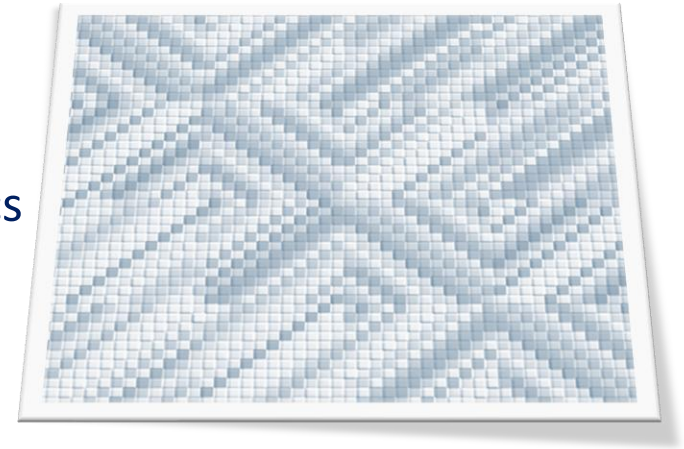
FURTHER INSIGHTS INTO THE EMBEDDING PROPERTIES OF HADAMARD MATRICES AND D-OPTIMAL DESIGNS

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Further insights into the embedding properties of Hadamard matrices and D-optimal designs

Topics:

1. Hadamard matrices – Basic properties and applications.
2. Remarks on the construction of Hadamard matrices.
3. Minors of (± 1)-matrices and the spectrum of the determinant function.
4. Hadamard equivalence and D-optimal designs.
5. Embeddability of Hadamard matrices of order $n-k$ in Hadamard matrices of order n.
6. Embeddability of D-optimal designs of order k in Hadamard matrices of order n.
7. Embeddability of D-optimal designs of order $n-1$ in D-optimal designs of order n.
8. Conclusions and further research.

Hadamard matrices

Hadamard was interested in finding the maximal determinant of square matrices with entries from the unit disc.

He showed (*Bull. Sciences Math.* 1893) that this maximal determinant, $n^{n/2}$, was achieved by matrices $X = [x_{ij}]_{n\times n}$ with entries ± 1 which satisfied the equality of the inequality:

$$|\det X|^2 \le \prod_{i=1}^{n}\sum_{j=1}^{n}|x_{ij}|^2 \quad , \quad XX^{\mathsf{T}} = I_n$$



Jacques Salomon Hadamard
1865 – 1963

J. Hadamard, Résolution d'une question relative aux déterminants. *Bull. Sci. Math.* 17 (1893), 240–246.

Hadamard matrices

A square matrix with elements ±1 and size n, whose distinct row vectors are orthogonal is an ***Hadamard matrix of order n***.

$n = 1$ $n = 2$ $n = 4$

H_1 H_2 H_4

$$[1], \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$



Jacques Salomon Hadamard
1865 – 1963

Basic properties of Hadamard matrices:

a) $H H^{\mathsf{T}} = n I_n$ b) $|\det H| = n^{n/2}$ c) $H H^{\mathsf{T}} = H^{\mathsf{T}} H$

The Hadamard conjecture

Furthermore, Hadamard observed that such matrices could exist only if n was 1, 2, or a multiple of 4.

This observation has formed the basis of one of the greatest unsolved mathematical problems.

The Hadamard Conjecture
There is a Hadamard matrix of order n for any natural number n multiple of 4.



Jacques Salomon Hadamard
1865 – 1963

Despite the efforts of several mathematicians, this observation remains unproven, even though it is widely believed that it is true.

Sylvester-Hadamard matrices

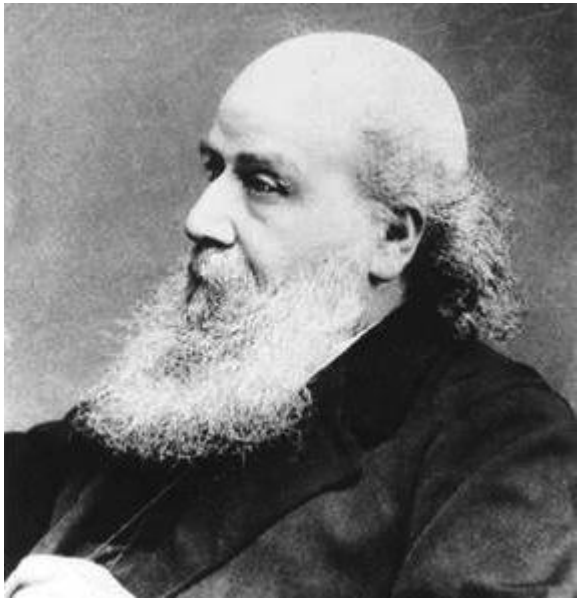
However, such matrices were first studied by Sylvester (*Phil. Mag.* 1867).

Now, if H is an Hadamard-type matrix, then

$$\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

is also an Hadamard-type matrix.

The matrices of order $\mathbf{2^k}$ constructed using Sylvester's construction are usually referred to as ***Sylvester-Hadamard*** matrices.

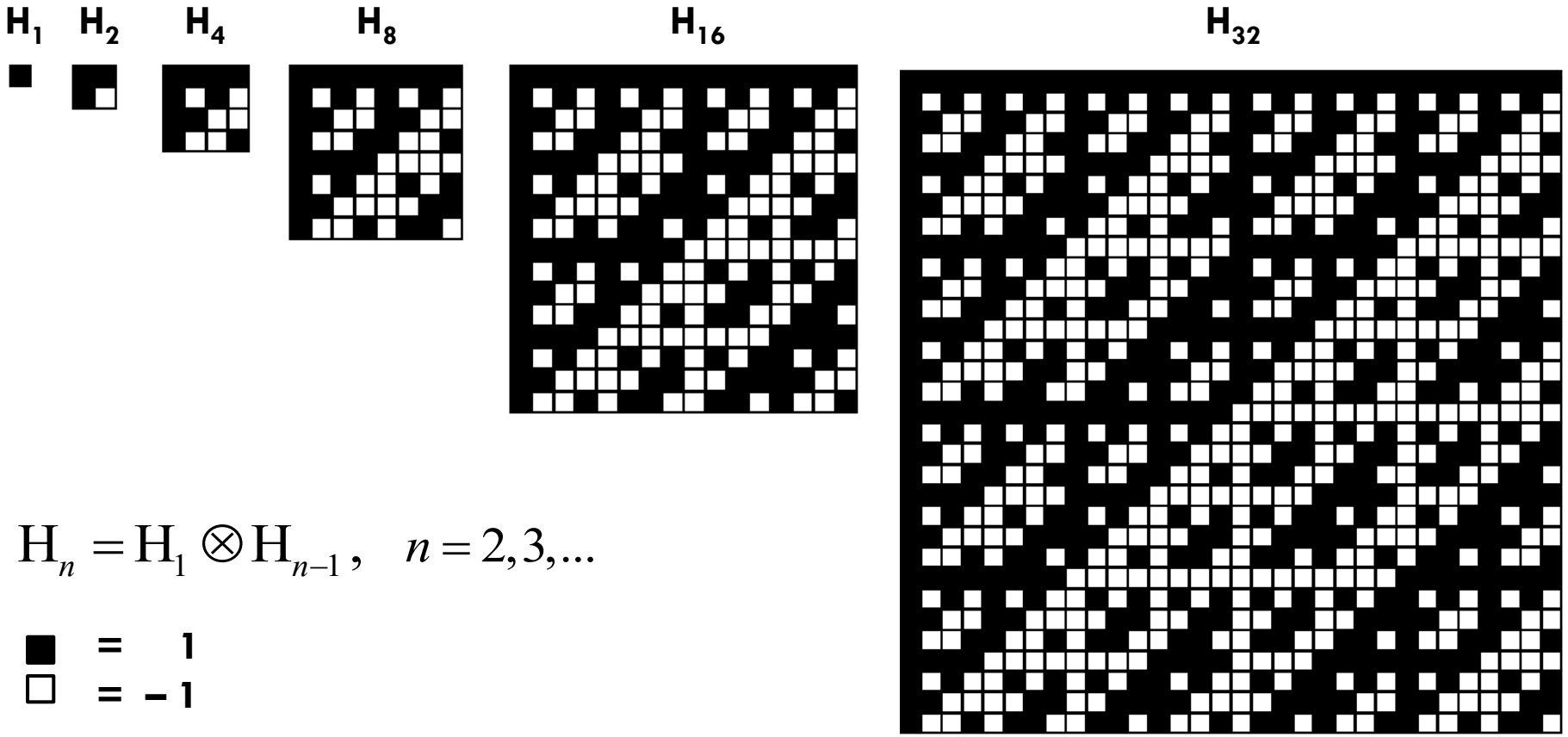


James Joseph Sylvester
1814 – 1897

***Lemma* (*Sylvester* 1867)**

There is an Hadamard-type matrix of order 2^k for all natural numbers k.

Visualization of Sylvester-Hadamard matrices



$$H_n = H_1 \otimes H_{n-1}, \quad n = 2,3,\ldots$$

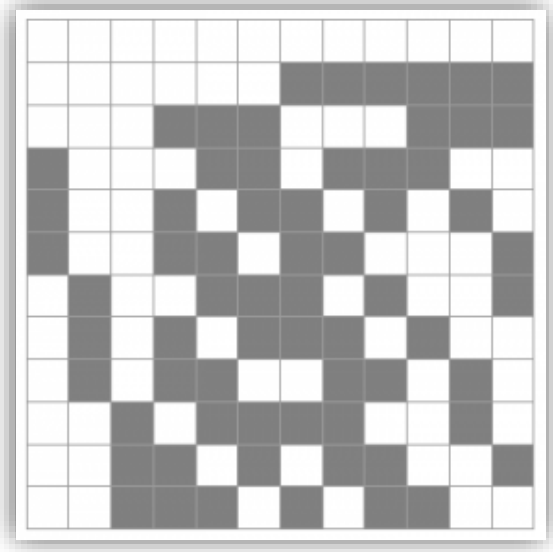
■ = 1

□ = −1

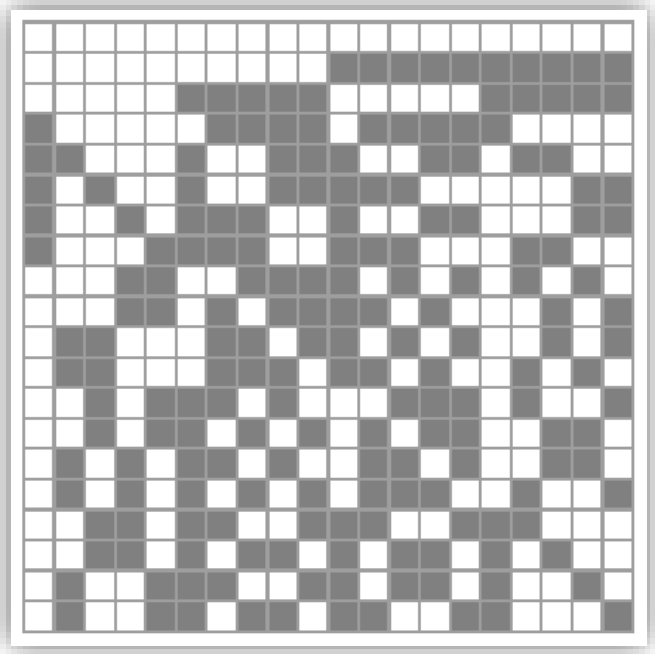
Hadamard's matrices

Sylvester's construction (1867) yields Hadamard matrices of order 1, 2, 4, 8, 16, 32, etc. Hadamard matrices of orders 12 and 20 were subsequently constructed by Hadamard in 1893.

12 × 12 Hadamard matrix



20 × 20 Hadamard matrix

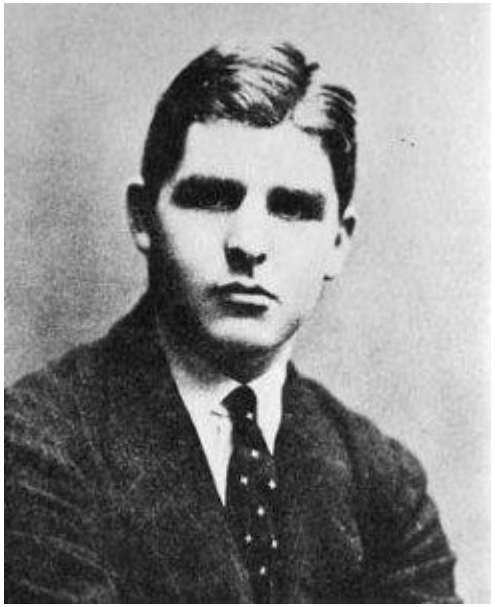


White square = 1
Gray square = –1

Construction of Hadamard matrices

In 1933, Raymond Paley discovered a construction that produces a Hadamard matrix of order $q + 1$ when q is any prime power that is congruent to 3 (mod 4) and that produces a Hadamard matrix of order $2(q+1)$ when q is a prime power that is congruent to 1 (mod 4).

The Hadamard conjecture should probably be attributed to Paley.



Raymond Paley
1907– 1933

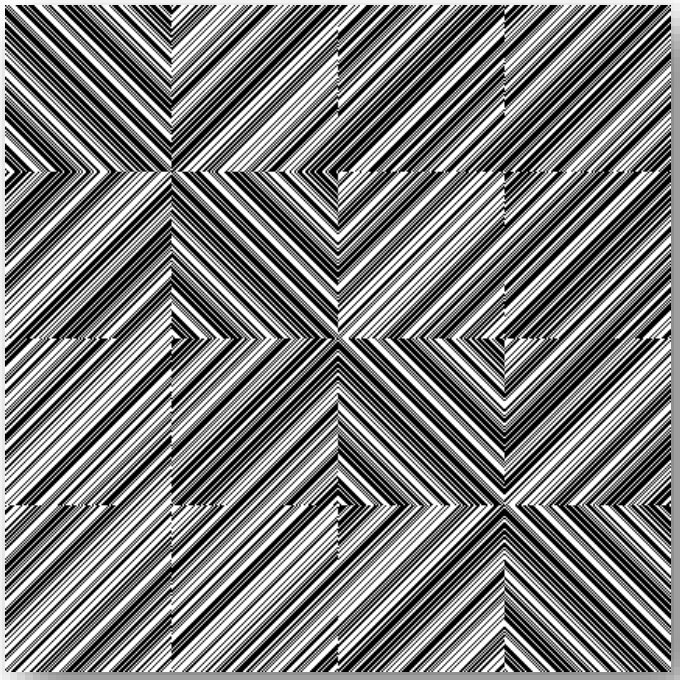
The smallest order that cannot be constructed by a combination of Sylvester's and Paley's methods is 92. An Hadamard matrix of this order was found using a computer by Baumert, Golomb, and Hall in 1962 at JPL. They used a construction, due to Williamson, that has yielded many additional orders.

Construction of Hadamard matrices

Different ***construction techniques*** of Hadamard matrices have been developed for a wide variety of applications:

- ***Sylvester's*** technique (1867)
- ***Paley's*** technique (1933)
- ***Williamson's*** technique (1944)
- ***Ahmed & Rao's*** technique (1975)
- ***Henderson's*** technique (1978)
- ***Golay's*** technique (1982)
- ***Lee & Kaveh's*** technique (1986)

…and others



428 × 428 Hadamard matrix
H. Kharaghani and B. Tayfeh-Rezaie, 2005

Construction of Hadamard matrices

Facts for Hadamard matrices:

- ✓ In 2005, Hadi Kharaghani and Behruz Tayfeh-Rezaie published their construction of an Hadamard matrix of order 428. As a result, the smallest order for which no Hadamard matrix is presently known is 668.

- ✓ As of 2008, there are 12 multiples of 4 less than or equal to 2000 for which no Hadamard matrix of that order is known. They are:
  668, 716, 892, 1132, 1244, 1388, 1436, 1676, 1772, 1916, 1948, and 1964.

- ✓ Every Hadamard matrix of order $n > 4$ contains a submatrix equivalent to:

- ✓ For any order n, $\boldsymbol{H_4 \in H_n}$ and $\boldsymbol{H_n \in H_{2n}}$.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ \hline 1 & 1 & - & - \\ 1 & - & - & 1 \end{array}\right]$$

Applications of Hadamard matrices

- **Signal processing, Coding and Cryptography**
  - Design of experiments
  - Object recognition
  - Coding of digital signals (CDMA telecommunications)
- **Spectral analysis or signal separation**
  - Mass spectroscopy
  - Polymer chemistry
  - Signal and information processing
  - Geophysics
  - Acoustics
  - Nuclear medicine and nuclear physics
- **Other novel applications**

  Digital logic design, pattern recognition, data compression, magnetic resonance imaging, neuroscience and quantum computing

Hadamard equivalence and D-optimal designs

Definition 1 (H-equivalence). *We call two Hadamard matrices Hadamard equivalent or H-equivalent if one can be obtained from the other by a sequence of the operations:*

i) interchange any pair of rows and/or columns,
ii) multiply any row and/or column through by-1.

Definition 2. *A $k \times k$ (± 1)-matrix with maximum determinant (in magnitude) is called a D-optimal design of order k and is denoted D_k. We also denote its determinant by $d_k = |det(D_k)|$. If the first row and the first column of D_k consists of all 1's is said to be a normalized form.*

Hadamard equivalence and D-optimal designs

A D-optimal design of order k **multiple of 4** is a Hadamard matrix of the same order, i.e., $\boldsymbol{D_k \equiv H_k}$ .Therefore, it achieves the maximum determinant value $\boldsymbol{d_k = k^{\frac{k}{2}}}$.

$$D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix}, \; d_4 = 16 \qquad D_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & 1 & - & - & - \\ 1 & 1 & - & 1 & - & 1 & - & - \\ 1 & 1 & - & - & - & - & 1 & 1 \\ 1 & - & 1 & 1 & - & - & 1 & - \\ 1 & - & 1 & - & - & 1 & - & 1 \\ 1 & - & - & 1 & 1 & - & - & 1 \\ 1 & - & - & - & 1 & 1 & 1 & - \end{bmatrix}, \; d_8 = 4096$$

Hadamard equivalence and D-optimal designs

A D-optimal design is a design that maximizes the determinant of the information matrix $X^T X$, where X is the design matrix.

Maximizing the determinant leads to minimizing the generalized variance of the estimated regression coefficients. Thus, the estimates will be as precise as possible, given the constraints of the design.

$$D_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & - \\ 1 & 1 & - & - & - \\ 1 & - & - & 1 & - \\ 1 & - & - & - & 1 \end{bmatrix}, \quad D_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & - & 1 \\ 1 & 1 & - & - & - & 1 \\ 1 & - & - & 1 & - & 1 \\ 1 & 1 & 1 & 1 & - & - \\ 1 & - & - & - & 1 & - \end{bmatrix}, \quad D_7 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & - & 1 & 1 \\ 1 & - & 1 & - & - & 1 & 1 \\ 1 & - & - & 1 & 1 & - & 1 \\ 1 & - & - & 1 & 1 & 1 & - \\ 1 & 1 & 1 & - & 1 & - & - \\ 1 & 1 & 1 & 1 & - & - & \end{bmatrix}$$

$$d_5 = 48, \quad d_6 = 160, \quad d_7 = 576$$

Minors of (±1)-matrices

Lemma (*Day & Peterson*, 1988)

Let A be an n×n matrix with elements ±1.
It holds that:

a) detA is an integer and $\mathbf{2^{n-1}}$ *divides detA*

b) when $n \leq 6$*, the only possible values for detA are the following, and they do all occur:*

n	det*A*
1	1
2	0, 2
3	0, 4
4	0, 8, 16
5	0, 16, 32, 48
6	0, 32, 64, 96, 128, 160

If M_m denotes the absolute value of an $m \times m$ minor of a (±1)-matrix of order $n \geq m$, then $\boldsymbol{M_m = p\ 2^{m-1}}$, where p is either a positive integer, or zero.

Definition 3. *The* spectrum of the determinant function for (±1)-matrices *is defined to be the set of values taken by* $p = 2^{1-m}\,|\det(R)|$ *as the matrix R ranges over all* $m \times m$ (±1)*-matrices. We denote this by* spec(m).

Spectrum of the determinant function

m	Confirmed spectrum spec$\{m\}$
1	1
2	[0, 1]
3	[0, 1]
4	[0, 2]
5	[0, 3]
6	[0, 5]
7	[0, 9]
8	[0, 18], 20, 24, 32
9	[0, 40], 42, [44, 45], 48, 56
10	[0, 102], [104, 105], 108, 110, 112, [116, 117], 120, 125, 128, 144
11	[0, 268], [270, 276], [278, 280], [282, 286], 288, 291, [294, 297], 304, 312, 315, 320
13	[0, 2172], [2174, 2185], [2187, 2196], [2199, 2202], 2205, 2208, 2210, 2211, [2214, 2218], [2220, 2226], 2228, 2229, 2230, 2232, 2233, 2235, 2238, 2240, 2241, [2243, 2245], 2247, 2248, 2250, 2253, 2256, [2258, 2260], 2262, 2264, 2265, 2267, 2268, 2271, 2272, 2274, 2277, 2280, 2283, 2286, 2288, 2292, 2295, 2296, 2304, 2307, 2312, 2313, 2316, 2319, 2320, 2322, 2325, 2328, 2331, 2334, 2336, 2340, 2343, 2344, 2349, 2352, 2355, 2360, 2361, 2367, 2368, 2370, 2373, 2376, 2385, 2394, 2400, 2403, 2406, 2421, 2430, 2432, 2439, 2457, 2472, 2484, 2496, 2511, 2520, 2538, 2560, 2583, 2592, 2619, 2646, 2673, 2835, 2916, 3159, 3645

$$\max p = \hat{p} = 2\left(\frac{m}{4}\right)^{\frac{m}{2}}$$

For $m = 8$, $\max p = 2 \cdot 2^4 = 32$

$M_8 = |H_8| = 32 \cdot 2^7 = 4096$

Minors of Hadamard matrices

The current approach is based on the analysis of the result derived from the next proposition by simply using calculus techniques.

Proposition (*Williamson 1944, Szöllősi 2010*)
Let M_k the absolute value of a $k \times k$ minor of the Hadamard matrix H_n of order n,

$$M_k = \left| \det H_{k,n} \right|$$

where $H_{k,n}$ denotes the $k \times k$ submatrix of H_n.

*Then, there is a one-to-one correspondence between the minors of size **k** and **n-k** described by the equation:*

$$M_{n-k} = n^{\frac{n}{2}-k} M_k$$

The above result readily applies to D-optimal designs of order $n > 1$.

Embedded Hadamard matrices

Theorem (*Cohn, 1965*)

If an H_{n+m} *exists and* $n > m$, *then no n-rowed minor is an* H_n.

Example: $n = 20$ and $m = 8$, then H_{20} is not embedded in H_{28}

Theorem (*Brent & Osborn, 2013*)

Let H_n *be an Hadamard matrix of order n having a Hadamard submatrix M of order* $m < n$. *Then,* $m \leq \frac{n}{2}$.

Example: $n = 20$ and $m = 8$, then H_8 is embedded in H_{20}

Embedded Hadamard matrices and D-optimal designs
Problem motivation

Problem 1:

Can an Hadamard matrix (D-optimal design) of order $\boldsymbol{n-4}$ or $\boldsymbol{n-8}$ *exist embedded in an Hadamard matrix of order n, for* $\boldsymbol{n=4t}$ *with integer* $\boldsymbol{t>2}$ *?*

$$H_8 \in H_{12} \;,\; H_{12} \in H_{16} \;,\; H_{16} \in H_{20} \;,\ldots,\; H_{n-4} \in H_n$$

$$H_{12} \in H_{20} \;,\; H_{16} \in H_{24} \;,\; H_{20} \in H_{28} \;,\ldots,\; H_{n-8} \in H_n$$



Problem 2 (Generalization)

Can an Hadamard matrix (D-optimal design) of order $\boldsymbol{n-k}$ *exist embedded in an Hadamard matrix of order n, for* $\boldsymbol{n=4t}$ and $\boldsymbol{k=4r}$ with $\boldsymbol{0<2r<t}$ *?*

What are the characteristics of such an embedding property?

Is H_{n-4} embedded in H_n?

Result 1

For every $n = 4t$ with integer $t > 2$,

$$\left|\det H_{n-4}\right| \neq n^{\frac{n}{2}-4} \cdot M_4$$

Therefore, a Hadamard matrix of order $n - 4$ cannot be embedded in an Hadamard matrix of order $n > 4$.

$$H_{n-4} \notin H_n$$

D. Christou, M. Mitrouli, and J. Seberry. Embedding and extension properties of Hadamard matrices revisited. *Special Matrices*, 6 (2018), 155–165.

Is $H_{n\text{-}8}$ embedded in H_n?

Result 2

For every $n = 4t$ with integer $t > 4$,

$$\left|\det H_{n-8}\right| \neq n^{\frac{n}{2}-8} \cdot M_8$$

Therefore, a Hadamard matrix of order $n - 8$ cannot be embedded in an Hadamard matrix of order $n > 16$.

$$H_{n-8} \notin H_n$$

D. Christou, M. Mitrouli, and J. Seberry. Embedding and extension properties of Hadamard matrices revisited. *Special Matrices*, 6 (2018), 155–165.

Generalization

Results

- ✓ *For every order* $\boldsymbol{n = 4t}$ *with integer* $\boldsymbol{t > 2}$, *it holds* $H_{n-4} \notin H_n$
- ✓ *For every order* $\boldsymbol{n = 4t}$ *with integer* $\boldsymbol{t > 4}$, *it holds* $H_{n-8} \notin H_n$

Research problem 2:

Is it possible that

$$H_{n-k} \notin H_n$$

for every order $\boldsymbol{n}$ and integer $\boldsymbol{k}$, such that $\boldsymbol{n = 4t}$ and $\boldsymbol{k = 4r}$ with $\boldsymbol{r > 0}$ and $\boldsymbol{t > 2r}$ **?**

Is H_{n-k} embedded in H_n?

Given a positive integer k, it is known that $M_k \le k^{\frac{k}{2}}$ and $M_k = p\,2^{k-1}$.

Therefore, if $k = 4r$, where $r > 0$, the maximum value of the integer p, denoted by $\hat{p}$ is given by

$$\hat{p} = 2\left(\frac{k}{4}\right)^{\frac{k}{2}} = 2r^{2r}$$

For $\boldsymbol{n = 4t}$, $\boldsymbol{k = 4r}$ for integers $\boldsymbol{r > 0}$ and $\boldsymbol{t > r}$ it holds:

$$\left|\det H_{n-k}\right| = n^{\frac{n}{2}-k} \cdot p \cdot 2^{k-1} \quad \Leftrightarrow \quad p = 2t^{2r}\left(\frac{t-r}{t}\right)^{2(t-r)}$$

A necessary condition for the general embedding problem is $\boldsymbol{p \le \hat{p}}$.

Is H_{n-k} embedded in H_n?

Let $\theta = \frac{r}{t} = \frac{4r}{4t} = \frac{k}{n}$. Since $n > k$, then $0 < \theta < 1$ and $0 < 1-\theta < 1$. Hence,

$$p \le \hat{p} \iff 2t^{2r}\left(\frac{t-r}{t}\right)^{2(t-r)} \le 2r^{2r} \overset{\text{Using Calculus}}{\iff} (1-\theta)\ln(1-\theta) - \theta\ln\theta \le 0$$

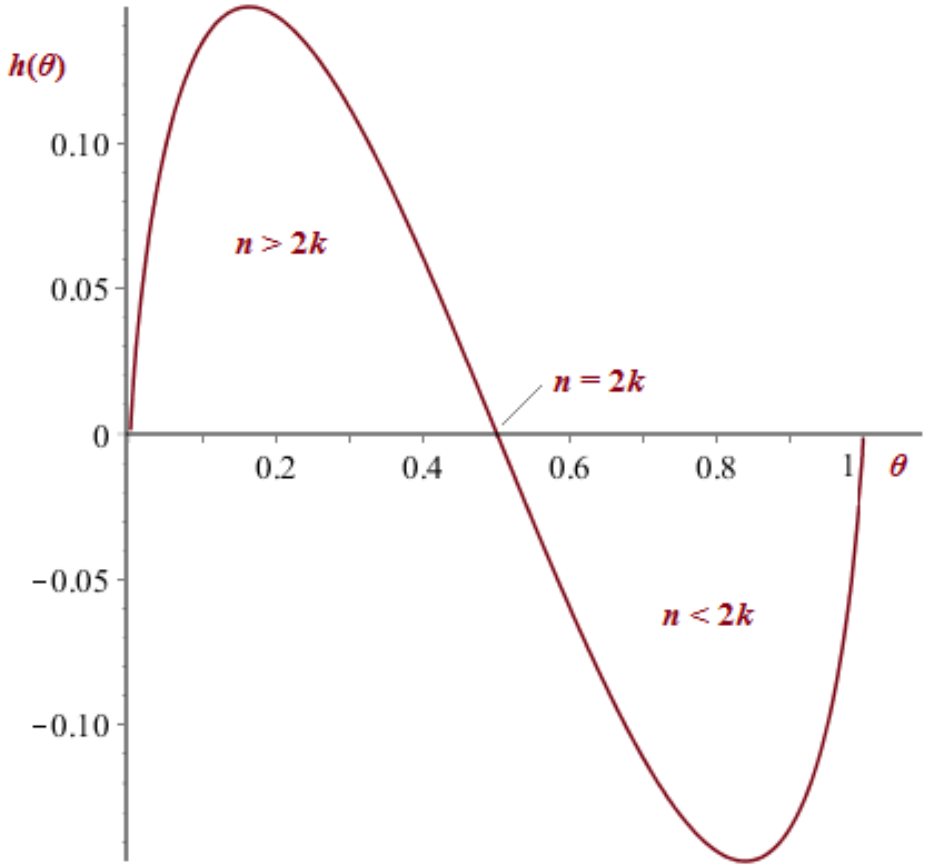
Studying the sign of the real function

$$h(\theta) = (1-\theta)\ln(1-\theta) - \theta\ln\theta$$

provides very important information about the behavior of p for the various values of the integers n and k when $n > k$.

Is H_{n-k} embedded in H_n?

$$h(\theta) = (1-\theta)\ln(1-\theta) - \theta\ln\theta$$



- $n > 2k \Leftrightarrow h(\theta) > 0 \Leftrightarrow p > \hat{p}$

  $H_{n-k} \notin H_n$ 

- $n = 2k \Leftrightarrow h(\theta) = 0 \Leftrightarrow p = \hat{p}$

  $H_k \in H_{2k}$ 

- $n < 2k \Leftrightarrow h(\theta) < 0 \Leftrightarrow p < \hat{p}$

  $H_{n-k} \in H_n$ 

But for what values of p is true?

Is H_{n-k} embedded in H_n?

Embeddability of Hadamard matrices H_{n-k} for $4 \leq k \leq 20$ and $8 \leq n \leq 40$.

Order	$k = 4$	$k = 8$	$k = 12$	$k = 16$	$k = 20$
$n = 8$	$H_4 \in H_8$				
$n = 12$	$H_8 \notin H_{12}$	$H_4 \in H_{12}$			
$n = 16$	$H_{12} \notin H_{16}$	$H_8 \in H_{16}$	$H_4 \in H_{16}$		
$n = 20$	$H_{16} \notin H_{20}$	$H_{12} \notin H_{20}$	$H_8 \in H_{20}$	$H_4 \in H_{20}$	
$n = 24$	$H_{20} \notin H_{24}$	$H_{16} \notin H_{24}$	$H_{12} \in H_{24}$	$H_8 \in H_{24}$	$H_4 \in H_{24}$
$n = 28$	$H_{24} \notin H_{28}$	$H_{20} \notin H_{28}$	$H_{16} \notin H_{28}$	$H_{12} \in H_{28}$	$H_8 \in H_{28}$
$n = 32$	$H_{28} \notin H_{32}$	$H_{24} \notin H_{32}$	$H_{20} \notin H_{32}$	$H_{16} \in H_{32}$	$H_{12} \in H_{32}$
$n = 36$	$H_{32} \notin H_{36}$	$H_{28} \notin H_{36}$	$H_{24} \notin H_{36}$	$H_{20} \notin H_{36}$	$H_{16} \in H_{36}$
$n = 40$	$H_{36} \notin H_{40}$	$H_{32} \notin H_{40}$	$H_{28} \notin H_{40}$	$H_{24} \notin H_{40}$	$H_{20} \in H_{40}$

Is H_{n-k} embedded in H_n?

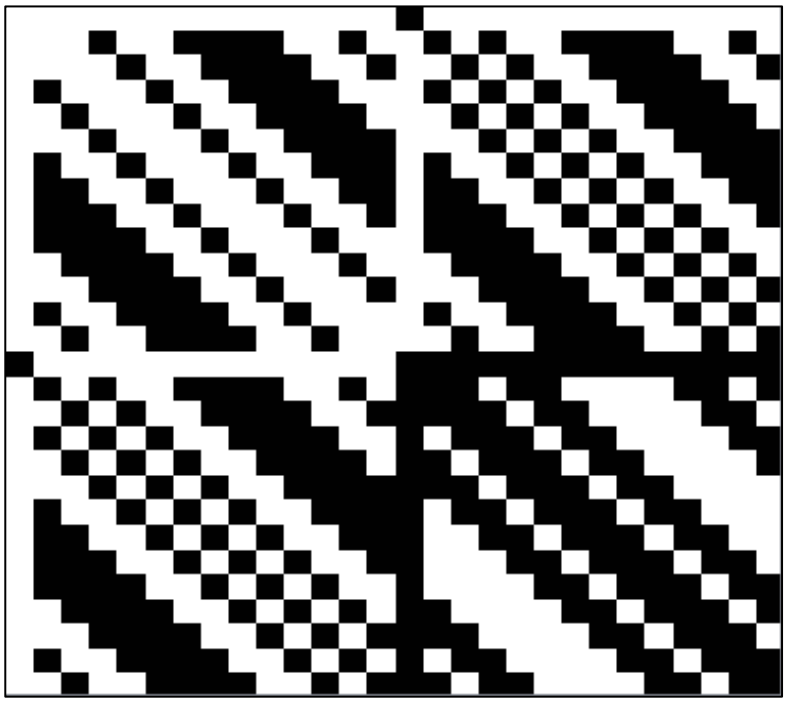
Embeddability of Hadamard matrices H_{n-k} for $4 \leq k \leq 20$ and $8 \leq n \leq 40$.

Order	$k = 4$	$k = 8$	$k = 12$	$k = 16$	$k = 20$
$n = 8$	$H_4 \in H_8$				
$n = 12$	$H_8 \notin H_{12}$	$H_4 \in H_{12}$			
$n = 16$	$H_{12} \notin H_{16}$	$H_8 \in H_{16}$	$H_4 \in H_{16}$		
$n = 20$	$H_{16} \notin H_{20}$	$H_{12} \notin H_{20}$	$H_8 \in H_{20}$	$H_4 \in H_{20}$	
$n = 24$	$H_{20} \notin H_{24}$	$H_{16} \notin H_{24}$	$H_{12} \in H_{24}$	$H_8 \in H_{24}$	$H_4 \in H_{24}$
$n = 28$	$H_{24} \notin H_{28}$	$H_{20} \notin H_{28}$	$H_{16} \notin H_{28}$	$H_{12} \in H_{28}$	$H_8 \in H_{28}$
$n = 32$	$H_{28} \notin H_{32}$	$H_{24} \notin H_{32}$	$H_{20} \notin H_{32}$	$H_{16} \in H_{32}$	$H_{12} \in H_{32}$
$n = 36$	$H_{32} \notin H_{36}$	$H_{28} \notin H_{36}$	$H_{24} \notin H_{36}$	$H_{20} \notin H_{36}$	$H_{16} \in H_{36}$
$n = 40$	$H_{36} \notin H_{40}$	$H_{32} \notin H_{40}$	$H_{28} \notin H_{40}$	$H_{24} \notin H_{40}$	$H_{20} \in H_{40}$

Example 1: $H_8 \in H_{28}$

$$n = 28,\ k = 20,\ \text{and}\ p = \mathcal{P}(28, 20) = 3764768$$

H_{28} (2^{nd} *Paley type*)



Argument: If $p = 3764768$ exists in the spectrum, meaning that H_{28} has a 20×20 submatrix with minor:

$$|\det H_{28}^{(20)}| = 3764768 \cdot 2^{19} = 1973822685184$$

then it may H_8 exist embedded in H_{28}.

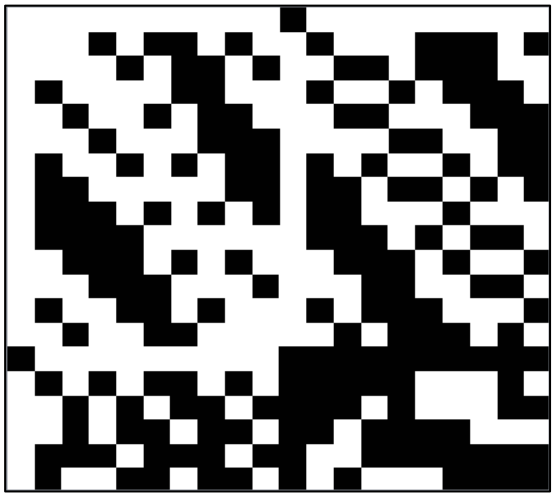
□ = 1

■ = −1

N. J. A. Sloane. A library of Hadamard matrices. http://neilsloane.com/hadamard/index.html

Example 1: $H_8 \in H_{28}$

$H_{28}^{(20)}$ *submatrix of* H_{28}



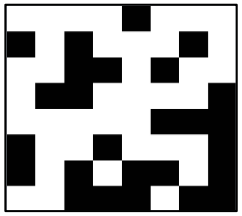
$$n = 28,\ k = 20, \text{ and } p = \mathcal{P}(28, 20) = 3764768$$

$H_{28}^{(20)} = [a_{ij}]$ of H_{28} where

$$i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 23, 27\},$$
$$j \in \{1, 2, 3, 4, 5, 7, 9, 11, 13, 14, 15, 16, 17, 19, 20, 22, 23, 24, 26, 27\}$$

$$|\det H_{28}^{(20)}| = 3764768 \cdot 2^{19} = 1973822685184$$

$H_8 \in H_{28}$



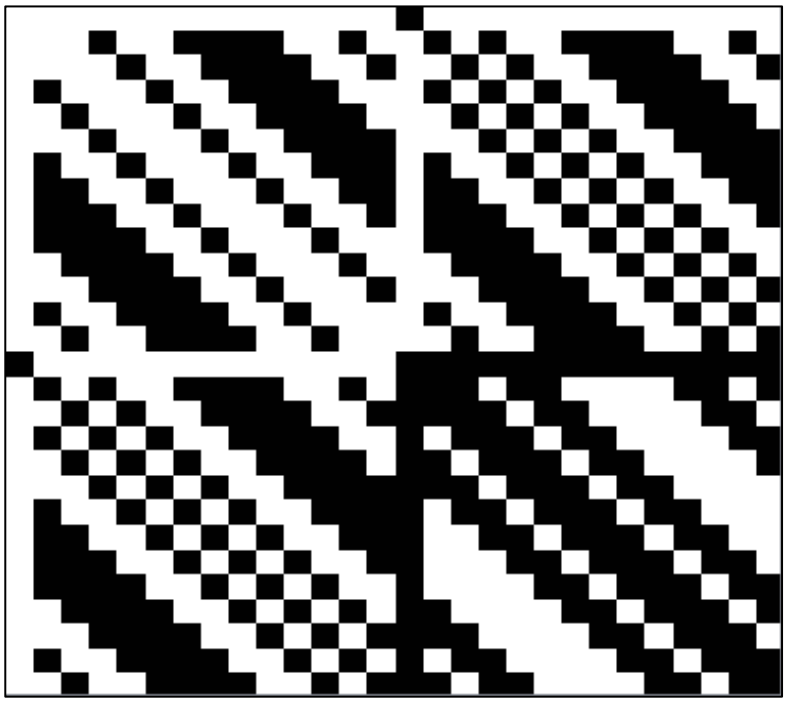
$A = [a_{ij}]$ of H_{28} where $i \in \{1, 2, 3, 4, 5, 6, 16, 17\}$, $j \in \{4, 6, 9, 14, 15, 19, 21, 26\}$

$$|\det A| = 2^{(28-20)-1} \cdot \left(\frac{28}{4}\right)^{\frac{28}{2}-20} \cdot 3764768 = 4096 = |\det H_8|$$

Example 2: $H_{12} \in H_{28}$

$$n = 28,\ k = 16,\ \text{and}\ p = \mathcal{P}(28, 16) = 71442$$

H_{28} (2^{nd} *Paley type*)



Argument: If $p = 71442$ exists in the spectrum, meaning that H_{28} has a 16×16 submatrix with minor:

$$|\det H_{28}^{(16)}| = 71442 \cdot 2^{15} = 2341011456$$

then it may H_{12} exist embedded in H_{28}.

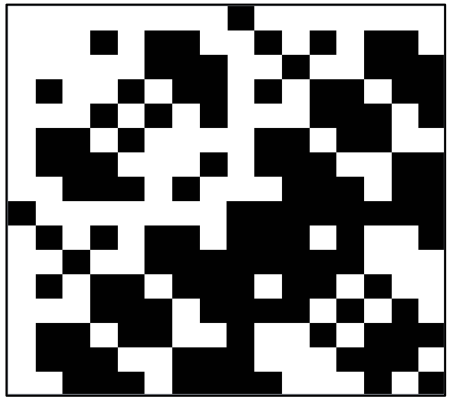
□ = 1

■ = −1

N. J. A. Sloane. A library of Hadamard matrices. http://neilsloane.com/hadamard/index.html

Example 2: $H_{12} \in H_{28}$

$H_{28}^{(16)}$ *submatrix of* H_{28}



$$n = 28,\ k = 16, \text{ and } p = \mathcal{P}(28, 16) = 71442$$

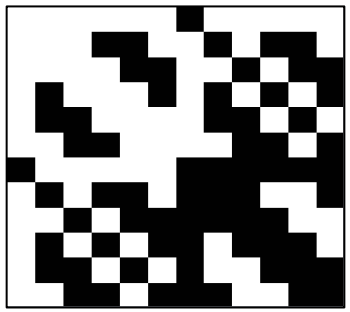
$$H_{28}^{(16)} = [a_{ij}] \text{ of } H_{28} \text{ where}$$

$$i \in \{1, 2, 3, 4, 6, 8, 9, 11, 15, 16, 17, 18, 20, 22, 23, 25\},$$

$$j \in \{1, 2, 3, 4, 6, 8, 9, 11, 15, 16, 17, 18, 20, 22, 23, 25\}$$

$$|\det H_{28}^{(16)}| = 71442 \cdot 2^{15} = 2341011456$$

$H_{12} \in H_{28}$

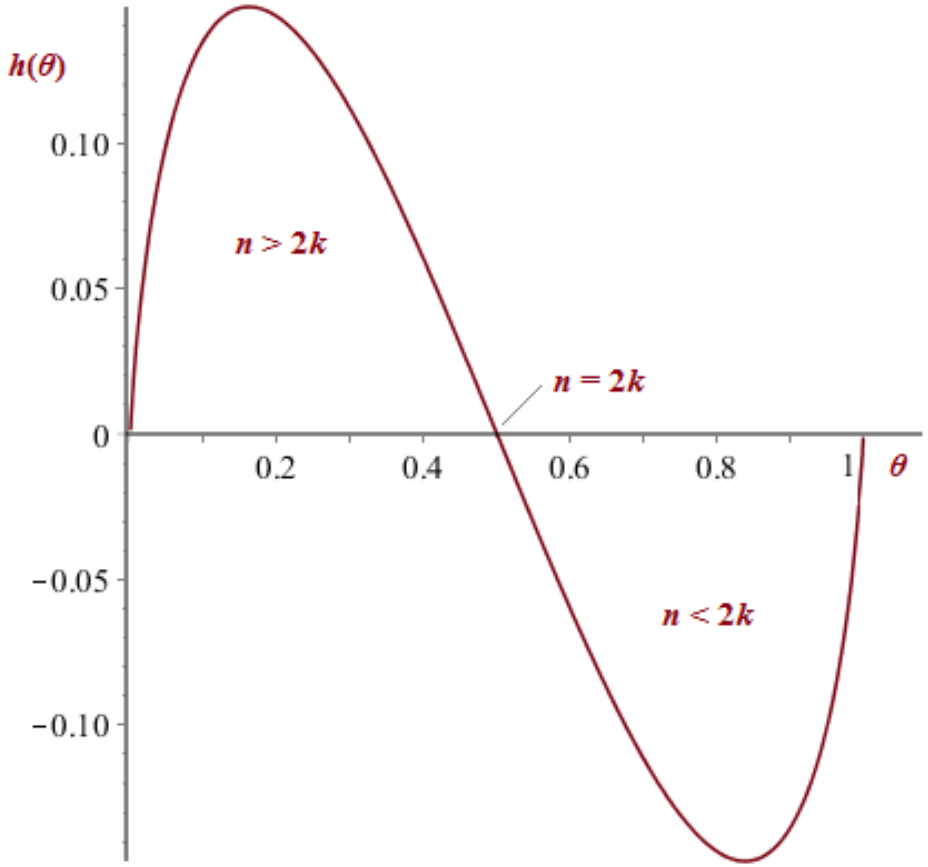


$$A = [a_{ij}] \text{ of } H_{28} \text{ where } \begin{array}{l} i \in \{1, 2, 3, 4, 8, 11, 15, 16, 17, 18, 22, 25\}, \\ j \in \{1, 2, 3, 4, 8, 11, 15, 16, 17, 18, 22, 25\} \end{array}$$

$$|\det A| = 2^{(28-16)-1} \cdot \left(\frac{28}{4}\right)^{\frac{28}{2}-16} \cdot 71442 = 2985984 = |\det H_{12}|$$

Is D_k embedded in H_n?

$$h(\theta) = (1-\theta)\ln(1-\theta) - \theta\ln\theta$$



$$\theta = \frac{r}{t} = \frac{4r}{4t} = \frac{k}{n} \in (0,1)$$

t	r	θ	Result
3	2	0.67	$D_8 \notin H_{12}$
4	3	0.75	$D_{12} \notin H_{16}$
5	3	0.60	$D_{12} \notin H_{20}$
5	4	0.80	$D_{16} \notin H_{20}$
6	4	0.67	$D_{16} \notin H_{24}$

Embedding D-optimal designs of order $k > 4$

Let us consider a D-optimal design D_k of order $k > 4$. If it can exist embedded in an Hadamard matrix of order $n > k$, then:

$$M_{n-k} = n^{\frac{n-2k}{2}} d_k \Leftrightarrow$$

$$p_{n-k}\, 2^{n-k-1} = n^{\frac{n-2k}{2}}\, \hat{p}_k\, 2^{k-1} \Leftrightarrow$$

$$p_{n-k} = 2^{k-2n}\, n^{\frac{n-2k}{2}}\, \hat{p}_k$$

This equation holds if $p_{n-k} \in spec\{m\}$. For positive integers n, k such that $4 < k < n$, we denote the computed value of p_{n-k} as p^* and form the following equation:

$$p^* = \left(\frac{\sqrt{n}}{2}\right)^{n-2k} \hat{p}_k \qquad \textbf{[Eq. (1)]}$$

Is D_k embedded in H_n?

n	k	$\hat{p}_k$	$m = n - k$	p^*	$p^* \in \text{spec}\{m\}$	Result
8	5	3	3	1.5	NO	$D_5 \notin H_8$
8	6	5	2	1.25	NO	$D_6 \notin H_8$
8	7	9	1	1.125	NO	$D_7 \notin H_8$
12	5	3	7	9	YES	$D_5 \in H_{12}$
12	6	5	6	5	YES	$D_6 \in H_{12}$
12	7	9	5	3	YES	$D_7 \in H_{12}$
12	8	32	4	3.5556	NO	$D_8 \notin H_{12}$
12	9	56	3	2.0741	NO	$D_9 \notin H_{12}$
12	10	144	2	1.7778	NO	$D_{10} \notin H_{12}$
12	11	320	1	1.3169	NO	$D_{11} \notin H_{12}$

Is D_k embedded in H_n?

n	k	$\hat{p}_k$	$m = n - k$	p^*	$p^* \in \text{spec}\{m\}$	Result
16	5	3	11	192	YES	$D_5 \in H_{16}$
16	6	5	10	80	YES	$D_6 \in H_{16}$
16	7	9	9	36	YES	$D_7 \in H_{16}$
16	8	32	8	32	YES	$D_8 \in H_{16}$
16	9	56	7	14	NO	$D_9 \notin H_{16}$
16	10	144	6	9	NO	$D_{10} \notin H_{16}$
16	11	320	5	5	NO	$D_{11} \notin H_{16}$
16	12	1458	4	5.6953	NO	$D_{12} \notin H_{16}$
16	13	3645	3	3.5596	NO	$D_{13} \notin H_{16}$
16	14	9477	2	2.3137	NO	$D_{14} \notin H_{16}$
16	15	25515	1	1.5573	NO	$D_{15} \notin H_{16}$

Is D_k embedded in H_n?

n	k	$\hat{p}_k$	$m = n - k$	p^*	$p^* \in \text{spec}\{m\}$	Result
20	7	9	13	1125	YES	$D_7 \in H_{20}$
20	8	32	12	800	YES	$D_8 \in H_{20}$
20	9	56	11	280	YES	$D_9 \in H_{20}$
20	10	144	10	144	YES	$D_{10} \in H_{20}$
20	11	320	9	64	NO	$D_{11} \notin H_{20}$
20	12	1458	8	58.32	NO	$D_{12} \notin H_{20}$
20	13	3645	7	29.16	NO	$D_{13} \notin H_{20}$
20	14	9477	6	15.163	NO	$D_{14} \notin H_{20}$
20	15	25515	5	8.1648	NO	$D_{15} \notin H_{20}$
20	16	131070	4	8.3886	NO	$D_{16} \notin H_{20}$
20	17	327680	3	4.1943	NO	$D_{17} \notin H_{20}$
20	18	1114100	2	2.8521	NO	$D_{18} \notin H_{20}$
20	19	3412000	1	1.7469	NO	$D_{19} \notin H_{20}$

Is D_k embedded in H_n?

Theorem: *For positive integers n, k multiples of 4 such that $4 < k < n$, a D-optimal design D_k of order k cannot exist embedded, up to Hadamard equivalence, in a Hadamard matrix of order n when $n < 2k$.*

$$D_k \notin H_n, \text{ for } n < 2k \text{ and } n, k \equiv 0 \pmod 4$$

Conjecture: Let H_n be a Hadamard matrix of order n. If D_k is a $k \times k$ D-optimal design, where $n \geq 8$ and $k \geq 4$ are multiples of 4 with $n > k$, then D_k is embedded in H_n, up to Hadamard equivalence, whenever $n > 2k$.

$$D_k \in H_n, \text{ for } 4 \leq k < \frac{n}{2} \text{ and } n, k \equiv 0 \pmod 4$$

Embedding D-optimal designs of order $n-1$ or $n+1$

An intriguing direction for further investigation is the potential existence of Hadamard matrices embedded within higher-order D-optimal designs.

Understanding such embeddings could reveal deeper structural properties and symmetries in optimal design theory.

Lemma 2 . *For $n = 2, 3, \ldots, 7$, if an $n \times n$ maximal-determinant (± 1)-matrix is D_n, then a D_{n-1} must be embedded in it, i.e. $D_{n-1} \in D_n$.*

- A. Edelman and W. Mascarenhas. On the complete pivoting conjecture for a Hadamard matrix of order 12. *Linear Multilinear Algebra*, 38(3):181–187, 1995.
- J. Williamson. Determinants whose elements are 0 and 1. *Am. Math. Mon.*, 56(8):427–434, 1946.

Immediate results: $H_4 \in D_5$, $D_5 \in D_6$, $D_6 \in D_7$

Embedding D-optimal designs of order $n-1$ or $n+1$

Definition 4. *If H is a Hadamard matrix of order m, then the sum of all its entries is called* excess of H *and is denoted by $\sigma(H)$. The* maximal excess *of Hadamard matrices of order m is denoted by $\sigma(m)$.*

$\sigma(2)=4$, $\sigma(4)=8$, $\sigma(8)=20$

(*K. W. Schmidt and E. T. H. Wang*, 1977)

$\sigma(12)=36$, $\sigma(16)=64$, $\sigma(20)=80$, $\sigma(24)=112$, $\sigma(28)=140$

(*M.R. Best*, 1977)

$\sigma(40)=244$, $\sigma(44)=280$, $\sigma(48)=324$, $\sigma(52)=364$, $\sigma(80)=704$, $\sigma(84)=756$

(*N. Farmakis and S. Kounias*, 1987)

Embedding D-optimal designs of order $n-1$ or $n+1$

In 1987, Farmakis and Kounias studied the problem of maximizing the excess of H, trying to find a $\sigma(m)$ for all Hadamard matrices H of a specific order m. They focused on finding a Hadamard matrix of order m so that the determinant of the $(m+1) \times (m+1)$ matrix R is maximized.

$$R = \begin{bmatrix} 1 & -e^{\top} \\ e & H \end{bmatrix}$$

Then, the determinant of R was proved to be equal to

$$\det(R) = m^{\frac{m}{2}-1}(m + e^{\top}He) \qquad \textbf{[Eq. (2)]}$$

where e is a row-vector of 1's of size m and $e^{\top}He$ is the excess of H.

Embedding D-optimal designs of order $n-1$ or $n+1$

Proposition 2. *A D-optimal design of order $n = 9$ or 17 has a Hadamard matrix of order $n-1$ embedded in it, i.e.,*

$$H_8 \in D_9 \text{ and } H_{16} \in D_{17}.$$

Proof. For $n = 9$, we know that

$$d_9 = 56 \cdot 2^{9-1} = 56 \cdot 256 = 14336.$$

$$\det(R) = 8^{\frac{8}{2}-1}(8 + \sigma(8)) = 8^3 \cdot 28 = 512 \cdot 28 = 14336.$$

Similarly, for $n = 17$, $d_{17} = 327680 \cdot 2^{16} = 21474836480$

$$\det(R) = 16^{\frac{16}{2}-1}(16 + \sigma(16)) = 16^7 \cdot 80 = 21474836480.$$

Embedding D-optimal designs of order $n-1$ or $n+1$

n	Result for $n-1$	Proof	n	Result for $n-1$	Proof
4	$D_3 \in H_4$	Lemma 2	5	$H_4 \in D_5$	Lemma 2
6	$D_5 \in D_6$	Lemma 2	7	$D_6 \in D_7$	Lemma 2
8	$D_7 \notin H_8$	Eq. (20)	9	$H_8 \in D_9$	Prop. 2
10	$D_9 \; ? \; D_{10}$	N/A	11	$D_{10} \; ? \; D_{11}$	N/A
12	$D_{11} \notin H_{12}$	Eq. (20)	13	$H_{12} \notin D_{13}$	Eq. (2)
14	$D_{13} \; ? \; D_{14}$	N/A	15	$D_{14} \; ? \; D_{15}$	N/A
16	$D_{15} \notin H_{16}$	Eq. (20)	17	$H_{16} \in D_{17}$	Prop. 2
18	$D_{17} \; ? \; D_{18}$	N/A	19	$D_{18} \; ? \; D_{19}$	N/A
20	$D_{19} \notin H_{20}$	Eq. (20)	21	$H_{20} \notin D_{21}$	Eq. (2)
22	$D_{21} \; ? \; D_{22}$	N/A	23	$D_{22} \; ? \; D_{23}$	N/A
24	$D_{23} \notin H_{24}$	Eq. (20)	25	$H_{24} \notin D_{25}$	Eq. (2)
26	$D_{25} \; ? \; D_{26}$	N/A	27	$D_{26} \; ? \; D_{27}$	N/A
28	$D_{27} \notin H_{28}$	Eq. (20)	29	$H_{28} \notin D_{29}$	Eq. (2)

Embedding D-optimal designs of order $n-1$ or $n+1$

Research problem:

Construct the D-optimal designs D_{10}, D_{11}, D_{14}, and D_{15} and search for submatrices that may satisfy the following embedding relations:

a) $D_9 \in D_{10}$ and $D_{10} \in D_{11}$

b) $D_{13} \in D_{14}$ and $D_{14} \in D_{15}$

If such submatrices exist, determine:

- how many arise for each case,
- the size of their determinant,
- the parameter p associated with their determinant.

$$d_9 = 58 \cdot 2^8$$
$$d_{10} = 144 \cdot 2^9$$
$$d_{11} = 320 \cdot 2^{10}$$
$$d_{13} = 3645 \cdot 2^{12}$$
$$d_{14} = 9477 \cdot 2^{13}$$
$$d_{15} = 25515 \cdot 2^{15}$$
$$d_{17} = 327680 \cdot 2^{16}$$
$$d_{18} = 1114212 \cdot 2^{17}$$
$$d_{19} = 3411968 \cdot 2^{18}$$

c) Continue the search for $D_{n-1} \in D_n$ and $D_n \in D_{n+1}$ for larger n.

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Thank you
for your attention