Μάθημα 17ο

18/05/2015

Υπενθύμιση (Μάθημα 03)

Παραμετρημένη επιφάνεια

$\vec{r} : I \times J \longrightarrow \mathbb{R}^3$, $(u,v) \longrightarrow \vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$

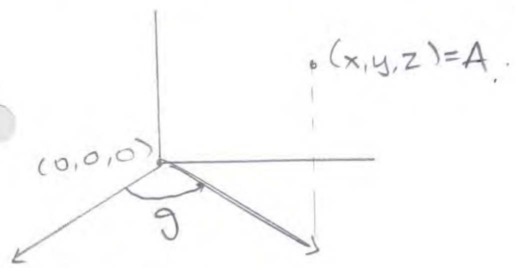
Συνήθως $I, J \subseteq \mathbb{R}$ διαστήματα

- $u = u_0$ / $\vec{r}(u_0, v)$ παραμετρημένη καμπύλη στον $\mathbb{R}^3$
- $v = v_0$ / $\vec{r}(u, v_0)$ " " " "

Κυλινδρικός Μετασχηματισμός

$\vec{T}(\theta, z) = (r\sigma\upsilon\nu\theta, r\eta\mu\theta, z)$ $(r, \theta, z) \in (0, +\infty) \times [0, 2\pi) \times \mathbb{R}$

1-1, επί του $\mathbb{R}^3 \setminus \{(0,0,z), z \in \mathbb{R}\}$



$r = \sqrt{x^2 + y^2}$

Παραδείγματα επιφανειών σε Κυλινδρικές, Καρτεσιανές εξισώσεις.

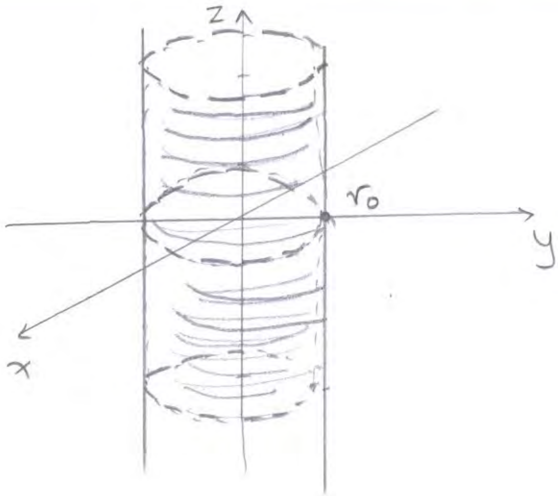
- Επιφάνεια του $\mathbb{R}^3$ $r = r_0$ $(r_0 = \sigma\tau\alpha\theta, r_0 > 0)$

$\vec{r}(\theta, z) = (r_0 \sigma\upsilon\nu\theta, r\eta\mu\theta, z)$, $\theta \in [0, 2\pi)$, $z \in \mathbb{R}$

$$\begin{cases} x = r_0 \sigma\upsilon\nu\theta \\ y = r_0 \eta\mu\theta \\ z = z \end{cases} \Longrightarrow \begin{cases} \sqrt{x^2+y^2} = r_0 \\ z \in \mathbb{R} \end{cases}$$

$\{(x,y,z) : \sqrt{x^2+y^2} = r_0, z \in \mathbb{R}\}$

$= \{(x,y,z) : -r_0 \le x \le r_0, -\sqrt{r_0^2 - x^2} \le y \le \sqrt{r_0^2 - x^2}, z \in \mathbb{R}\}$



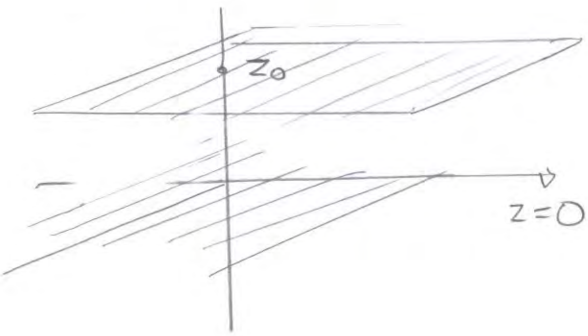
Εσωτερικό της επιφάνειας:

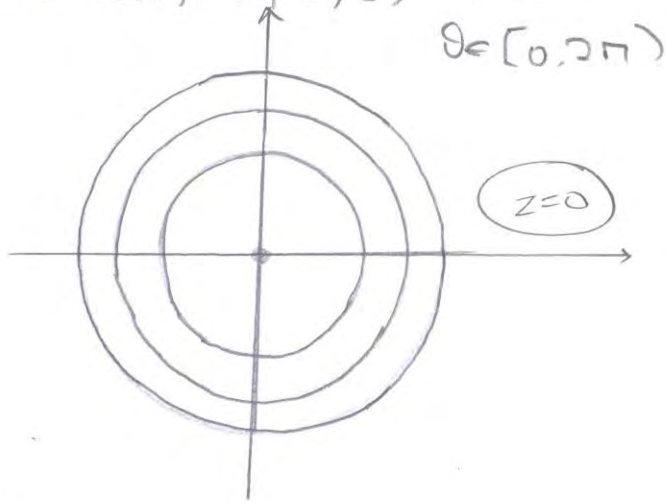
$\{(x,y,z): \sqrt{x^2+y^2} \leq r_0,\ z\in\mathbb{R}\}$

• <u>Επιφάνεια $z=0$ στον $\mathbb{R}^3$</u>

$\{(x,y,z)\in\mathbb{R}^3 : x,y\in\mathbb{R} : z=0\}$

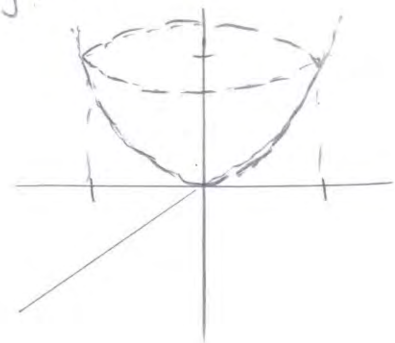
Διανυσματική εξίσωση $\vec{r}(r,\theta)=(r\,\text{συν}\theta,\ r\,\text{ημ}\theta,\ 0)$ $r>0$, $\theta\in[0,2\pi)$





• <u>Επιφάνεια $\vec{r}(r,\theta)=(r\,\text{συν}\theta,\ r\,\text{ημ}\theta,\ r^2)$, $r>0$, $\theta\in[0,2\pi)$</u>

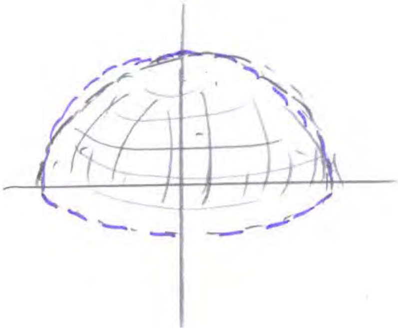
$$\left.\begin{cases} \left.\begin{aligned} x &= r\,\text{συν}\theta \\ y &= r\,\text{ημ}\theta \end{aligned}\right\} \Rightarrow x^2+y^2=r^2 \\ z = r^2 \end{cases}\right\} \Rightarrow x^2+y^2=z$$



$\{(x,y,z) : x\in\mathbb{R},\ y\in\mathbb{R},\ z=x^2+y^2\}$

• Επιφάνεια $\vec{r}(r,\vartheta)=(r\sigma\upsilon\nu\vartheta, r\eta\mu\vartheta, \sqrt{a^2-r^2})$, $r\in(0,a]$, $\vartheta\in[0,2\pi)$ του $\mathbb{R}^3$ $(a>0)$.

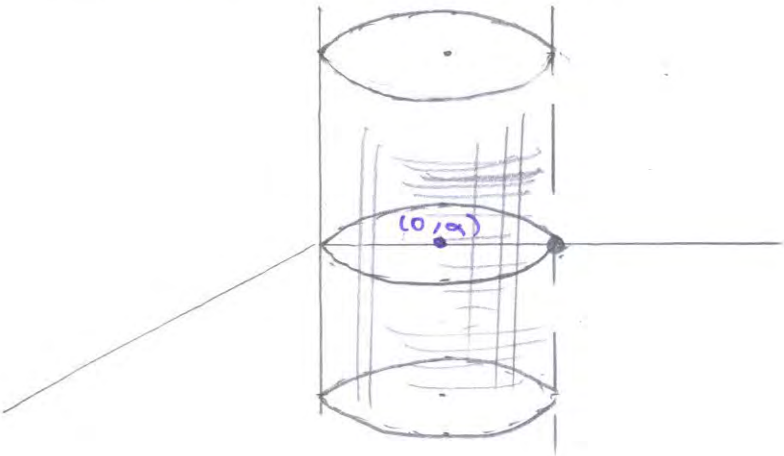
$$\left.\begin{array}{l}\left.\begin{array}{l}x=r\sigma\upsilon\nu\vartheta\\ y=r\eta\mu\vartheta\end{array}\right\}\Longrightarrow x^2+y^2=r^2\\ z=\sqrt{a^2-r^2}\end{array}\right\}\Longrightarrow \begin{array}{l}z=\sqrt{a^2-(x^2+y^2)}\\ x^2+y^2+z^2=a^2,\ z\geq 0\end{array}$$



• Επιφάνεια $\{(x,y,z):\ x^2+y^2=2ay,\ z\in\mathbb{R}\}$ $(a>0)$ $\quad x^2+y^2-2ay=0$, $\quad x^2+(y-a)^2=a^2$

$$\left\{\begin{array}{l}x=r\sigma\upsilon\nu\vartheta\\ y=r\eta\mu\vartheta\\ z=z\end{array}\right\}\ x^2+y^2=r^2=2a(r\eta\mu\vartheta),\quad \begin{array}{|l}r=2a\eta\mu\vartheta\\ \hline z=z\end{array}$$

$$\vec{r}(\vartheta,z)=\big((2a\eta\mu\vartheta)\sigma\upsilon\nu\vartheta,\ (2a\eta\mu\vartheta)\eta\mu\vartheta,\ z\big)$$



Ασκήσεις (Παλιά Θέματα)

1) $I = \iiint_B \sqrt{x^2+y^2}\, dx\,dy\,dz$, $V(B)$?

$$B = \{ (x,y,z) : -1 \le x \le 1,\ \underbrace{-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}}_{\hookrightarrow\ x^2+y^2 \le 1,\ r^2 \le 1,\ 0 < r \le 1},\ 0 \le z \le 1-(x^2+y^2) \}$$

Λύση. $\begin{cases} x = r\,\text{συν}\theta \\ y = r\,\text{ημ}\theta \\ z = z \end{cases}$

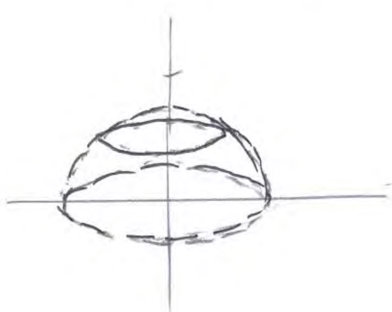
$0 \le z \le 1 - r^2$

$$I = \int_0^{2\pi}\left(\int_0^1\left(\int_0^{1-r^2} \sqrt{r^2}\cdot r\, dz\right) dr\right) d\theta = 2\pi \int_0^1 r^2(1-r^2)\,dr =$$
$$= 2\pi\left(\frac{1}{3}-\frac{1}{5}\right)$$

$$V(B) = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} 1\cdot r\, dz\,dr\,d\theta$$

Με σχήμα:

$z = 1-(x^2+y^2)$



$$\{ (r,\theta,z) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 1-r^2 \}$$

2) $V(B)$, B στην 1η στερεά γωνία $(x,y,z\geq 0)$ φράσσεται από τις επιφάνειες του $\mathbb{R}^3$, $x^2+y^2=1$, $x^2+y^2=4$, $z^2=x^2+y^2$, $z=0$

Λύση $\quad x=r\cos\theta,\ y=r\sin\theta,\ z=z$

Επιφάνειες

$x^2+y^2=1$ Κ.Σ $r=1$

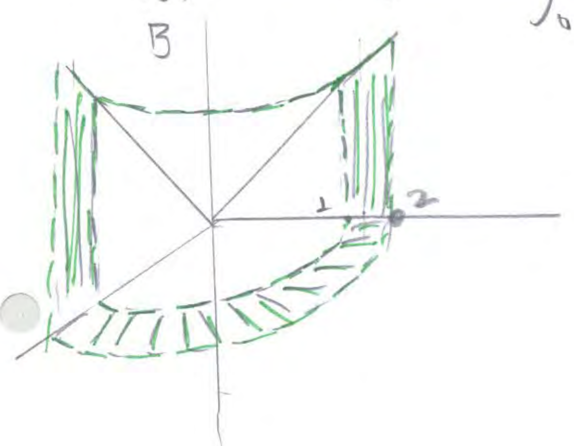
$x^2+y^2=4$ Κ.Σ $r=2$

$z^2=x^2+y^2$ Κ.Σ $z^2=r^2,\ z=r$

$z=0$

$$\left.\begin{cases} x\geq 0 & \cos\theta\geq 0 \\ y\geq 0 & \sin\theta\geq 0 \end{cases}\right\} \Longleftrightarrow \theta\in[0,\pi/2].$$

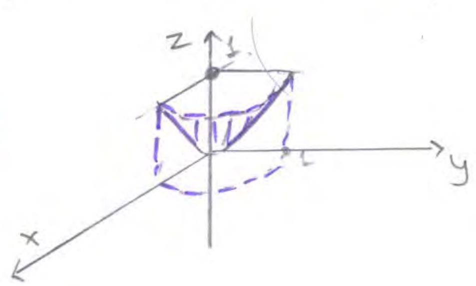
$$V(B)=\iiint_B 1\,dx\,dy\,dz=\int_0^{\frac{\pi}{2}}\int_1^2\left(\int_0^r r\,dz\right)dr\,d\theta=\frac{7\pi}{6}.$$



3) $$I=\int_0^1\int_0^{\sqrt{1-x^2}}\left(\int_{\sqrt{x^2+y^2}}^1 dz\right)dy\,dx$$

Λύση $\left\{(x,y,z):\ 0\leq x\leq 1,\ 0\leq y\leq\sqrt{1-x^2},\ \sqrt{x^2+y^2}\leq z\leq 1\right\}$

Με σχήμα



$$\begin{cases} 0\leq\theta\leq\frac{\pi}{2} \\ 0\leq r\leq 1 \\ r\leq z\leq 1. \end{cases}$$

$x = r\,\sigma\upsilon\nu\vartheta$

$y = r\,\eta\mu\vartheta$

$z = z$

$x, y \geq 0 \Longrightarrow \sigma\upsilon\nu\vartheta, \eta\mu\vartheta \geq 0 \Rightarrow \vartheta \in [0, \frac{\pi}{2}]$

$0 \leq y \leq \sqrt{1-x^2}$, $y^2 \leq 1-x^2$, $x^2+y^2 \leq 1$, $r^2 \leq 1$, $r \leq 1$

$\sqrt{r^2} \leq z \leq 1$, $r \leq z \leq 1$

$$I = \int_0^{\frac{\pi}{2}} \int_0^1 \left(\int_r^1 r\,dz \right) dr\,d\vartheta$$

4) i) Όγκος Σφαίρας $x^2+y^2+z^2 \leq a^2$ $(a > 0)$

ii) $V(B)$, B φράσσεται από τις επιφάνειες

$x^2+y^2+z^2 = 4$, $x^2+y^2 = 1$, $z = 0$, $x, y, z \geq 0$

Λύση

i) $x^2+y^2+z^2 = a^2$

$x = r \cdot \sigma\upsilon\nu\vartheta$

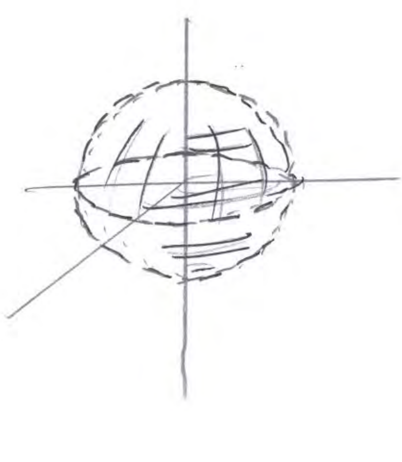
$y = r \cdot \eta\mu\vartheta$

$z = z$

$r^2 + z^2 = a^2$ / $-\sqrt{a^2+r^2} \leq z \leq \sqrt{a^2+r^2}$, $r \in (0, a]$

$$V(B) = \int_0^{2\pi} \left(\int_0^a \left(\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r \cdot dz \right) dr \right) d\vartheta =$$

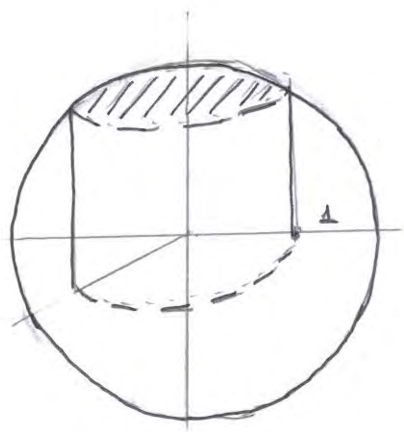
$$2\pi \left(\int_0^a 2r\sqrt{a^2-r^2} \right) dr = 2\pi \left[-\frac{2}{3} (a^2-r^2)^{3/2} \Big|_0^a \right] = \frac{4\pi}{3} a^3$$



ii)

$$\left.\begin{array}{lcl} x^2+y^2+z^2=4 & — & r^2+z^2=4 \\ x^2+y^2=1 & — & r^2=1\ ,\ r=1 \\ z=0 & & \\ x,y,z\geq 0, & & x,y\geq 0\ ,\ z\geq 0 \end{array}\right\} \Longrightarrow \begin{array}{l} 0\leq z\leq \sqrt{4-r^2} \\ 0\leq r\leq 1 \\ \vartheta\in[0,\frac{\pi}{2}] \end{array}$$

$$V(B)=\int_0^{\frac{\pi}{2}}\int_0^1\left(\int_0^{\sqrt{4-r^2}} r\,dz\right)dr\,d\vartheta=\frac{\pi}{2}$$



5) $V(B)$

B εντός της επιφάνειας $x^2+y^2=2x$

φράσσεται από $z=x^2+y^2$

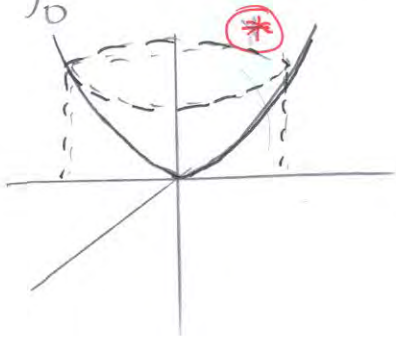
$z=0$

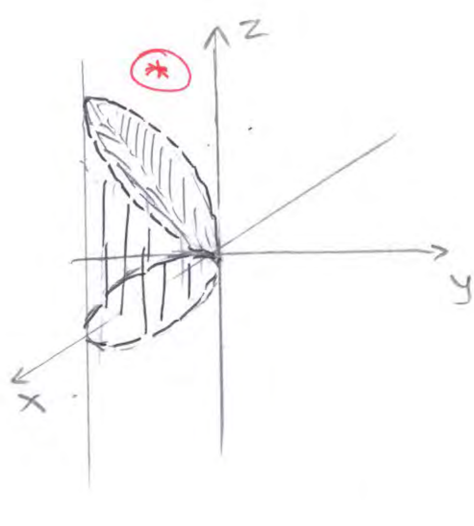
Λύση

$x^2+y^2=2x$, $r^2=2r\sigma\upsilon\nu\vartheta$, $r=2\sigma\upsilon\nu\vartheta$, $\sigma\upsilon\nu\vartheta\geq 0$.

$z=x^2+y^2$, $z=r^2$

$$V(B)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\sigma\upsilon\nu\vartheta}\int_0^{r^2} r\,dz\,dr\,d\vartheta=\cdots$$





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$$V(B)=2\int_0^{\pi/2}\int_0^{2\sigma\upsilon\nu\vartheta}\int_0^{r^2} r\,dz\,dr\,d\vartheta$$

6) $J=\iiint_{\Omega}\sqrt{x^2+y^2}\,dx\,dy\,dz$

Ω φράσσεται από $x^2+y^2=2y$, $z=x^2+y^2$, $z\geq 0$

Λύση

$x^2+y^2=2y$ $\quad r^2=2r\eta\mu\vartheta$, $r=2\eta\mu\vartheta$, $\eta\mu\vartheta\geq 0$, $\vartheta\in[0,\pi]$

$(x^2+(y-1)^2=1)$

$z=x^2+y^2$, $z=r^2$

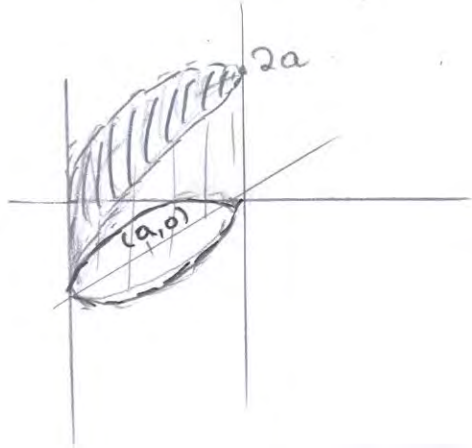
$$I=\int_0^{\pi}\int_0^{2\eta\mu\vartheta}\left(\int_0^{r^2} r\cdot r\,dz\right)dr\,d\vartheta$$

7) $V(B)$, $B=\{(x,y,z): x^2+y^2\leq 2ax,\ x^2+y^2+z^2\leq 4a^2,\ z\geq 0\}$ $(\alpha>0)$

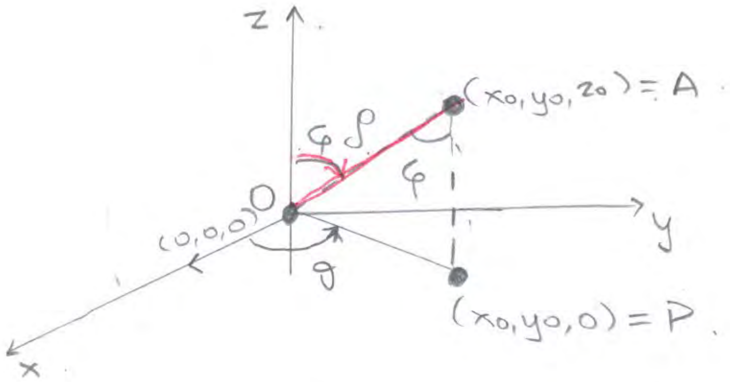
Λύση $\quad x^2+y^2=2ax$, $r^2=2ar\sigma\upsilon\nu\vartheta$, $r=2a\sigma\upsilon\nu\vartheta$, $\sigma\upsilon\nu\vartheta\geq 0$

$x^2+y^2+z^2=4a^2$, $r^2+z^2=4a^2$, $z=\sqrt{4a^2-r^2}$

$$V(B)=2\int_0^{\pi/2}\int_0^{2a\sigma\upsilon\nu\vartheta}\left(\int_0^{\sqrt{4a^2-r^2}}\cdot r\,dz\right)dr\,d\vartheta=\frac{8a^3}{3}\left(\pi-\frac{4}{3}\right)$$



Σφαιρικός Μετασχηματισμός



$\rho = \sqrt{x^2+y^2+z^2} > 0$

$P = (x_0, y_0, 0)$, $\theta = \sphericalangle(O\vec{x}, O\vec{P})$

$\varphi = \sphericalangle(O\vec{z}, O\vec{A})$

$ημφ = \frac{(OP)}{\rho}$

$x = (OP)\,συνθ = (\rho\,ημφ)\,συνθ$

$y = (OP)\,ημθ = (\rho\,ημφ)\,ημθ$

$z = \rho\,συνφ$, $\varphi \in [0, \pi]$

$\theta \in [0, 2\pi)$

$\rho > 0$

$$\vec{T}(\rho, \theta, \varphi) = (\rho\,ημφ\,συνθ, \rho\,ημφ\,ημθ, \rho\,συνφ)$$

1–1, επί του $\mathbb{R}^3 \setminus \{(0,0,Z)\}$

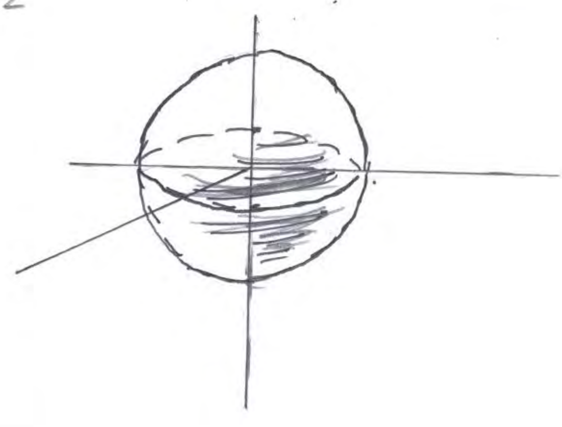
$(\rho, \theta, \varphi) \in (0, +\infty) \cup [0, 2\pi) \cup (0, \pi)$

Παραδείγματα Επιφανειών σε Σφαιρικές, Καρτεσιανές συντετ.

- $\rho = \rho_0$ ($\rho_0 > 0$, σταθερός)

$$\vec{r}(\theta, \varphi) = (\underbrace{\rho_0\,ημφ\,συνθ}_{x''}, \underbrace{\rho_0\,ημφ\,ημθ}_{y''}, \underbrace{\rho_0\,συνφ}_{z''})$$ (Παραμετρημένη Επιφάνεια)

$$\left.\begin{aligned} x^2+y^2 &= \rho_0^2\,ημ^2φ \\ z^2 &= \rho_0^2\,συν^2φ \end{aligned}\right\} \Longrightarrow x^2+y^2+z^2 = \rho_0^2$$



• $\varphi = \varphi_0 \in [0, \pi]$.

$\vec{r}(\rho, \theta) = (\rho\,\text{συν}\theta\,\text{ημ}\varphi_0,\ \rho\,\text{ημ}\theta\,\text{ημ}\varphi_0,\ \rho\,\text{συν}\varphi_0)$

(Παραμετρημένη εξίσωση επιφάνειας)

$$\begin{cases} x = \rho\,\text{συν}\theta\,\text{ημ}\varphi_0 \\ y = \rho\,\text{ημ}\theta\,\text{ημ}\varphi_0 \\ z = \rho\,\text{συν} \end{cases} \qquad \begin{matrix} \rho > 0 \\ \theta \in [0, 2\pi) \end{matrix}$$

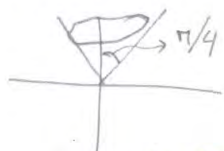
• $\varphi_0 \in (0, \frac{\pi}{2})$, $z > 0$

$$\left.\begin{matrix} x^2 + y^2 = \rho^2 \text{ημ}^2 \varphi_0 \\ z = \rho \cdot \text{συν}\varphi_0 \end{matrix}\right| \left.\rho = \frac{z}{\text{συν}\varphi_0}\right\} \Longrightarrow x^2 + y^2 = (\text{εφ}^2\varphi_0) z^2$$

$$z = \frac{1}{(\text{εφ}\varphi_0)} \sqrt{(x^2 + y^2)}$$

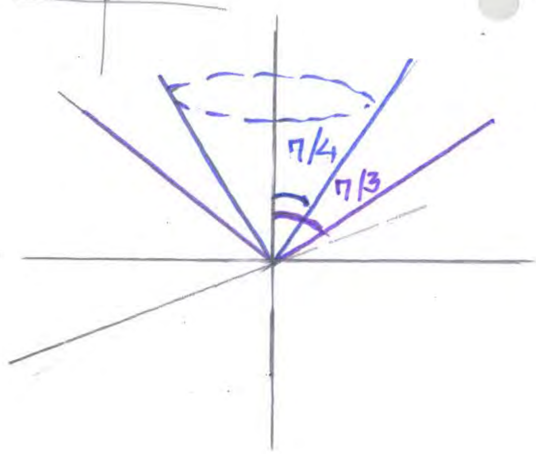
(Κώνος $z \geq 0$)

• $\varphi_0 \in (-\frac{\pi}{2}, 0)$: $z = + \frac{1}{(\text{εφ}\varphi_0)} \sqrt{x^2 + y^2}$ $\left(\begin{matrix}\text{κώνος} \\ z \leq 0\end{matrix}\right)$



πχ. $\varphi = \frac{\pi}{4}$: $z = \sqrt{x^2 + y^2}$

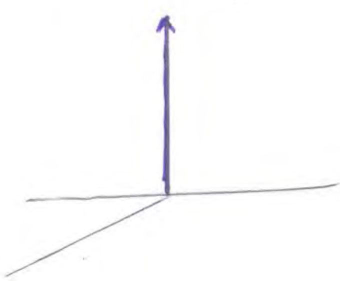
$\varphi = \pi - \frac{\pi}{4}$: $z = -\sqrt{x^2 + y^2}$



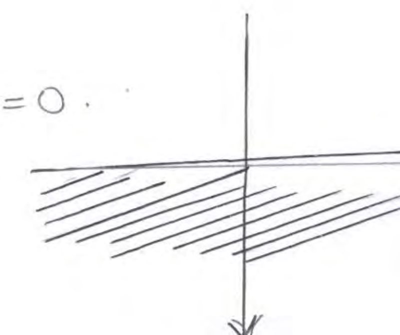
$\varphi = \frac{\pi}{6}$, $\text{εφ}\frac{\pi}{6} = \frac{1}{\sqrt{3}}$: $z = \sqrt{3(x^2 + y^2)}$

$\varphi = \frac{\pi}{3}$, $\text{εφ}\frac{\pi}{3} = \sqrt{3}$, $z = \sqrt{\frac{x^2 + y^2}{3}}$

$\varphi = 0$, $z = \rho > 0$



$\varphi = \frac{\pi}{2}$, $z = 0$.



• $cz = \sqrt{x^2+y^2}$ $\quad (c>0)$

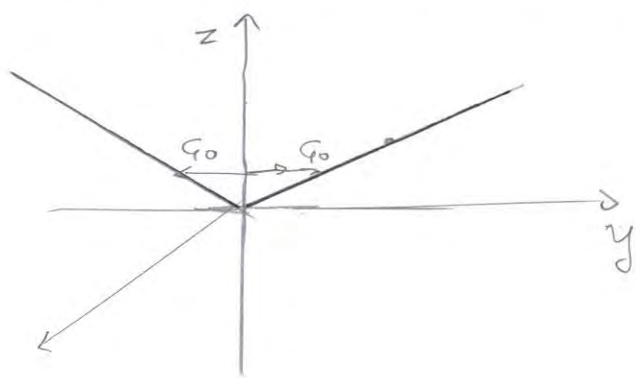
$$\begin{aligned} x &= ρ\,\text{συν}θ\,\text{ημ}φ \\ y &= ρ\,\text{ημ}θ\,\text{ημ}φ \\ z &= ρ\,\text{συν}φ \end{aligned}$$

$c\,ρ\,\text{συν}φ = \sqrt{ρ^2\text{ημ}^2φ(\text{συν}^2θ + \text{ημ}^2θ)} =$

$= \sqrt{ρ^2\text{ημ}^2φ} = ρ\,\text{ημ}φ$

$\Longrightarrow \text{εφ}φ = c \;,\quad φ = \text{τοξεφ}(c)\,(=φ_0)$

$φ = φ_0$



Ορίζουσα του Σφαιρικού Μετασχηματισμού

$(\Delta x\,\Delta y\,\Delta z = |\det J_{\vec{T}}|\,\Delta ρ\,\Delta θ\,\Delta φ$

$\vec{T}(ρ,θ,φ) = (ρ\,\text{συν}θ\,\text{ημ}φ,\ ρ\,\text{ημ}θ\,\text{ημ}φ,\ ρ\,\text{συν}φ)$

$$\det J_{\vec{T}}(ρ,θ,φ) = \begin{vmatrix} \text{συν}θ\,\text{ημ}φ & -ρ\,\text{ημ}θ\,\text{συν}φ & ρ\,\text{συν}θ\,\text{συν}φ \\ \text{ημ}θ\,\text{ημ}φ & ρ\,\text{συν}θ\,\text{ημ}φ & ρ\,\text{ημ}θ\,\text{συν}φ \\ \text{συν}φ & 0 & -ρ\,\text{ημ}φ \end{vmatrix}$$

$= \text{συν}θ\,\text{ημ}φ\,(-ρ^2\text{συν}θ\,\text{ημ}^2θ) - \text{ημ}θ\,\text{ημ}φ\,(+ρ^2\text{ημ}θ\,\text{ημ}^2φ) +$

$+ (-ρ^2\text{ημ}^2θ\,\text{ημ}φ\,\text{συν}φ - ρ^2\text{συν}^2θ\,\text{ημ}φ\,\text{συν}φ) =$

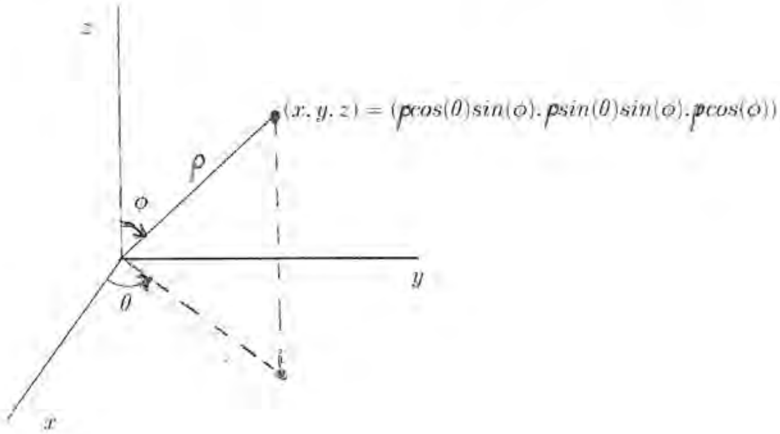
$= -ρ^2\left[\text{συν}^2θ\,\text{ημ}^3φ + \text{ημ}^2θ\,\text{ημ}^3φ + \text{συν}φ\,(\text{ημ}φ\,\text{συν}φ)\right] =$

$= -ρ^2\left[\text{ημ}^3φ + \text{ημ}φ\,\text{συν}^2φ\right] = -ρ^2\text{ημ}φ$. Άρα $|\det J_{\vec{T}}(ρ,θ,φ)| = ρ^2\text{ημ}φ > 0$

II. Σφαιρικές συντεταγμένες (ρ, θ, φ)

Ο $\vec{T}: [0,\infty) \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}^3$ (επί), με $\vec{T}(\rho,\theta,\varphi) = (\rho\sigma\upsilon\nu\theta\ \eta\mu\varphi, \rho\eta\mu\theta\ \eta\mu\varphi, \rho\sigma\upsilon\nu\varphi)$ καλείται σφαιρικός μετασχηματισμός και τα ρ, θ, φ σφαιρικές συντεταγμένες.

Ο περιορισμός του $\vec{T}: (0,\infty) \times [0,2\pi) \times (0,\pi) \to \boldsymbol{R}^3 \setminus \{(0,0,z), z \in \boldsymbol{R}\}$ είναι 1-1 και επί.

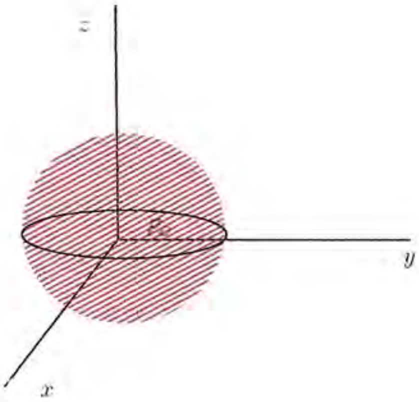


Σχέση Καρτεσιανών Σφαιρικών συντεταγμένων

$$\begin{cases} x = \rho\sigma\upsilon\nu\theta\eta\mu\varphi \\ y = \rho\eta\mu\theta\eta\mu\varphi \\ z = \rho\sigma\upsilon\nu\varphi \end{cases}, (\rho,\theta,\varphi) \in [0,\infty) \times \boldsymbol{R} \times \boldsymbol{R}$$

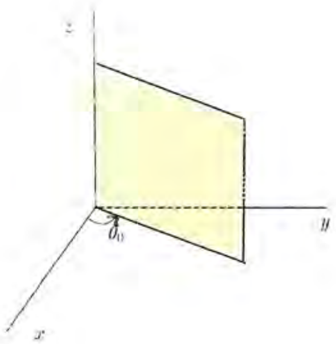
$$\begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \varepsilon\varphi\theta = \dfrac{y}{x}\ \alpha\nu\ x \neq 0. A\nu\ x = 0{:}\ \theta = \dfrac{\pi}{2}\ \gamma\iota\alpha\ y > 0, \theta = \dfrac{3\pi}{2}\ \gamma\iota\alpha\ y < 0 \qquad (x,y,z) \in \boldsymbol{R}^3 \setminus \{(0,0,z){:}\ z \in \boldsymbol{R}\} \\ \sigma\upsilon\nu\varphi = \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{cases}$$

Σφαιρικές επιφάνειες $\rho=\rho_0>0, \theta=\theta_0, \varphi=\varphi_0\in(0,\pi)$ (στο καρτεσιανό σύστημα)



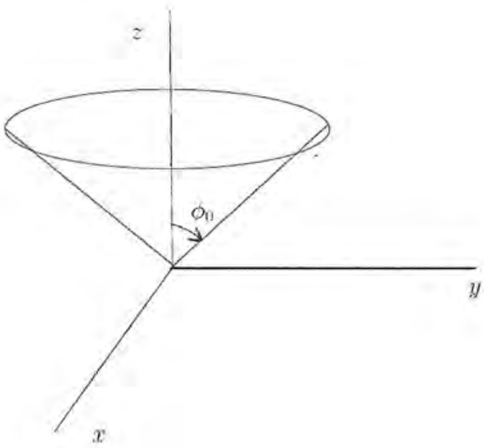
$S_1: \rho=\rho_0, σφαίρα\ \{(x,y,z)\in\boldsymbol{R}^3: x^2+y^2+z^2=\rho_0^2\}$

$$\left\{\begin{matrix}\vec{r}_1(\theta,\varphi)=(\rho_0\, συνθ\, ημφ, \rho_0\, ημθ\, ημφ, \rho_0\, συνφ), \\ \theta\in[0,2\pi], \varphi\in[0,\pi]\end{matrix}\right\}$$



$S_2: \theta=\theta_0, ημιεπίπεδο$

$$\left\{\begin{matrix}\vec{r}_2(\rho,\varphi)=(\rho συνθ_0\, ημφ, \rho ημθ_0 ημφ, \rho συνφ), \\ \rho\geq\ 0, \varphi\in[0,\pi]\end{matrix}\right\}$$

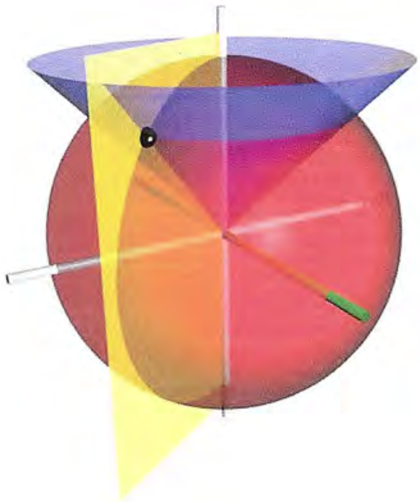


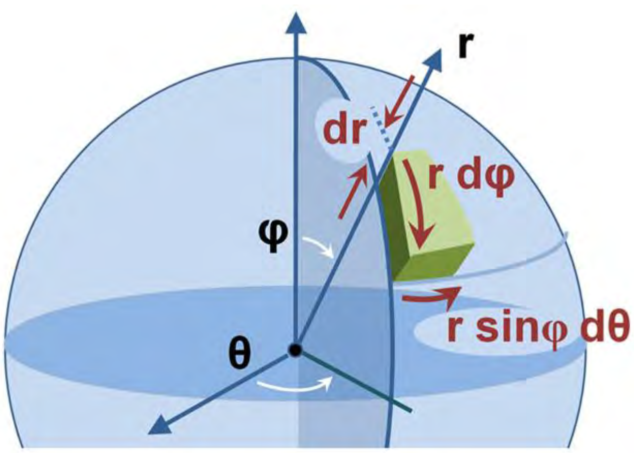
$S_3: \varphi=\varphi_0, κώνος$

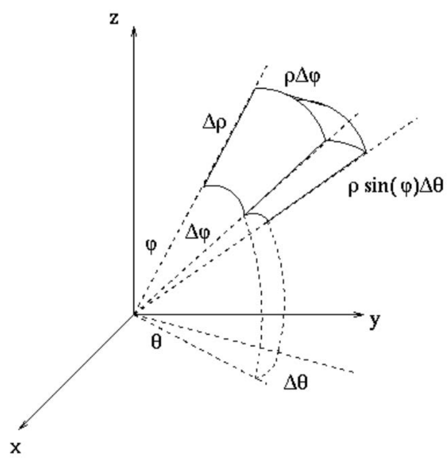
$$\left\{\begin{matrix}\vec{r}_3(\rho,\theta)=(\rho συνθ\, ημφ_0, \rho ημθ\, ημφ_0, \rho συνφ_0), \\ \rho\geq\ 0, \theta\in[0,2\pi]\end{matrix}\right\}$$

Το $(x_0, y_0, z_0) = (\rho_0 \sigma\upsilon\nu\theta_0\ \eta\mu\varphi_0, \rho_0\ \eta\mu\theta_0\ \eta\mu\varphi_0, \rho_0\ \sigma\upsilon\nu\varphi_0)$ είναι το σημείο τομής των επιφανειών

$$S_1 : \rho = \rho_0, S_2 : \theta = \theta_0, S_3 : \varphi = \varphi_0$$







Στοιχειώδεις Όγκοι
ΔxΔyΔz≈ρ²ημφdρdθdφ

Ασκήσεις

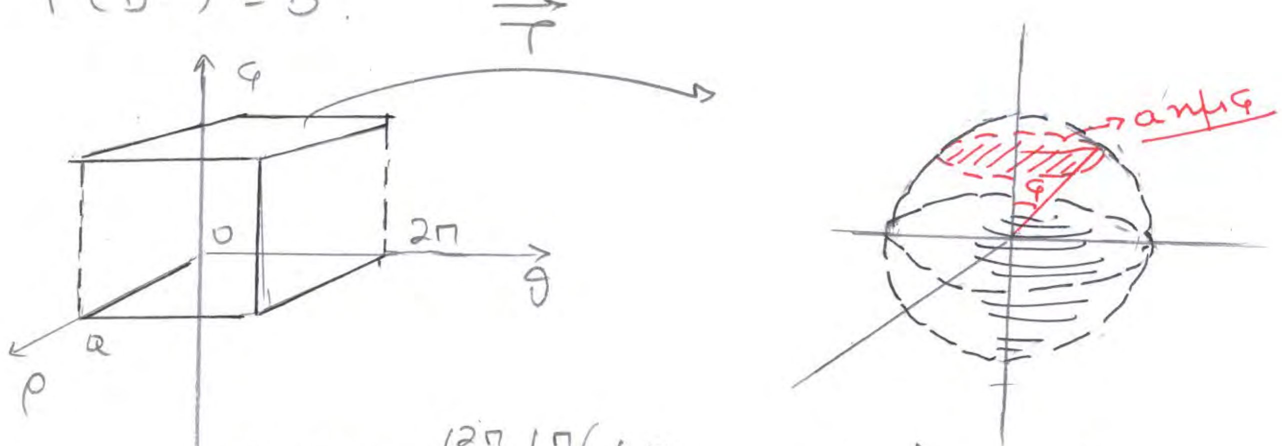
1. Να υπολογιστεί ο όγκος της σφαίρας $B: x^2+y^2 \leq a^2 \ (a>0)$

Λύση $x = \rho\, \sigma\upsilon\nu\theta\, \eta\mu\varphi$ $\quad x^2+y^2+z^2 = \rho^2 \leq a^2$

$y = \rho\, \eta\mu\theta\, \eta\mu\varphi$

$z = \rho\, \sigma\upsilon\nu\varphi$ $\quad B^* = \{(\rho,\theta,\varphi) : 0 < \rho \leq a^2,\ 0 \leq \theta < 2\pi,\ 0 \leq \varphi \leq 2\pi\}$

$\vec{T}(B^*) = B$



$$V(B) = \int_0^{2\pi}\int_0^{\pi}\left(\int_0^{\alpha} \rho^2 \eta\mu\varphi \, d\rho\right) d\varphi\, d\theta =$$

$$= 2\pi \int_0^{\pi} \left(\frac{a^3}{3}\eta\mu\varphi\right) d\varphi = \frac{2\pi a^3}{3}\left[-\sigma\upsilon\nu\varphi\Big|_0^{\pi}\right] =$$

$$= \frac{2\pi a^3}{3}\left(-(-1)+1\right) = \frac{4\pi a^3}{3}$$