k}^{1/2}\max_{x\in X}|T_k(\ell(x))|. \tag{9}$$

Suppose that $\ell(x)\le 1$ for all $x\in P$ and hence $|\ell(x)|\le 1$ for all $x\in X$. Then by (2) we have $|T_k(\ell(x))|\le 1$ for all $x\in X$. If for some $x\in C$, we have $\ell(x)>\tau$, then by (3) we have

$$|T_k(\ell(x))| > \frac{(\tau-\sqrt{\tau^2-1})^k+(\tau+\sqrt{\tau^2-1})^k}{2} \ge 3\binom{d+k}{k}^{1/2},$$

which contradicts (9). Therefore,

$$\max_{x\in P}\ell(x) \le \max_{x\in C}\ell(x) \le \tau\max_{x\in P}\ell(x), \tag{10}$$

for every linear function $\ell:\mathbb{R}^d \longrightarrow \mathbb{R}$, which proves that $C\subset \tau P$. ■

3.1 Remarks

One can sharpen the bounds somewhat by noticing that the polynomial T_k is even for even k and odd for odd k. Consequently, the map $\phi: V\longrightarrow W$ can be replaced by

$$\phi_e(x) = 1\oplus x^{\otimes 2}\oplus\cdots\oplus x^{\otimes k-2}\oplus x^{\otimes k},$$

for even k and by

$$\phi_o(x) = x\oplus x^{\otimes 3}\oplus\cdots\oplus x^{\otimes k-2}\oplus x^{\otimes k},$$

for odd k. This allows us to replace $\binom{d+k}{k}$ by $D(d,k)$ defined by (1) throughout the statement of Theorem 1.1.

Instead of using Theorem 2.2, one can choose a random sparsification of the John decomposition as in [14]. This will produce a slightly weaker bound for N in Theorem 1.1 by introducing an extra logarithmic factor but may appear to be useful for some applications.

Proof of Theorem 1.4. As in the proof of Theorem 1.1, we construct the space W, the map ϕ, the set $\hat{C}$, and the subset $X\subset C$ so that (8) holds. We then define P as the convex hull

$$P = \operatorname{conv}(X\cup(-1/\mu)X).$$

Clearly, $P \subset C$ and P has at most $8\binom{d+k}{k}$ vertices. To conclude that P approximates C with the desired accuracy, we compare the maxima of linear functions $\ell : \mathbb{R}^d \longrightarrow \mathbb{R}$ on C and on P.

Let T_k be the Chebyshev polynomial of degree k. We define a polynomial S_k by

$$S_k(t) = T_k\left(\frac{2}{\mu+1}t + \frac{\mu-1}{\mu+1}\right).$$

Hence, $\deg S_k(t) = k$. Moreover,

$$|S_k(t)| \leq 1 \quad \text{provided } -\mu \leq t \leq 1 \tag{11}$$

and

$$|S_k(t)| > \frac{(\lambda - \sqrt{\lambda^2-1})^k + (\lambda + \sqrt{\lambda^2-1})^k}{2} \quad \text{provided } t > \tau. \tag{12}$$

Given a linear function $\ell : \mathbb{R}^d \longrightarrow \mathbb{R}$,

$$\ell(x) = \langle y, x\rangle \quad \text{for some } y \in V,$$

we define a linear function $\mathcal{L} : W \longrightarrow \mathbb{R}$ by

$$\mathcal{L}(w) = \langle S_k^{\otimes}(y),\ w\rangle \quad \text{for all } w \in W.$$

Then

$$\mathcal{L}(\phi(x)) = S_k(\langle y, x\rangle).$$

Hence, from (8) we obtain

$$\max_{x\in X} |S_k(\ell(x))| \leq \max_{x\in C} |S_k(\ell(x))| \leq 3\binom{d+k}{k}^{1/2} \max_{x\in X} |S_k(\ell(x))|. \tag{13}$$

Suppose that $\ell(x) \leq 1$ for all $x \in P$. Then, necessarily, $1 \geq \ell(x) \geq -\mu$ for all $x \in X$ and hence by (11) we have $|S_k(\ell(x))| \leq 1$ for all $x \in X$. If for some $x \in C$, we have $\ell(x) > \tau$, then by (12)

$$|S_k(\ell(x))| > \frac{(\lambda - \sqrt{\lambda^2-1})^k + (\lambda + \sqrt{\lambda^2-1})^k}{2} \geq 3\binom{d+k}{k}^{1/2},$$

which contradicts (13). Hence, (10) holds for every linear function $\ell : \mathbb{R}^d \longrightarrow \mathbb{R}$ and, therefore, $C \subset \tau P$. ■

Proof of Corollary 1.2. Let us choose $\tau=1+\epsilon$ in Theorem 1.1. We use the standard estimate

$$\binom{d+k}{k} \leq \left(\frac{d+k}{k}\right)^k \left(\frac{d+k}{d}\right)^d \leq e^d \left(1+\frac{k}{d}\right)^d. \tag{14}$$

Let us choose

$$k = \left\lceil \frac{\beta d}{\sqrt{\epsilon}} \ln \frac{1}{\epsilon} \right\rceil, \tag{15}$$

where $\beta > 0$ is a constant. Then

$$\frac{1}{k} \ln\left(6 \binom{d+k}{d}^{1/2}\right) \leq \frac{\sqrt{\epsilon}}{4\beta}(1+o(1)), \tag{16}$$

where "$o(1)$" stands for a term that converges to 0 uniformly on d as $\epsilon \longrightarrow 0$.

On the other hand,

$$\ln(\tau + \sqrt{\tau^2-1}) = \sqrt{2\epsilon}(1+o(1)),$$

where "$o(1)$" stands for a term that converges to 0 as $\epsilon \longrightarrow 0$. Then, as long as

$$\beta > \frac{1}{4\sqrt{2}},$$

the condition of Theorem 1.1 is satisfied for all sufficiently small $0<\epsilon<\epsilon_0(\beta)$. The proof now follows by (14). ■

Proof of Corollary 1.5. To prove Part (1), we observe that $\lambda = 1+\delta$, and hence,

$$\ln(\lambda + \sqrt{\lambda^2-1}) = \sqrt{2\delta}(1+o(1)), \tag{17}$$

where "$o(1)$" stands for a term that converges to 0 as $\delta \longrightarrow 0$. In Theorem 1.4, let us choose k defined by (15) with ϵ replaced by δ. Comparing (16) with ϵ replaced by δ and (17), we conclude the proof as in the proof of Corollary 1.2.

To prove Part (2), in Theorem 1.4, we choose $\tau = 1+\epsilon$ and k defined by (15). Then

$$\ln(\lambda + \sqrt{\lambda^2-1}) = 2\left(\frac{\epsilon}{\mu+1}\right)^{1/2}(1+o(1)), \tag{18}$$

where "$o(1)$" stands for a term that converges to 0 uniformly on $\mu \geq 1$ as $\epsilon \longrightarrow 0$. Comparing (18) and (16), we conclude that the condition of Theorem 1.4 is satisfied for all sufficiently small $0 < \epsilon < \epsilon_0(\beta)$ as long as

$$\beta > \frac{\sqrt{\mu+1}}{8}.$$

The proof now follows by (14). ■

Proof of Corollary 1.3. Let us choose $\tau = \gamma\sqrt{d/k}$ in Theorem 1.1, where $\gamma > 0$ is a constant. Using Stirling's formula, we conclude that for each k

$$\lim_{d\longrightarrow\infty} \frac{1}{\sqrt{d}} 6^{1/k} \binom{d+k}{d}^{1/2k} = \sqrt{\frac{e}{k}}(1+o(1)),$$

where "$o(1)$" stands for a term that converges to 0 as k grows.

On the other hand, for each k

$$\lim_{d\longrightarrow\infty} \frac{\tau + \sqrt{\tau^2-1}}{\sqrt{d}} = \frac{2\gamma}{\sqrt{k}}.$$

The proof now follows by Theorem 1.1. ■

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