Chapter 6
Representations of Modelling in Mathematics Education

Helen M. Doerr, Jonas B. Ärlebäck, and Morten Misfeldt

Abstract Mathematical models have a substantial impact at all levels of society, and hence mathematical modelling stands as an important topic in mathematics education. Mathematical modelling has a particular pedagogical/didactical discourse as modelling continues to garner attention in educational research. Diagrammatic representations of mathematical modelling processes are increasingly being used in curriculum documents on national and transnational levels. In this chapter, we critically discuss one of the most frequently used representations of modelling processes in the literature, namely, that of the *modelling cycle*, and offer alternative representations to more fully capture multiple aspects of modelling in mathematics education.

Keywords Modelling cycle • Modelling competences • Technology • Social-critical education • Mathematical modelling • Prescriptive models

6.1 Introduction

Both in society more broadly and in the workplace in particular, mathematical models are used to control processes, to design products, to monitor and influence economic systems, to enhance human agency, and to structure and understand the natural world. Given the widespread use and impact of mathematical models (Niss

H.M. Doerr (✉)
Mathematics Department, Syracuse University, 215 Carnegie Hall,
Syracuse, NY 13244, USA
e-mail: hmdoerr@syr.edu

J.B. Ärlebäck
Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden
e-mail: jonas.bergman.arleback@liu.se

M. Misfeldt
Department of Learning and Philosophy, Aalborg University Copenhagen,
AC Meyers Vænge 15, 2450 Copenhagen, Denmark
e-mail: misfeldt@learning.aau.dk

G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_6

2015), it is not surprising to find mathematical modelling competencies as an educational goal in various curriculum standards documents on national and transnational levels. Prominent examples are the PISA 2012 framework and the recently adopted Common Core State Standards in Mathematics (CCSSM) in the United States (Council of Chief State School Officers [CCSSO] 2010). The 2012 PISA framework defines mathematical literacy as "an individual's capacity to *formulate*, *employ*, and *interpret* mathematics in a variety of contexts. It includes *reasoning* mathematically and *using* mathematical concepts, procedures, facts and tools to *describe*, *explain,* and *predict* phenomena" (OECD 2013, p. 25, italics added). In CCSSM, modelling with mathematics is one of eight standards for mathematical practices that teachers should seek to develop in their students at all grade levels, K-12. Modelling is described in terms of what students are able to do:

> Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society and the workplace. …[They] are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts, and formulas. They can analyse those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose. (CCSSO 2010, p. 7)

Both the PISA (OECD 2013) and the CCSSM (CCSSO 2010) standards documents include representations of mathematical modelling that are intended to convey to stakeholders and practitioners the key elements involved in learning to do mathematical modelling and in learning about mathematical models and their role in society. As images of modelling, these representations necessarily convey some important aspects of modelling, but as with all images and representations, other important aspects of modelling are pushed into the background or left out in some way. Hence, our concern with the dominance of particular images of modelling is with the influence that dominant images will have as modelling is taken up by writers of curriculum materials, by textbook authors, by teachers, by teacher educators and others involved in professional development and by developers of large-scale and high-stakes assessments. One of the most frequently used representations of mathematical modelling in curricular documents and in the research literature is that of the *modelling cycle*. Our goal in this chapter is to critically examine the question of what important aspects of modelling are pushed to the background or omitted by widely used representations of the modelling cycle.

6.2 The Modelling Cycle

We begin our analysis of the cyclic representations of modelling with the PISA framework (OECD 2013), followed by the CCSSM (CCSSO 2010) and then the research literature. The PISA document situates modelling in real-world contexts,

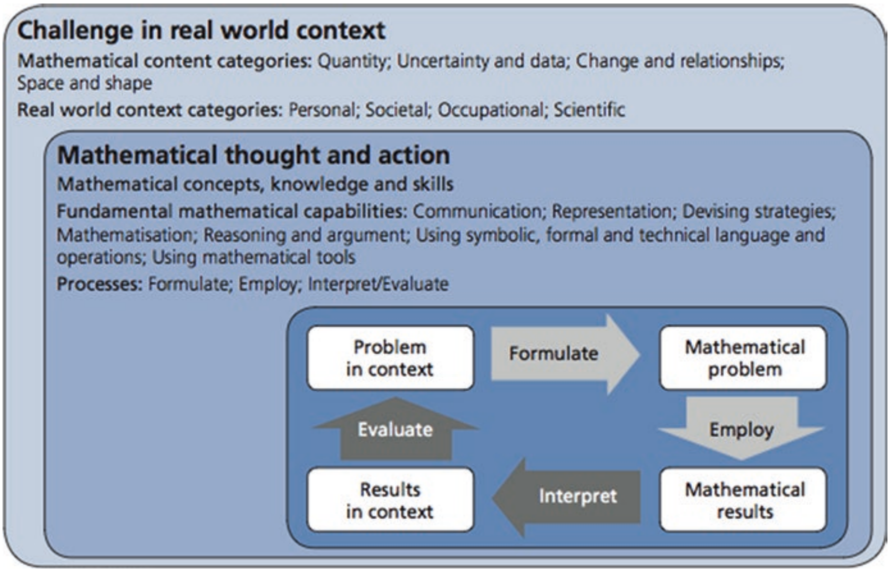

Fig. 6.1 Representation of modelling in the 2012 PISA framework (OECD 2013, p. 26)

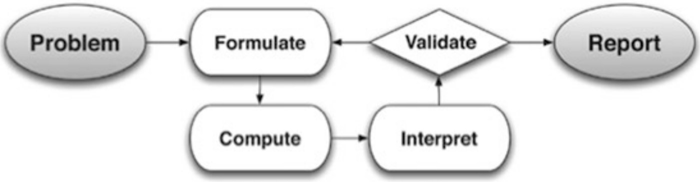

Fig. 6.2 Representation of modelling in the Common Core (CCSSO 2010, p. 72)

noting that this includes four contexts: personal, societal, occupational and scientific. Mathematical concepts, knowledge and skill are drawn upon in order to engage in the four processes of *formulating* the model, *employing* mathematical skills to obtain mathematical results, *interpreting* those results in context and *evaluating* the goodness of the solution (Fig. 6.1).

In the CCSSM (CCSSO 2010), modelling is both a standard of mathematical practices at all grade levels and a content standard in high school (grades 9 through 12). As with the PISA framework (OECD 2013), modelling is about analysing empirical situations: "Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modelled using mathematical and statistical methods" (CCSSO 2010, p. 72), as shown in Fig. 6.2. The vision of modelling includes both descriptive models (such as graphs of observations) and analytic models that seek to explain phenomena. Computational technology (such as graphing utilities, spreadsheets, computer algebra systems, dynamic geometry software) plays a role in "varying assumptions, exploring consequences, and comparing

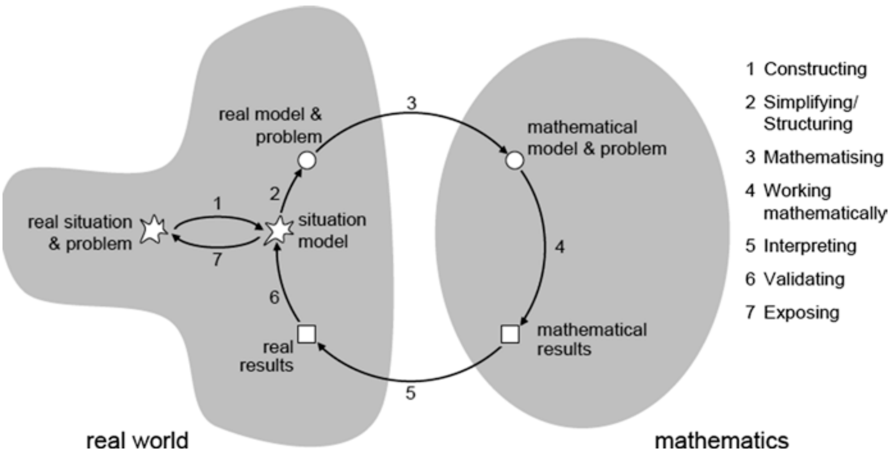

Fig. 6.3 The modelling cycle depicted by Blum and Leiß (2007, p. 225)

predictions with data" (CCSSO 2010, p. 72). The CCSSM elaborates each of these modelling processes, including a clarification that "compute" does not mean to "calculate" per se, but rather means to analyse, to perform operations on relationships between variables and to draw conclusions.

In the research literature on modelling, there are several variants of the modelling cycle, such as the widely cited image of Blum and Leiß (2007) shown in Fig. 6.3. A similar image has been developed by Blomhøj and Jensen (2007), where modelling competency is defined as "someone's insightful readiness to carry through all parts of a mathematical modelling process in a certain context" (p. 48). All representations of modelling have their strengths and weaknesses, a point also made by Blum (2015). There are some striking similarities among many of these cyclic representations, even when the specific words chosen to describe the subprocesses of modelling differ. All of these representations capture some sense that a mathematical model is a simplified version of some aspect of the real world that is formalized in mathematics for the purpose of solving a problem situation in the real world.

Given the recent manifestations and importance of these representations of modelling in curriculum standards documents for policy-makers, curriculum developers, teachers and researchers, we put forward four important aspects of mathematical modelling that are *not* well captured by the images we have shown: the non-linearity of modelling, the role of multiple models and pre-existing models within modelling activity, the social and critical aspects of modelling and the role of computational media in modelling.

6.3 The Non-linearity of the Modelling Process

These widely used representations of mathematical modelling processes share the same problem: they provide a useful analytical abstraction of the processes involved when engaged in the creative thinking when an individual (or a group of individuals) maps a real problem situation onto some subset of mathematics for some particular purpose. However, all the individual differences that occur when students engage in doing mathematical tasks make the transition from an abstract analytical representation of modelling to a more normative tool for planning teaching and learning of modelling at best problematic. Teaching approaches that would guide students through predetermined boxes would be inadequate for embracing the multitude of learning pathways that are known to occur in the classroom (Borromeo Ferri 2007; Lesh and Doerr 2012). In her work, for example, Borromeo Ferri illustrates both the non-linearity (in terms of following steps or sub-competencies shown in the modelling cycles) and the differences between two pupils in their individual modelling routes or pathways, as shown in Fig. 6.4.

Just as importantly, when digital technologies are introduced into modelling tasks, the non-linearity of students' actual modelling pathways becomes more dynamic and stochastic. As illustrated by Lesh and Doerr (2012), students' actual modelling activity does not move in a linear path through the boxes and subprocesses of the modelling cycle. As students work, they "bounce around" as they attend to different aspects of the problem situation (sometimes re-defining the problem), their mathematical work (revising the relationships between objects), the data and their representations (selecting new objects to represent) and their interpretations of their outcomes in terms of perceived criteria (Doerr and Pratt 2008). We suggest that an image of moving between "nodes" or multiple paths in a network (as shown in Fig. 6.5) might offer teachers and researchers new ways of thinking about both teaching and researching mathematical modelling.

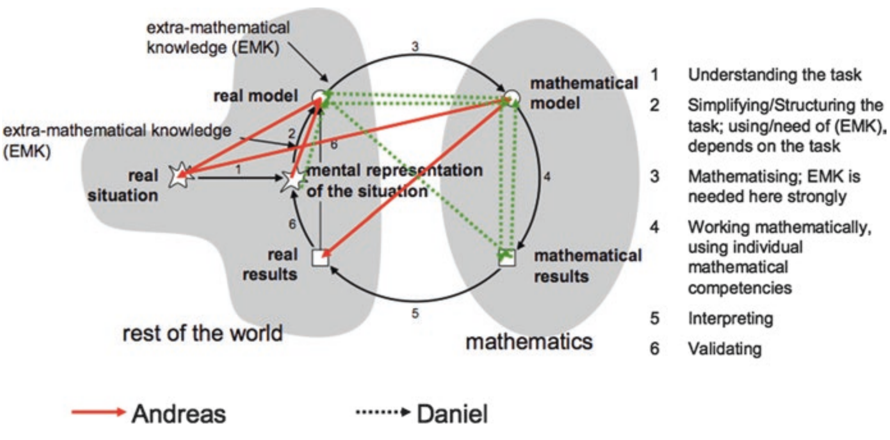

Fig. 6.4 Individual modelling pathways (Borromeo Ferri 2007, p. 2087)

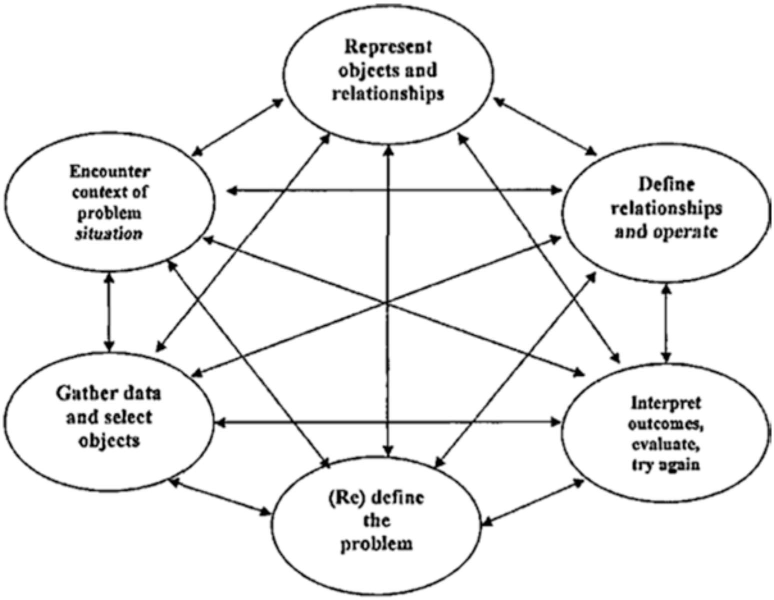

Fig. 6.5 The nodes of the modelling process (Doerr and Pratt 2008, p. 264)

6.4 The Role of Multiple Models or Pre-existing Models

As noted earlier, models serve many purposes in society and the workplace. Models sometimes serve descriptive purposes, where the modeller wants to describe or predict the behaviour of some real phenomena. Both the PISA framework (OECD 2013) and the CCSSM (CCSSO 2010) point to the role of graphs in describing physical phenomena. However, as Hestenes (2010) and others have pointed out, models often need to serve explanatory purposes. To accomplish this, the modeller may need to draw on multiple models within the modelling process or on other pre-existing models, whose structure may need to be explored and understood. Consider, for example, the well-known problem of modelling light intensity as a function of distance from a light source. The graph of this relationship can readily be found to follow an inverse square relationship, but this leaves an important question unanswered: why is this an inverse square relationship? A graph is *descriptive* but not *explanatory*. To understand why light behaves in this way, another model is needed, namely, the geometry of the sphere (see Ärlebäck and Doerr 2015). Most representations of the modelling cycle do not include how these two models (one descriptive and the other explanatory) are brought together in the modelling process.

6.5 The Social and Critical Aspects of Modelling

We know from the work of Barbosa (2006), Niss (2015) and many others that models are projected back into the world. Recent years have provided us with numerous examples in governance and finance, as well as in science and engineering. For example, macroeconomic models of the development of state finances and welfare increasingly control political decision-making. New public management structures that encourage people to deliver more work and output on certain measurable parameters can be seen as the result of underlying models on how to increase worker productivity. In finance, the complexity of the models that govern the stock exchange (Johansen and Sørensen 2014), and the large losses that occurred as a consequence of these models, places new kinds of responsibilities on the mathematicians and financial analysts for the major economic losses that occurred during the dramatic events in the financial crisis. Our claim here is simple: models have a huge impact on our world; but the social and critical aspects of the role of models in such areas as governance, management and finance are not captured by the modelling cycle. As Barbosa (2006) noted, “mathematical models are not neutral descriptions about an independent reality” (p. 294). Barbosa described the kinds of critical mathematical modelling activity that occurs when pupils investigate a real social problem as “quite removed from the characterization of modelling as involving diagrammatic representations” (p. 294). Rosa and Orey (2015) have recently put forward a representation (Fig. 6.6) that captures some of the dynamic and humanized aspects of

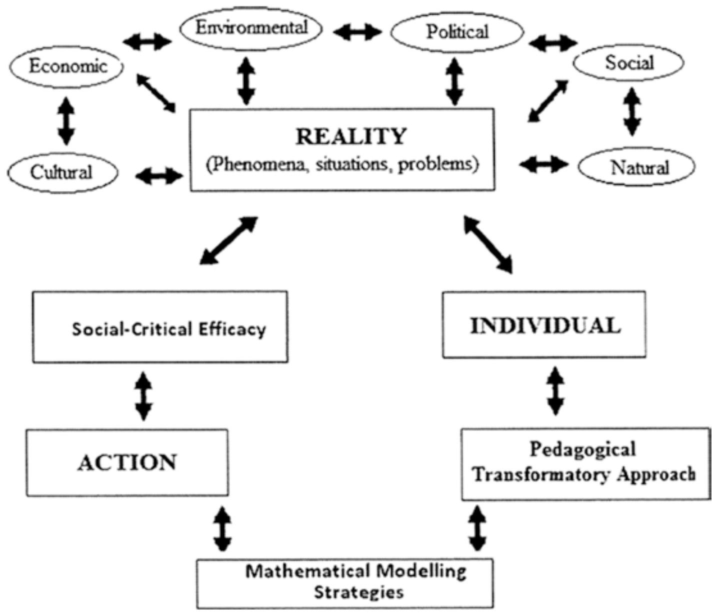

Fig. 6.6 Representation of socio-critical modelling (Rosa and Orey 2015, p. 394)

modelling, capturing the role of the individual modeller, the transformative nature of the pedagogy involved and the orientation towards action as models are projected back into a social context.

6.6 The Role of Digital Technologies

Some attempts to characterize the role of computational media have aimed at augmenting the modelling cycle (Greefrath et al. 2011). For instance, the representation shown in Fig. 6.7 depicts the "computer model" as distinctly separated from the mathematical model and suggests a sense in which technology becomes a medium for helping in the process of moving from mathematical problem (model) to mathematical results.

The interplay between the world and the mathematics that are shown in the modelling cycle (Figs. 6.1, 6.2 and 6.3) might have described the mathematical modelling done in an era when many crucial insights were gained from the interplay between mathematical analysis and real-world experiments. However, advances in computational media have changed this situation because a new kind of "experimental" work is now done through computational models of various phenomena. Moreover, these computational models often involve mathematics (particularly in the case of stochastic phenomena) that is simply not possible with the closed form solutions suggested by the image of the modelling cycle. Validation of such computational models is often far more complex than a mapping back to the problem situation would suggest. With computational media we often have several types of models involved in much modelling work. Indeed, we have only to look at the role of mathematical modelling in biology to see the role that computational experiments play. One representation that captures this interplay between physical phenomena (or empirical data), simulation (or computational data) and analysis (or explanatory theory) is shown in Fig. 6.8.

Research has shown that modern mathematical software can be a powerful tool in supporting a multitude of mathematical work processes and can act as a tool

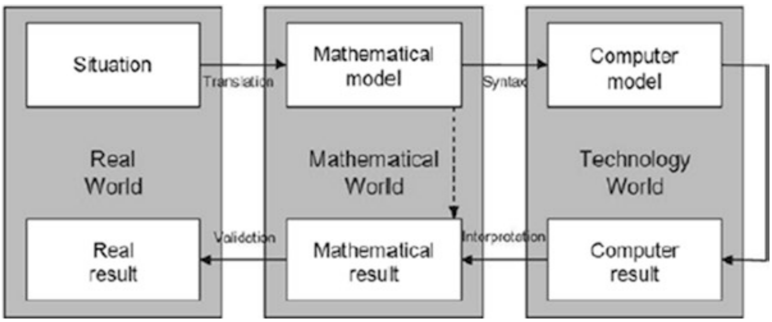

Fig. 6.7 Modelling cycle augmented with technology (Greefrath et al. 2011, p. 316)

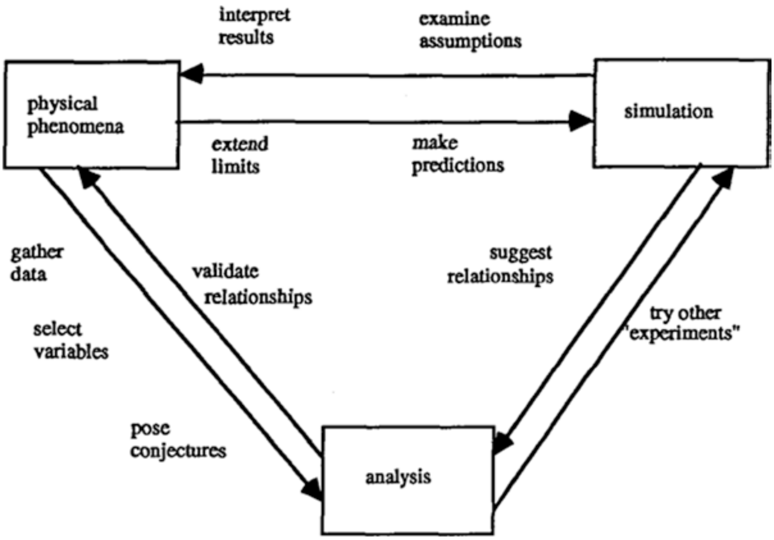

Fig. 6.8 The interplay of phenomena, simulation and analysis (Doerr 1997, p. 269)

towards enhancing the mathematical abilities of their users (Guin et al. 2005; Laborde 2005). But computational media have also been described as a new "universe" for mathematical activity, in the sense of a new type of mathematical reality. This has been articulated as computational media offering mathematical "microworlds" for students to tinker with in order to develop their mathematical curiosity and start mathematical investigations. Hence, modern computational media allow for new mathematical venues to be investigated and also allow professional mathematicians to investigate types of mathematical realities that previously were inaccessible (Borwein and Devlin 2009). Furthermore, the computational speed of computers allows mathematical models to project their results back into the world in real time, hence shaping the real world. In other words, computational media both empower the mathematical processes involved in modelling activities by providing new "worlds" to explore and potentially shape the world we try to model. These different roles can be summed up in a representation focusing on the roles of computational media in modelling activities rather than the modelling process as such, if we think of them as overlapping spheres of influence, as shown in Fig. 6.9.

6.7 Conclusion: The Necessity of Multiple Representations

The issue addressed in this chapter is the dominance of the one single image of mathematical modelling that is shown by the *modelling cycle* in international and national curriculum documents such as PISA (OECD 2013) and the Common Core Standards (CCSSO 2010). As noted earlier, any one representation of modelling has its strengths

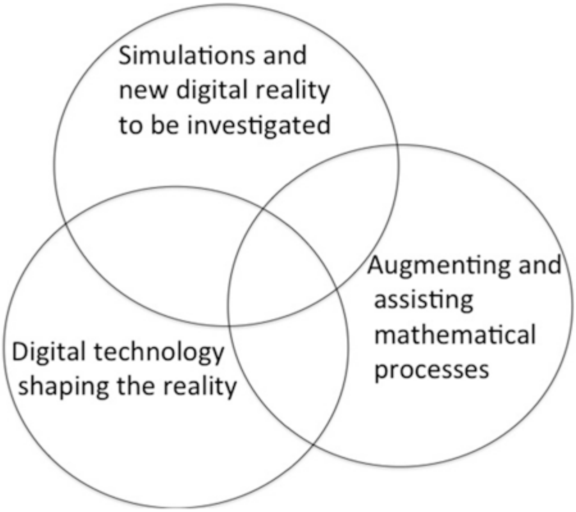

Fig. 6.9 Spheres of digital technology influence on the modelling process

and weaknesses, and hence we argue that we need multiple representations and images to capture and convey the richness of modelling for mathematics education for policy-makers, curriculum developers and teachers. Curriculum materials that would guide students through predetermined steps in a modelling cycle would be inadequate for conveying to teachers the non-linearity of the multiple learning pathways that would occur in a classroom. Similarly, modelling activities for students need to move beyond creating descriptive models that can be validated by comparison to empirical data to working with a full range of models including those with explanatory power, those with social and political implications and those using computational media. Representations of these aspects of modelling imply modelling tasks that explore and bring to bear existing models, that are socially relevant and engage students in action as the models are projected back into the world and that open up new realms of mathematical venues. Our recommendation is not that we should improve or revise the modelling cycle to encompass these important aspects of modelling. Rather, we suggest that a complex process such as mathematical modelling should be conveyed in policy and curriculum documents by multiple images that accommodate the aspects addressed in this chapter and through future research.

References

Ärlebäck, J. B., & Doerr, H. M. (2015). At the core of modelling: Connecting, coordinating and integrating models. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the 9th congress of European research in mathematics education*. Prague: Charles University. Retrieved from http://www.cerme9.org.

Barbosa, J. C. (2006). Mathematical modelling in classroom – a socio-critical and discursive perspective. *ZDM: The International Journal on Mathematics Education, 38*(3), 293–301.

Blomhøj, M., & Jensen, T. H. (2007). What's all the fuss about competencies? In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 45–56). New York: Springer.

Blum, W. (2015). Quality teaching of mathematical modelling: What do we know, what can we do? In S. J. Cho (Ed.), *The Proceedings of the 12th international congress on mathematical education* (pp. 73–96). Cham: Springer. doi:10.1007/978-3-319-12688-3_9.

Blum, W., & Leiß, D. (2007). How do students and teachers deal with modelling problems? In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling: Education, engineering and economics* (pp. 222–231). Chichester: Horwood.

Borromeo Ferri, R. (2007). Personal experiences and extra-mathematical knowledge as an influence factor on modelling routes of pupils. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of CERME 5* (pp. 2020–2079). Larnaca.

Borwein, J. M., & Devlin, K. J. (2009). *The computer as crucible: An introduction to experimental mathematics*. Wellesley: A.K. Peters.

Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Council of Chief State School Officers: Washington, DC. Retrieved from http://www.corestandards.org/Math/.

Doerr, H. M. (1997). Experiment, simulation and analysis: An integrated instructional approach to the concept of force. *International Journal of Science Education, 19*(3), 265–282.

Doerr, H. M., & Pratt, D. (2008). The learning of mathematics and mathematical modeling. In M. K. Heid & G. W. Blume (Eds.), *Research on technology in the teaching and learning of mathematics: Syntheses and perspectives: Vol. 1. Mathematics learning, teaching and policy* (pp. 259–285). Charlotte: Information Age.

Greefrath, G., Siller, H.-S., & Weitendorf, J. (2011). Modelling considering the influence of technology. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *Trends in teaching and learning mathematical modelling* (pp. 315–329). Dordrecht: Springer.

Guin, D., Ruthven, K., & Trouche, L. (2005). *The didactical challenge of symbolic calculators turning a computational device into a mathematical instrument*. New York: Springer.

Hestenes, D. (2010). Modeling theory for math and science education. In R. Lesh, R. Galbraith, C. Haines, & A. Huford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 13–41). New York: Springer.

Johansen, M. W., & Kragh, S. H. (2014). *Invitation til matematikkens videnskabsteori*. Frederiksberg: Samfundslitteratur.

Laborde, C. (2005). The hidden role of diagrams in students' construction of meaning in geometry. In J. Kilpatrick, C. Hoyles, O. Skovsmose, & P. Valero (Eds.), *Meaning in mathematics education* (Vol. 37, pp. 4–25). New York: Springer US. doi:10.1007/0-387-24040-3_11.

Lesh, R. A., & Doerr, H. M. (2012). Alternatives to trajectories and pathways to describe development in modeling and problem solving. In W. Blum, R. Borromeo Ferri, & K. Maaß (Eds.), *Mathematikunterricht im Kontext von Realität, Kultur un Lehrerprofessionalität* (pp. 138–147). Fachmedien Wiesbaden: Springer Spektrum.

Niss, M. (2015). Prescriptive modelling – challenges and opportunities. In G. A. Stillman, W. Blum, & M. Biembengut (Eds.), *Mathematical modelling in education research and practice: Cultural, social and cognitive influences* (pp. 67–79). Cham: Springer.

OECD. (2013). *PISA 2012 assessment and analytical framework: Mathematics, reading, science, problem solving and financial literacy*. Paris: OECD Publishing.

Rosa, M., & Orey, D. (2015). Social-critical dimension of mathematical modelling. In G. A. Stillman, W. Blum, & M. Biembengut (Eds.), *Mathematical modelling in education research and practice: Cultural, social and cognitive influences* (pp. 385–395). Cham: Springer.