Παράδειγμα 2 (Διαδικασία Poisson – Διαδικασία Γέννησης)

$S = \{0, 1, 2, 3, \dots\}$

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[r, bend left, "\lambda"] & 2 \arrow[r, bend left, "\lambda"] & 3 \arrow[r, bend left, "\lambda"] & \dots \end{tikzcd}$$

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \dots \end{array} \\ \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ 0 & -\lambda & \lambda & 0 & \dots \\ 0 & 0 & -\lambda & \lambda & \dots \\ 0 & 0 & 0 & -\lambda & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

$$E[X(t)] = \sum_{j=0}^{\infty} j\,p_j(t) = ;$$
$$= \sum_{j=1}^{\infty} j\,p_j(t)$$

$$p'(t) = p(t)\cdot Q \Rightarrow$$

$$[p_0'(t) \;\; p_1'(t) \;\; p_2'(t) \;\; p_3'(t) \;\; \dots] = [p_0(t) \;\; p_1(t) \;\; p_2(t) \;\; \dots] \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ 0 & -\lambda & \lambda & 0 & \dots \\ 0 & 0 & -\lambda & \lambda & \dots \\ 0 & 0 & 0 & -\lambda & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\left.\begin{aligned} p_0'(t) &= -\lambda p_0(t) \\ p_1'(t) &= \lambda p_0(t) - \lambda p_1(t) \\ p_2'(t) &= \lambda p_1(t) - \lambda p_2(t) \\ p_3'(t) &= \lambda p_2(t) - \lambda p_3(t) \\ &\vdots \\ p_j'(t) &= \lambda p_{j-1}(t) - \lambda p_j(t), \quad j \geq 1 \end{aligned}\right\} \sum_{j=0}^{\infty} j \times (\;)$$

$$\Longrightarrow \sum_{j=0}^{\infty} j\,p_j'(t) = -\lambda \sum_{j=0}^{\infty} j\,p_j(t) + \lambda \sum_{j=1}^{\infty} j\,p_{j-1}(t) \Rightarrow$$

$$\frac{d}{dt}E[X(t)] = -\lambda E[X(t)] + \lambda \sum_{j=1}^{\infty}(j-1+1)\,p_{j-1}(t) \Rightarrow$$

$$\frac{d}{dt}E[X(t)] = -\lambda E[X(t)] + \lambda \underbrace{\sum_{j=1}^{\infty}(j-1)\,p_{j-1}(t)}_{E[X(t)]} + \lambda \underbrace{\sum_{j=1}^{\infty} p_{j-1}(t)}_{\sum_{j=0}^{\infty} p_j(t) \,=\, 1} \Rightarrow$$

$$\frac{d}{dt}E[X(t)] = \cancel{-\lambda E[X(t)]} + \cancel{\lambda E[X(t)]} + \lambda \Rightarrow$$

$$\frac{d}{dt}E[X(t)] = \lambda \Rightarrow$$

$$E[X(t)] = \lambda t + c$$

$$E[X(0)] = 0 \Rightarrow \lambda \cdot 0 + c = 0 \Rightarrow c = 0$$

Οπότε $E[X(t)] = \lambda t$.