Time Series

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MSc in Statistics and Operations Research

Outline of the Course

This course provides theory and practice of time series analysis

- Presents the basic theory of stationary/non-stationary processes - unit root testing
- Describes and presents analytically AR(I)MA models and the Box-Jenkins methodology
- Introduces the class of conditional heteroscedastic models (ARCH/GARCH)
- Presents time series forecasting techniques
- Illustrative examples applying time series models/techniques to actual economic and financial data

Part 1: Introduction and Unit Root Testing

- Introduction: modeling approaches
- Basic concepts: Autocorrelation and stationarity
- Properties of stationary and non-stationary processes
- Unit root testing: Augmented Dickey-Fuller test
- Illustration of unit root testing using R to economic and financial data sets
  - Example 1: unit root testing to financial time series, e.g. stocks and indices (application and useful conclusions)
  - Example 2: Unit root testing to exchange rate series (application and useful conclusions)

Introduction: Data

- Types of data
  - Time series data, $y_t, t = 1, \ldots, T$, is a sequence of random variables taking values at specific time periods (daily, weekly, monthly, etc.)
  - Cross-sectional data, $y_i, i = 1, \ldots, N$ refer to one or more characteristics (variables) being observed at the same point in time
  - Pooled data/panel data/longitudinal data, $y_{it}, i = 1, \ldots, N$ and $t = 1, \ldots, T$ refer to measurements on one or more characteristics collected at specific time periods (weekly, monthly, yearly, etc.)

Introduction: Aims of Time Series Analysis

- Construct appropriate models that are able to capture the characteristics of the observed data.
- Describe the relationship between different variables in time or between subsequent/lagged values of the time series.
- Use historical data and advanced statistical techniques in order to confirm the assertions of economic/financial theory.
- Obtain predictions of future values/forecasts.

Time Series Analysis aims to unreveal the data generating process (DGP) that governs the dynamics of observed time series of interest.

Introduction: Modeling Approaches

- Regression-type models: models that use explanatory variables, based on the economic/financial theory, or the problem at hand.
- Time series models: models that use the behavior - characteristics of the series under consideration at previous time periods.
- Regression models with time series components.

Further, we may consider:

- Univariate models
- Multivariate models

In this course we will focus on constructing and estimating univariate models for time series data.

Introduction: Regression Models

Use explanatory variables, based on the economic - financial theory, or the problem at hand.

Explanatory Models - Asset Pricing: built models with the aim to identify important explanatory variables (risk factors) that explain financial series.

$$y_t = \alpha + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \ldots + \beta_k x_{k,t} + \varepsilon_t$$

Forecasting Models - Return Predictability: built models with the aim to identify important predictive variables that have the ability to forecast financial returns.

$$y_t = \alpha + \beta_1 x_{1,t-1} + \beta_2 x_{2,t-1} + \ldots + \beta_k x_{k,t-1} + \varepsilon_t$$

Assuming (a) uncorrelated errors, (b) constant variance -

Introduction: Regression Models

If the standard assumptions on the error terms are violated:

- Point estimation of model parameters is valid [e.g. least squares, maximum likelihood].
- Statistical inference, which is theoretically based on the above assumptions is not valid [e.g. hypothesis testing, CIs].

Consequences:

- We can not identify accurately which risk factors are really important to explain financial returns and to predict future returns [model selection problem].
- We can not accurately infer the constant α in the regression model (test its statistical significance), which is a measure of the performance or skill of a manager, and the regression coefficients, which quantify the relationship between y_t and the risk factors or predictors.

Introduction: Time Series models

Use lagged values of the series or/and lagged error terms.

Autoregressive models [AR(p)]

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + \varepsilon_t$$

Moving Average models [MA(q)]

$$y_t = \mu + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \ldots + \theta_q \varepsilon_{t-q} + \varepsilon_t$$

Autoregressive Moving Average models [ARMA(p,q)]

$$y_t = \delta + \phi_1 y_{t-1} + \ldots + \phi_p y_{t-p} + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q} + \varepsilon_t$$

Assuming (a) uncorrelated errors, (b) constant variance - homoscedastic errors, (c) normal errors.

Introduction: Regression - Time Series Models

Models that use both explanatory variables and time series components [due to autocorrelated regression errors].

$$y_t = \alpha + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \ldots + \beta_k x_{k,t} + u_t$$

$$u_t = \delta + \phi_1 u_{t-1} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim N(0, \sigma^2)$$

These models are able to account for autocorrelation, assuming homoscedastic and normally distributed error terms.

Introduction: Regression - Time Series - Volatility Models

Models that use explanatory variables, time series components [due to autocorrelated regression errors] and volatility models [due to heteroscedastic errors].

$$y_t = \alpha + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \ldots + \beta_k x_{k,t} + u_t$$

$$u_t = \delta + \phi_1 u_{t-1} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim N(0, \sigma_t^2)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \sigma_{t-1}^2$$

Assuming (a)autocorrelated errors, (b) heteroscedasticity (e.g. volatility clustering, fat tails, excess kurtosis).

Basic concepts: Stationarity

- Strictly Stationary process: the joint distribution of $(y_i, y_{i+1}, \ldots, y_{i+k})$ and $(y_{i+m}, y_{i+m+1}, \ldots, y_{i+m+k})$ are the same for all i, k, m.
- Weakly Stationary process: the mean, the variance and the autocovariance do not depend on time t.

More rigorously, a process is said to be weakly stationary if:

$E(y_t) = \mu$, for all t,

$V(y_t) = E(y_t - \mu)^2 = \sigma^2$, for all t,

$\gamma_k = Cov(y_t, y_{t-k}) = E[(y_t - \mu)(y_{t-k} - \mu)]$, for all t and any k.

Basic concepts: Autocorrelation

Autocorrelation shows the interdependence - correlation between the values of the series at different time periods.

$$\rho_k = Corr(y_t, y_{t-k}) = \frac{Cov(y_t, y_{t-k})}{\sigma_{y_t}\sigma_{y_{t-k}}} = \frac{\gamma_k}{\gamma_0}$$

$$\rho_k = \frac{E[(y_t-\mu)(y_{t-k}-\mu)]}{\sqrt{E(y_t-\mu)^2}\sqrt{E(y_{t-k}-\mu)^2}}$$

Properties of autocorrelation:

$$\rho_k = \rho_{-k}$$

$$-1 \leq \rho_k \leq 1$$

Sample estimate of autocorrelation:

$$\hat{\rho}_k = \frac{\sum_{t=1}^{T-k}(y_t-\bar{y})(y_{t-k}-\bar{y})}{\sum_{t=1}^{T}(y_t-\bar{y})^2}$$

Significance Test for the Autocorrelation

Bartlett's test (for a particular lag k):
$H_0 : \rho_k = 0$
$H_1 : \rho_k \neq 0$

If the time series is random (white noise), then the sampling distribution of $\hat{\rho}_k$ is approximately normal, i.e. $\hat{\rho}_k \sim N(0, \frac{1}{T})$.

test statistic: $Z = \frac{\hat{\rho}_k - 0}{\sqrt{1/T}} \sim N(0, 1)$

Reject H_0, at level of significance α, if the observed value of the test statistic $Z < -Z_{1-\alpha/2}$ or $Z > Z_{1-\alpha/2}$.

$100(1-\alpha)\%$ Confidence interval for ρ_k:
$(\hat{\rho}_k - Z_{1-\alpha/2}\sqrt{1/T}, \hat{\rho}_k + Z_{1-\alpha/2}\sqrt{1/T})$.

Significance Test for all Autocorrelations

$H_0 : \rho_1 = \rho_2 = \ldots = \rho_m = 0$, for a fixed value of m
$H_1 : \rho_i \neq 0$, for at least one $i \leq m$

Box-Pierce test statistic: $Q = T \sum_{k=1}^{m} \hat{\rho_k}^2 \sim \chi^2_m$

Ljung-Box test statistic: $LB = T(T+2) \sum_{k=1}^{m} \frac{\hat{\rho_k}^2}{T-k} \sim \chi^2_m$

The Ljung-Box test has better small sample properties.

Reject H_0, at level of significance α, if the observed value of the test statistic $Q > \chi^2_{m,1-\alpha}$ ($LB > \chi^2_{m,1-\alpha}$).

Understanding stationarity

Consider a time series y_t, and assume an AR(1) model of the form: $y_t = \mu + \rho y_{t-1} + \epsilon_t$, where ϵ_t are uncorrelated with mean zero and variance σ^2.

$t = 1$: $y_1 = \mu + \rho y_0 + \epsilon_1$

$t = 2$:

$y_2 = \mu + \rho y_1 + \epsilon_2 = \mu + \rho(\mu + \rho y_0 + \epsilon_1) + \epsilon_2 = \mu + \rho\mu + \rho^2 y_0 + \rho\epsilon_1 + \epsilon_2$

$t = 3$: $y_3 = \mu + \rho\mu + \rho^2\mu + \rho^3 y_0 + \rho^2\epsilon_1 + \rho\epsilon_2 + \epsilon_3$

...

$t = t$:

$y_t = \mu + \rho\mu + \rho^2\mu + \ldots + \rho^{t-1}\mu + \rho^t y_0 + \rho^{t-1}\epsilon_1 + \rho^{t-2}\epsilon_2 + \ldots + \epsilon_t$

$y_t = \rho^t y_0 + \mu \sum_{s=0}^{t-1} \rho^s + \sum_{s=1}^{t} \rho^{t-s}\epsilon_s$

Understanding stationarity

The ϵ_t's are the shocks at time t. The parameter ρ shows if the shocks are permanent or temporary. Assume that at time $t = 1$ the shock is ϵ_1. Which is the effect of ϵ_1 on the value of the time series at time t, y_t?

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The effect is given by: $\frac{\partial y_t}{\partial \epsilon_1} = \rho^{t-1}$

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$t = 1$: $\frac{\partial y_1}{\partial \epsilon_1} = \rho^{1-1} = \rho^0 = 1$
$t = 2$: $\frac{\partial y_2}{\partial \epsilon_1} = \rho^{2-1} = \rho$
$t = 3$: $\frac{\partial y_3}{\partial \epsilon_1} = \rho^{3-1} = \rho^2 \ldots$

Understanding stationarity

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If $|\rho| < 1$, then $\frac{\partial y_t}{\partial \epsilon_1} \to 0$, as $t \to \infty$: not permanent shocks, i.e. the effect of ϵ_1 vanishes after some period of time.

Understanding stationarity

The ϵ_t's are the shocks at time t. The parameter ρ shows if the shocks are permanent or temporary. Assume that at time $t = 1$ the shock is ϵ_1. Which is the effect of ϵ_1 on the value of the time series at time t, y_t?

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$t = 1$: $\frac{\partial y_1}{\partial \epsilon_1} = \rho^{1-1} = \rho^0 = 1$
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$t = 3$: $\frac{\partial y_3}{\partial \epsilon_1} = \rho^{3-1} = \rho^2 \ \ldots$

If $|\rho| < 1$, then $\frac{\partial y_t}{\partial \epsilon_1} \to 0$, as $t \to \infty$: not permanent shocks, i.e. the effect of ϵ_1 vanishes after some period of time.

If $\rho = 1$, then $\frac{\partial y_t}{\partial \epsilon_1} = 1$: permanent shocks, i.e. the random term at time $t = 1$, ϵ_1, affects the series y_t permanently.

Non-stationary process I: Random walk with drift

For $\rho = 1$ i.e. when the shocks are permanent, the model takes the form: $y_t = \mu + y_{t-1} + \epsilon_t$ [Random walk with drift].

We will write down the model in an equivalent form:

$t = 1$: $y_1 = \mu + y_0 + \epsilon_1$

$t = 2$:
$y_2 = \mu + y_1 + \epsilon_2 = \mu + (\mu + y_0 + \epsilon_1) + \epsilon_2 = \mu + \mu + y_0 + \epsilon_1 + \epsilon_2$

$t = 3$: $y_3 = \mu + y_2 + \epsilon_3 = \mu + \mu + \mu + y_0 + \epsilon_1 + \epsilon_2 + \epsilon_3$

...

$t = t$: $y_t = t\mu + y_0 + \sum_{s=1}^{t} \epsilon_s$

Non-stationary process I: Random walk with drift

Random walk with drift: $y_t = \mu + y_{t-1} + \epsilon_t = t\mu + y_0 + \sum_{s=1}^{t} \epsilon_s$

We will prove that in this case, the series y_t is a non-stationary process. For simplicity assume that $y_0 = 0$ (in general, we take y_0 fixed).

Non-stationary process I: Random walk with drift

Random walk with drift: $y_t = \mu + y_{t-1} + \epsilon_t = t\mu + y_0 + \sum_{s=1}^{t} \epsilon_s$

We will prove that in this case, the series y_t is a non-stationary process. For simplicity assume that $y_0 = 0$ (in general, we take y_0 fixed).

$$E(y_t) = E(t\mu + y_0 + \sum_{s=1}^{t} \epsilon_s) = E(t\mu) + E(y_0) + E(\sum_{s=1}^{t} \epsilon_s) = t\mu$$

Non-stationary process I: Random walk with drift

Random walk with drift: $y_t = \mu + y_{t-1} + \epsilon_t = t\mu + y_0 + \sum_{s=1}^{t} \epsilon_s$

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$$E(y_t) = E(t\mu + y_0 + \sum_{s=1}^{t} \epsilon_s) = E(t\mu) + E(y_0) + E(\sum_{s=1}^{t} \epsilon_s) = t\mu$$

$$V(y_t) = V(t\mu + y_0 + \sum_{s=1}^{t} \epsilon_s) = V(t\mu) + V(y_0) + V(\sum_{s=1}^{t} \epsilon_s) = t\sigma^2$$

Non-stationary process I: Random walk with drift

Random walk with drift: $y_t = \mu + y_{t-1} + \epsilon_t = t\mu + y_0 + \sum_{s=1}^{t} \epsilon_s$

We will prove that in this case, the series y_t is a non-stationary process. For simplicity assume that $y_0 = 0$ (in general, we take y_0 fixed).

$E(y_t) = E(t\mu + y_0 + \sum_{s=1}^{t} \epsilon_s) = E(t\mu) + E(y_0) + E(\sum_{s=1}^{t} \epsilon_s) = t\mu$

$V(y_t) = V(t\mu + y_0 + \sum_{s=1}^{t} \epsilon_s) = V(t\mu) + V(y_0) + V(\sum_{s=1}^{t} \epsilon_s) = t\sigma^2$

$\gamma_k = Cov(y_t, y_{t-k}) = E[(y_t - E(y_t))(y_{t-k} - E(y_{t-k}))]$

Non-stationary process I: Random walk with drift

Random walk with drift: $y_t = \mu + y_{t-1} + \epsilon_t = t\mu + y_0 + \sum_{s=1}^{t} \epsilon_s$

We will prove that in this case, the series y_t is a non-stationary process. For simplicity assume that $y_0 = 0$ (in general, we take y_0 fixed).

$E(y_t) = E(t\mu + y_0 + \sum_{s=1}^{t} \epsilon_s) = E(t\mu) + E(y_0) + E(\sum_{s=1}^{t} \epsilon_s) = t\mu$

$V(y_t) = V(t\mu + y_0 + \sum_{s=1}^{t} \epsilon_s) = V(t\mu) + V(y_0) + V(\sum_{s=1}^{t} \epsilon_s) = t\sigma^2$

$\gamma_k = Cov(y_t, y_{t-k}) = E[(y_t - E(y_t))(y_{t-k} - E(y_{t-k}))]$

$= E[(y_t - t\mu)(y_{t-k} - (t-k)\mu)] = E[(y_0 + \sum_{s=1}^{t} \epsilon_s)(y_0 + \sum_{s=1}^{t-k} \epsilon_s)]$

$= E[(\sum_{s=1}^{t} \epsilon_s)(\sum_{s=1}^{t-k} \epsilon_s)] = (t-k)\sigma^2$

Non-stationary process I: Random walk with drift

Therefore, the Random walk with drift model: $y_t = \mu + y_{t-1} + \epsilon_t$

- is a non-stationary process
- has permanent shocks
- its mean is not constant over time, $E(Y_t) = t\mu$, i.e. it has a linear trend
- its variance is not constant over time, $V(y_t) = t\sigma^2$, i.e. it increases over time
- its covariance, i.e. the way the lagged values affect future values, changes over time

Non-stationary process II: Random walk without drift

Consider a time series y_t and assume a model of the form: $y_t = \rho y_{t-1} + \epsilon_t$, where ϵ_t are uncorrelated with mean zero and variance σ^2.

For $\rho = 1$ i.e. when the shocks are permanent, the model takes the form: $y_t = y_{t-1} + \epsilon_t$ [Random walk without drift]

We will write the model in an equivalent form:

$t = 1$: $y_1 = y_0 + \epsilon_1$

$t = 2$: $y_2 = y_1 + \epsilon_2 = (y_0 + \epsilon_1) + \epsilon_2 = y_0 + \epsilon_1 + \epsilon_2$

$t = 3$: $y_3 = y_2 + \epsilon_3 = y_0 + \epsilon_1 + \epsilon_2 + \epsilon_3$

...

$t = t$: $y_t = y_0 + \sum_{s=1}^{t} \epsilon_s$

Non-stationary process II: Random walk without drift

Random walk without drift: $y_t = y_{t-1} + \epsilon_t = y_0 + \sum_{s=1}^{t} \epsilon_s$

We will prove that the series y_t is a non-stationary process. For simplicity assume that $y_0 = 0$.

Non-stationary process II: Random walk without drift

Random walk without drift: $y_t = y_{t-1} + \epsilon_t = y_0 + \sum_{s=1}^{t} \epsilon_s$

We will prove that the series y_t is a non-stationary process. For simplicity assume that $y_0 = 0$.

$$E(y_t) = E(y_0 + \sum_{s=1}^{t} \epsilon_s) = E(y_0) + E(\sum_{s=1}^{t} \epsilon_s) = 0$$

Non-stationary process II: Random walk without drift

Random walk without drift: $y_t = y_{t-1} + \epsilon_t = y_0 + \sum_{s=1}^{t} \epsilon_s$

We will prove that the series y_t is a non-stationary process. For simplicity assume that $y_0 = 0$.

$$E(y_t) = E(y_0 + \sum_{s=1}^{t} \epsilon_s) = E(y_0) + E(\sum_{s=1}^{t} \epsilon_s) = 0$$

$$V(y_t) = V(y_0 + \sum_{s=1}^{t} \epsilon_s) = V(y_0) + V(\sum_{s=1}^{t} \epsilon_s) = t\sigma^2$$

Non-stationary process II: Random walk without drift

Random walk without drift: $y_t = y_{t-1} + \epsilon_t = y_0 + \sum_{s=1}^{t} \epsilon_s$

We will prove that the series y_t is a non-stationary process. For simplicity assume that $y_0 = 0$.

$$E(y_t) = E(y_0 + \sum_{s=1}^{t} \epsilon_s) = E(y_0) + E(\sum_{s=1}^{t} \epsilon_s) = 0$$

$$V(y_t) = V(y_0 + \sum_{s=1}^{t} \epsilon_s) = V(y_0) + V(\sum_{s=1}^{t} \epsilon_s) = t\sigma^2$$

$$\gamma_k = Cov(y_t, y_{t-k}) = E[(y_t - E(y_t))(y_{t-k} - E(y_{t-k}))]$$

$$= E[y_t y_{t-k}] = E[(y_0 + \sum_{s=1}^{t} \epsilon_s)(y_0 + \sum_{s=1}^{t-k} \epsilon_s)]$$

$$= E[(\sum_{s=1}^{t} \epsilon_s)(\sum_{s=1}^{t-k} \epsilon_s)] = (t-k)\sigma^2$$

Non-stationary process II: Random walk without drift

Therefore, the Random walk without drift model: $y_t = y_{t-1} + \epsilon_t$

- is a non-stationary process
- has permanent shocks
- its mean is constant through time, $E(Y_t) = 0$, i.e. y_t moves around zero
- its variance is not constant over time, $V(y_t) = t\sigma^2$, i.e. it increases over time
- its covariance, i.e. the way the lagged values affect future values, changes over time

Stationarity through Differencing I

Consider a non-stationary process y_t which follows a Random walk model with drift, i.e. $y_t = \mu + y_{t-1} + \epsilon_t$

By subtracting y_{t-1} we obtain:

$$y_t = \mu + y_{t-1} + \epsilon_t \Rightarrow y_t - y_{t-1} = \mu + y_{t-1} + \epsilon_t - y_{t-1} \Rightarrow$$

$$Z_t = \Delta y_t = \mu + \epsilon_t$$

$$E(Z_t) = E(\Delta y_t) = E(\mu + \epsilon_t) = E(\mu) + E(\epsilon_t) = \mu$$

$$V(Z_t) = V(\Delta y_t) = V(\mu + \epsilon_t) = V(\mu) + V(\epsilon_t) = \sigma^2$$

$$\gamma_k = Cov(Z_t, Z_{t-k}) = E[(Z_t - E(Z_t))(Z_{t-k} - E(Z_{t-k}))]$$

$$= E[(Z_t - \mu)(Z_{t-k} - \mu)] = E[\epsilon_t \epsilon_{t-k}] = 0$$

That is $Z_t = \Delta y_t$ is a stationary process.

Stationarity through Differencing II

Consider a non-stationary process y_t which follows a Random walk model without drift, i.e. $y_t = y_{t-1} + \epsilon_t$

By subtracting y_{t-1} we obtain:

$$y_t = y_{t-1} + \epsilon_t \Rightarrow y_t - y_{t-1} = \epsilon_t \Rightarrow Z_t = \Delta y_t = \epsilon_t$$

$$E(Z_t) = E(\Delta y_t) = E(\epsilon_t) = 0$$

$$V(Z_t) = V(\Delta y_t) = V(\epsilon_t) = \sigma^2$$

$$\gamma_k = Cov(Z_t, Z_{t-k}) = E[(Z_t - E(Z_t))(Z_{t-k} - E(Z_{t-k}))]$$

$$= E[Z_t Z_{t-k}] = E[\epsilon_t \epsilon_{t-k}] = 0$$

That is $Z_t = \Delta y_t$ is a stationary process

Stationarity through Differencing: Definitions

Consider a non-stationary process y_t

If $\Delta y_t = y_t - y_{t-1}$ is a stationary process, then y_t is called Integrated of order one [$I(1)$].

Generally, if y_t is non-stationary and by taking iteratively d differences y_t becomes stationary, then y_t is called Integrated of order d, $I(d)$.

If y_t is stationary, then it is an $I(0)$ process.

Stationary process: The AR(1) model

Consider a time series y_t and assume an AR(1) model of the form: $y_t = \mu + \rho y_{t-1} + \epsilon_t$, where ϵ_t are uncorrelated with mean zero and variance σ^2.

Recall that for $|\rho| < 1$, the shocks are not permanent and the effect of ϵ_1, or generally of ϵ_t, vanishes after some period of time.

Furthermore, recall that y_t can be written as

$y_t = \rho^t y_0 + \mu \sum_{s=0}^{t-1} \rho^s + \sum_{s=1}^{t} \rho^{t-s} \epsilon_s$

Assuming that $y_0 = 0$, the mean, variance and autocovariance at lag k of y_t are given by

$E(y_t) = \frac{\mu}{1-\rho}$

$V(y_t) = \frac{\sigma^2}{1-\rho^2}$

$\gamma_k = Cov(y_t, y_{t-k}) = \rho^k \gamma_0 = \rho^k \frac{\sigma^2}{1-\rho^2}$

Stationary process: The AR(1) model without constant

Consider a time series y_t and assume an AR(1) model of the form: $y_t = \rho y_{t-1} + \epsilon_t$, where ϵ_t are uncorrelated with mean zero and variance σ^2.

Again, for $|\rho| < 1$, the shocks are not permanent and the the effect of ϵ_1, or generally of ϵ_t, vanishes after some period of time. This is a special case of the AR(1) model, with $\mu = 0$.

Assuming that $y_0 = 0$, the mean, variance and autocovariance at lag k of $y_t$are given by

$E(y_t) = 0$

$V(y_t) = \frac{\sigma^2}{1-\rho^2}$

$\gamma_k = Cov(y_t, y_{t-k}) = \rho^k \gamma_0 = \rho^k \frac{\sigma^2}{1-\rho^2}$

Unit-Root test of Stationarity: Different tests

The hypothesis test of interest (test for stationarity) is:
$H_0 : \rho = 1$
$H_1 : |\rho| < 1$ usually $H_1 : \rho < 1$

Under H_0, the process is non-stationary, the variance of the process increases over time, therefore a standard t-test is not valid.

Different testing approaches have been proposed in the literature:

- Dickey - Fuller test (Augmented Dickey-Fuller)
- Phillips - Perron test
- Kwiatkowski - Phillips - Schmidt - Shin test
- Ng - Perron test

The main problem of the tests for stationarity is that the power of the tests is not large.

Unit-Root test of Stationarity: Different models

The stationary test of interest is:

$H_0 : \rho = 1$

$H_1 : |\rho| < 1$ usually $H_1 : \rho < 1$

Different modeling approaches have been proposed in the literature:

- AR(1) model with constant: $y_t = \mu + \rho y_{t-1} + \epsilon_t$
- AR(1) model without constant: $y_t = \rho y_{t-1} + \epsilon_t$
- AR(1) model with constant and linear trend: $y_t = \mu + \rho y_{t-1} + \gamma t + \epsilon_t$
- AR(p) model with/without constant/trend
- AR(p) models with structural breaks , etc.

The idea is that in order to test if a process is stationary or not, one needs to use a model that fits the data well.

Dickey-Fuller test - Model with constant

Model under consideration: $y_t = \mu + \rho y_{t-1} + \epsilon_t$
$H_0 : \rho = 1$ [Non-stationary process: Random walk with drift]
$H_1 : \rho < 1$ [Stationary process: AR(1) with constant]

The model can be reparametrized as follows:

$$y_t = \mu + \rho y_{t-1} + \epsilon_t \Rightarrow y_t - y_{t-1} = \mu + \rho y_{t-1} + \epsilon_t - y_{t-1} \Rightarrow$$

$$\Delta y_t = \mu + (\rho - 1) y_{t-1} + \epsilon_t \Rightarrow$$

$\Delta y_t = \mu + \beta y_{t-1} + \epsilon_t$, where $\beta = \rho - 1$

$H_0 : \beta = 0$ [Non-stationary process]
$H_1 : \beta < 0$ [Stationary process]

The reparametrized model is used, but the test examines stationarity of the y_t process, not of the Δy_t process!!!

Dickey-Fuller test - Model with constant

- Similar in spirit with an one-tailed regression-type test
- The test statistic is of the form: $\frac{\widehat{\beta}}{s.e.(\widehat{\beta})}$
- Due to non-stationarity under H_0, the distribution of the test statistic is not Student-t
- Dickey - Fuller have provided 'corrected' critical values
- Reject H_0 if the test statistic is smaller than the critical value in the left tail of the distribution
- Reject H_0 if the significance level α is larger than the corresponding p-value

Dickey-Fuller test - Model without constant

Model under consideration: $y_t = \rho y_{t-1} + \epsilon_t$
$H_0 : \rho = 1$ [Non-stationary process: Random walk without drift]
$H_1 : \rho < 1$ [Stationary process: AR(1) without constant]

The model can be reparametrized as follows:

$$y_t = \rho y_{t-1} + \epsilon_t \Rightarrow y_t - y_{t-1} = \rho y_{t-1} + \epsilon_t - y_{t-1} \Rightarrow$$

$$\Delta y_t = (\rho - 1) y_{t-1} + \epsilon_t \Rightarrow$$

$\Delta y_t = \beta y_{t-1} + \epsilon_t$, where $\beta = \rho - 1$

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Dickey-Fuller test - Model with constant and trend

Model under consideration: $y_t = \mu + \rho y_{t-1} + \gamma t + \epsilon_t$
$H_0 : \rho = 1 (\gamma = 0)$ [Non-stationary process: Stochastic trend]
$H_1 : \rho < 1 (\gamma \neq 0)$ [Stationary process: Deterministic trend]

The model can be reparametrized as follows:

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$$\Delta y_t = \mu + (\rho - 1) y_{t-1} + \gamma t + \epsilon_t \Rightarrow$$

$\Delta y_t = \mu + \beta y_{t-1} + \gamma t + \epsilon_t$, where $\beta = \rho - 1$

$H_0 : \beta = 0 (\gamma = 0)$ [Non-stationary process]
$H_1 : \beta < 0 (\gamma \neq 0)$ [Stationary process]

The reparametrized model is used, but the test examines stationarity of the y_t process, not of the Δy_t process!!!

Dickey-Fuller test - Model with constant and trend

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$$\Delta y_t = \mu + (\rho - 1) y_{t-1} + \gamma t + \epsilon_t \Rightarrow$$

$$\Delta y_t = \mu + \beta y_{t-1} + \gamma t + \epsilon_t, \text{ where } \beta = \rho - 1$$

$H_0 : \beta = 0 (\gamma = 0)$ [Non-stationary process]
$H_1 : \beta < 0 (\gamma \neq 0)$ [Stationary process]

The reparametrized model is used, but the test examines stationarity of the y_t process, not of the Δy_t process!!!

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Augmented Dickey-Fuller test

$H_0 : \beta = 0$ [Non-stationary process]
$H_1 : \beta < 0$ [Stationary process]

If the errors $\widehat{\epsilon_t}$ in the model under consideration are correlated, we use the Augmented Dickey-Fuller test (ADF) to examine stationarity. That is, the model takes the form:

$$\Delta y_t = \mu + \beta y_{t-1} + \sum_{j=1}^{p} \lambda_j \Delta y_{t-j} + \epsilon_t$$

$$\Delta y_t = \beta y_{t-1} + \sum_{j=1}^{p} \lambda_j \Delta y_{t-j} + \epsilon_t$$

$$\Delta y_t = \mu + \beta y_{t-1} + \gamma t + \sum_{j=1}^{p} \lambda_j \Delta y_{t-j} + \epsilon_t$$

Part 1: Application to financial and economic series

- Example 1: unit root testing to Johnson & Johnson quarterly data
- Example 2: unit root testing to exchange rate series
- Detailed empirical analysis is presented in the applications using R