Ιούνιος 2016. Θέμα 4 (ii)

Να λυθεί το πρόβλημα

$$x u u_x + y u u_y = -xy$$

$$u\left(x, \frac{1}{x}\right) = 5 \qquad \forall x>0$$

Συνοπτική λύση

Τη γράφουμε ως $x u_x + y u_y = -\frac{xy}{u}$

Χαρακτηριστικές

$$\left.\begin{aligned} x'(s) &= x(s) \\ y'(s) &= y(s) \end{aligned}\right\} \quad \text{παίρνουμε} \quad \begin{aligned} x(s) &= e^s \\ y(s) &= A e^s, \ s \in \mathbb{R} \end{aligned}$$

$$z'(s) = -\frac{x(s)\,y(s)}{z(s)}$$

$$z \cdot z' = -A e^{2s} \Rightarrow \left(z^2(s)\right)' = -2A e^{2s}$$

$$\Rightarrow z^2(s) = -A e^{2s} + B$$

Για $x, y > 0$ σταθερά παίρνουμε $A = \frac{y}{x}$

Για s_1: $x(s_1)\,y(s_1) = 1$ ($\Leftrightarrow A e^{2s_1} = 1$) πρέπει $z(s_1) = 5$.

$$\Rightarrow 25 = -1 + B \Rightarrow B = 26$$

Για s_0: $x(s_0) = x$ παίρνουμε

$$u(x,y) = z(s_0) = -A x^2 + B = -\frac{y}{x} x^2 + 26 = 26 - xy$$

Η $u(x,y) = \sqrt{26 - xy}$ είναι λύση στο $\{(x,y) \in \mathbb{R}^2 : xy < 26\}$

Άσκηση 1.22

Χαρακτηριστικές οι $\{(x, x+c) : x \in \mathbb{R}\}$, $c \in \mathbb{R}$

λύση $u(x,y) = \dfrac{2(y-x)}{x^2-y^2-4}$

στο $\{(x,y) \in \mathbb{R}^2 : x^2-y^2<4\}$

Ιούνιος 2016 Θέμα 1.

$u_t = k u_{xx} - \mu u \qquad (x,t) \in (0,L) \times (0,\infty)$

$k, \mu > 0$

$u(x,0) = \varphi(x)$

$\varphi(0) = \varphi(L) = 0, \quad \varphi(x) > 0$ για $x \in [0,L]$

$E(t) = \int_0^L u^2(x,t)\,dx$

i, ii/ Αν ισχύει $u(0,t) = u(L,t) = 0 \quad \forall t > 0$

ή $u_x(0,t) = u_x(L,t) = 0 \quad \forall t > 0$

τότε E είναι γνησίως φθίνουσα

(iii) $\lim_{t\to\infty} E(t) = 0$

λύση

το (iii) το είδαμε στην τάξη

Για τα (i), (ii).

$E'(t) = 2\int_0^L u u_t\,dx = 2k\int_0^L u u_{xx}\,dx - 2\mu\int_0^L u^2\,dx$

$= 2k\, u(x,t)\, u_x(x,t)\Big|_{x=0}^{x=L} - 2k\int_0^L u_x^2\,dx - 2\mu\int_0^L u^2\,dx$

$= -2k\int_0^L u_x^2\,dx - 2\mu\int_0^L u^2\,dx$

ισχυριζόμαστε ότι $\int_0^L u^2(x,t)\,dx > 0 \quad \forall t>0$

οπότε $E'(t)<0$ κλπ.

Έστω ότι $\exists\, t_0>0$ με $\int_0^L u^2(x,t_0)\,dx=0$

τότε $u(x,t_0)=0 \quad \forall x\in[0,L]$

Θέτουμε $v(x,t)=e^{\mu t}u(x,t)$

τότε η v ικανοποιεί την

$$v_t = k v_{xx} \qquad (x,t)\in(0,L)\times(0,a)$$

και αν $u(0,t)=u(L,t)=0$ τότε το ίδιο ισχύει και για την v

ομοίως αν $u_x(0,t)=u_x(L,t)=0$

Επειδή $v(x,t_0)=0 \quad \forall x\in[0,L]$

το θεώρημα μοναδικότητας* προς τα πίσω για την εξίσωση θερμότητας σε περιορισμένο διάστημα δίνει

$v(x,0)=0 \quad \forall x\in[0,L]$. Άτοπο αφού $v(x,0)=\varphi(x)$

Δες Θεώρημα 4, σελ 4 στις σημειώσεις που ακολουθούν.

Το θεώρημα ισχύει και με τις συνθήκες $v_x(0,t)=v_x(L,t)=0 \quad \forall t>0$ εκτός από την $v(0,t)=v(L,t)=0 \quad \forall t>0$ για την οποία γίνεται η απόδειξη.

Ένα τυπογραφικό: $\ddot{E}(t) = -4\int_0^L w_{tx}\, w_x \,dx$

Comments on the heat equation

Gyu Eun Lee

These notes were written for my personal use for teaching purposes for the course Math 136 running in the Spring of 2016 at UCLA.

1 Uniqueness for the heat equation on the line

Over the past few weeks we have discussed the heat equation $u_t - u_{xx} = 0$ in two contexts.[1] The first is the Cauchy problem for the heat equation on a finite interval:

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{for } x \in [0,l], t > 0, \\ u(0,t) &= f(t), \\ u(l,t) &= g(t), \\ u(x,0) &= \phi(x). \end{aligned} \tag{1}$$

The second is the Cauchy problem on the entire real line:

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{for } x \in (-\infty,\infty), t > 0, \\ u(x,0) &= \phi(x). \end{aligned} \tag{2}$$

For (1), we proved (using either the maximum principle or the energy method) the following uniqueness theorem:

Theorem 1. *Suppose u and v are both solutions to the heat equation on the interval $[0,l]$ with $u(0,t) = v(0,t)$, $u(l,t) = v(l,t)$, and $u_t(x,0) = v_t(x,0)$. Then $u \equiv v$.*

However, for (2), we do not have the maximum principle, and in contrast to the energy on the finite interval,

$$E(t) = \int_0^l u(x,t)^2 \, dx,$$

on the real line the energy

$$E(t) = \int_{-\infty}^{\infty} u(x,t)^2 \, dx$$

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[1] In these notes we take $k = 1$ without loss of generality.

is no longer necessarily well-defined, since the improper integral may diverge. In fact, if in (2) we take $\phi(x) \equiv 1$, then $u(x,t) \equiv 1$ is a solution to (2), and clearly the energy for this solution is infinite. Therefore we cannot hope to prove uniqueness for (2) by the same methods we have used for (1).

In fact, the situation is even worse: if we make no further assumptions on the solutions u, then we do not have uniqueness for (2). (Strauss claims that we do have uniqueness in 2.4, but in fact his "proof" sneaks in an extra assumption. More on this later.) One example of how this happens is

$$u(x,t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(t)}{(2n)!} x^{2n}, \tag{3}$$

where g is the infinitely differentiable function

$$g(t) = \begin{cases} e^{-1/t^2} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

The power series converges uniformly on all closed bounded subsets of the xt-plane with $t \geq 0$,[2] so u is well-defined, and formally differentiating gives us

$$u_t(x,t) = \sum_{n=0}^{\infty} \frac{g^{(n+1)}(t)}{(2n)!} x^{2n},$$

$$u_{xx}(x,t) = \sum_{n=2}^{\infty} \frac{g^{(n)}(t)}{(2n)!}(2n)(2n-1)x^{2n-2} = \sum_{n=2}^{\infty} \frac{g^{(n)}(t)}{(2n-2)!} x^{2n-2} = \sum_{n=0}^{\infty} \frac{g^{(n+1)}(t)}{(2n)!} x^{2n}.$$

Therefore u satisfies $u_t - u_{xx} = 0$. (Note that the exact form for g was not used. This series is a good way to come up with several solutions to the heat equation, by varying g appropriately.) As for the initial condition, it is not difficult to show that $g^{(n)}(0) = 0$ for all n; therefore $u(x,0) = 0$. So u solves (2) with initial condition $\phi(x) = 0$. But clearly $v(x,t) \equiv 0$ is another solution to this initial-value problem. Thus we see that uniqueness is violated.

What went so badly wrong? The issue is that $u(x,t)$ grows too rapidly as $|x| \to \infty$; in fact, there exist no constants $C > 0$, $\lambda > 0$ such that

$$|u(x,t)| \leq Ce^{\lambda x^2}$$

for all $x \in (-\infty, \infty)$ and all $t > 0$. That is, $u(x,t)$ beats all square-exponential growth (but, in some sense, just barely; see the references for details). This growth is a little too fast for the heat equation, whose fundamental solution

$$S_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

exhibits square-exponential decay. In fact, this square-exponential growth is something of a critical barrier between uniqueness and nonuniqueness, in the sense of the following theorem:

[2]To see this, it is sufficient to show that all derivatives of g are bounded on closed bounded subsets. This can be argued using the exponential decay of g as $t \searrow 0$.

Theorem 2. *Suppose u and v both solve*

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{for } x \in (-\infty, \infty),\ 0 < t \le T, \\ u(x,0) &= \phi(x). \end{aligned}$$

Suppose also that there exist constants $C, \lambda > 0$ *such that*

$$|u(x,t)| \le Ce^{\lambda x^2},\ |v(x,t)| \le Ce^{\lambda x^2}. (0 < t \le T) \tag{4}$$

Then $u \equiv v$ *in* $(-\infty, \infty) \times [0, T]$.

(For the proof of a more general assertion, see the references. This is, in essence, a type of maximum principle on an unbounded domain.) If the estimates in (4) hold for all $t > 0$, then we can send $T \to \infty$ as well.

Thus we see that it is possible to impose some sort of growth restrictions on the solutions u in order to obtain uniqueness. This uniqueness result holds only in the class of functions obeying these growth restrictions. Strauss alludes to this in his "proof" of uniqueness for the heat equation on the line, where he assumes ϕ is for $|y|$ sufficiently large - a condition commonly referred to as *compact support*. Though this is not proved, the fact is that assuming ϕ has compact support implies that the solution u satisfies (4) automatically (in fact, even better, u will be of "rapid decay," aka "Schwartz class;" see section 3 of these notes.).

There is another kind of restriction we can put on the solutions to obtain uniqueness. It no longer has to do with growth, but with integrability, and is an interesting application of the energy.

Definition. Given a function f defined on $(-\infty, \infty)$, we say f is *square-integrable* if

$$\int_{-\infty}^{\infty} f(x)^2\, dx < \infty.$$

Theorem 3. *Let u and v both be solutions of* (2) *with* $u(x,0) = v(x,0)$. *Suppose that u, v, and their derivatives* $u_t, v_t, u_x, v_x, u_{xx}, v_{xx}$ *are all square-integrable. Suppose also that* $u(x,t) \to 0$ *as* $|x| \to \infty$, *and* $v(x,t) \to 0$ *as* $|x| \to \infty$. *Then* $u \equiv v$.

Proof. Define the energy

$$E(u)(t) = \int_{-\infty}^{\infty} u(x,t)^2\, dx,\ E(v)(t) = \int_{-\infty}^{\infty} v(x,t)^2\, dx.$$

These are well-defined since we are assuming u and v are square-integrable. Let $w = u - v$. Then w solves (2) with zero initial conditions, so

$$E(w)(0) = \int_{-\infty}^{\infty} w(x,0)^2\, dx = 0.$$

One can show that if f^2 and g^2 are integrable, then

$$\left(\int_{-\infty}^{\infty} (f-g)^2\, dx\right)^{1/2} \le \left(\int_{-\infty}^{\infty} f^2\, dx\right)^{1/2} + \left(\int_{-\infty}^{\infty} g^2\, dx\right)^{1/2}.$$

(This is the triangle inequality for square-integrable functions.) Therefore for all $t > 0$, we have

$$E(w)(t)^{1/2} = E(u)(t)^{1/2} + E(v)(t)^{1/2} < \infty.$$

So $E(w)(t)$ is well-defined. Now recall that the energy of a solution to the heat equation is decreasing:

$$\frac{d}{dt}E(w)(t) = \int_{-\infty}^{\infty} w_t w \, dx = \int_{-\infty}^{\infty} w_{xx} w \, dx = -\int_{-\infty}^{\infty} w_x^2 \, dx \leq 0.$$

(To see that these integrals are well-defined, apply the Cauchy-Schwarz integral inequality and use our assumptions on u, v, and their derivatives.) The last equality is obtained by integration by parts; the assumption $u, v \to 0$ as $|x| \to \infty$ takes care of the boundary terms. So $E(w)(t)$ is a decreasing function of t. But $E(w)(0) = 0$, and clearly $E(w)(t) \geq 0$. Therefore $E(w)(t)$ must be constantly equal to 0. But this implies that $w(x,t)^2 \equiv 0$, and therefore $u(x,t) \equiv v(x,t)$. □

Therefore we obtain uniqueness for the heat equation if we restrict our attention to solutions in the class of square-integrable functions that decay at infinity.

In a physical scenario, we always expect uniqueness to solutions of the heat equation. The failure of uniqueness in these scenarios is a reflection of the non-physical nature of the domain: in reality, an infinite rod is an absurd object. However, the infinite-line model can be a useful approximation under certain physical circumstances. In practice, under these physical circumstances, we usually expect to have at least one of the above two uniqueness theorems available.

2 Backward uniqueness for the heat equation on a finite interval

I had intended to do this in discussion, but I ran out of time, so I will do this here.

In section 2.5, Strauss gives two examples to show that the heat equation on the finite interval is not well-posed, and in particular unstable, in backward time. That is, two solutions u and v that are very close to each other at time T need not be close at $T - \varepsilon$, where $\varepsilon > 0$. In fact, they may be infinitely far apart. That being said, it may surprise you to learn that the heat equation still has a backward uniqueness property:

Theorem 4. *Suppose u and v are both solutions of $u_t - u_{xx} = 0$ in $[0,l] \times [0,T]$ with $u(0,t) = v(0,t)$, $u(l,t) = v(l,t)$, and $u(x,T) = v(x,T)$. (Note we are not assuming anything about $t = 0$.) Then $u(x,t) = v(x,t)$ for all $0 \leq x \leq l$, $0 \leq t \leq T$.*

The proof is an adaptation of a more general proof from the book of Evans, which deals with the higher-dimensional case. It is another interesting application of the energy.

Proof. Let $w = u - v$. Then w solves the heat equation, and satisfies $w(0,t) = 0$, $w(l,t) = 0$, and $w(x,T) = 0$. Define as usual the energy

$$E(t) = \int_0^l w(x,t)^2 \, dx,$$

and denote its time derivatives by

$$\dot{E}(t) = \frac{d}{dt}E(t),\ \ddot{E}(t) = \frac{d^2}{dt^2}E(t).$$

Then taking the derivative inside the integral, we have

$$\dot{E}(t) = 2\int_0^l w_t w\ dx = 2\int_0^l w_{xx} w\ dx = -2\int_0^l w_x^2\ dx;$$

here we have used integration by parts and the fact that $w(0,t) = w(l,t) = 0$ to cancel the boundary term. Differentiating again, we have

$$\ddot{E}(t) = -4\int_0^l w_{tx} w\ dx = -4\int_0^1 w_{xxx} w\ dx = 4\int_0^l w_{xx}^2\ dx.$$

(Again, the boundary terms cancel in the integration by parts.) Now, by the Cauchy-Schwarz inequality for integrals,

$$\int_0^l w_x^2\ dx = -\int_0^l w_{xx} w\ dx \leq \left(\int_0^l w^2\ dx\right)^{1/2} \left(\int_0^l w_{xx}^2\ dx\right)^{1/2}.$$

Combining our formulas,

$$\dot{E}(t)^2 = 4\left(\int_0^l w_x^2\ dx\right) \leq \left(\int_0^l w^2\ dx\right)\left(4\int_0^l w_{xx}^2\ dx\right) = E(t)\ddot{E}(t),$$

or in summary

$$\dot{E}(t)^2 \leq E(t)\ddot{E}(t).$$

Now, if $E(t) = 0$ for all $0 \leq t \leq T$, then this implies that $w \equiv 0$, and hence $u(x,t) = v(x,t)$ for all $0 \leq t \leq T$, as claimed. Otherwise, there is an interval $[t_1,t_2] \subset [0,T]$ such that $E(t) > 0$ for $t_1 \leq t < t_2$, and $E(t_2) = 0$. (This last equality holds for some t_2 since $E(T) = 0$.) Define

$$f(t) = \log E(t),\ t_1 \leq t < t_2.$$

f is well-defined since $E(t) > 0$ for $t_1 \leq t < t_2$. Differentiating twice gives us

$$\ddot{f}(t) = \frac{\ddot{E}(t)}{E(t)} - \frac{\dot{E}(t)^2}{E(t)^2} \geq 0.$$

Therefore f is a convex function on (t_1,t_2). So for all $0 < \tau < 1$ and $t_1 < t < t_2$,

$$f((1-\tau)t_1 + \tau t) \leq (1-\tau)f(t_1) + \tau f(t).$$

Exponentiating,

$$E((1-\tau)t_1 + \tau t) \leq E(t_1)^{1-\tau}E(t)^{\tau}.$$

Sending t to t_2, we obtain

$$0 \leq E((1-\tau)t_1 + \tau t_2) \leq E(t_1)^{1-\tau}E(t_2)^{\tau} = 0.$$

But then $E(t) = 0$ for all $t_1 \leq t \leq t_2$, contradiction. □

3 Regularity for the heat equation on the line; Schwartz functions

In discussion we discussed when solutions to (2), the Cauchy problem for the heat equation on the real line, has smooth solutions. After some work, we eventually derived the sufficient condition that the initial condition ϕ decays fast enough to "beat all polynomial growth." More precisely, for every polynomial degree $n \geq 0$, there exists a constant C_n depending only on n such that

$$|x|^n |\phi(x)| \leq C_n \text{ for all } x \in (-\infty, \infty). \tag{5}$$

In particular, when ϕ is bounded and vanishes outside a finite interval (i.e. ϕ has compact support) this condition is satisfied trivially. Under this assumption, we saw in class how to differentiate under the integral sign to show that the solution u given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} \phi(y)\, dy$$

is infinitely differentiable in both x and t. If the above inequality is satisfies only up to $n \leq N$, then instead we obtain differentiability up to order N.

In passing, I alluded to the class of *Schwartz functions* briefly. Schwartz functions are smooth functions ϕ that satisfy not only the inequality (5) for all n, but also the inequality for all derivatives of ϕ. In other words, Schwartz functions are very nicely behaved functions: not only are they smooth, but they and all of their derivatives decay faster than any polynomial. They are also known as function of *rapid decay*.

Naturally every Schwartz function satisfies (5), and hence gives rise to a smooth function of the heat equation, but not all functions satisfying (5) for all n are Schwartz functions. (For instance, the function may fail to be smooth.) The prototypical examples of Schwartz functions are the Gaussian

$$f(x) = e^{-x^2}$$

and the standard mollifier (aka "bump function")

$$g(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & |x| < 1, \\ 0 & |x| \geq 1. \end{cases}$$

Schwartz functions are a very useful class of functions in PDE and analysis; in particular, they are the natural setting for the technique known as the Fourier transform.

References

[1] L. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island, USA, Second edition, 2010.

[2] X. Yu, *Heat equation - maximum principles*, Accessed 2016 April 27, [`http://www.math.ualberta.ca/~xinweiyu/527.1.08f/lec13.pdf`.]