



The multiplicative meaning conveyed by visual representations

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ABSTRACT

Research and practitioner articles advocate the use of visual representations in scaffolding elementary students' learning of multiplication and division. Prior research suggests students use different strategies when provided with different visualized representations of multiplication and division. However, there is relatively little study examining how children's multiplicative reasoning corresponds with different representations. The present study collected data from 182 elementary students responding to set, area, and length representations of multiplication/division. Rasch modeling was used to estimate item difficulty statistics to measure differences between visual representations. Results suggest that visual representations differed primarily in how unit was represented and quantified, and not regarding the form of representation (set, area, length).

1. Introduction

Multiplicative reasoning is important in elementary children's mathematical learning in early and later years of schooling (Hackenberg & Tillema, 2009; Norton et al., 2015). For decades, visual representations have been advocated as fundamental for facilitating multiplicative reasoning (Davydov, 1991; Gilbert & Bush, 1988; Kinzer & Stanford, 2014; Lay, 1963), and researchers have examined both children's and teachers' interactions with such representations (Anghileri, 1989; Bolden, Barmby, Raine, & Gardner, 2015; Clark & Kamii, 1996; Dabic & Milinkovic, 2015; Hurst, 2017; Kosko, 2018; Mulligan & Mitchelmore, 2009; Nunes & Bryant, 1995).

Current research on multiplicative reasoning has generally focused on describing children's reasoning in response to various tasks, where such tasks may incorporate various representations. However, such research seldom focuses on the role such representations take in promoting or limiting engagement in various levels of multiplicative reasoning. For example, a common finding with the use of traditional arrays is that many students will rely on count-by-1s strategies, while others attend to the column-row structure (Anghileri, 1995; Barmby, Harries, Higgins, & Suggate, 2009; Barrett & Clements, 2003; Davydov, 1991; Huang, 2014). Although this describes reasonings different children may use with the same representation, such research does not describe the reasonings that different representations of multiplicative structure may encourage. Therefore, it is the purpose of the current study to explore this relationship by examining the interaction between variations in multiplicative representations with variations in children's multiplicative reasoning.

2. Theoretical perspective

The present study considers multiplicative reasoning from the perspective of Scheme Theory (Empson & Turner, 2006; Hackenberg, 2010; Lamon, 1994; Steffe, 1994), with particular emphasis on the multiplicative concepts described by Hackenberg

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(2010) and others. Multiplicative reasoning, as described by scheme theory, is fundamentally concerned with unit coordination (Hackenberg, 2010; Steffe, 1994), where a *unit*, in the context of whole numbers, is a re-presentation of a counting act or acts (Ulrich, 2015). Depending on a child's multiplicative reasoning, they may construct a unit *in activity*, (i.e., counting each 1 within a 3 by 5 array). However, such coordination is described as *anticipatory* "when recognition of a situation leads to a re-presentation of the activity and/or results" (Steffe, 1992, p. 307–308). For example, a child may anticipate the unit 5, rather than constructing a 5 through counts of 1, when finding the total in a 3 by 5 array. Multiplicative Scheme Theory, as described by Steffe (1994), Hackenberg (2010), and others, provides a theoretical description of the development of multiplicative reasoning from children's initial actions of counting by 1s to multiplication/division of numbers that does not rely on counts, but on more conceptual multiplicative models.

Prior to meaningful engagement in multiplication or division, children engage in arithmetic strategies that incorporate variations of counting by 1s (Park & Nunes, 2001; Steffe, 1994; Ulrich & Wilkins, 2017). Typically, this form of reasoning is described as pre-multiplicative (Steffe, 1994) or additive reasoning (Park & Nunes, 2001), since it does not incorporate skip-counting. A subset of actions grouped in the larger frame of additive reasoning includes counting by 1s to create the composite unit being iterated in a multiplication problem. For example, finding the total in a 3 by 5 array, a child may count the 1s in each 5-row but use the row structure to account for progress in their direct counting. I refer to this construction of one level of unit in activity as *emergent multiplicative reasoning* (EMR)¹. Specifically, students at EMR consider the second level of units in solving multiplication/division tasks, but do so through construction of one level of unit (i.e., the composite unit). Count-by-one approaches outside this description of EMR may best be described as additive reasoning (2018, Kosko, 2019).

When students coordinate two levels of units in activity, they operate at what Hackenberg (2010) describes as the *first multiplicative concept* (MC1). MC1 involves anticipating one level of units and operating on such units to construct a second level of units. A student operating at MC1 asked to find the total in a three by six array may anticipate the composite unit of 3 (anticipating the unit of 3 as a unit of three 1s) to skip-count 3, 6, 9, 12, 15, 18. The distinction between EMR and MC1 is similar to descriptions by other researchers who differentiate between arithmetic and multiplicative schemes (Davydov, 1991; Park & Nunes, 2001). Note that the description of skip-counting provided here does not include repeated addition as equivalent. Rather, skip-counting anticipates the next count in an iterated sequence (5, 10, 15, and so on) while repeated addition constructs the next count in an iterated sequence (5, $5 + 5 = 10$, $5 + 10 = 15$, and so on).

The *second multiplicative concept* (MC2) involves anticipating two levels of units to coordinate three levels of units in activity (Hackenberg, 2010; Tillema, 2013). Coordination of three levels of units essentially engages individuals in dealing with composites of composite units. For example, asked to find the total in a 3×16 array, a student at MC2 could use the anticipated (known) fact $3 \times 6 = 18$ and skip count by 3s from the sixth row (i.e., from 18) up to 48. Thus, a student at MC2 can anticipate units of units (knowing the 3×6 portion of the array is 18), allowing them to partition the 3×6 sub-array from the larger 3×16 array. The same student can then construct the 3×10 sub-array by iterating 3s in reference to both the 3×6 sub-array. This particular example also illustrates disembedding, an action that emerges in MC2 (Hackenberg & Tillema, 2009; Kosko, 2018). *Disembedding* involves "the identification of a subunit...from a larger unit without destroying the initial whole" (Kosko, 2018, p. 55), and use of this subunit in reference to the initial whole. Emerging as a main component of a scheme constructed in activity in MC2, disembedding is used in an anticipatory manner in the third multiplicative concept (Hackenberg, 2010). Students who are not capable of disembedding would "destroy" the whole by essentially having to reconstruct it in activity (i.e., construct 3×6 by skip-counting 6 three times).

A student at the *third multiplicative concept* (MC3) anticipates three levels of units (Hackenberg, 2010; Tillema, 2013). Such a student finding the total in a 3×16 array may disembed 3×6 and 3×10 and then add $18 + 30$ to find the solution. Such a student anticipates 3×16 as not only including three units of 16 (or sixteen units of 3), but as also including units of units of units. Thus, an individual at MC3 can anticipate 3×16 as being composed of other units of units including, $3 \times 6 + 3 \times 10$ (where 3×6 and 3×10 can be considered as units of units in themselves). Students at MC3 are able to consider multiplicative contexts more flexibly and efficiently than students at MC2 or MC1. These four key stages are summarized in Table 1.

3. Visual representations of multiplicative contexts

There are numerous ways of visually representing multiplication and division, along with variations for classifying such representations and models (Anghileri, 1989; Bolden et al., 2015; Davydov, 1991; Lay, 1963). For example, Finesilver (2017) refers to *unit containers* as "groups of two or more units enclosed by visible boundaries" (p. 219). Carpenter, Fennema, Franke, Levi, and Empson, (2015) refer to such representations as *grouping problems*, and Downton and Sullivan (2017) describes them as *equal groups* tasks. The present article focuses on the term *representation* to refer to three primary visual representations of multiplication and division: set, length, and area representations. Although additional means of visually representing multiplication and division exist, these appear to be the most common in elementary school mathematics. Furthermore, distinctions between these three primary visual representations follow current norms in practitioner and research-oriented literature. For example, the second set representation and first area representation in Table 2 can similarly be solved by attending to the row and column structure of each. However, set model arrays are described in this paper as distinct from area models with all square units illustrated due to the distance between each illustrated unit. Similar distinctions were attended to in all three visual representations.

¹ Statistical findings by Author (2019, 2018) indicate that such schemes are part of the same overarching construct for multiplicative reasoning. Thus, while not descriptive of students who can engage in multiplication, students at EMR can demonstrate actions that lead to this development.

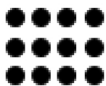
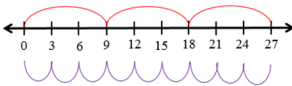
Table 1

Overview of key stages in multiplicative reasoning.

	Description	Example
Emergent Multiplicative Reasoning [EMR]	Constructs one level of units in activity.	Solves 3×6 by counting to 18 by 1s.
First Multiplicative Concept [MC1]	Anticipates one level of units to construct two levels of units in activity.	Solves 3×6 by counting to 18 by 3s.
Second Multiplicative Concept [MC2]	Anticipates two levels of units to construct three levels of units in activity.	Solves 3×16 by counting from 18 to 48 by ten 3s.
Third Multiplicative Concept [MC3]	Anticipates three levels of units to construct multiple levels of units.	Solves 3×16 by adding $3 \times 6 + 3 \times 10$.

Table 2

Example set, length, and area representations.

Set Representations	Length Representations	Area Representations
 	 	 

3.1. Set representations

Set representations convey discrete units that allow for interpreting these units as organized into sets (or groups). For example, the first and second representations in [Table 2](#) might be interpreted as three sets of four units, 12 individual units, or both, depending on the individual. Although the multiplicative structure may appear obvious to an adult, it is not a structure necessarily inferred by children ([Anghileri, 1995](#); [Barmby et al., 2009](#); [Barmby, Bolden, Raine, & Thompson, 2011](#); [Bolden et al., 2015](#); [Dabic & Milinkovic, 2015](#); [Davydov, 1991](#)). Children's strategies can vary dramatically on the same visual representation. The most well documented strategy across groups-of and array models is children's counting by 1s ([Anghileri, 1989](#); [Clark & Kamii, 1996](#); [Downton & Sullivan, 2017](#)). In particular, Barmby and Singhs have conducted several studies of students' use of set representations and have repeatedly documented the use of counting by 1s ([Barmby et al., 2009, 2011](#); [Bolden et al., 2015](#)). One such study incorporated use of eye tracking software and tasked fifth graders with matching the representation (groups-of, array, or number line model) with one of two symbolic expressions. Data from the eye tracking software revealed that these students attended to the grouping (multiplicative) structure in 48 % of the groups-of tasks and 50 % of the array tasks. However, the prevalence of counting by 1s was observed in 33 % of groups-of and 30 % of arrays tasks. Confounding these findings is the observation that children were largely inconsistent in their strategy use across the tasks incorporated. No child relied solely on counting by 1s or attending to multiplicative structure.

Nearly every study identified that examined children's use of set representations observed children's use of counting by 1s strategies. However, additional strategies have been observed by various researchers. Both [Anghileri \(1995\)](#) and [Clark and Kamii \(1996\)](#) documented variations of emergent multiplicative strategies including an application of adding onto a given composite, and skip counting with miscues. These strategies have also been described by [Steffe \(1994\)](#) in articulating distinctions between what actions of children were or were not multiplicative.

[Barmby et al. \(2009\)](#) provide extensive detail with nine different strategies for calculating the total in an array. [Barmby et al.'s \(2009\)](#) array based tasks embedded 5×5 arrays within whichever larger array was visually represented to children. Thus, a child viewing a 14×7 array would see two complete 5×5 arrays, a 4×5 array, a 4×2 array, and two 5×2 arrays, given the computer application used by the researchers. [Barmby et al. \(2009\)](#) described children's strategies, in order of sophistication, as including: counting by 1s; to counting by groups of 2s, 3s, or 4s; counting by 5s; noting the 25s (5×5 arrays) and then adding onto by 1s; noting the 25s and then adding onto by skip counting; noting the 25s and adding onto by skip counting with 5s; rearranging any arrays to complete 5×5 arrays and then adding the remaining units; and completing more complex calculations symbolically.

3.2. Area representations

Area representations convey multiplicative relationships in two dimensions. Area representations are similar to array models of set representations in that an individual must consider the row and column dimensions to infer a multiplicative relationship. However, area representations can appear similar to length representations when conveying continuous rather than discrete information. The degree to which an area representation is successful at conveying either continuous information or the length by width relationship interacts with how such representations are conveyed. The first area example in [Table 2](#) illustrates each square unit, while the second area example conveys only the designated length and width. When students are provided with individual square units, researchers have observed a tendency to rely on counting by 1s instead of attending to the row or column structure in the representation ([Battista, Clements, Arnoff, Battista, & Van Auken Borrow, 1998](#); [Ben-Haim, Lappan, & Houang, 1985](#); [Finesilver, 2017](#); [Huang & Witz, 2013](#);

Huang, 2014; Izsák, 2005; Outhred & Mitchelmore, 2000). Similar to research examining array models, various other strategies have been observed by researchers.

Battista et al. (1998) examined elementary students' construction of area and observed several distinct levels related to multiplicative reasoning. Initially, students did not make "use of a row or column of squares as a composite unit" (p. 508), but counted individual tiled squares by 1s. Next, students begin to use rows or columns as composite units, "but this composite is not used to cover the whole rectangle" (p. 510). Rather, some rows (or columns) are directly counted and then directly inferred to estimate the total area. In the next level, students consider the rectangle as being completely covered by row or column units, but may miscount the area when attempting skip counting. At the final observed level, students use and count the row/column composite units appropriately.

The trajectory observed by Battista et al. (1998) has been confirmed by other researchers investigating area. Huang and Witz (2013) found that many of the fourth grade students in their sample "did not explain the structure of rectangular arrays on the basis of coordinating two dimensions of units" (p. 15), but instead considered area representations as if they were set models, and used repeated addition to explain their calculations. Huang (2014) later confirmed these findings, noting that "some children are less likely to spontaneously connect multiplicative calculations with meaning, even if they had memorized multiplication facts" (p. 460). Similarly, Izsák (2005) observed that some students learning to represent multiplication with area models on dot paper tended to count the dots rather than the intervals between the dots to construct their representation. Izsák (2005) noted such an approach does not attend to rows or columns as composites, as students tend to focus on perimeter rather than area.

3.3. Length representations

Length representations convey quantities in relation to one another. For instance, the first length representation example in Table 2 includes a length of 3 juxtaposed with an unknown length. In this case, it is implied there are seven lengths of 3 that are the same length as the unknown. Number lines, such as the one illustrated in Table 2, are another model of length representations. The "jumps" illustrated in Table 2 convey nine 3s below the number line, or three 9s above the number line. In both examples, the representation conveys relationships between at least two distances, or lengths. In some instantiations, such juxtaposition is embedded within the same length. Whereas set representations include discrete counts, length representations typically, but not necessarily, convey continuous information.

Variation in strategies similar to those observed across studies of set and area representations have been observed in studies involving length representations. Barrett and Clements' (2003) examination of how two fourth grade children developed a concept of length measurement provides one means of describing the variety of strategies, and their multiplicative nature. Specifically, Barrett and Clements (2003) observed that, over time, the students engaged in "an iterative building up of longer segments as collections of smaller, unitary segments" (p. 512). Finding any length requires developing a relationship between the number sequence and a line segment. This relationship initially involves coordination of counts of 1, but eventually transitions into consideration of other units (smaller or larger). This process interrelates with the development of multiplicative relationships. Although students may develop a rich conception of length as observed by Barrett and Clements (2003), students without such a developed conception interact with length representations much differently. Bolden et al.'s (2015) use of eye tracking software revealed that fifth grade students who did not attend to multiplicative structure when using number lines tended to focus on the number considered to be the product. Diezmann and Lowrie (2006) distinguish such students by use of a number line through counting versus through measurement. Specifically, Diezmann and Lowrie (2006) found that some students counted the markings on a number line as discrete elements rather than considering the relationship between certain lengths. Kosko and Singh (2018a) observed that children will create their own discrete units in length representations that do not already include partitions (or marked intervals). Students demonstrating EMR may count these units as units of 1, while students at higher levels of multiplicative reasoning will treat such units as composites (Kosko & Singh, 2018a). Rather, the coordination between a number sequence and a line segment is less evident in students demonstrating EMR and more evident in students with higher levels of multiplicative reasoning.

Working with length representations, Kosko (2019, 2018) suggested a particular hierarchy of sophistication for unit coordination that is similar to the strategies Barmby et al. (2009) observed when examining arrays. Kosko (2019, 2018) observed that students operating at EMR were more likely to solve tasks in which a length unit of 1 was provided than if the length unit was a composite. Analysis of these students' written work confirmed a tendency to rely on counting by 1s when provided tasks aligned with higher multiplicative reasoning levels (Kosko & Singh, 2018a). Similar to EMR, students operating at MC1 have been observed to use less sophisticated strategies when solving length representation tasks designated at MC2 or MC3 (Kosko & Singh, 2018a; Kosko, 2019). Ulrich and Wilkins (2017) observed a similar trend when studying sixth and seventh grade students' written work on bar models of length representations. On some tasks, Ulrich and Wilkins (2017) partitioned a larger comparison length juxtaposed to a smaller length. Each partition on the larger length was identical in length (and juxtaposed to emphasize this relationship) to the smaller length. Students who were identified as not anticipating a composite unit (i.e., relying on counts of 1s) were unable to solve such tasks.

3.4. Overview of multiplicative representations

The literature reviewed in the prior sections for set, length, and area representations suggests that for each representation, there are particular ways students may interpret them multiplicatively, and particular ways they may engage with such representations in ways that are not multiplicative. There are aspects of each representation that allow for potential uses that are not multiplicative. For

Table 3

Visual comparison of three different hierarchies of strategies and schemes aligned with levels of multiplicative reasoning.

Set	Area	Length
EMR - Barmby et al., 2009 - Counting in 1s.	Battista et al., 1998 - Complete lack of Row- or Column-Structuring. - Partial Row- or Column-Structuring.	Hackenberg, 2010 as described by Kosko, 2018 Given a unit of 1, find a composite unit length.
MC1 - Counting in groups of 2s, 3s, 4s, etc. - Counting in 5s.	- Structuring as a Set of Row- or Column-Composites. - Visual Row- or Column-Iteration.*	- Given a composite unit length, find a unit of 1. - Given a smaller composite unit, find a larger composite unit.
MC2 Picking out the 25s and then counting in 1s.	- Visual Row- or Column-Iteration.^a - Row-by-Column Structuring.	- Given a larger composite unit, find a smaller composite unit. - Given a composite unit, disembed with implicit 1s to find another unit.
MC3 - Picking out the 25s and then counting by 5s and other variations. - Completing 25 arrays by rearranging other arrays. - Completing larger calculations.	n/a	Given a composite unit, disembed with implicit composite lengths to find another unit.

^a Aspects of this described scheme may appear in both MC1 and MC2.

example, various models of set, length, and area representations may be solved through some form of counting by 1s. Some researchers have created stages of students' strategies in operating on specific representations of multiplication or division (e.g., Barmby et al., 2009; Battista et al., 1998; Hackenberg, 2010; Kosko, 2018). Three such descriptions are summarized in Table 3, and aligned with the level of multiplicative reasoning that I hypothesize they exemplify.

Although the strategies and schemes summarized in Table 3 appear to align, there has been surprisingly little research to study such alignment (see Ulrich & Wilkins, 2017 for an example exception). Many comparisons of students' strategies and demonstrated reasoning when solving multiplicative tasks across representations often does not discuss the potential multiplicative structure that is illustrated in such representations. For example, Anghileri's (1989) comparison of students' strategies when solving different tasks (including groups of, array, and number line models) did not attend to the similarities or differences in the visual structure of these representations. Within Bolden et al.'s (2015) eye tracking study, all groups of and array model tasks included discrete units with a value of 1. However, the number line tasks included intervals of composite unit lengths; not intervals of one unit.

The present study seeks to build on and extend findings of prior researchers by examining the role of the illustrated unit in different representations in multiplicative tasks. To do this, visual structures identified in prior studies with length representations (Kosko & Singh, 2018a; Kosko, 2018, 2019) were adapted to set and area representations. This was meant to provide a means for comparing across representation type, with purposeful attention to how the unit was interpreted by students' engagement with such tasks. Patterns in students' responses, analysis of their written strategies, and a subsample of interview data were used to examine how students' multiplicative reasoning interacted with various representations. This data was used to answer the following research questions:

- 1 How do students respond to different illustrations of unit across different representation types (set, length, area)?
- 2 Do differences in how unit is represented affect students' interaction with such tasks?

4. Methods

4.1. Sample and measures

Data include a convenience sample collected from 182 primary grades students in May 2017 enrolled in third (typical ages 8–9; $n = 82$) and fourth grade (typical ages 9–10; $n = 100$). Students were enrolled in two elementary schools within the same predominately White (92 %) suburban school district in a Midwestern U.S. state. Additionally, the school included a higher percentage of students qualifying for free/reduced lunch (32 %).

Participants were asked to show their written work for how they solved each problem, and all written work was digitally scanned for analysis. In addition to analysis of written work, a subsample of students participated in interviews ($n = 9$). Participants completed the Multiplicative Reasoning Assessment (MRA) prior to cognitive interviews using a subset of eight items from the second assessment (Set_03, Set_05, Set_06, Set_08, Area_02, Area_03, Area_04, Area_06). Participants were selected to represent a range of scores representing EMR, MC1, MC2, and MC3.

4.2. Measures

Participants completed a variation of the MRA, which has a validity argument presented across multiple studies (Kosko & Singh, 2018a; Kosko, 2018, 2019). Whereas prior versions of the MRA provided validity evidence for length models of multiplication, this

Table 4

Item design blueprint for length items.

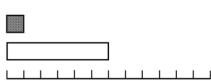
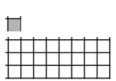
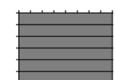
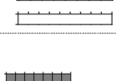
Level	Scheme	Actions Characterizing Scheme	Item Parameters
EMR	1	Iterate 1s to find a collection of 1s.	Given a unit of 1, determine the value of a composite unit.
MC1	2	Given a composite unit, determine the number of 1s within the composite (such as through partitioning).	Given a composite unit, find the value of a unit of 1.
	3	Given a composite unit, iterate the unit to count to a larger number.	Given a smaller composite unit, find a larger composite unit (multiple).
MC2	4	Given a composite unit, partition it into smaller composite units.	Given a larger composite unit, find a smaller composite unit (factor).
	5	Disembed a composite unit using collections of 1s.	Given a composite unit, disembed with implicit 1s to find another unit.
MC3	6	Disembed a composite unit using composites.	Given a composite unit, disembed with implicit composite lengths to find another unit.

study includes 40 items which include length, set, and area models (19 length, 10 set, 11 area). Thus, the item design blueprint used for length models was also used for set and area models (see Table 4). Multiplicative Scheme Theory suggests that students at each multiplicative concept may use different schemes that correspond with different levels of unit coordination. For the MRA, six schemes were used in the item design blueprint, based on descriptions by Hackenberg (2010); Steffe (1994), and others (Norton et al., 2015; Tillema, 2013). There are other specific schemes identified in the literature that correspond with the multiplicative concepts (e.g., Simon, Placa, & Avitzur, 2016; Tzur et al., 2011). However, these six schemes were selected as focus schemes that, as described by Hackenberg (2010) and Norton et al. (2015), provide a useful means of distinguishing between the multiplicative concepts.

Table 5 provides an illustrative comparison of example items across the three models incorporated in the current study. For example, Scheme 1 items label a given unit as having a value of 1 and prompt the participant for a composite whole number. Scheme 6 items provide a representation with an assigned composite whole number value, and a juxtaposed representation that, when solved correctly, yields a different composite whole number. The most efficient strategy hypothesized for such items involves partitioning the given composite into composite subunits, and using these new subunits to determine the value of the unknown. As with Schemes 1 and 6, all other items correspond with the item design blueprint specified in Table 4, and the overview of the multiplicative

Table 5

Example items designed to assess each scheme in the item design blueprint.

	Length	Area	Set
Scheme 1	 = 1 = _____	 = 1 = _____	 = 1 = _____
Scheme 2	 = 6 = _____	n/a	 = 6 = _____
Scheme 3	 = 5 = _____	 = 4 = _____	 = 6 = _____
Scheme 4	 = 8 = _____	 = 72 = _____	 = 72 = _____
Scheme 5	 = 2 = _____	 = 21 = _____	 = 5 = _____
Scheme 6	 = 24 = _____	 = 16 = _____	 = 180 = _____

concepts illustrated in Table 1. Important to consider in examining Table 5 and all items within the present study is that they are considered as a lower bound indicator for a scheme. Thus, participants answering a Scheme 2 item correctly are more likely to be at this level of reasoning or above, but someone who has not constructed this scheme yet may, however unlikely, be able to answer it correctly (Kosko & Singh, 2018a). These probabilities are taken into consideration with the Rasch modeling approach used in this paper.

A key point worth noting for the items included in this study is that not all possible variations of every representation type could be pragmatically included. Some items included the same or similar operands, such as the length and set examples shown in Table 5 for Scheme 2. This allowed for more direct comparison of the effect specific models may have. Other items included similar expected actions but with different magnitude of numbers. The length and area items for Scheme 4 are useful examples of this type of comparison.

4.3. Analysis

This study focuses on examining how students respond to different representations for multiplicative contexts (set, area, length), as well as differences in how unit is conveyed in these representations. A Rasch modeling approach was used to examine these differences. Rasch modeling is fundamentally concerned with understanding the differences between a person's ability, an item's difficulty, and the relationship between the two (Bond & Fox, 2015). In Rasch, a person's ability, designated by the theta statistic (θ), is defined as their probabilistic likelihood of responding in a certain way for items with a particular logit value. Likewise, item difficulty, designated by the logit-based statistic delta (δ), is defined by the likelihood individuals with different theta statistics will respond correctly to that item. This relationship is described mathematically in Eq. (1), where the probability (P) of person n responding correctly to item i is defined as the difference between person n 's theta score and item i 's delta statistic.² Following Eq. (1), a person with a theta score of 1.00 has a 50 % chance of responding correctly to an item with a delta statistic of 1.00 (Bond & Fox, 2015; Wilson, 2005). Participants' theta scores in this study is interpreted as a measure of their multiplicative reasoning ± 0.33 logits, based on prior statistical analysis of the MRA and students observed actions within and outside of the assessment (Kosko, 2018, 2019).

$$P_{ni}(x_{ni} = 1/\theta_n, \delta_i) = \frac{e^{(\theta_n - \delta_i)}}{1 + e^{(\theta_n - \delta_i)}} \quad (1)$$

Estimation of theta and delta statistics effectively transform ordinal data (i.e., raw responses of correct/incorrect) into continuous data on a logit-based scale (Bond & Fox, 2015). This conversion allows for measuring the distance between persons and/or items, with the confidence interval in the precision of the measure decreasing with larger sample sizes (Linacre, 1994). To help illustrate the measurable differences between items' delta and persons' theta scores, Rasch scholars have utilized the Wright Map (Wilson, 2005). A Wright Map provides a graphical representation that juxtaposes item difficulties with participants' ability measures.

The present study uses the delta statistics of items to compare items incorporating different representations via a modified Wright Map. To do this, participants' responses were modeled using the Rasch approach. Items were then plotted on a Wright Map to allow for a visual comparison of delta statistics across the items. Children's theta statistics represent an indicator of their multiplicative reasoning (ability), while items' delta statistics are indicative of the level of multiplicative reasoning targeted by each item (difficulty).

In addition to examining these statistics and their visual position on the Wright Map, supplemental data from one-on-one interviews and students' written work are incorporated to assist in interpreting the delta statistics. Results are separated into four sections corresponding with each level of multiplicative reasoning (see Table 1).

5. Results

5.1. Overview of Rasch model results

A Rasch model of the MRA was constructed using Winsteps 3.92.1. Item reliability (.98) and person reliability (.87) were observed. Following Bond and Fox's (2015) recommendations, item reliability above .90 suggests the MRA distinguishes between easier and more difficult items, and person reliability above .80 suggests it distinguishes between individuals with higher or lower scores. All but one item was found to have sufficient mean square infit statistics. Area_08 (MNSQ = 1.38) had mean square statistics outside of the accepted range of 0.75 – 1.33 (Bond & Fox, 2015). However, additional analysis recommended by Linacre (2010) suggest the item be retained. To confirm unidimensionality, a principal components analysis (PCA) of the standardized response residuals was conducted (Bond & Fox, 2015; Wright & Stone, 2004). The Rasch PCA indicated the assessment explained 50.5 % of the variance, with the first contrast reporting an eigenvalue of 2.90. This suggests the potential of a second construct. Examination of the disattenuated Pearson correlation, 0.47, and item loadings to the hypothetical factor supports maintaining one construct for the items in the assessment (Table 6).

Item delta statistics indicate a wide range of difficulty from -6.14 to 4.79 ($M = .38$, $SD = 2.37$). Given the sample size, these estimates are accurate with ± 0.50 logits (Linacre, 1994). An initial Wright Map of all items is illustrated in Fig. 1. Notably most

² Since theta and delta statistics are logit-based scores, the equation converts the difference in logits to a probability using e .

Table 6
Descriptive statistics for the Rasch model.

Item	Point Biserial	Delta	SE	Infit		Outfit	
				M.S.	<i>t</i>	M.S.	<i>t</i>
Set_01	0.41	-3.93	0.29	0.82	-0.9	1.70	1.2
Set_02	0.41	-2.44	0.21	1.09	0.9	1.89	2.3
Set_03	0.52	-1.04	0.18	1.05	0.7	0.95	-0.2
Set_04	0.64	-0.41	0.18	0.92	-0.9	0.84	-0.8
Set_05	0.53	-3.54	0.26	0.87	-0.7	1.08	0.3
Set_06	0.44	-0.06	0.19	1.17	1.7	1.11	0.6
Set_07	0.49	-1.65	0.19	1.13	1.6	1.28	1.1
Set_08	0.66	0.70	0.20	0.83	-1.6	0.59	-1.6
Set_09	0.45	-1.04	0.18	1.13	1.7	1.32	1.5
Set_10	0.69	1.34	0.23	0.74	-2.2	0.43	-2.0
Area_01	0.56	-2.66	0.21	0.87	-1.1	0.97	0.1
Area_02	0.47	-1.97	0.19	0.94	-0.6	0.78	-0.8
Area_03	0.48	-1.04	0.18	1.16	2.0	1.33	1.5
Area_04	0.67	1.39	0.23	0.78	-1.8	0.46	-1.8
Area_05	0.54	-1.31	0.18	0.97	-0.3	0.91	-0.3
Area_06	0.41	0.23	0.19	1.30	2.8	2.23	4.1
Area_07	0.69	0.27	0.19	0.74	-2.8	0.54	-2.3
Area_08	0.46	0.87	0.21	1.38	3.0	1.30	1.0
Area_09	0.41	4.04	0.46	1.26	0.8	4.75	2.8
Area_10	0.45	0.66	0.20	1.20	1.7	1.02	0.2
Area_11	0.61	2.54	0.29	.95	-0.2	0.46	-1.1
Length_01	0.55	-2.02	0.17	1.04	0.5	2.79	3.5
Length_02	0.63	-1.59	0.17	1.05	0.6	0.84	-0.6
Length_03	0.64	-1.82	0.17	0.91	-1.1	0.96	0.0
Length_04	0.72	-1.26	0.16	0.80	-2.7	0.72	-1.2
Length_05	0.67	-1.00	0.16	0.91	-1.1	0.75	-1.2
Length_06	0.72	-1.05	0.16	0.91	-1.1	0.84	-0.7
Length_07	0.66	-0.59	0.16	1.05	0.6	1.18	1.1
Length_08	0.77	0.31	0.16	0.78	-2.7	0.64	-2.4
Length_09	0.76	0.31	0.16	0.80	-2.4	0.67	-2.2
Length_10	0.73	-0.07	0.16	0.84	-1.9	1.12	0.8
Length_11	0.64	-0.02	0.16	1.16	1.8	1.50	2.7
Length_12	0.69	0.49	0.16	1.13	1.5	1.58	2.9
Length_13	0.68	0.21	0.16	1.06	0.7	1.22	1.3
Length_14	0.69	0.64	0.16	0.98	-0.2	1.03	0.2
Length_15	0.72	1.15	0.16	0.84	-1.9	0.93	-0.2
Length_16	0.66	1.43	0.16	1.00	0.0	0.94	-0.1
Length_17	0.67	1.64	0.16	0.99	-0.1	0.94	-0.1
Length_18	0.55	2.23	0.17	1.13	1.5	3.05	3.8
Length_19	0.60	1.02	0.16	1.17	1.8	1.43	1.9

items fall approximately within the ± 2.5 logits range, with three items appearing as outliers (Set_01, Set_05, Area_09). For the remaining discussion of results, items were divided into four sections for pragmatic comparison, following recommendations by Kosko (2019). Kosko (2019) used cognitive interview data and student written work to estimate ranges representative of participants' multiplicative reasoning (EMR, $\theta < -1.50$; MC1, $-1.50 \leq \theta < -0.50$; MC2, $-0.50 \leq \theta < 1.00$; MC3, $1.00 \leq \theta$). Recalling that theta statistics in this paper are estimated to be precise within ± 0.50 logits, these ranges should be considered approximations. Items, and persons, with scores near each threshold were examined for placement. Item analysis in this regard is included within the following sections. For participants, response processes were examined using person fit statistics (an identical statistical process as done with items). All participants' response processes were found to fit the Rasch model.

5.2. Representations eliciting EMR

Fig. 2 illustrates the section of the scale from -2.70 to -1.50, representing items assessing students' use of EMR. Set_01 ($\delta = -3.93$) and Set_05 ($\delta = -3.54$) are EMR items but not illustrated due to their extremely low delta statistics. Common across all types of representation (set, area, length) considered eliciting EMR is that a unit of 1 is explicitly illustrated in each item in such a way as to allow for direct counts of 1s to find a solution. This is most apparent in items where the discrete unit of 1 is explicitly identified (Set_01, Set_05, Area_01, Length_01, Length_02, Length_03). When considering the other three EMR items, illustrated in Fig. 2 (Set_02, Set_07, Area_02), every discrete unit is explicitly illustrated, but not explicitly named (i.e., not identified as being 1). Thus, the visually discrete set of 1s allows for a discrete unit of 1 to be implicitly inferred.

Data from cognitive interviews support results from the Rasch model. For example, when responding to Area_02, many students were observed to touch or label each individual square unit in the unshaded length of four and count by 1s. This happened regardless

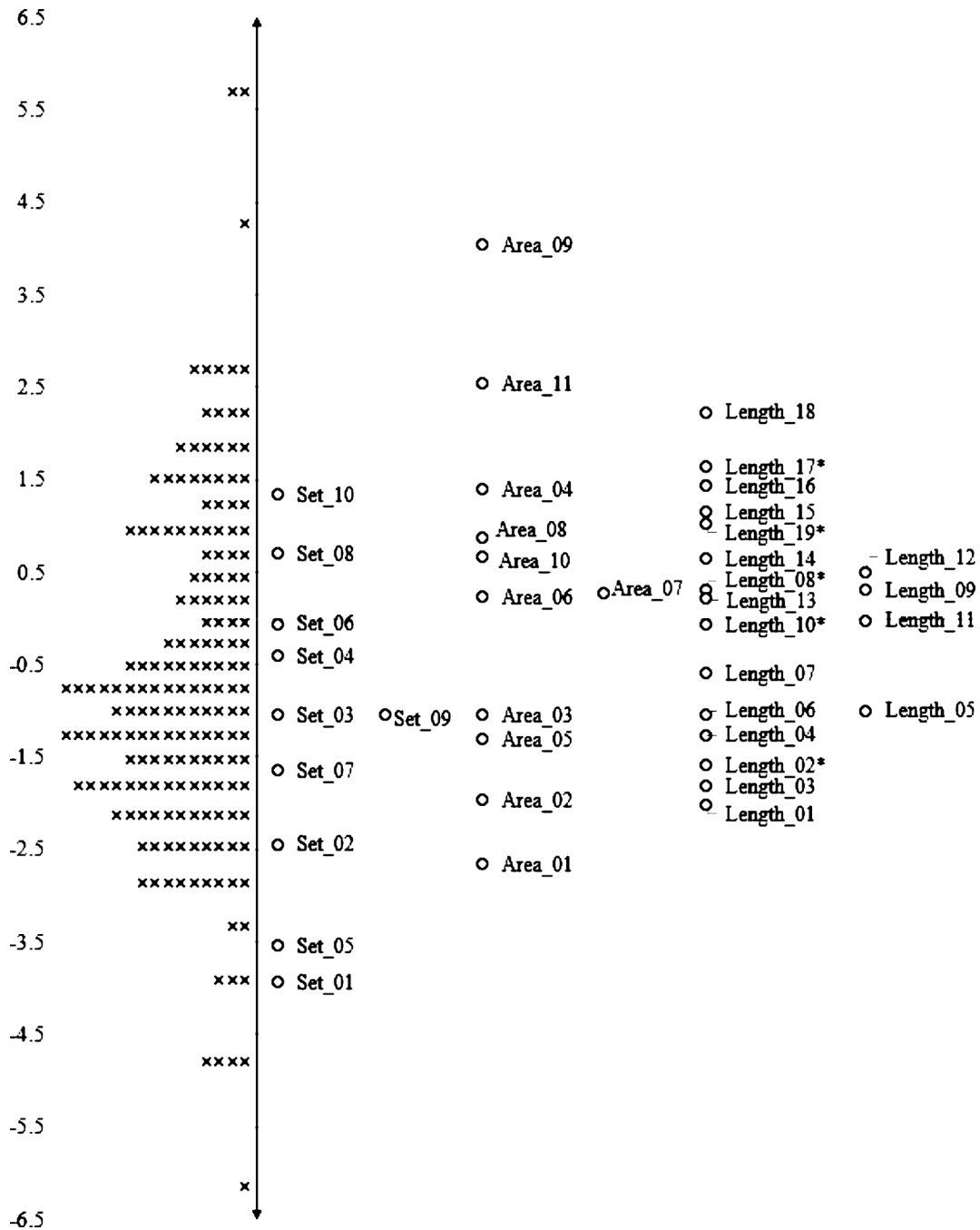


Fig. 1. Wright map comparing set, area, and length model items. Anchor items are indicated with an asterisk. Each x in the adjacent line plot represents one student of the 182 included in the current analysis.

of the child's observed multiplicative reasoning. For example, Edward ($\Theta = 1.33$) was observed to operate at MC3. When solving Set_05 (see Fig. 2), Edward stated that he could explain it as 4×3 , but that he solved it by counting each unit because it was simpler. Evidence of similar strategies was also observed in students' written work across EMR items, indicating that if discrete 1-units were explicitly provided, they were likely to be explicitly used as 1s.

5.3. Representations eliciting MC1

Fig. 3 illustrates the modified Wright Map between -1.50 and -0.50 logits. There are three distinct ways unit is conveyed across these items: composites composed of 1s (Length_04, Length_05, Length_06), representing discrete sets of 1s (Set_09, Area_03), and

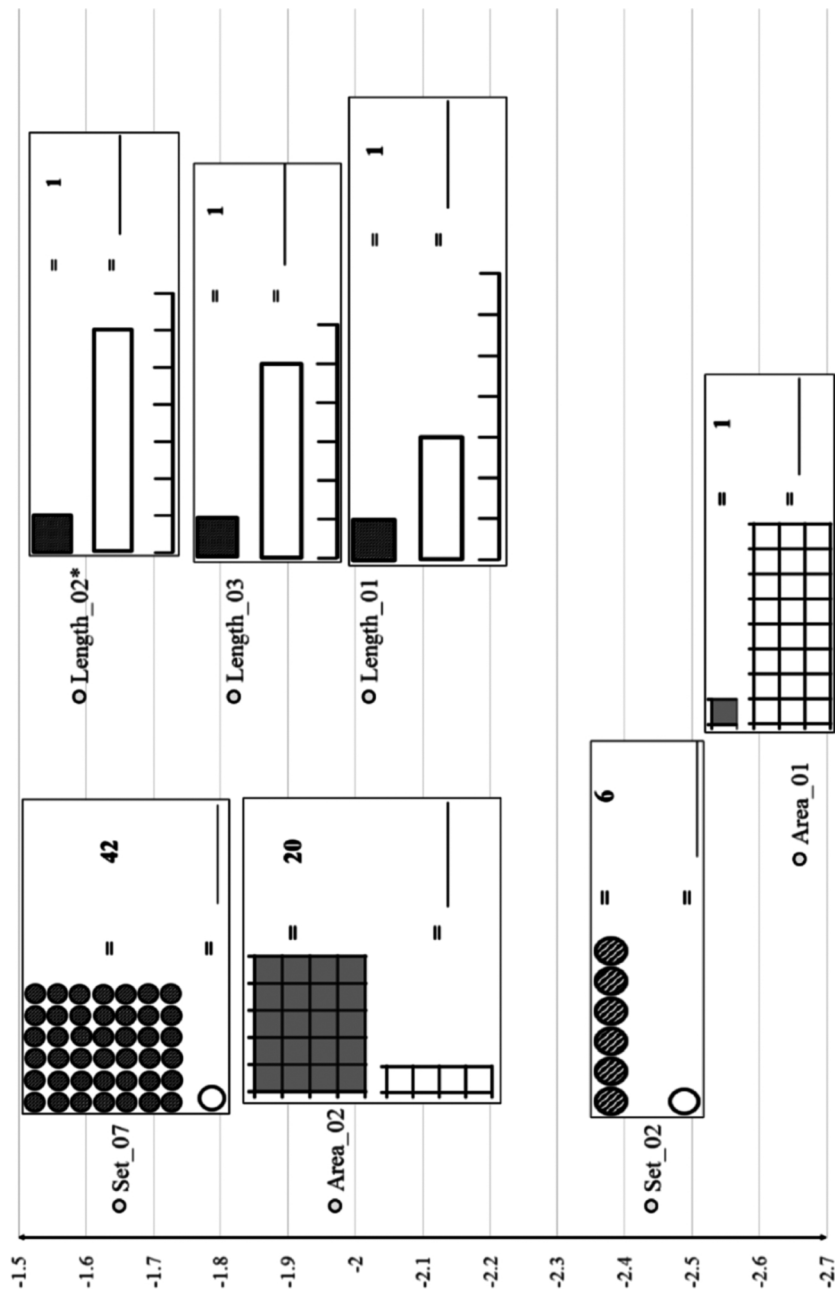


Fig. 2. Modified Wright Map for deltas between -2.7 and -1.5.

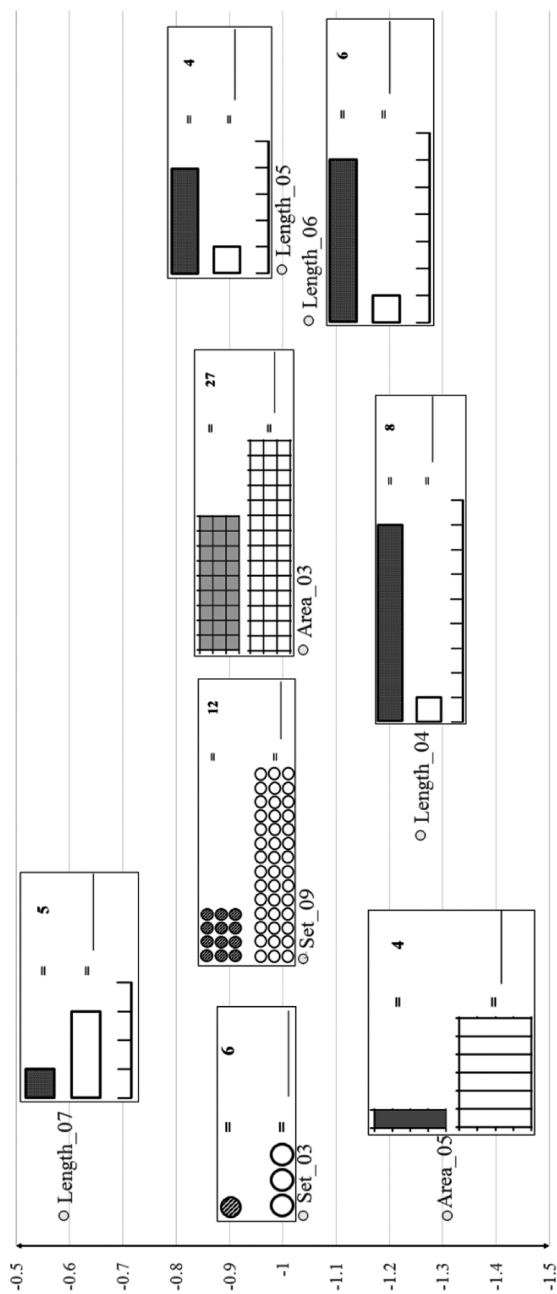


Fig. 3. Modified Wright Map for deltas between -1.5 and -0.5.

discrete composites used to compose other composites (Set_03, Area_05, Length_07). The first illustration type provides a given composite length, but the unknown length has an implicit, not given, value of 1. This distinguishes such items from counterparts at the EMR level where 1s explicitly illustrated and labeled. For example, consider the difference between EMR item Set_02 ($\delta = -2.44$) and MC1 item Length_06 ($\delta = -1.05$). Both items illustrate a given composite of 6 and solicit a value of 1 for the unknown, but participants' response data indicates Length_06 is significantly more difficult than Set_02. Thus, the distinction between a unit being explicitly illustrated versus needing to be inferred, or deduced, is quite significant.

Other items provide explicit, discrete 1s, but provide them in sets or groupings for both Set_09 and Area_03. In these items, finding a unit of 1 is neither the given nor unknown. For example, Set_09 illustrates an array of 12 discrete units and requests the value of an array with 42 discrete units. Examining students' written work and interviews, students responding correctly typically considered the larger (unknown) array as composed of three 12s and a 6. Important in understanding this written work is that although the three 12s were considered in relation to the given, the 6 was not. For students at MC1, the 6 was added onto the 36 created from three 12s. A similar approach was taken by Taylor, a participant at MC1 in the cognitive interviews, when solving Area_03. Taylor added onto 27 by counting the remaining 15 units by 1s. By contrast, Sherry, a student at MC2 solved Area_03 also by adding onto 27. However, Sherry determined the unknown partition by multiplying 3×5 and adding that to 27. This observational evidence, taken together with examination of written work and the location of these items on the Wright Map, indicate that MC1 students solving such items were able to accept and use the given composite, but had difficulty incorporating additional composite units that could be used to solve the tasks.

In the last type of MC1 item, the given unit is illustrated as a discrete composite unit, and the unknown is a multiple of this composite (see Set_03, Area_05, Length_07 in Fig. 3). As will be discussed in the section pertaining to MC2 items, there are items of this description that appear to elicit MC2 reasoning (see Length_08, Length_09 in Fig. 4). The key difference in the MC1 and MC2 variants of such representations is in the number of times a given composite need be iterated. MC2 variants have larger number of iterations and MC1 has fewer number of such iterations.

5.4. Representations eliciting MC2

Fig. 4 illustrates items from 0.50 to 1.00, and includes items considered to assess MC2 reasoning. This portion of the modified Wright Map includes the largest sample of items (14), with four observed types of illustrations. The first type was previously described in the prior MC1 description, and includes items (Length_08, Length_09) where the given is a visually discrete composite, and the unknown is a multiple of this composite. As previously described, the magnitude of iterations appears to differ from MC1 items of the same illustration type. To better understand this discrepancy in item location on the Wright Map, consider items Length_07 ($\delta = 0.59$) and Length_08 ($\delta = 0.31$). For Length_07, many participants who were able to correctly consider three units of 5 as 15 did not appear to consider six units of 3 as 18. Rather, many such participants identified the number of iterations (6). Further examination of item response statistics and students' written work suggests that other students made similar attempts but miscounted the number of iterations for Length_08 (responding 5, 7, or 8) or miscounted in their skip counts for Length_08. These strategies accounted for 73.8 % of incorrect responses (where a correct response was provided for Length_07 but not Length_08). This evidence suggests that multiplication with operands of certain magnitudes may require coordination of three levels of units. However, this finding needs further research to examine the nuances that may underlie the observed differences found here.

The second illustration type evident in MC2 items provides the given as a composite unit, and the unknown is composite unit that is a factor of the given (see Set_04, Area_06, Length_10, Length_11, Length_12 in Fig. 4). Area_06 ($\delta = 0.23$) and Length_12 ($\delta = 0.49$) have comparatively larger delta statistics than the other three items of this illustration type (Set_04 = -0.41 ; Length_10 = -0.07 ; Length_11 = -0.02). One potential explanation for this difference is the magnitude of the number of partitions between the higher and lower items of this type. Specifically, Area_06 and Length_12 require nine and six partitions, respectively, while the other three items require between three and four partitions.

The third illustration type observed involved MC2 items that illustrate the given as a composite that must be disembedded into units of 1 to find the unknown composite (Area_07, Area_08, Area_10, Length_13, Length_14). Recall that Kosko's (2018) differentiation between disembedding at MC2 and MC3 suggested that, at MC2, disembedding involves "a reliance on disembedding and/or iterating [perceived] units of 1" (p. 10). The five items identified in Fig. 4 include a given composite unit in which units of 1 can be derived to determine the unknown unit. For example, Area_10 does not explicitly illustrate rows or columns in either the shaded or unshaded figure. However, units of 1 can be deduced given the markings on the exterior edges of each rectangle, as is evident from examples of participants' written work (see Fig. 5). These markings can be used to derive the row and column structure in a particular way. Specifically, it allows for a illustrated square unit to be considered as having a value of 1. The length items (Length_13, Length_14) included implicitly identifiable units of 1 in a slightly different manner. Considering Length_14, the unshaded length is juxtaposed against a length of 10, and a reference line that visually marks the halfway point on the given 10. The marked halfway length of 10, in reference to the unshaded length, visually marks the portion of the unshaded length that is 5 and that which is not. This allows for units of 1 to be deduced given the reference line and shaded length that are juxtaposed (see Kosko, 2018 for a detailed analysis of these length items).

The fourth illustration type involves arrays in which a single unit within the array is a composite (see examples 1 and 2 in Fig. 5). Thus, a visually discrete composite is juxtaposed to a composite of composites. An inspection of students' written work on these items revealed two primary approaches for obtaining the correct solutions. Focusing on Set_08 ($\delta = 0.70$), students with theta scores over 1.00 typically included a numeric expression to explain their reasoning (i.e., $24 \times 3 = 72$, $72 \div 24 = 3$). However, students with theta scores ranging from approximately 0.40–1.00 (the range of the delta statistics for these items) typically used repeated addition

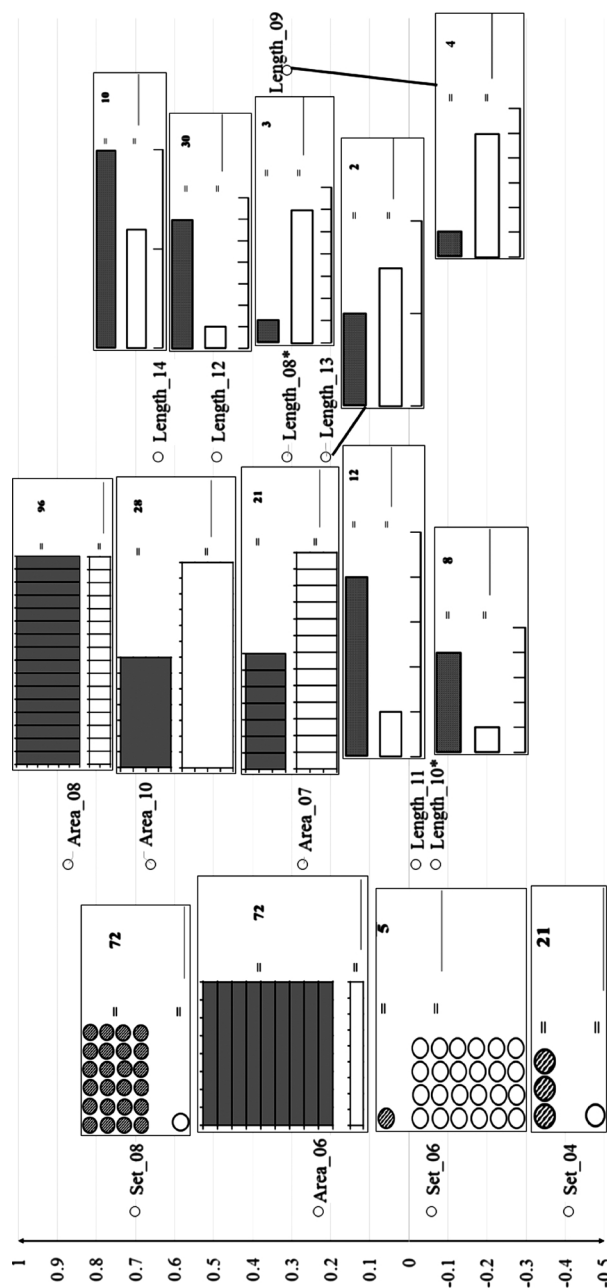


Fig. 4. Modified Wright Map for deltas between -0.5 and 1.0.

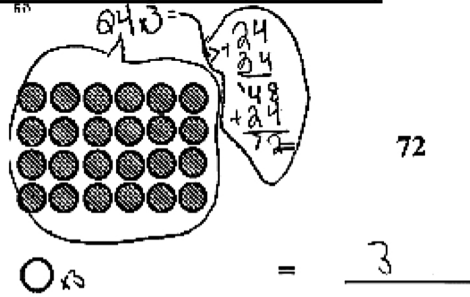
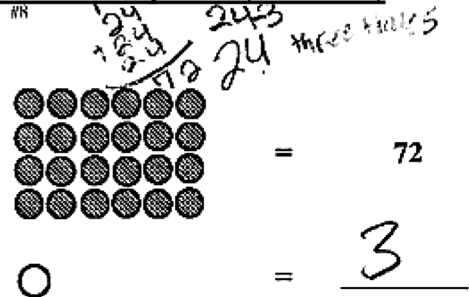
Student Example #1 ($\theta = 0.44$)**Student Example #2 ($\theta = 0.94$)**

Fig. 5. Example student work for Set_08.

to solve the item (see Fig. 5). The first student example in Fig. 5 includes the expression 24×3 and a written $\times 3$ beside the discrete count, suggesting the student recognized the need to multiply $4 \times 6 \times 3$ to solve the problem. However, to solve the problem, the child used repeated addition for $24 + 24 + 24$. This suggests a need to construct the third level of units in activity, which is characteristic of students at MC2.

5.5. Representations eliciting MC3

Fig. 6 illustrates the section of the scale from 1.00–2.60, and includes items considered to assess MC3 reasoning. This portion of the modified Wright Map includes two observed types of illustration. All MC3 items assessed disembedding in some manner, but with different visual elements. The first illustration type illustrated both the given and unknown as a composite of composites (Set_10, Area_04), where discrete units were illustrated. These items differed from MC2 items that also illustrated each unit (Set_06, Set_08) in that the MC2 items illustrated units with values of 1 and the MC3 items illustrated units that were composite whole numbers greater than 1. For example, Area_04 includes a 2 by 2 rectangle, identifying an area of 36, but the value of each discrete square unit must be deduced.

The second illustration type is similar to the first, with the key difference being whether the composite units illustrated the individual units (Area_09, Area_11, Length_15, Length_16, Length_17, Length_18, Length_19). For example, Area_11 includes a composite unit of 16, but does not explicitly identify the discrete square unit value of 4. This value must be deduced from the larger composite, and can be used to identify the value of the unknown.

Participant interviews confirmed that only students with theta scores above 1.00 logits were able to demonstrate anticipating three levels of units in these items. For example, one student with a theta of 4.29 solved Area_04 by recognizing the second figure was slightly twice the area of the first and doubled 36 to find 72 and then added 9. Another student at MC3, but with a lower theta statistics ($\theta = 1.30$) was able to solve the same task by dividing 36 by 4 to determine the value of the square unit, and then multiplied by 9 to find the solution. By comparison, a student at MC2 ($\theta = -0.32$) was unable to anticipate that each individual unit had a value greater than one, and disembedded 1s by adding $9-36$ to find the incorrect solution of 45 to Area_04. Thus, evidence from interview data suggests that students at different multiplicative levels interpreted the illustrated units differently.

6. Discussion

Prior research has investigated how students engage with set (Anghileri, 1989; Barmby et al., 2009; Clark & Kamii, 1996), area (Battista et al., 1998; Outhred & Mitchelmore, 2000), and length representations (Barrett & Clements, 2003; Kosko, 2018; Ulrich & Wilkins, 2017). However, relatively few studies have examined similarities and differences across these different types of representations. Those studies including such comparisons (e.g., Anghileri, 1989; Bolden et al., 2015) tend not to attend to how unit is visually conveyed in such representations, but rather focus on comparison of set versus length versus area, and so forth. By contrast, the current study found empirical evidence to suggest the manner a unit is visually conveyed may influence students' interactions with multiplication/division tasks more than whether a representation is a length model, set model, or area model. Although evidence for the results presented here is based on strong statistical evidence and supported by evidence from participant interviews and analysis of written work, this investigation limits itself to only those item formats examined. Therefore, analysis of other illustration types and variations in how those visual models are presented to students is warranted.

Each form of representation examined here (set, length, area) was found to elicit multiplicative reasoning at every level assessed (EMR, MC1, MC2, MC3). However, aspects of each kind of representation differed significantly from one level of reasoning to another. One significant finding from the present study is the visual nature of items at the EMR level. For example, Set_05, Set_07, Area_01, and Area_02 include array and area models that can be found in third grade textbooks covering multiplication and division (Kosko & Singh, 2018b). An extensive body of literature has indicated that many children engaging with similar representations use count by 1s strategies to solve such tasks, while other children may describe the same tasks multiplicatively (Anghileri, 1995; Barmby et al., 2009, 2011; Battista et al., 1998; Ben-Haim et al., 1985; Bolden et al., 2015; Clark & Kamii, 1996; Dabic & Milinkovic, 2015; Davydov, 1991; Finesilver, 2017; Huang, 2014; Huang & Witz, 2013; Izsák, 2005; Outhred & Mitchelmore, 2000). The results

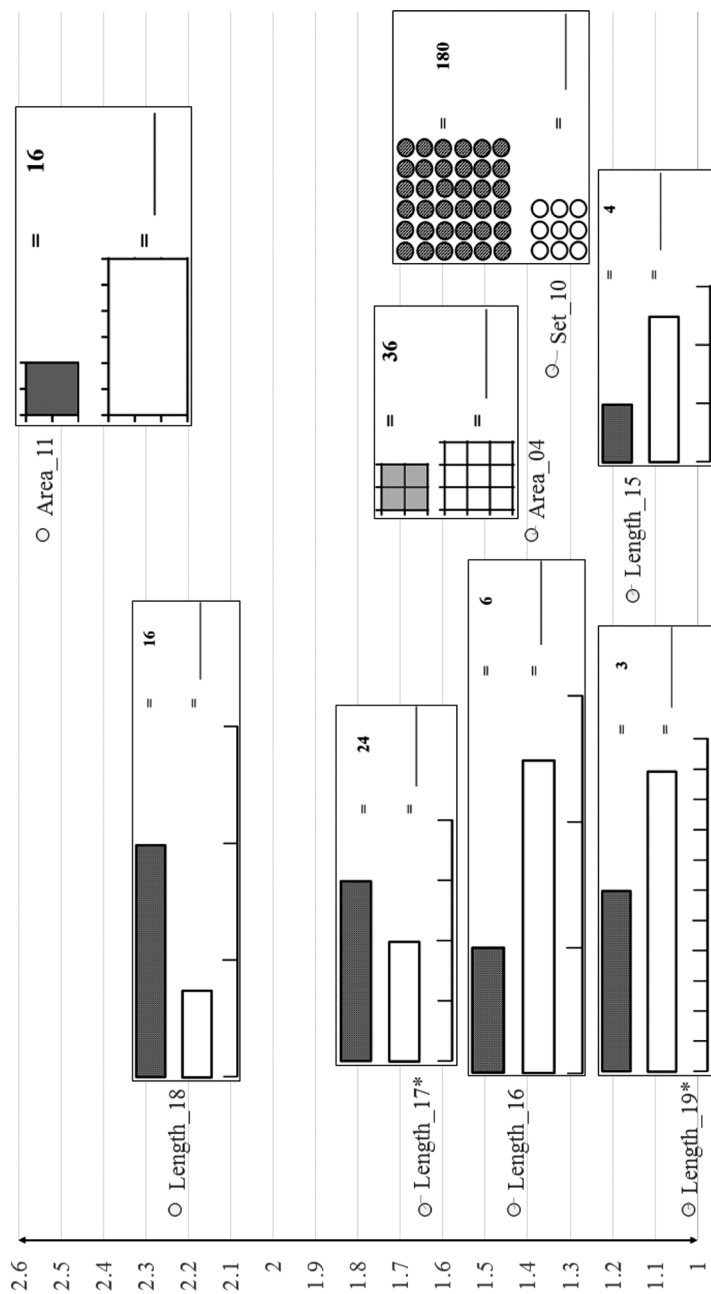


Fig. 6. Modified Wright Map for deltas between 1.0 and 2.6.

presented here support and extend that of prior research, but with a particular reframing of how students' interactions with such representations might best be considered.

Many of the representations that illustrate explicit 1s units, such as items found to elicit EMR in this paper, are commonly used in U.S. school textbooks and assessments (Kosko & Singh, 2018b). However, these representations may not *necessarily* engage our students in multiplicative reasoning. By *necessarily* engage, I suggest that such items could engage some students in multiplication or division, but such engagement likely stems more from the child's reasoning and less from the scaffolding provided from the representation itself. Thus, findings of the present study suggest that when representations allow for direct counting of explicit 1s, multiplicative reasoning at MC1 or higher is not required.

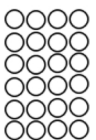
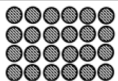
Prior research has found that the particular operands in multiplication tasks affects the strategies individuals use (Jost, Beinhoff, Henninghausen, & Rosler, 2004; Sherin & Fuson, 2005). When both single-digit operands are greater than 5, individuals will consider the tasks differently. Yet, larger factors do not necessarily increase difficulty (i.e., multiples of 10). The present study supports and extends such findings, with examples of how such multiplication and division may be visually conveyed. The effect of operand size appears to manifest at MC2 and MC3. A notable development for students who operate at MC2 is the ability to disembed (Hackenberg, 2010; Kosko, 2018). Recall that disembedding involves taking and using a subunit from an initial unit. For multiplication and division, this operation is often observable in students' formal and informal application of the associative and distributive properties. For example, students who solve 4×13 by adding 4×10 and 4×3 are disembedding. Likewise, those who solve 4×8 by doubling 4×4 are also disembedding, by recognizing 4×4 as two subunits within 4×8 . Children's informal use of such strategies has been well documented (Sherin & Fuson, 2005), but there is significant need for organizing the use of such strategies more cohesively into a learning progression that is useable by classroom teachers. Findings presented in this study suggest that Multiplicative Scheme Theory may serve as one means of organizing such strategies, but significant study is needed in this area.

Findings from the present study provide a statistically robust comparison of various illustration types and the degree to which such representations elicit different levels of multiplicative reasoning, with inferences based on responses from 182 third and fourth grade students. However, this study is not without its limitations. Rasch provides statistical estimations, but such estimates should not be considered absolute and are accurate within ± 0.25 logits in this study. Additionally, items were designed with specific criteria that allowed for comparisons across representation types, but not all variations in illustrations could pragmatically be included. Variations in the specific multiplication and division facts were also limited. The present study provides several useful findings regarding how different representations of multiplication and division may elicit different levels of reasoning, but there were limitations in the design that allowed for only certain forms and numbers of comparison. There are important facets of visual representation that need further study. Additionally, the present study did not examine some of the many contexts and framings for incorporating particular multiplicative representations. For example, the three by eight area model in Area_01 was found to elicit EMR in the present study, but certain pedagogical framings of such a three by eight rectangle may elicit higher levels of multiplicative reasoning (e.g., showing one row while covering the other two).

7. Conclusion

The manner unit is illustrated in representations of multiplication and division matters. When representations illustrate 1s units explicitly, they do not require reasoning beyond counting by 1s. Similarly, representations that do not illustrate any units or rows/columns may require higher levels of multiplicative reasoning for students to engage conceptually. Misuse of representations can significantly affect observations of researchers and teachers alike. Careful attention is needed in how unit is conveyed, but also the particular operands involved. Regarding both research and practice, further study is needed to better understand how unit and quantity interact with both the tasks students engage for multiplication and division, and students' learning of this important concept. The findings presented here provide initial guidelines that can be used to design and interpret visual representations for both researchers and practitioners.

Appendix A

Item	Delta	Image
Set_01	-3.93	 = 1  = _____
Set_02	-2.44	 = 6  = _____
Set_03	-1.04	 = 6  = _____
Set_04	-0.41	 = 21  = _____
Set_05	-3.54	 = 1  = _____
Set_06	-0.06	 = 5  = _____
Set_07	-1.65	 = 42  = _____
Set_08	0.70	 = 72  = _____
Set_09	-1.04	 = 12  = _____

Appendix B. Supplementary data

Supplementary material related to this article can be found, in the online version, at doi:<https://doi.org/10.1016/j.jmathb.2020.100800>.

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