8

Realization Theory and Algorithms

8.1 Introduction

In this chapter the following problem is being addressed: Given an external description of a linear system, specifically, its transfer function or its impulse response, determine an internal, state-space description for the system that generates the given transfer function. This is the problem of *system realization.* The name reflects the fact that if a (continuous-time) state-space description is known, an operational amplifier circuit can be built in a straightforward manner to realize (actually simulate) the system response.

There are many ways, an infinite number in fact, of realizing a given transfer function. Presently, we are interested in realizations that contain the least possible number of energy or memory storage elements, i.e., in realizations of least order (in terms of differential or difference equations). To accomplish this, the concepts of controllability and observability play a central role. Indeed, it turns out that realizations of transfer functions of least order are both controllable and observable. In Section 8.2, the problem of state-space realizations of input–output descriptions is defined and the existence of such realizations is addressed. The minimality of realizations of $H(s)$ is studied in Section 8.3, culminating in two results, Theorem 8.9 and Theorem 8.10, where it is first shown that a realization is minimal if and only if it is controllable and observable, and next, that if a realization is minimal, all other minimal realizations of a given $H(s)$ can be found via similarity transformations. It is also shown how to determine the order of minimal realizations directly from $H(s)$. Several realization algorithms are presented in Section 8.4, and the role of duality is emphasized in Subsection 8.4.1.

8.2 State-Space Realizations of External Descriptions

In this section, state-space realizations of impulse responses and of transfer functions for time-invariant systems are introduced. Continuous-time systems are discussed first in Subsection 8.2.1, followed by discrete-time systems in Subsection 8.2.2.

8.2.1 Continuous-Time Systems

Before formally defining the problem of system realization, we first review some of the relations that were derived in Chapter 3.

We consider a time-invariant system described by equations of the form

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{8.1}$$

where $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}$, and $D \in R^{p\times m}$. The response of this system is given by

$$y(t) = Ce^{At}x_0 + \int_0^t H(t,\tau)u(\tau)d\tau, \tag{8.2}$$

where, without loss of generality, the initial time t_0 was taken to be zero. The impulse response is now given by the expression

$$H(t,\tau) = \begin{cases} Ce^{A(t-\tau)}B + D\delta(t-\tau), & \text{for} t \geq \tau, \\ 0, & \text{for} t < \tau. \end{cases} \tag{8.3}$$

Recall that the time invariance of system (8.1) implies that $H(t,\tau) = H(t-\tau, 0)$, and therefore, τ, which is the time at which a unit impulse input is applied to the system, can be taken to equal zero ($\tau = 0$), without loss of generality, to yield $H(t,0)$. The transfer function matrix of the system is the (one-sided) Laplace transform of $H(t,0)$, namely,

$$H(s) = \mathcal{L}[H(t,0)] = C(sI-A)^{-1}B + D. \tag{8.4}$$

In the time-invariant case, a realization is commonly defined in terms of the transfer function matrix. We let $\{A,B,C,D\}$ denote the system description given in (8.1), and we let $H(s)$ be a $p\times m$ matrix with entries that are functions of s.

Definition 8.1. *A* realization *of $H(s)$ is any set $\{A,B,C,D\}$, the transfer function matrix of which is $H(s)$; i.e., $\{A,B,C,D\}$ is a realization of $H(s)$ if (8.4) is satisfied. (See Figure 8.1.)* ■

As will be shown in the next section, given $H(s)$, a condition for a realization $\{A,B,C,D\}$ of $H(s)$ to exist is that all entries in $H(s)$ are proper, rational functions. Alternative conditions under which a given set $\{A,B,C,D\}$ is a realization of some $H(s)$ can easily be derived. To this end, we expand $H(s)$ in a Laurent series to obtain

$$H(s) = H_0 + H_1 s^{-1} + H_2 s^{-2} + \cdots. \tag{8.5}$$

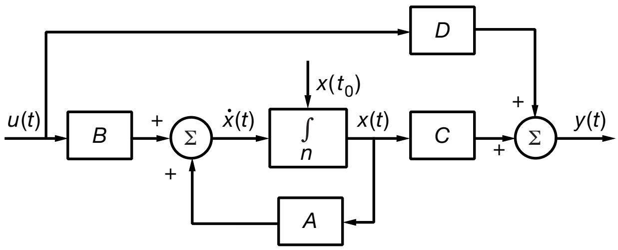

Figure 8.1. Block diagram realization of $\{A, B, C, D\}$

Definition 8.2. *The terms H_i, $i = 0, 1, 2, \dots$, in (8.5) are the* Markov parameters *of the system.* ■

The Markov parameters can be determined by the formulas

$$H_0 = \lim_{s\to\infty} H(s),\ H_1 = \lim_{s\to\infty} s(H(s)-H_0),\ H_2 = \lim_{s\to\infty} s^2(H(s)-H_0-H_1 s^{-1}),$$

and so forth. Recall that relations involving the Markov parameters were used in Exercise 3.34 of Chapter 3.

Theorem 8.3. *The set $\{A, B, C, D\}$ is a realization of $H(s)$ if and only if*

$$H_0 = D \text{ and } H_i = CA^{i-1}B, \quad i = 1, 2, \dots. \tag{8.6}$$

Proof. $H(s) = D + C(sI - A)^{-1}B = D + Cs^{-1}(I - s^{-1}A)^{-1}B = D + Cs^{-1}[\sum_{i=0}^{\infty}(s^{-1}A)^i]B = D + \sum_{i=1}^{\infty}[CA^{i-1}B]s^{-i}$, from which (8.6) is derived in view of (8.5). ■

8.2.2 Discrete-Time Systems

The realization theory in the discrete-time case essentially parallels the continuous-time case. There are of course certain notable differences because in the present case the realizations are difference equations instead of differential equations. We point to these differences in the subsequent sections.

Some of the relations derived in Section 3.4 will be recalled next. We consider systems described by equations of the form

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) + Du(k), \tag{8.7}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, and $D \in R^{p\times m}$. The response of this system is given by

$$y(k) = CA^k x_0 + \sum_{i=0}^{k-1} H(k,i)u(i), \quad k > 0, \tag{8.8}$$

where, without loss of generality, k_0 was taken to be zero. The unit pulse (discrete impulse) response is now given by

$$H(k,i) = \begin{cases} CA^{k-(i+1)}B, & k > i, \\ D, & k = i, \\ 0, & k < i. \end{cases} \tag{8.9}$$

Recall that since the system (8.7) is time-invariant, $H(k,i) = H(k-i,0)$ and i, the time the pulse input is applied, can be taken to be zero, to yield $H(k,0)$ as the external system description. The transfer function matrix for (8.7) is now the (one-sided) z-transform of $H(k,0)$. We have

$$H(z) = \mathcal{Z}\{H(k,0)\} = C(zI-A)^{-1}B + D. \tag{8.10}$$

Now let $\{A,B,C,D\}$ denote the system description (8.7) and let $H(z)$ be a $p \times m$ matrix with functions of z as entries.

Definition 8.4. *A* realization of $H(z)$ *is any set* $\{A,B,C,D\}$*, the transfer function matrix of which is* $H(z)$*; i.e., it satisfies (8.10).* ■

A result that is analogous to Theorem 8.3 is also valid in the discrete-time case [with $H(s)$ replaced by $H(z)$].

8.3 Existence and Minimality of Realizations

The existence of realizations is examined first. Given a $p \times m$ matrix $H(s)$, conditions for $H(s)$ to be the transfer function matrix of a system described by equations of the form $\dot{x} = Ax + Bu$, $y = Cx + Du$ are given in Theorem 8.5. It is shown that such realizations exist if and only if $H(s)$ is a matrix of rational functions with the property that $\lim_{s\to\infty} H(s)$ is finite. The corresponding results for discrete-time systems are also presented.

Realizations of least order, also called minimal or irreducible realizations, are of interest to us since they realize a system, using the least number of dynamical elements (minimum number of elements with memory). The principal results are given in Theorems 8.9 and 8.10, where it is shown that minimal realizations are controllable (-from-the-origin) and observable and that all minimal realizations of $H(s)$ are equivalent representations. The order of any minimal realization can be determined directly without first determining a minimal realization, and this can be accomplished by using the characteristic polynomial and the degree of $H(s)$ (Theorem 8.12) or from the rank of a Hankel matrix (Theorem 8.16). All the results on minimality of realizations apply to the discrete-time case as well with no substantial changes. This is discussed at the end of the section.

8.3.1 Existence of Realizations

Continuous-Time Systems. Given a $p \times m$ matrix $H(s)$, the following result establishes necessary and sufficient conditions for the existence of time-invariant realizations.

Theorem 8.5. *$H(s)$ is realizable as the transfer function matrix of a time-invariant system described by (8.1) if and only if $H(s)$ is a matrix of rational functions and satisfies*

$$\lim_{s\to\infty} H(s) < \infty, \tag{8.11}$$

i.e., if and only if $H(s)$ is a proper rational matrix.

Proof. (Necessity) If the system $\dot{x} = Ax + Bu, y = Cx + Du$ is a realization of $H(s)$, then $C(sI - A)^{-1}B + D = H(s)$, which shows that $H(s)$ must be a rational matrix. Furthermore,

$$\lim_{s\to\infty} H(s) = D, \tag{8.12}$$

which is a real finite matrix.

(Sufficiency) If $H(s)$ is a proper rational matrix, then any of the algorithms discussed in the next section can be applied to derive a realization. ∎

Discrete-Time Systems. Given a $p \times m$ matrix $H(z)$, the next theorem establishes necessary and sufficient conditions for time-invariant realizations. This result corresponds to Theorem 8.5 for the continuous-time case. Notice that the conditions in these results are identical.

Theorem 8.6. *$H(z)$ is realizable as the transfer function matrix of a time-invariant system described by (8.7) if and only if $H(z)$ is a matrix of rational functions and satisfies the condition that*

$$\lim_{z\to\infty} H(z) < \infty. \tag{8.13}$$

Proof. Similar to the proof of Theorem 8.5. ∎

8.3.2 Minimality of Realizations

Realizations of a transfer function matrix $H(s)$ can be expected to generate only the *zero-state response* of a system, since the external description $H(s)$ has, by definition, no information about the initial conditions and the zero-input response of the system.

A second important point is the fact that *if a realization of a given $H(s)$ exists, then there exists an infinite number of realizations.* If (8.1) is a realization of the $p \times m$ matrix $H(s)$, then realizations of the same order n, i.e., of the same dimension n of the state vector, can readily be generated by an equivalence transformation. There are, of course, other ways of generating alternative realizations. In particular, if (8.1) is a realization of $H(s)$, then, for example, the system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad y = Cx + Du, \\ \dot{z} &= Fz + Gu \end{aligned} \tag{8.14}$$

is also a realization. This was accomplished by adding to (8.1) a state equation $\dot{z} = Fz + Gu$ that does not affect the system output. The dimension of $F, \dim F$, and consequently the order of the realization, $n + \dim F$, can be larger than any given finite number. In other words, *there may be no upper bound to the order of the realizations of a given* $H(s)$. There exists, however, a lower bound, and a realization of such lowest order is called a least-order minimal or irreducible realization.

Definition 8.7. *A realization*

$$\dot{x} = Ax + Bu, \quad y = Cx + Du \tag{8.15}$$

of the transfer function matrix $H(s)$ of least order n ($A \in R^{n\times n}$) is called a least-order*, or* a minimal*, or* an irreducible realization *of $H(s)$.* ■

Theorems 8.9 and 8.10 below completely solve the minimal realization problem. The first of these results shows that a realization is minimal if and only if it is controllable (-from-the-origin or reachable) and observable, whereas the second result shows that if a minimal realization has been found, then all other minimal realizations can be obtained from the determined realization, using equivalence of representations.

Controllability (-from-the-origin, or reachability) and observability play an important role in the minimality of realizations. Indeed, it was shown in Section 7.2 that only that part of a system that is both controllable and observable appears in $H(s)$. In other words, $H(s)$ *contains no information about the uncontrollable and/or unobservable parts of the system.* To illustrate this, consider the following specific case.

Example 8.8. Let $H(s) = 1/(s+1)$. Four different realizations of $H(s)$ are given by

(i) $\{A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = [-1, 1],\ D = 0\}$,
(ii) $\{A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ C = [0, 1],\ D = 0\}$,
(iii) $\{A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = [0, 1],\ D = 0\}$,
(iv) $\{A = -1,\ B = 1,\ C = 1,\ D = 0\}$.

The eigenvalue $+1$ in (i) is unobservable, in (ii) is uncontrollable, and in (iii) is both uncontrollable and unobservable and does not appear in $H(s)$ at all. Realization (iv), which is of order 1, is a minimal realization. It is controllable and observable.

Theorem 8.9. *An n-dimensional realization $\{A, B, C, D\}$ of $H(s)$ is minimal (irreducible, of least order) if and only if it is both controllable and observable.*

Proof. (*Necessity*) Assume that $\{A, B, C, D\}$ is a minimal realization but is not both controllable and observable. Using Kalman's Canonical Decomposition (see Subsection 6.2.3), one may find another realization of lower dimension that is both controllable and observable. This contradicts the assumption that $\{A, B, C, D\}$ is a minimal realization. Therefore, it must be both controllable and observable.

(*Sufficiency*) Assume that the realization $\{A, B, C, D\}$ is controllable and observable, but there exists another realization, say, $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ of order $\bar{n} < n$. Since they are both realizations of $H(s)$, or of the impulse response $H(t, 0)$, then

$$Ce^{At}B + D\delta(t) = \bar{C}e^{\bar{A}t}\bar{B} + \bar{D}\delta(t) \tag{8.16}$$

for all $t \geq 0$. Clearly, $D = \bar{D} = \lim_{s\to\infty} H(s)$. Using the power series expansion of the exponential and equating coefficients of the same power of t, we obtain

$$CA^kB = \bar{C}\bar{A}^k\bar{B}, \quad k = 0, 1, 2, \ldots; \tag{8.17}$$

i.e., the Markov parameters of the two representations are the same (see Theorem 8.3). Let

$$\mathcal{C}_n \triangleq [B, AB, \ldots, A^{n-1}B] \in R^{n\times mn}$$

and

$$\mathcal{O}_n \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in R^{pn\times n}. \tag{8.18}$$

Then the $pn \times mn$ matrix product $\mathcal{O}_n\mathcal{C}_n$ assumes the form

$$\begin{aligned} \mathcal{O}_n\mathcal{C}_n &= \begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \\ CAB & CA^2B & \cdots & CA^nB \\ \vdots & \vdots & & \vdots \\ CA^{n-1}B & CA^nB & \cdots & CA^{2n-2}B \end{bmatrix} \\ &= \begin{bmatrix} \bar{C}\bar{B} & \bar{C}\bar{A}\bar{B} & \cdots & \bar{C}\bar{A}^{n-1}\bar{B} \\ \bar{C}\bar{A}\bar{B} & \bar{C}\bar{A}^2\bar{B} & \cdots & \bar{C}\bar{A}^n\bar{B} \\ \vdots & \vdots & & \vdots \\ \bar{C}\bar{A}^{n-1}\bar{B} & \bar{C}\bar{A}^n\bar{B} & \cdots & \bar{C}\bar{A}^{2n-2}\bar{B} \end{bmatrix} = \bar{\mathcal{O}}_n\bar{\mathcal{C}}_n. \end{aligned} \tag{8.19}$$

In view of *Sylvester's Rank Inequality*, which relates the rank of the product of two matrices to the rank of its factors, we have

$$\operatorname{rank}\mathcal{O}_n + \operatorname{rank}\mathcal{C}_n - n \leq \operatorname{rank}(\bar{\mathcal{O}}_n\bar{\mathcal{C}}_n) \leq \min(\operatorname{rank}\mathcal{O}_n, \operatorname{rank}\mathcal{C}_n) \tag{8.20}$$

and we obtain that $\operatorname{rank}\mathcal{O}_n = \operatorname{rank}\mathcal{C}_n = n$, $\operatorname{rank}(\bar{\mathcal{O}}_n\bar{\mathcal{C}}_n) = n$. This result, however, contradicts our assumptions, since $n = \operatorname{rank}(\bar{\mathcal{O}}_n\bar{\mathcal{C}}_n) \leq \min(\operatorname{rank}\bar{\mathcal{O}}_n, \operatorname{rank}\bar{\mathcal{C}}_n) \leq \bar{n}$ because $\bar{n}$ is the order of $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$. Therefore $n \leq \bar{n}$. Hence, $\bar{n}$ cannot be less than n and they can only be equal. Thus, $n = \bar{n}$ and $\{A, B, C, D\}$ is indeed a minimal realization. ∎

Theorem 8.9 suggests the following procedure to realize $H(s)$. First, we obtain a controllable (observable) realization of $H(s)$. Next, using a similarity transformation, we obtain an observable standard form to separate the observable from the unobservable parts (controllable from the uncontrollable parts), using the approach of Subsection 6.2.1. Finally, we take the observable (controllable) part that will also be controllable (observable) as the minimal realization. We shall use this procedure in the next section.

Is the minimal realization unique? The answer to this question is of course "no" since we know that equivalent representations, which are of the same order, give the same transfer function matrix. The following theorem shows how to obtain *all* minimal realizations of $H(s)$.

Theorem 8.10. *Let $\{A, B, C, D\}$ and $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ be realizations of $H(s)$. If $\{A, B, C, D\}$ is a minimal realization, then $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ is also a minimal realization if and only if the two realizations are equivalent, i.e., if and only if $\bar{D} = D$ and there exists a nonsingular matrix P such that*

$$\bar{A} = PAP^{-1}, \quad \bar{B} = PB, \text{ and } \bar{C} = CP^{-1}. \tag{8.21}$$

Furthermore, if P exists, it is given by

$$P = \mathcal{C}\bar{\mathcal{C}}^T(\bar{\mathcal{C}}\bar{\mathcal{C}}^T)^{-1} \text{ or } P = (\bar{\mathcal{O}}^T\bar{\mathcal{O}})^{-1}\bar{\mathcal{O}}^T\mathcal{O}. \tag{8.22}$$

Proof. (*Sufficiency*) Let the realizations be equivalent. Since $\{A, B, C, D\}$ is minimal, it is controllable and observable and its equivalent representation $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ is also controllable and observable and, therefore, minimal. Alternatively, since equivalence preserves the dimension of A, the equivalent realization $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ is also minimal.

(*Necessity*) Suppose $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ is also minimal. We shall show that it is equivalent to $\{A, B, C, D\}$. Since they are both realizations of $H(s)$, they satisfy $D = \bar{D}$ and

$$CA^kB = \bar{C}\bar{A}^k\bar{B}, \overline{k} = 0, 1, 2 \ldots, \tag{8.23}$$

as was shown in the proof of Theorem 8.9. Here, both realizations are minimal, and therefore, they are both of the same order n and are both controllable and observable.

Define $\mathcal{C} = \mathcal{C}_n$ and $\mathcal{O} = \mathcal{O}_n$, as in (8.18). Then, in view of (8.19), $\mathcal{O}\mathcal{C} = \bar{\mathcal{O}}\bar{\mathcal{C}}$ and premultiplying by $\bar{\mathcal{O}}^T$, we obtain $\bar{\mathcal{O}}^T\mathcal{O}\mathcal{C} = \bar{\mathcal{O}}^T\bar{\mathcal{O}}\bar{\mathcal{C}}$. Using Sylvester's Inequality, we obtain $\operatorname{rank} \bar{\mathcal{O}}^T\bar{\mathcal{O}} = n$, and therefore,

$$\bar{\mathcal{C}} = [(\bar{\mathcal{O}}^T\bar{\mathcal{O}})^{-1}\bar{\mathcal{O}}^T\mathcal{O}]\mathcal{C} = P\mathcal{C}, \tag{8.24}$$

where $P \triangleq (\bar{\mathcal{O}}^T\bar{\mathcal{O}})^{-1}\bar{\mathcal{O}}^T\mathcal{O} \in R^{n\times n}$. Note that $\operatorname{rank} P = n$ since $\operatorname{rank} \bar{\mathcal{O}}^T\mathcal{O}$ is also equal to n as can be seen from $\operatorname{rank} \bar{\mathcal{O}}^T\mathcal{O}\mathcal{C} = n$ and from Sylvester's Inequality. Therefore, P qualifies as a similarity transformation. Similarly, $\mathcal{O}\mathcal{C} = \bar{\mathcal{O}}\bar{\mathcal{C}}$ implies that $\mathcal{O}\mathcal{C}\mathcal{C}^T = \bar{\mathcal{O}}\bar{\mathcal{C}}\mathcal{C}^T$, and

$$\mathcal{O} = \bar{\mathcal{O}}[\bar{\mathcal{C}}\mathcal{C}^T(\mathcal{C}\mathcal{C}^T)^{-1}] = \bar{\mathcal{O}}\bar{P}, \tag{8.25}$$

where $\bar{P} \triangleq \bar{\mathcal{C}}\mathcal{C}^T(\mathcal{C}\mathcal{C}^T)^{-1} \in R^{n\times n}$ with rank $\bar{P} = n$. Note that $P = (\bar{\mathcal{O}}^T\bar{\mathcal{O}})^{-1}\bar{\mathcal{O}}^T(\bar{\mathcal{O}}\bar{P}) = \bar{P}$. To show that P is the equivalence transformation given in (8.21), we note that $\mathcal{O}A\mathcal{C} = \bar{\mathcal{O}}\bar{A}\bar{\mathcal{C}}$ from (8.19). Premultiplying by $\bar{\mathcal{O}}^T$ and postmultiplying by $\mathcal{C}^T$, we obtain $PA = \bar{A}P$, in view of (8.24) and (8.25). To show that $PB = \bar{B}$ and $C = \bar{C}P$, we simply use the relations $P\mathcal{C} = \bar{\mathcal{C}}$ and $\mathcal{O} = \bar{\mathcal{O}}P$, respectively. ■

8.3.3 The Order of Minimal Realizations

One could ask the question whether the order of a minimal realization of $H(s)$ can be determined directly, without having to actually derive a minimal realization. The answer to this question is yes, and in the following we will show how this can be accomplished.

Determination via the Characteristic or Pole Polynomial of $H(s)$.

The *characteristic polynomial (or pole polynomial), $p_H(s)$, of a transfer function* matrix $H(s)$ was defined in Section 7.4 using the Smith–McMillan form of $H(s)$. The polynomial $p_H(s)$ is equal to the monic least common denominator of all nonzero minors of $H(s)$. *The minimal polynomial of a transfer function* matrix $H(s)$, $m_H(s)$, was defined as the monic least common denominator of all nonzero first-order minors (entries) of $H(s)$.

Definition 8.11. *The* McMillan degree *of $H(s)$ is the degree of $p_H(s)$.* ■

The number of poles in $H(s)$, which are defined as the zeros of $p_H(s)$, is equal to the McMillan degree of $H(s)$. The degree of $H(s)$ is in fact the order of any minimal realization of $H(s)$, as the following result shows.

Theorem 8.12. *Let $\{A, B, C, D\}$ be a minimal realization of $H(s)$. Then the characteristic polynomial of $H(s), p_H(s)$, is equal to the characteristic polynomial of $A, \alpha(s) \triangleq |sI - A|$; i.e., $p_H(s) = \alpha(s)$. Therefore, the McMillan degree of $H(s)$ equals the order of any minimal realization.*

Proof. See [1, p. 397, Chapter 5, Theorem 3.11]. ■

It can also be shown that the minimal polynomial of $H(s), m_H(s)$, is equal to the minimal polynomial of $A, \alpha_m(s)$, where $\{A, B, C, D\}$ is any controllable and observable realization of $H(s)$. This is illustrated in the following example.

Example 8.13. Let $H(s) = \begin{bmatrix} 1/s & 2/s \\ 0 & -1/s \end{bmatrix}$. The first-order minors, the entries of $H(s)$, have denominators s, s, and s, and therefore, $m_H(s) = s$. The only second-order minor is $-1/s^2$ and $p_H(s) = s^2$ with $\deg p_H(s) = 2$. Therefore, the order of a minimal realization is 2. Such a realization is given by $\dot{x} = Ax + Bu$ and $y = Cx$ with $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We verify first that this system is a realization of $H(s)$ and then that it is controllable and observable and, therefore, minimal. Notice that the characteristic polynomial of A is $\alpha(s) = s^2 = p_H(s)$ and that its minimal polynomial is $\alpha_m(s) = s = m_H(s)$.

In the case when $H(s)$ is a scalar, the roots of $m_H = p_H$ are the eigenvalues of any minimal realization of $H(s)$.

Corollary 8.14. *Let $H(s) = n(s)/d(s)$ be a scalar proper rational function. If $\{A, B, C, D\}$ is a minimal realization of $H(s)$, then*

$$kd(s) = \alpha(s) = \alpha_m(s), \tag{8.26}$$

where $\alpha(s) = \det(sI - A)$ and $\alpha_m(s)$ are the characteristic and minimal polynomials of A, respectively, and k is a real scalar so that $kd(s)$ is a monic polynomial.

Proof. The characteristic and minimal polynomials of $H(s), p_H(s)$, and $m_H(s)$ are by definition equal to $d(s)$ in the scalar case. Applying Theorem 8.12 proves the result. ■

Determination via the Hankel Matrix

There is an alternative way of determining the order of a minimal realization of $H(s)$. This is accomplished via the Hankel matrix, associated with $H(s)$.

Given $H(s)$, we express $H(s)$ as a Laurent series expansion to obtain

$$H(s) = H_0 + \widehat{H}(s) = H_0 + H_1 s^{-1} + H_2 s^{-2} + H_3 s^{-3} + \dots, \tag{8.27}$$

where $\widehat{H}(s)$ is strictly proper and the real $p \times m$ matrices $H_0, H_1, \dots$ are the Markov parameters of the system. They can be determined by the formulas

$$H_0 = \lim_{s\to\infty} H(s),\ H_1 = \lim_{s\to\infty} s(H(s) - H_0),\ H_2 = \lim_{s\to\infty} s^2(H(s) - H_0 - H_1 s^{-1}),$$

and so forth.

Definition 8.15. *The* Hankel matrix $M_H(i,j)$ *of order (i,j) corresponding to the (Markov parameter) sequence $H_1, H_2, \dots$ is defined as the $ip \times jm$ matrix given by*

$$M_H(i,j) \triangleq \begin{bmatrix} H_1 & H_2 & \cdots & H_j \\ H_2 & H_3 & \cdots & H_{j+1} \\ \vdots & \vdots & & \vdots \\ H_i & H_{i+1} & \cdots & H_{i+j-1} \end{bmatrix}. \tag{8.28}$$

■

Theorem 8.16. *The order of a minimal realization of $H(s)$ is the rank of $M_H(r,r)$ where r is the degree of the least common denominator of the entries of $H(s)$; i.e., $r = \deg m_H(s)$.*

Proof. See [1, p. 399, Chapter 5, Theorem 3.13]. ■

Example 8.17. Let $H(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{-1}{(s+1)(s+2)} & \frac{1}{s+2} \end{bmatrix}$. Here the minimal polynomial is $m_H(s) = (s+1)(s+2)$, and therefore, $r = \deg m_H(s) = 2$. The Hankel matrix $M_H(r,r)$ is then

$$M_H(r,r) = M_H(2,2) = \begin{bmatrix} H_1 & H_2 \\ H_2 & H_3 \end{bmatrix},$$

an $rp \times rm = 4 \times 4$ matrix, and $H_1 = \lim_{s\to\infty} sH(s) = \lim_{s\to\infty} \begin{bmatrix} \frac{s}{s+1} & \frac{2s}{s+1} \\ \frac{-s}{(s+1)(s+2)} & \frac{s}{s+2} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $H_2 = \lim_{s\to\infty} s^2(H(s) - H_1 s^{-1}) = \lim_{s\to\infty} \begin{bmatrix} \frac{s^2}{s+1} - s & \frac{2s^2}{s+1} - 2s \\ \frac{-s^2}{(s+1)(s+2)} & \frac{s^2}{s+2} - s \end{bmatrix} = \lim_{s\to\infty} \begin{bmatrix} \frac{-s}{s+1} & \frac{-2s}{s+1} \\ \frac{-s^2}{(s+1)(s+2)} & \frac{-2s}{s+2} \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix}$. Similarly, $H_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Now

$$\operatorname{rank} M_H(2,2) = \operatorname{rank} \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -2 \\ -1 & -2 & 1 & 2 \\ -1 & -2 & 3 & 4 \end{bmatrix} = 3,$$

which is the order of any minimal realization, in view of Theorem 8.16. The reader should verify this result, using Theorem 8.12.

Example 8.18. Consider the transfer function matrix $H(s) = \begin{bmatrix} 1/s & 2/s \\ 0 & -1/s \end{bmatrix}$, as in Example 8.13. Here $r = \deg m_H(s) = \deg s = 1$. Now, the Hankel matrix $M_H(r,r) = M_H(1,1) = H_1 = \lim_{s\to\infty} sH(s) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$. Its rank is 2, which is the order of a minimal realization of $H(s)$. This agrees with the results in Example 8.13.

8.3.4 Minimality of Realizations: Discrete-Time Systems

The fact that the results on minimality of realizations in the discrete-time case are essentially identical to the corresponding results for the continuous-time case is not surprising since we are concentrating here on the time-invariant cases for which the transfer function matrices have the same forms: $H(s) = C(sI-A)^{-1}B+D$ and $H(z) = C(zI-A)^{-1}B+D$. Accordingly, the results on how to generate 4-tuples $\{A, B, C, D\}$ to satisfy these relations are, of course, also the same.

8.4 Realization Algorithms

In this section, algorithms for generating time-invariant state-space realizations of external system descriptions are introduced. A brief outline of the contents of this section follows.

Realizations of $H(s)$ can often be derived in an easier manner if duality is used, and this is demonstrated first in this section. Realizations of minimal order are both controllable and observable, as was shown in the previous section. To derive a minimal realization of $H(s)$, one typically derives a realization that is controllable (observable) and then extracts the part that is also observable (controllable). This involves in general a two-step procedure. However, in certain cases, a minimal realization can be derived in one step, as for example, when $H(s)$ is a scalar transfer function. Algorithms for realizations in a controller/observer form are discussed first. In the interest of clarity, the SISO case is presented separately, thus providing an introduction to the general MIMO case. Realization algorithms, where A is diagonal, are introduced next. Finally, balanced realizations are addressed.

It is not difficult to see that the above algorithms can also be used to derive realizations described by equations of the form $x(k+1) = Ax(k) + Bu(k), y(k) = Cx(k) + Du(k)$ of transfer function matrices $H(z)$ for discrete-time time-invariant systems. Accordingly, the discrete-time case will not be treated separately in this section. Additional details, algorithms, and proofs may be found in [1, Section 5.4].

8.4.1 Realizations Using Duality

If the system described by the equations $\dot{x} = Ax + Bu$, $y = Cx + Du$ is a realization of $H(s)$, then

$$H(s) = C(sI - A)^{-1}B + D. \tag{8.29}$$

If $\widetilde{H}(s) \triangleq H^T(s)$, then $\dot{\tilde{x}} = \widetilde{A}\tilde{x} + \widetilde{B}\tilde{u}$ and $\tilde{y} = \widetilde{C}\tilde{x} + \widetilde{D}\tilde{u}$, where $\widetilde{A} = A^T, \widetilde{B} = C^T, \widetilde{C} = B^T$, and $\widetilde{D} = D^T$, is a realization of $\widetilde{H}(s)$ since in view of (8.29),

$$\begin{aligned}\widetilde{H}(s) &= H^T(s)\\ &= B^T(sI - A^T)^{-1}C^T + D^T\\ &= \widetilde{C}(sI - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D}.\end{aligned} \tag{8.30}$$

The representation $\{\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}\}$ is *the dual representation* to $\{A, B, C, D\}$, and if $\{A, B, C, D\}$ is controllable (observable), then $\{\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}\}$ is observable (controllable) (see Section 5.2.3). In other words, if a controllable (observable) realization $\{A, B, C, D\}$ of the $p \times m$ transfer function matrix $H(s)$ is known, then an observable (controllable) realization of the $m \times p$ transfer function matrix $\widetilde{H}(s) = H^T(s)$ can be derived immediately: It is the dual representation, namely, $\{\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}\} = \{A^T, C^T, B^T, D^T\}$. This fact is used to advantage in deriving realizations in the MIMO case, since obtaining first a realization of $H^T(s)$ instead of $H(s)$ and then using duality leads sometimes to simpler, lower order, realizations.

Duality is very useful in realizations of symmetric transfer functions, which have the property that $H(s) = H^T(s)$, as, e.g., in the case of SISO systems where $H(s)$ is a scalar. Under these conditions, if $\{A, B, C, D\}$ is a controllable (observable) realization of $H(s)$, then $\{A^T, C^T, B^T, D^T\}$ is an observable (controllable) realization of the same $H(s)$. Note that in this case,

$$H(s) = C(sI - A)^{-1}B + D = H^T(s) = B^T(sI - A^T)^{-1}C^T + D^T.$$

In realization algorithms of MIMO systems, a realization that is either controllable or observable is typically obtained first. Next, this realization is reduced to a minimal one by extracting the part of the system that is both controllable and observable, using the methods of Subsection 6.2.1. Dual representations may simplify this process considerably. In the following discussion, we summarize the process of deriving minimal realizations for the reader's convenience.

Given a proper rational $p \times m$ transfer function matrix $H(s)$, with $\lim_{s\to\infty} H(s) < \infty$, we consider the strictly proper part $\widehat{H}(s) = H(s) - \lim_{s\to\infty} H(s) = H(s) - D$ [noting that working with $\widehat{H}(s)$ instead of $H(s)$ is optional].

1. If a realization algorithm leading to a controllable realization is used, then the following steps are taken:
$$\widehat{H}(s) \to (\widetilde{H}(s) = \widehat{H}^T(s)) \to \{\widetilde{A}, \widetilde{B}, \tilde{C}\} \to \{A = \widetilde{A}^T, B = \widetilde{C}^T, C = \widetilde{B}^T\}, \tag{8.31a}$$
where $\{\widetilde{A}, \widetilde{B}, \widetilde{C}\}$ is a controllable realization of $\widetilde{H}(s)$ and $\{A, B, C\}$ is an observable realization of $\widehat{H}(s)$.
2. To obtain a minimal realization,
$$\{A, B, C\} \to \left\{ \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, [C_1, C_2] \right\}, \tag{8.31b}$$
where $\{A, B, C\}$ is an observable realization of $\widehat{H}(s)$ obtained from step (1), and (A_1, B_1) is controllable (derived by using the method of Subsection 6.2.1), then $\{A_1, B_1, C_1\}$ is a controllable and observable, and therefore, a minimal realization of $\widehat{H}(s)$, and furthermore, $\{A_1, B_1, C_1, D\}$, is a minimal realization of $H(s)$.

8.4.2 Realizations in Controller/Observer Form

We shall first consider realizations of scalar transfer functions $H(s)$.

Single-Input/Single-Output (SISO) Systems ($p = m = 1$)

Let

$$H(s) = \frac{n(s)}{d(s)} = \frac{b_n s^n + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \tag{8.32}$$

where $n(s)$ and $d(s)$ are prime polynomials. This is the general form of a proper transfer function of (McMillan) degree n. Note that if the leading coefficient in the numerator $n(s)$ is zero, i.e., $b_n = 0$, then $H(s)$ is strictly proper. Also, recall that

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = b_n u^{(n)} + \cdots + b_1 u^{(1)} + b_0 u \quad (8.33a)$$

or

$$\begin{aligned} d(q)y(t) &= (q^n + a_{n-1}q^{n-1} + \cdots + a_1 q + a_0)y(t) \\ &= (b_n q^n + \ldots b_1 q + b_0)u(t) = n(q)u(t), \end{aligned} \quad (8.33b)$$

where $q \triangleq d/dt$, the differential operator. This is the corresponding nth-order differential equation that directly gives rise to the map $\hat{y}(s) = H(s)\hat{u}(s)$ if the Laplace transform of both sides is taken, assuming that all variables and their derivatives are zero at $t = 0$.

Controller Form Realizations

Given $n(s)$ and $d(s)$, we proceed as follows to derive a realization in controller form.

1. Determine $C_c^T \in R^n$ and $D_c \in R$ so that

   $$n(s) = C_c S(s) + D_c d(s), \tag{8.34}$$

   where $S(s) \triangleq [1, s, \ldots, s^{n-1}]^T$ is an $n \times 1$ vector of polynomials. Equation (8.34) implies that

   $$D_c = \lim_{s \to \infty} H(s) = b_n. \tag{8.35}$$

   Then $n(s) - b_n d(s)$ is in general a polynomial of degree $n-1$, which shows that a real vector C_c that satisfies (8.34) always exists.
   If $b_n = 0$, i.e., if $H(s)$ is strictly proper, then from (8.34) we obtain $C_c = [b_0, \ldots, b_{n-1}]$; i.e., C_c consists of the coefficients of the $n-1$ degree numerator.
   If $b_n \neq 0$, then (8.34) implies that the entries of C_c are a combination of the coefficients b_i and a_i. In particular,

   $$C_c = [b_0 - b_n a_0, b_1 - b_n a_1, \ldots, b_{n-1} - b_n a_{n-1}]. \tag{8.36}$$

2. A realization of $H(s)$ in controller form is given by the equations

$$\dot{x}_c = A_c x_c + B_c u = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} x_c + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u,$$
$$y = C_c x_c + D_c u. \tag{8.37}$$

The n states of the realization in (8.37) are related by $x_{i+1} = \dot{x}_i$, $i = 1, \ldots, n-1$, or $x_{i+1} = x_1^{(i)}$, $i = 1, \ldots, n-1$, and $\dot{x}_n = -a_0 x_1 - \sum_{i=1}^{n-1} a_i x_{i+1} + u = -a_0 x_1 - \sum_{i=1}^{n-1} a_i x_1^{(i)} + u$. It can now be shown that x_1 satisfies the relationship

$$d(q)x_1(t) = u(t), \quad y(t) = n(q)x_1(t), \tag{8.38}$$

where $q \triangleq d/dt$, the differential operator. Note that $d(q)x_1(t) = u(t)$ because $\dot{x}_n = -\sum_{i=0}^{n-1} a_i x_1^{(i)} + x_1^{(n)} + u = -d(q)x_1 + u + x_1^{(n)}$, which in view of $\dot{x}_n = x_1^{(n)}$, derived from $x_n = x_1^{(n-1)}$, implies that $-d(q)x_i + u = 0$. The relation $y(t) = n(q)x_1(t)$ can easily be verified by multiplying both sides of $n(q) = C_c S(q) + D_c d(q)$ given in (8.34) by x_1.

Lemma 8.19. *The representation (8.37) is a minimal realization of $H(s)$ given in (8.32).*

Proof. We must first show that (8.37) is indeed a realization, i.e., that it satisfies (8.29). This is of course true in view of the Structure Theorem in Subsection 6.4.1. Presently, this will be shown directly, using (8.38).

Relation $d(q)x_1(t) = u(t)$ implies that $\hat{x}_1(s) = (d(s))^{-1}\hat{u}(s)$. This yields for the state that $\hat{x}(s) = [\hat{x}_1(s), \ldots, \hat{x}_n(s)]^T = [1, s, \ldots, s^{n-1}]^T \ \hat{x}_1(s) = S(s)(d(s))^{-1}\hat{u}(s)$. However, we also have $\hat{x}(s) = (sI - A_c)^{-1} B_c \hat{u}(s)$. Therefore,

$$(sI - A_c)S(s) = B_c d(s). \tag{8.39}$$

Now $C_c(sI - A_c)^{-1}B_c + D_c = C_c S(s)(d(s))^{-1} + D_c = (C_c S(s) + D_c d(s))(d(s))^{-1} = \frac{n(s)}{d(s)} = H(s)$; i.e., (8.37) is indeed a realization.

System (8.37) is of order n and is therefore, a minimal, controllable, and observable realization. This is because the degree of $H(s)$ is n, which in view of Theorem 8.12, is the order of any minimal realization. Controllability and observability can also be established directly by forming the controllability and observability matrices. The reader is encouraged to pursue this approach. ■

According to Definition 8.11, the McMillan degree of a rational scalar transfer function $H(s) = n(s)/d(s)$ is the degree of $d(s)$ only when $n(s)$ and $d(s)$ are prime polynomials; if they are not, all cancellations must first take place before the degree can be determined. If $n(s)$ and $d(s)$ are not prime, then the above algorithm will yield a realization that is not observable. Notice that realization (8.37) is always controllable, since it is in controller form. This can also be seen directly from the expression

$$[B_c, A_cB_c, \ldots, A_c^{n-1}B_c] = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \cdots & \times \\ 1 & \times & \cdots & \times \end{bmatrix}, \tag{8.40}$$

which is of full rank. The realization (8.37) is observable if and only if the polynomials $d(s)$ and $n(s)$ are prime.

In Figure 8.2 a block realization diagram of the form (8.37) for a second-order transfer function is shown. Note that the states $x_1(t)$ and $x_2(t)$ are taken to be the voltages at the outputs of the integrators.

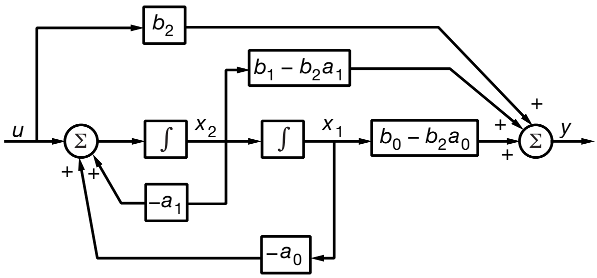

Figure 8.2. Block realization of $H(s)$ in controller form of the system $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$, $y = [b_0 - b_2a_0, b_1 - b_2a_1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_2u$; $H(s) = \frac{b_2s^2 + b_1s + b_0}{s^2 + a_1s + a_0}$

Observer Form Realizations

Given the transfer function (8.32), the nth-order realization in observer form is given by

$$\begin{aligned} \dot{x}_o &= A_ox_o + B_ou \\ &= \begin{bmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & & 0 & -a_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} x_o + \begin{bmatrix} b_0 - b_na_0 \\ b_1 - b_na_1 \\ \vdots \\ b_{n-1} - b_na_{n-1} \end{bmatrix} u, \\ y &= C_ox_o + D_ou = [0, 0, \ldots, 0,\ 1]x_o + b_nu. \end{aligned} \tag{8.41}$$

This realization was derived by taking the dual of realization (8.37). Notice that $A_o = A_c^T, B_o = C_c^T, C_o = B_c^T$, and $D_o = D_c^T$.

Lemma 8.20. *The representation (8.41) is a minimal realization of $H(s)$ given in (8.32).*

Proof. Note that the observer form realization $\{A_o, B_o, C_o, D_o\}$ described by (8.41) is the dual of the controller form realization $\{A_c, B_c, C_c, D_c\}$ described by (8.37), used in Lemma 8.19. ■

The realization (8.41) can also be derived directly from $H(s)$, using defining relations similar to (8.34). In particular, B_o and D_o can be determined from the expression [see Subsection 6.4.2]

$$n(s) = \widetilde{S}(s)B_o + d(s)D_o, \tag{8.42}$$

where $\widetilde{S}(s) = [1, s, \ldots, s^{n-1}]$.

It can be shown (by taking transposes) that the corresponding relation to (8.39) is now given by

$$\widetilde{S}(s)(sI - A_o) = d(s)C_o \tag{8.43}$$

and that

$$d(q)z(t) = n(q)u(t), \quad y(t) = z(t) \tag{8.44}$$

corresponds to (8.38).

Figure 8.3 depicts a block realization diagram of the form (8.41) for a second-order transfer function.

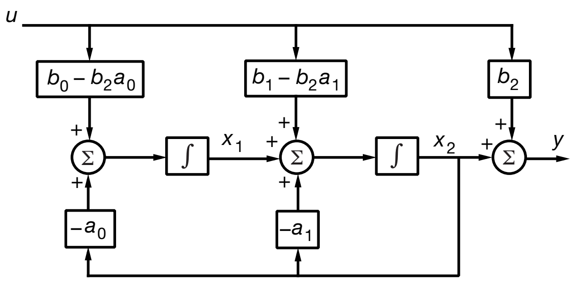

Figure 8.3. Block realization of $H(s)$ in observer form of the system $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -a_0\\ 1 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}b_0 - b_2a_0\\ b_1 - b_2a_1\end{bmatrix}u$, $y = [0,\ 1]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + b_2u$; $H(s) = \frac{b_2s^2 + b_1s + b_0}{s^2 + a_1s + a_0}$

Example 8.21. We wish to derive a minimal realization for the transfer function $H(s) = \frac{s^3+s-1}{s^3+2s^2-s-2}$. Consider a realization $\{A_c, B_c, C_c, D_c\}$, where (A_c, B_c) is in controller form. In view of (8.34) to (8.37), $D_c = \lim_{s\to\infty} H(s) = 1$ and $n(s) = s^3+s-1 = C_cS(s)+D_cd(s)$, from which we have

$C_cS(s) = (s^3+s-1)-(s^3+2s^2-s-2) = -2s^2+2s+1 = [1,2,-2][1,s,s^2]^T$. Therefore, a realization of $H(s)$ is $\dot{x}_c = A_cx_c + B_cu$, $y = C_cx_c + D_cu$, where

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_c = [1, 2, -2], \quad D_c = 1.$$

This is a minimal realization. Instead of solving $n(s) = C_cS(s) + D_cd(s)$ for C_c as was done above, it is possible to derive C_c by inspection after $H(s)$ is written as

$$H(s) = \widehat{H}(s) + \lim_{s\to\infty} H(s) = \widehat{H}(s) + D_c, \tag{8.45}$$

where $\widehat{H}(s)$ is now strictly proper. Notice that if $H(s)$ is given by (8.32), then $D_c = b_n$ and

$$\widehat{H}(s) = \frac{c_{n-1}s^{n-1} + \cdots + c_1s + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0}, \tag{8.46}$$

where in fact, $c_i = b_i - b_na_i$, $i = 0, \ldots, n-1$. The realization $\{A_c, B_c, C_c\}$ of $\widehat{H}(s)$ has (A_c, B_c) precisely the same as before; however, C_c can now be written directly as

$$C_c = [c_0, c_1, \ldots, c_{n-1}]; \tag{8.47}$$

i.e., given $H(s)$ there are three ways of determining C_c: (i) using formula (8.36), (ii) solving $C_cS(s) = n(s) - D_cd(s)$ as in (8.34), and (iii) calculating $\widehat{H}(s) = H(s) - \lim_{s\to\infty} H(s)$. The reader should verify that for this example, (i) and (iii) yield the same $C_c = [1, 2, -2]$ as in method (ii).

Suppose now that it is of interest to determine a minimal realization $\{A_o, B_o, C_o, D_o\}$, where (A_o, C_o) is in observer form. This can be accomplished in ways completely analogous to the methods used to derive realizations in controller form. Alternatively, one could use duality directly and show that

$$A_o = A_c^T = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}, B_o = C_c^T = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, C_o = B_c^T = [0, 0, 1], D_o = D_c^T = 1$$

is a minimal realization, where the pair (A_o, C_o) is in observer form.

Example 8.22. Consider now the transfer function $H(s) = \frac{s^3-1}{s^3+2s^2-s-2}$, where the numerator is $n(s) = s^3 - 1$ instead of $s^3 + s - 1$, as in Example 8.21. We wish to derive a minimal realization of $H(s)$. Using the same procedure as in the previous example, it is not difficult to derive the realization

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_c = [1, 1, -2], \quad D_c = 1.$$

This realization is controllable, since (A_c, B_c) is in controller form (see Exercise 8.4); however, it is not observable, since $\operatorname{rank}\mathcal{O} = 2 < 3 = n$, where $\mathcal{O}$ denotes the observability matrix given by

$$\mathcal{O} = \begin{bmatrix} C_c \\ C_c A_c \\ C_c A_c^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ -4 & -1 & 5 \\ 10 & 1 & -11 \end{bmatrix}.$$

Therefore, the above matrix is not a minimal realization. This has occurred because the numerator and denominator of $H(s)$ are not prime polynomials; i.e., $s-1$ is a common factor. Thus, strictly speaking, the $H(s)$ given above is not a transfer function, since it is assumed that in a transfer function all cancellations of common factors have taken place. (See also the discussion following Lemma 8.19.) Correspondingly, if the algorithm for deriving an observer form would be applied to the present case, the realization $\{A_o, B_o, C_o, D_o\}$ would be an observable realization, but not a controllable one, and would therefore not be a minimal realization.

To obtain a minimal realization of the above transfer function $H(s)$, one could either extract the part of the controllable realization $\{A_c, B_c, C_c, D_c\}$ that is also observable or simply cancel the factor $s-1$ in $H(s)$ and apply the algorithm again. The former approach of reducing a controllable realization will be illustrated when discussing the MIMO case. The latter approach is perhaps the easiest one to apply in this case. We have

$$H(s) = \frac{s^3 - 1}{s^3 + 2s^2 - s - 2} = \frac{s^2 + s + 1}{s^2 + 3s + 2} = \frac{-2s - 1}{s^2 + 3s + 2} + 1,$$

and a minimal realization of this is then determined as

$$A_c = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_c = [-1, \ -2], \quad D_c = 1.$$

Multi-Input/Multi-Output (MIMO) Systems ($pm > 1$)

Let a $(p \times m)$ proper rational matrix $H(s)$ be given with $\lim_{s\to\infty} H(s) < \infty$. We now present alogrithms to obtain realizations $\{A_c, B_c, C_c, D_c\}$ of $H(s)$ in controller form and realizations $\{A_o, B_o, C_o, D_o\}$ of $H(s)$ in observer form. Minimal realizations can then be obtained by separating the observable (controllable) part of the controllable (observable) realization.

Controller Form Realizations

Consider a transfer function matrix $H(s) = [n_{ij}(s)/d_{ij}(s)]$, $i = 1, \ldots, p$, $j = 1, \ldots, m$, and let $\ell_j(s)$ denote the (monic) least common denominator of all entries in the jth column of $H(s)$. The $\ell_j(s)$ is the least degree polynomial divisible by all $d_{ij}(s)$, $i = 1, \ldots, p$. Then $H(s)$ can be written as

$$H(s) = N(s)D^{-1}(s), \tag{8.48}$$

a ratio of two polynomial matrices, where $N(s) \triangleq [\bar{n}_{ij}(s)]$ and $D(s) \triangleq \operatorname{diag}[\ell_1(s), \ldots, \ell_m(s)]$. Note that $\bar{n}_{ij}(s)/\ell_j(s) = n_{ij}(s)/d_{ij}(s)$ for $i = 1, \ldots, p$, and all $j = 1, \ldots, m$. Let $d_j \triangleq \deg \ell_j(s)$, and assume that $d_j \geq 1$. Define

$$\Lambda(s) \triangleq \operatorname{diag}(s^{d_1}, \ldots, s^{d_m})$$

and

$$S(s) \triangleq \text{block diag}\left(\begin{bmatrix} 1 \\ s \\ \vdots \\ s^{d_j - 1} \end{bmatrix} j = 1, \ldots, m \right), \tag{8.49}$$

and note that $S(s)$ is an $n \left(\triangleq \sum_{j=1}^{m} d_j \right) \times m$ polynomial matrix. Write

$$D(s) = D_h \Lambda(s) + D_\ell S(s), \tag{8.50}$$

and note that D_h is the highest column degree coefficient matrix of $D(s)$. Here $D(s)$ is diagonal with monic polynomial entries, and therefore, $D_h = I_m$. If, for example, $D(s) = \begin{bmatrix} 3s^2 + 1 & 2s \\ 2s & s \end{bmatrix}$, then the highest column degree coefficient matrix $D_h = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, and $D_\ell S(s)$ given in (8.50) accounts for the remaining lower column degree terms in $D(s)$, with D_ℓ being a matrix of coefficients.

Observe that $|D_h| \neq 0$, and define the $m \times m$ and $m \times n$ matrices

$$B_m = D_h^{-1}, \quad A_m = -D_h^{-1} D_\ell, \tag{8.51}$$

respectively. Also, determine C_c and D_c such that

$$N(s) = C_c S(s) + D_c D(s), \tag{8.52}$$

and note that

$$D_c = \lim_{s \to \infty} H(s). \tag{8.53}$$

We have $H(s) = N(s)D^{-1}(s) = C_c S(s) D^{-1}(s) + D_c$ with $C_c S(s) D^{-1}(s)$ being strictly proper (show this). Therefore, only C_c needs to be determined from (8.52).

A controllable realization of $H(s)$ in controller form is now given by the equations

$$\dot{x}_c = A_c x_c + B_c u, \quad y = C_c x_c + D_c u.$$

Here C_c and D_c were defined in (8.52) and (8.53), respectively,

$$A_c = \bar{A}_c + \bar{B}_c A_m, \quad B_c = \bar{B}_c B_m, \tag{8.54}$$

where $\bar{A}_c = \text{block diag}[A_1, A_2 \ldots, A_m]$ with

$$A_j = \begin{bmatrix} 0 & \\ \vdots & I_{d_{j-1}} \\ & \\ 0 & 0 \cdots 0 \end{bmatrix} \in R^{d_j \times d_j},$$

$$\bar{B}_c = \text{block diag}\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R^{d_j},\ j = 1, \ldots, m \right),$$

and A_m, B_m were defined in (8.51). Note that if $d_j = \mu_j$, $j = 1, \ldots, m$, the controllability indices, then (8.54) is precisely the relation (6.56) of Section 6.4.

Lemma 8.23. *The system $\{A_c, B_c, C_c, D_c\}$ is an $n(=\sum_{j=1}^{m} d_j)$-th-order controllable realization of $H(s)$ with (A_c, B_c) in controller form.*

Proof. First, to show that $\{A_c, B_c, C_c, D_c\}$ is a realization of $H(s)$, we note that in view of the Structure Theorem given in Subsection 6.4.1, we have $C_c(sI - A_c)^{-1}B_c + D_c = \bar{N}(s)\bar{D}(s)^{-1}$, where

$$\bar{D}(s) \triangleq B_m^{-1}[\Lambda(s) - A_m S(s)], \quad \bar{N}(s) \triangleq C_c S(s) + D_c D(s).$$

However, $\bar{D}(s) = D(s)$ and $\bar{N}(s) = N(s)$, in view of (8.50) to (8.52). Therefore, $C_c(sI - A_c)^{-1}B_c + D_c = N(s)D^{-1}(s) = H(s)$, in view of (8.48).

It is now shown that (A_c, B_c) is controllable. We write

$$\begin{aligned} [sI - A_c, B_c] &= [sI - \bar{A}_c - \bar{B}_c A_m, \bar{B}_c B_m] \\ &= [sI - \bar{A}_c, \bar{B}_c] \begin{bmatrix} I & 0 \\ -A_m & B_m \end{bmatrix} \end{aligned} \tag{8.55}$$

and notice that $\text{rank}[s_j I - \bar{A}_c, \bar{B}_c] = n$ for any complex s_j. This is so because of the special form of $\bar{A}_c, \bar{B}_c$. (This is, in fact, the Brunovski canonical form.) Now since $|B_m| \neq 0$, Sylvester's Rank Inequality implies that $\text{rank}[s_j I - A_c, B_c] = n$ for any complex s_j, which in view of Section 6.3 implies that (A_c, B_c) is controllable. In addition, since $B_m = I_m$, it follows that (A_c, B_c) is of the form (6.55) of Section 6.4. With $d_j = \mu_i$, the pair (A_c, B_c) is in controller form. ■

An alternative way of determining C_c is to first write $H(s)$ in the form

$$H(s) = \widehat{H}(s) + \lim_{s \to \infty} H(s) = \widehat{H}(s) + D_c, \tag{8.56}$$

where $\widehat{H}(s) \triangleq H(s) - D_c$ is strictly proper. Now applying the above algorithm to $\widehat{H}(s)$, one obtains $\widehat{H}(s) = \widehat{N}(s)D^{-1}(s)$, where $D(s)$ is precisely equal to the expression given in (8.50). We note, however, that $\widehat{N}(s)$ is different. In fact, $\widehat{N}(s) = N(s) - D_c D(s)$. In view of (8.52) the matrix C_c is now found to be of the form

$$\widehat{N}(s) = C_c S(s). \tag{8.57}$$

Note that this is a generalization of the scalar case discussed in Example 8.21 [see (8.45) to (8.47)].

In the above algorithm the assumption that $d_j \geq 1$ for all $j = 1, \ldots, m$, was made. If for some j, $d_j = 0$, this would mean that the jth column of $H(s)$ will be a real $m \times 1$ vector that will be equal to the jth column of D_c [recall that $D_c = \lim_{s\to\infty} H(s)$]. The strictly proper $\widehat{H}(s)$ in (8.56) will then have its jth column equal to zero, and this zero column can be generated by a realization where the jth column of B_c is set to zero. Therefore, the zero column (the jth column) of $\widehat{H}(s)$ is ignored in this case and the algorithm is applied to obtain a controllable realization. A zero column is then added to B_c. (See Example 8.26 below.)

Observer Form Realizations

These realizations are dual to the controller form realizations and can be obtained by duality arguments. In the following discussion, observer form realizations are obtained directly for completeness of exposition.

We consider the transfer function matrix $H(s) = [n_{ij}(s)/d_{ij}(s)]$, $i = 1, \ldots, p$, $j = 1, \ldots, m$, and let $\tilde{\ell}_i(s)$ be the (monic) least common denominator of all entries in the ith row of $H(s)$. Then $H(s)$ can be written as

$$H(s) = \widetilde{D}^{-1}(s)\widetilde{N}(s), \tag{8.58}$$

where $\widetilde{N}(s) \triangleq [\bar{n}_{ij}(s)]$ and $\widetilde{D}(s) \triangleq \operatorname{diag}[\tilde{\ell}_1(s), \ldots, \tilde{\ell}_p(s)]$. Note that $\bar{n}_{ij}(s)/\tilde{\ell}_i(s) = n_{ij}(s)/d_{ij}(s)$ for $j = 1, \ldots, m$, and all $i = 1, \ldots, p$.

Let $\tilde{d}_i \triangleq \deg \ell_i(s)$, assume that $\tilde{d}_i \geq 1$, define

$$\tilde{\Lambda}(s) \triangleq \operatorname{diag}(s^{\tilde{d}_1}, \ldots, s^{\tilde{d}_p}), \widetilde{S}(s) \triangleq \text{block diag}([1, s, \ldots, s^{\tilde{d}_i - 1}], i = 1, \ldots, p), \tag{8.59}$$

and note that $\widetilde{S}(s)$ is a $p \times n (\triangleq \sum_{i=1}^{p} \tilde{d}_i)$ polynomial matrix. Now, write

$$\widetilde{D}(s) = \tilde{\Lambda}(s)\widetilde{D}_h + \widetilde{S}(s)\widetilde{D}_\ell \tag{8.60}$$

and note that $\widetilde{D}_h$ is the highest row degree coefficient matrix of $\widetilde{D}(s)$. Note that $\widetilde{D}(s)$ is diagonal, with entries monic polynomials, so that $\widetilde{D}_h = I_p$, the $p \times p$ identity matrix. If for example, $\widetilde{D}(s) = \begin{bmatrix} 3s^2+1 & 2s \\ 2s & s \end{bmatrix}$, then the highest row degree coefficient matrix is $\widetilde{D}_h = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$ and $\widetilde{S}(s)\widetilde{D}_\ell$ in (8.60) accounts for the remaining lower row degree terms of $\widetilde{D}(s)$, with $\widetilde{D}_\ell$ a matrix of coefficients.

Observe that $|\widetilde{D}_h| \neq 0$, in fact $\widetilde{D}_h = I_p$. Define the $p \times p$ and $n \times p$ matrices

$$C_p = \widetilde{D}_h^{-1} \quad \text{and} \quad A_p = -\widetilde{D}_\ell \widetilde{D}_h^{-1}, \tag{8.61}$$

respectively. Also, determine B_o and D_o such that

$$\widetilde{N}(s) = \widetilde{S}(s) B_o + \widetilde{D}(s) D_o. \tag{8.62}$$

Note that

$$D_o = \lim_{s \to \infty} H(s), \tag{8.63}$$

and therefore, only B_o needs to be determined from (8.62).

An observable realization of $H(s)$ in observer form is now given by

$$\dot{x}_o = A_o x_o + B_o u, \quad y = C_o x_o + D_o u,$$

where B_o and D_o were defined in (8.62) and (8.63), respectively, and

$$A_o = \bar{A}_o + A_p \bar{C}_o, \quad C_o = C_p \bar{C}_o, \tag{8.64}$$

where $\bar{A}_o = \text{block diag}[A_1, A_2, \ldots, A_p]$ with

$$A_i = \begin{bmatrix} 0 \cdots 0 & 0 \\ & 0 \\ I_{\tilde{d}_i - 1} & \vdots \\ & 0 \end{bmatrix} \in R^{\tilde{d}_i \times \tilde{d}_i},$$

$\bar{C}_o = \text{block diag}([0, \ldots, 0, 1] \in R^{1 \times \tilde{d}_i}\ i = 1, \ldots, p)$, and A_p, C_p is defined in (8.61). Note that (8.64) is exactly relation (6.76) of Section 6.4 if $\tilde{d}_i = \nu_i$, $i = 1, \ldots, p$, the observability indices.

Lemma 8.24. *The system $\{A_o, B_o, C_o, D_o\}$ is an $n(\triangleq \sum_{i=1}^p \tilde{d}_i)$-th-order observable realization of $H(s)$ with (A_o, C_o) in observer form.*

Proof. This is the dual result to Lemma 8.23. The proof is completely analogous and is omitted. ■

We conclude by noting that results dual to the results discussed after Lemma 8.23 are also valid here, i.e., results involving (i) a strictly proper $\widehat{H}(s)$, (ii) an $H(s)$ with $\tilde{d}_i = 0$ for some row i, and (iii) $H(s) = \widetilde{D}^{-1}(s)\widetilde{N}(s)$, where $\widetilde{D}(s), \widetilde{N}(s)$ are not necessarily determined using (8.58) (refer to the following examples).

Example 8.25. Let $H(s) = \left[\frac{s^2+1}{s^2}, \frac{s+1}{s^3}\right]$. We wish to derive a minimal realization for $H(s)$. To this end we consider realizations $\{A_c, B_c, C_c, D_c\}$, where (A_c, B_c) is in controller form. Here $\ell_1(s) = s^2, \ell_2(s) = s^3$ and $H(s)$ can therefore be written in the form (8.48) as

$$H(s) = N(s)D^{-1}(s) = [s^2+1, s+1]\begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix}^{-1}.$$

Here $d_1 = 2$, $d_2 = 3$ and $\Lambda(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix}$ 4, $S(s) = \begin{bmatrix} 1 & s & 0 & 0 & 0 \\ 0 & 0 & 1 & s & s^2 \end{bmatrix}^T$. Note that $n = d_1 + d_2 = 5$, and therefore, the realization will be of order 5. Write $D(s) = D_h\Lambda(s) + D_\ell S(s)$, and note that $D_h = I_2$, $D_\ell = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore, in view of (8.51),

$$B_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_m = -D_\ell = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here $D_c = \lim_{s\to\infty} H(s) = [1, 0]$ and (8.52) implies that $C_cS(s) = N(s) - D_cD(s) = [s^2+1, s+1] - [s^2, 0] = [1, s+1]$, from which we have $C_c = [1,\ 0,\ 1,\ 1,\ 0]$. A controllable realization in controller form is therefore given by $\dot{x} = A_cx_c + B_cu$ and $y = C_cx_c + D_cu$, where

$$A_c = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad B_c = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right],$$

$$C_c = [1,\ 0,\ 1,\ 1,\ 0], \quad \text{and} \quad D_c = [1, 0].$$

Note that the characteristic (pole) polynomial of $H(s)$ is s^3 and that the McMillan degree of $H(s)$ is 3. The order of any minimal realization of $H(s)$ is therefore 3. This implies that the controllable fifth-order realization derived above cannot be observable [verify that (A_c, C_c) is not observable]. To derive a minimal realization, the observable part of the system $\{A_c, B_c, C_c, D_c\}$ needs to be extracted, using the method described in Section 6.2. In particular, a transformation matrix P needs to be determined so that

$$\widehat{A} = PA_cP^{-1} = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix} \quad \text{and} \quad \widehat{C} = C_cP^{-1} = [C_1,\ 0],$$

where (A_1, C_1) is observable. If $\widehat{B} = PB_c = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, then $\{A_1, B_1, C_1, D_1\}$ is a minimal realization of $H(s)$. To reduce (A_c, C_c) to such a standard form for unobservable systems, we let $A_D = A^T, B_D = C_c^T$, and $C_D = B_c^T$ and we reduce (A_D, B_D) to a standard form for uncontrollable systems. Here the controllability matrix is

$$\mathcal{C}_D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Note that $\operatorname{rank}\mathcal{C}_D = 3$. Now if the first three columns of $Q_D = P_D^{-1}$ are taken to be the first three linearly independent columns of $\mathcal{C}_D$, whereas the rest are chosen so that $|Q_D| \neq 0$, then

$$Q_D = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q_D^{-1} = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 & 0 \end{bmatrix}.$$

This implies that

$$\widehat{A}_D = Q_D^{-1} A_D Q_D = \begin{bmatrix} A_{D1} & A_{D12} \\ 0 & A_{D2} \end{bmatrix} = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right],$$

$$\widehat{B}_D = Q_D^{-1} B_D = \begin{bmatrix} B_{D1} \\ B_{D2} \end{bmatrix} = \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right], \quad \text{and} \quad \widehat{C}_D = C_D Q_D = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then

$$\widehat{A} = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix} = \widehat{A}_D^T = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 \end{array}\right],$$

$$\widehat{B} = \widehat{C}_D^T = \left[\begin{array}{cc} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right], \quad \text{and} \quad \widehat{C} = \widehat{B}_D^T = [C_1, 0] = [1,\ 0,\ 0,\ \vdots\ 0,\ 0].$$

Clearly, $\widehat{A} = \widehat{A}_D^T, \widehat{C} = \widehat{B}_D^T$ is in standard form. Therefore, a controllable and observable realization, which is a minimal realization, is given by $\dot{x}_{co} = A_{co}x_{co} + B_{co}u$ and $y = C_{co}x_{co} + D_{co}u$, where

$$A_{co} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{co} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C_{co} = [1,\ 0,\ 0], \quad D_{co} = [1,\ 0].$$

A minimal realization could also have been derived directly in the present case if a realization $\{A_o, B_o, C_o, D_o\}$ of $H(s)$, where (A_o, B_o) is in observer form, had been considered first, as is shown next. Notice that the McMillan degree of $H(s)$ is 3, and therefore, any realization of order higher than 3 will not be minimal. Here, however, the degree of the least common denominator of the (only) row is 3, and therefore, it is known in advance that the realization in observer form, which is of order 3, will be minimal.

A realization $\{A_o, B_o, C_o, D_o\}$ of $H(s)$ in observer form can also be derived by considering $H^T(s)$ and deriving a realization in controller form. Presently, $\{A_o, B_o, C_o, D_o\}$ is derived directly. In particular, we write $H(s) = \widetilde{D}^{-1}(s)\widetilde{N}(s) = (s^3)^{-1}[s(s^2+1), s+1]$. Then $\tilde{d}_1 = 3$ [$= \deg \tilde{\ell}_1(s) = \deg s^3$] and $\tilde{\Lambda}(s) = s^3$, $\widetilde{S}(s) = [1, s, s^2]$. Then $\widetilde{D}(s) = s^3 = \tilde{\Lambda}(s)\widetilde{D}_h + \widetilde{S}(s)\widetilde{D}_\ell$ implies that $\widetilde{D}_h = 1$ and $\widetilde{D}_\ell = [0,\ 0,\ 0]^T$. In view of (8.61), we have

$$C_p = 1, \quad A_p = [0,\ 0,\ 0]^T,$$

$D_o = \lim_{s\to\infty} H(s) = [1, 0]$, and (8.62) implies that $\widetilde{S}(s)B_o = \widetilde{N}(s) - \widetilde{D}(s)D_o$ $= [s(s^2+1), s+1] - [s^3, 0] = [s, s+1]$, from which we have $B_o = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}^T$. An observable realization of $H(s)$ is the system $\dot{x} = A_o x_o + B_o u, y = C_o x_o + D_o u$, where

$$A_o = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_o = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_o = [0,\ 0,\ 1], \quad D_o = [1,\ 0],$$

with (A_o, C_o) in observer form (see Lemma 8.24). This realization is minimal since it is of order 3, which is the McMillan degree of $H(s)$. (The reader should verify this.) Note how much easier it was to derive a minimal realization, using the second approach.

Example 8.26. Let $H(s) = \begin{bmatrix} \frac{2}{s+1} & 1 \\ \frac{1}{s} & 0 \end{bmatrix}$. We wish to derive a minimal realization. Here $\ell_1(s) = s(s+1)$ with $d_1 = 2$ and $\ell_2(s) = 1$ with $d_2 = 0$. In view of the discussion following Lemma 8.23, we let $D_c = \lim_{s\to\infty} H(s) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\widehat{H}(s) = \begin{bmatrix} \frac{2}{s+1} & 0 \\ \frac{1}{s} & 0 \end{bmatrix}$. We now consider the transfer function $\widehat{\widehat{H}}(s) = \begin{bmatrix} \frac{2}{s+1} \\ \frac{1}{s} \end{bmatrix}$ and determine a minimal realization.

Note that the McMillan degree of $\widehat{\widehat{H}}(s)$ is 2, and therefore, any realization of order 2 will be minimal. Minimal realizations are now derived using two alternative approaches:

1. *Via a controller form realization.* Here $\ell_1(s) = s(s+1), d_1 = 2$, and $\widehat{\widehat{H}}(s) = \begin{bmatrix} 2s \\ s+1 \end{bmatrix} [s(s+1)]^{-1} = N(s)D^{-1}(s)$. Then $\Lambda(s) = s^2$ and $S(s) = [1,\quad s]^T, D(s) = s(s+1) = 1s^2 + [0,\ 1][1,\ s]^T = D_h\Lambda(s) + D_\ell S(s)$.

Therefore, $B_m = 1$ and $A_m = -[0,\ 1]$. Also, $C_c = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$, which follows from $N(s) = \begin{bmatrix} 2s \\ s+1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} = C_c S(s)$. Then a minimal realization for $H(s)$ is $A_c = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$, $B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_c = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$. Adding a zero column to B_c, a minimal realization of $H(s)$ is now derived as

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We ask the reader to verify that by adding a zero column to B_c, controllability is preserved.

2. *Via an observer form realization.* We consider $\widehat{\widehat{H}}^T(s) = [2/(s+1), 1/s]$ and derive a realization in controller form. In particular, $\ell_1 = s + 1, \ell_2 = s, \widehat{\widehat{H}}^T(s) = [2, 1] \begin{bmatrix} s+1 & 0 \\ 0 & s \end{bmatrix}^{-1}$, $d_1 = d_2 = 1$, $\Lambda(s) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$, and $S(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $D(s) = \begin{bmatrix} s+1 & 0 \\ 0 & s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = D_h \Lambda(s) + D_\ell S(s)$ and $B_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_m = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$. Also, $C_c = [2,\ 1]$, from which we obtain $N(s) = [2, 1] = [2,\ 1] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = C_c S(s)$. Therefore, a minimal realization $\{A, B, C\}$ of $\widehat{\widehat{H}}^T(s)$ is $\left\{ \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, [2, 1] \right\}$. The dual of this is a minimal realization of $\widehat{\widehat{H}}(s)$, namely, $A_o = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$, $B_o = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $C_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore, a minimal realization of $H(s)$ is

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

8.4.3 Realizations with Matrix A Diagonal

When the roots of the minimal polynomial $m_H(s)$ of $H(s)$ are distinct, there is a realization algorithm that provides a minimal realization of $H(s)$ with A diagonal. Let

$$m_H(s) = s^r + d_{r-1}s^{r-1} + \cdots + d_1 s + d_0 \tag{8.65}$$

be the (monic) least common denominator of all nonzero entries of the $p \times m$ matrix $H(s)$, which in view of Section 7.4, is the minimal polynomial of $H(s)$. We assume that its r roots λ_i are distinct, and we write

$$m_H(s) = \prod_{i=1}^{r}(s - \lambda_i). \tag{8.66}$$

Note that the pole polynomial of $H(s), p_H(s)$, will have repeated roots (poles) if $p_H(s) \neq m_H(s)$. We now consider the strictly proper matrix $\widehat{H}(s) \triangleq H(s) - \lim_{s\to\infty} H(s) = H(s) - D$, and we expand it into partial fractions to obtain

$$\widehat{H}(s) = \widehat{N}(s)/m_H(s) = \sum_{i=1}^{r} \frac{1}{s-\lambda_i} R_i. \tag{8.67}$$

The $p \times m$ residue matrices R_i can be found from the relation

$$R_i = \lim_{s\to\lambda_i} (s - \lambda_i)\widehat{H}(s). \tag{8.68}$$

We write

$$R_i = C_i B_i, \quad i = 1, \dots, r, \tag{8.69}$$

where C_i is a $p \times \rho_i$ and B_i is a $\rho_i \times m$ matrix with $\rho_i \triangleq \operatorname{rank} R_i \leq \min(p, m)$. Note that the above expression is always possible. Indeed, there is a systematic procedure of generating it, namely, by obtaining an LU decomposition of R_i. Then

$$A = \begin{bmatrix} \lambda_1 I_{\rho_1} & & & \\ & \lambda_2 I_{\rho_2} & & \\ & & \ddots & \\ & & & \lambda_r I_{\rho_r} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_r \end{bmatrix}, \tag{8.70}$$
$$C = [C_1, C_2, \dots, C_r], \quad D = \lim_{s\to\infty} H(s)$$

is a minimal realization of order $n \triangleq \sum_{i=1}^{r} \rho_i$.

Lemma 8.27. *Representation (8.70) is a minimal realization of $H(s)$.*

Proof. It can be verified directly that $C(sI - A)^{-1}B + D = H(s)$, i.e., that (8.70) is a realization of $H(s)$. To verify controllability, we write

$$\mathcal{C} = [B, AB, \dots, A^{n-1}B] = \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_r \end{bmatrix} \begin{bmatrix} I_m, & \lambda_1 I_m, & \dots, & \lambda_1^{n-1} I_m \\ \vdots & \vdots & & \vdots \\ I_m, & \lambda_r I_m, & \dots, & \lambda_r^{n-1} I_m \end{bmatrix}.$$

The second matrix in the product is a block Vandermonde matrix of dimensions $mr \times mn$. It can be shown that this matrix has full rank mr since all λ_i are assumed to be distinct. Also note that the $(n = \Sigma\rho_i) \times mr$ matrix block diag$[B_i]$ has rank equal to $\sum_{i=1}^{r} \operatorname{rank} B_i = \sum_{i=1}^{r} \rho_i = n \leq mr$. Now, in view of Sylvester's Rank Inequality, as applied to the above matrix product, we have $n + mr - mr \leq \operatorname{rank} \mathcal{C} \leq \min(n, mr)$, from which $\operatorname{rank} \mathcal{C} = n$. Therefore, $\{A, B, C, D\}$ is controllable. Observability is shown in a similar way. Therefore, representation (8.70) is minimal. ∎

Example 8.28. Let $H(s) = \begin{bmatrix} \frac{1}{s} & 0 \\ \frac{2}{s+1} & \frac{1}{s(s+1)} \end{bmatrix}$. Here $m_H(s) = s(s+1)$ with roots $\lambda_1 = 0, \lambda_2 = -1$ distinct. We write $H(s) = \frac{1}{s}R_1 + \frac{1}{s+1}R_2$, where $R_1 = \lim_{s\to 0} sH(s) = \lim_{s\to 0} \begin{bmatrix} 1 & 0 \\ \frac{2s}{s+1} & \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R_2 = \lim_{s\to -1}(s+1)H(s) = \lim_{s\to -1} \begin{bmatrix} \frac{s+1}{s} & 0 \\ 2 & \frac{1}{s} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix}$, $\rho_1 = \operatorname{rank} R_1 = 2$, and $\rho_2 = \operatorname{rank} R_2 = 1$; i.e., the order of a minimal realization is $n = \rho_1 + \rho_2 = 3$. We now write

$$R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = C_1 B_1,$$

$$R_2 = \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} [2 \ -1] = C_2 \ B_2.$$

Then

$$A = \begin{bmatrix} \lambda_1 I_2 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{bmatrix},$$

$$C = [C_1, \ C_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

is a minimal realization with A diagonal (show this). Note that the characteristic polynomial of $H(s)$ is $p_H(s) = s^2(s+1)$, and therefore, the McMillan degree, which is equal to the order of any minimal realization, is 3, as expected.

8.4.4 Realizations Using Singular-Value Decomposition

Internally Balanced Realizations. Given a proper $p \times m$ matrix $H(s)$, we let r denote the degree of its minimal polynomial $m_H(s)$, we write

$$H(s) = H_0 + H_1 s^{-1} + H_2 s^{-2} + \dots$$

to obtain the Markov parameters H_i, and we define

$$T \triangleq M_H(r,r) = \begin{bmatrix} H_1 \cdots H_r \\ \vdots \\ H_r \cdots H_{2r-1} \end{bmatrix}, \widehat{T} \triangleq \begin{bmatrix} H_2 \cdots H_{r+1} \\ \vdots \\ H_{r+1} \cdots H_{2r} \end{bmatrix}, \tag{8.71}$$

where $M_H(r,r)$ is the Hankel matrix (see Definition 8.15) and $T, \widehat{T}$ are real matrices of dimension $rp \times rm$.

Using *singular-value decomposition* (see Section A.9), we write

$$T = K \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} L, \tag{8.72}$$

where $\sum = \text{diag}[\lambda_1, \ldots, \lambda_n] \in R^{n \times n}$ with $n = \text{rank}\, T = \text{rank}\, M_H(r, r)$, which in view of Theorem 8.16 is the order of a minimal realization of $H(s)$. The λ_i with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ are the singular values of T, i.e., the nonzero eigenvalues of $T^T T$. Furthermore, $KK^T = K^T K = I_{pr}$ and $LL^T = L^T L = I_{mr}$. We write

$$T = K_1 \Sigma L_1 = (K_1 \Sigma^{1/2})(\Sigma^{1/2} L_1) = VU, \tag{8.73}$$

where K_1 denotes the first n columns of K, L_1 denotes the first n rows of L, $K_1^T K_1 = I_n$, and $L_1 L_1^T = I_n$. Also, $V \in R^{rp \times n}$ and $U \in R^{n \times rm}$.

We let V^+ and U^+ denote pseudoinverses of V and U, respectively (see the appendix); i.e.,

$$V^+ = \Sigma^{-1/2} K_1^T \quad \text{and} \quad U^+ = L_1^T \Sigma^{-1/2}, \tag{8.74}$$

where $V^+ V = I_n$ and $UU^+ = I_n$. Now define

$$A = V^+ \widehat{T} U^+, \quad B = U I_{m,mr}^T, \quad C = I_{p,pr} V, \quad D = H_0, \tag{8.75}$$

where $I_{k,\ell} \triangleq [I_k, 0_{\ell-k}]$, $k > \ell$; i.e., $I_{k,\ell}$ is a $k \times \ell$ matrix with its first k columns determining an identity matrix and the remaining $\ell - k$ columns being equal to zero. Thus, B is defined as the first m columns of U, and C is defined as the first p rows of V. Note that $A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{p \times n}$, and $D \in R^{p \times m}$.

Lemma 8.29. *The representation (8.75) is a minimal realization of $H(s)$.*

Proof. It can be shown that $CA^{i-1}B = H_i$, $i = 1, 2, \ldots$. Thus, $\{A, B, C, D\}$ is a realization. We note that V and U are the observability and controllability matrices, respectively, and that both are of full rank n. Therefore, the realization is minimal. Furthermore, we notice that $V^T V = UU^T = \Sigma$. Realizations of this type are called *internally balanced realizations.* ■

The term *internally balanced* emphasizes the fact that realizations of this type are "as much controllable as they are observable," since their controllability and observability Gramians are equal and diagonal. Using such representations, it is possible to construct reasonable reduced-order models of systems by deleting that part of the state space that is "least controllable" and therefore "least observable" in accordance with some criterion. In fact, the realization procedure described can be used to obtain a *reduced-order model* for a given system. Specifically, if the system is to be approximated by a q-dimensional model with $q < n$, then the reduced-order model can be obtained from

$$T = K_q \,\text{diag}[\lambda_1, \ldots, \lambda_q] L_q, \tag{8.76}$$

where K_q denotes the first q columns of K in (8.72) and L_q denotes the first q rows of L.

8.5 Polynomial Matrix Realizations

It is rather straightforward to derive a realization of H in PMD form [see Section 7.5]. In fact, realizations in right (left) Polynomial Matrix Fractional Description (PMFD) form were derived as a step toward determining a state-space realization in controller (observer) form (see Subsection 8.4.2). However, these realizations, of the form $\{D_R, I_m, N_R\}$ and $\{D_L, N_L, I_p\}$, are typically not of minimal order; i.e., they are not controllable and observable. This implies that the controllable realization $\{D_R, I_m, N_R\}$, for example, is not observable; i.e., D_R, N_R are not right coprime. Similarly, the observable realization $\{D_L, N_L, I_p\}$ is not controllable; i.e., D_L, N_L are not left coprime. To obtain a minimal realization, a greatest common right divisor (gcrd) must be extracted from D_R, N_R, and similarly, a gcld must be extracted from D_L, N_L. This leads to the following realization algorithm, which results in a minimal realization $\{D, I_m, N\}$ of H. A minimal realization of the form $\{D, N, I_p\}$ is obtained in an analogous (dual) manner.

Consider $H(s) = [n_{ij}(s)/d_{ij}(s)]$, $i = 1, \dots, p$, $j = 1, \dots, m$, and let $l_j(s)$ be the (monic) least common denominator of all entries in the jth column of $H(s)$. Note that $l_j(s)$ is the (monic) least degree polynomial divisible by all $d_{ij}(s)$, $i = 1, \dots, p$. Then $H(s)$ can be written as

$$H(s) = N_R(s) D_R^{-1}(s), \tag{8.77}$$

where $N_R(s) \triangleq =[\bar{n}_{ij}(s)]$ and $D_R(s) \triangleq = \mathrm{diag}(l_1(s), \dots, l_m(s))$. Note that $\bar{n}_{ij}/l_j(s) = n_{ij}(s)/d_{ij}(s)$ for $i = 1, \dots, p$ and all $j = 1, \dots, m$. Now

$$D_R(q) z_R(t) = u(t), \quad y(t) = N_R(q) z_R(t) \tag{8.78}$$

is a controllable realization of $H(s)$. If D_R, N_R are right coprime, it is observable as well and therefore minimal. If D_R and N_R are not right coprime, let G_R be a greatest common right divisor (gcrd) and let $D = D_R G_R^{-1}$ and $N = N_R G_R^{-1}$. Then

$$D(q) z(t) = u(t), \quad y(t) = N(q) z(t) \tag{8.79}$$

is a controllable and observable, and therefore, minimal realization of $H(s)$ since D, I and D, N are left and right coprime polynomial matrix pairs, respectively. Note that $ND^{-1} = (N_R G_R^{-1})(D_R G_R^{-1})^{-1} = (N_R G_R^{-1})(G_R D_R^{-1}) = N_R D_R^{-1} = H$.

There is a dual algorithm that extracts a left PMFD resulting in

$$H(s) = D_L^{-1}(s) N_L(s), \tag{8.80}$$

which corresponds to an observable realization of $H(s)$, given by

$$D_L(q) z_L(t) = N_L(q) u(t), y(t) = z_L(t). \tag{8.81}$$

The details of this procedure are completely analogous to the above procedure that led to (8.77). If D_L, N_L are not left coprime, let G_L be a greatest common left divisor and let $\widetilde{D} = G_L^{-1} D_L$ and $\widetilde{N} = G_L^{-1} N_L$. Then a controllable and observable and, therefore, minimal realization of $H(s)$ is given by

$$\widetilde{D}(q)\tilde{z}(t) = \widetilde{N}(q)u(t), \quad y(t) = \tilde{z}(t). \tag{8.82}$$

The following example illustrates the above realization algorithms.

Example 8.30. Let us derive a minimal realization for $H(s) = \left[\frac{s^2+1}{s^2}, \frac{s+1}{s^3}\right]$. Note that this is the same $H(s)$ as in Example 8.25 of Section 8.4.2, where minimal state-space realizations were derived. The reader is encouraged to compare those results with the realizations derived below. We shall begin with a controllable realization. In view of (8.77) $l_1 = s^2, l_2 = s^3$, and $H = N_R D_R^{-1} = [s^2+1, s+1]\begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix}^{-1}$. Therefore, $D_R z_R = u$ and $y = N_R z_R$ constitute a controllable realization. This realization is not observable since $\operatorname{rank}\begin{bmatrix} D_R(s) \\ N_R(s) \end{bmatrix}_{s=0} = \operatorname{rank}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = 1 < m = 2$; i.e., D_R and N_R are not right coprime.

Another way of determining that D_R and N_R are not right coprime would have been to observe that $\deg\det D(s) = 5 =$ order of the realization $\{D_R, I, N_R\}$. Now the McMillan degree of H, which is easily derived in the present case, is three. Therefore, the order of any minimal realization for this example is three. Since $\{D_R, I, N_R\}$ is of order five and is controllable, it cannot be observable; i.e., D_R and N_R cannot be right coprime.

We shall now extract a gcrd from D_R and N_R (using the procedure described in [1, Section 7.2D]). We have

$$\begin{bmatrix} D_R \\ N_R \end{bmatrix} = \begin{bmatrix} s^2 & 0 \\ 0 & s^3 \\ s^2+1 & s+1 \end{bmatrix} \longrightarrow \begin{bmatrix} s^2 & 0 \\ 0 & s^3 \\ 1 & s+1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & s+1 \\ s^2 & 0 \\ 0 & s^3 \end{bmatrix} \longrightarrow$$
$$\begin{bmatrix} 1 & s+1 \\ 0 & -s^3-s^2 \\ 0 & s^3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & s+1 \\ 0 & s^2 \\ 0 & s^3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & s+1 \\ 0 & s^2 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $G_R = \begin{bmatrix} 1 & s+1 \\ 0 & s^2 \end{bmatrix}$ is a gcrd. We now determine $D = D_R G_R^{-1}$ and $N = N_R G_R^{-1}$, using $D_R = \begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix} = \begin{bmatrix} s^2 & -(s+1) \\ 0 & s \end{bmatrix}\begin{bmatrix} 1 & s+1 \\ 0 & s^2 \end{bmatrix} = DG_R$ and $N_R = [s^2+1, s+1] = [s^2+1, -(s+1)]\begin{bmatrix} 1 & s+1 \\ 0 & s^2 \end{bmatrix} = NG_R$, and we verify that they are right coprime. Then

$$\{D_R, I, N_R\} = \left\{ \begin{bmatrix} q^2 & -(q+1) \\ 0 & q \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, [q^2+1, -(q+1)] \right\}$$

is a minimal realization of $H(s)$.

Alternatively, given H, we shall first derive an observable realization. In view of (8.80),

$$H = D_L^{-1} N_L = (s^3)^{-1}[s(s^2+1), s+1].$$

Here $D_L(q)$ and $N_L(q)$ are left coprime, and therefore, $\widetilde{D}(q)\tilde{z}(t) = \widetilde{N}(q)u(t)$ and $y(t) = \tilde{z}(t)$ with $\widetilde{D}(q) = D_L(q)$ and $\widetilde{N}(q) = N_L(q)$ is controllable and observable and is a minimal realization. Note that the order of this realization is $\deg \det D_L(s) = 3$, which equals the McMillan degree of $H(s)$.

8.6 Summary and Highlights

Realizations

- $\dot{x} = Ax + Bu$, $y = Cx + Du$ is a realization of $H(s)$ ($\hat{y} = H(s)\hat{u}$) if $\hat{y} = [C(sI - As^{-1}B + D]\hat{u}$.
- Realizations of $H(s)$ exist if and only if $H(s)$ is a proper rational matrix. $\lim_{s\to\infty} H(s) = D < \infty$. (See Theorem 8.5.)
- The Markov parameters H_i of the system in

  $$H(s) = H_0 + H_1 s^{-1} + H_2 s^{-2} + \ldots$$

  satisfy

  $$H_0 = D \text{ and } H_i = CA^{i-1}B, \quad i = 1, 2, \ldots.$$

  (See Theorem 8.3.)
- A realization $\{A, B, C, D\}$ of $H(s)$ is minimal if and only if it is both controllable and observable. (See Theorem 8.9.)
- Two realizations of $H(s)$ that are minimal must be equivalent representations. (See Theorem 8.10.)
- The order of a minimal realization of $H(s)$ equals its McMillan degree, the order of its characteritic or pole polynomial $p_H(s)$. (See Theorem 8.12.) The order of a minimal realization of $H(s)$ is also given by the rank of the Hankel matrix $M_H(r, r)$. (See Theorem 8.16.)
- Duality can be very useful in obtaining realizations. (See Subsection 8.4.1.)
- Realization algorithms are presented to obtain realizations in controller/observer form [Subsection 8.4.2], realizations with A diagonal [Subsection 8.4.3], and balanced realizations via singular-value decomposition [Subsection 8.4.4].

8.7 Notes

A clear understanding of the relationship between external and internal descriptions of systems is one of the principal contributions of systems theory. This topic was developed in the early sixties with original contributions by Gilbert [3] and Kalman [5]. The role of controllability and observability in minimal realizations is due to Kalman [5]. See also Kalman, Falb, and Arbib [6]. The first realization method for MIMO systems is attributed to Gilbert [3]. It was developed for systems where the matrix A can be taken to be diagonal. This method is presented in this chapter. For extensive historical comments concerning this topic, see Kailath [4]. Additional information concerning realizations for the time-varying case can be found, for example, in Brockett [2], Silverman [10], Kamen [7], Rugh [9], and the literature cited in these references. Balanced realizations were introduced in Moore [8].

References

1. P.J. Antsaklis and A.N. Michel, *Linear Systems*, Birkhäuser, Boston, MA, 2006.
2. R.W. Brockett, *Finite Dimensional Linear Systems*, Wiley, New York, NY, 1970.
3. E. Gilbert, "Controllability and observability in multivariable control systems," *SIAM J. Control*, Vol. 1, pp. 128–151, 1963.
4. T. Kailath, *Linear Systems*, Prentice Hall, Englewood Cliffs, NJ, 1980.
5. R.E. Kalman, "Mathematical description of linear systems," *SIAM J. Control*, Vol. 1, pp. 152–192, 1963.
6. R.E. Kalman, P.L. Falb, and M.A. Arbib, *Topics in Mathematical System Theory*, McGraw-Hill, New York, NY, 1969.
7. E.W. Kamen, "New results in realization theory for linear time-varying analytic systems," *IEEE Trans. Auto. Control*, Vol. AC-24, pp. 866–877, 1979.
8. B.C. Moore, "Principal component analysis in linear systems: controllability, observability and model reduction," *IEEE Trans. Auto. Control*, Vol. AC-26, pp. 17–32, 1981.
9. W.J. Rugh, *Linear System Theory, Second Edition*, Prentice-Hall, Englewood Cliffs, NJ, 1996.
10. L.M. Silverman, "Realization of linear dynamical systems," *IEEE Trans. Auto. Control*, Vol. AC-16, pp. 554–567, 1971.

Exercises

8.1. Consider a scalar proper rational transfer function $H(s) = n(s)/d(s)$, and let $\dot{x} = A_c x_c + B_c u$, $y = C_c x_c + D_c u$ be a realization of $H(s)$ in controller form.

(a) Show that the realization $\{A_c, B_c, C_c, D_c\}$ is always controllable.

(b) Show that $\{A_c, B_c, C_c, D_c\}$ is observable if and only if $n(s)$ and $d(s)$ do not have any factors in common; i.e., they are prime polynomials.
(c) State the dual results to (a) and (b) involving a realization in observer form.

8.2. Let $H(s) = \frac{n(s)}{d(s)} = \frac{s^2-s+1}{s^5-s^4+s^3-s^2+s-1}$. Determine a realization in controller form. Is your realization minimal? Explain your answer. *Hint*: Use the results of Exercise 8.1.

8.3. For the transfer function $H(s) = \frac{s+1}{s^2+2}$, find

(a) an uncontrollable realization,
(b) an unobservable realization,
(c) an uncontrollable and unobservable realization,
(d) a minimal realization.

8.4. Consider the transfer function matrix $H(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{1}{s^2-1} \\ 1 & 0 \end{bmatrix}$.

(a) Determine the pole polynomial and the McMillan degree of $H(s)$, using both the Smith–McMillan form and the Hankel matrix.
(b) Determine an observable realization of $H(s)$.
(c) Determine a minimal realization of $H(s)$. *Hint*: Obtain realizations for $\left[\frac{s-1}{s+1}, \frac{1}{s^2-1}\right]$.

8.5. Consider the transfer function matrix $H(s) = \left[\frac{(s+1)(-s+5)}{(s-1)(s^2-9)}, \frac{s}{s-1}\right]^T$, and determine for $H(s)$ a minimal realization in controller form.

8.6. Consider the transfer function $H(s) = \begin{bmatrix} \frac{1}{s} & \frac{s+3}{s+1} \\ \frac{1}{s+3} & \frac{s}{s+1} \end{bmatrix}$.

(a) Determine the pole polynomial of $H(s)$ and the McMillan degree of $H(s)$.
(b) Determine a minimal realization $\{A, B, C, D\}$ of $H(s)$, where A is a diagonal matrix.

8.7. Given is the system depicted in the block diagram of Figure 8.4, where $H(s) = \frac{s^2+1}{(s+1)(s+2)(s+3)}$. Determine a minimal state-space representation for the closed-loop system, using two approaches. In particular:

(a) First, determine a state-space realization for $H(s)$, and then, determine a minimal state-space representation for the closed-loop system;
(b) first, find the closed-loop transfer function, and then, determine a minimal state-space representation for the closed-loop system.

Compare the two approaches.

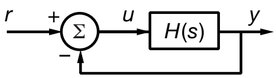

Figure 8.4. Block diagram of the system in Exercise 8.7

8.8. Consider the system depicted in the block diagram of Figure 8.5, where $H(s) = \frac{s+1}{s(s+3)}$ and $G(s) = \frac{k}{s+a}$ with $k, a \in R$. Presently, $H(s)$ could be viewed as the system to be controlled and $G(s)$ could be regarded as a feedback controller.

(a) Obtain a state-space representation of the closed-loop system by
   (i) first, determining realizations for $H(s)$ and $G(s)$ and then combining them;
   (ii) first, determining $H_c(s)$, the closed-loop transfer function.
(b) Are there any choices for the parameters k and a for which your closed-loop state-space representation is uncontrollable *and* unobservable? If your answer is affirmative, state why.

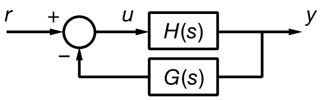

Figure 8.5. Block diagram of the system in Exercise 8.8

8.9. Consider the controllable and observable system given by $\dot{x} = Ax + Bu$, $y = Cx + Du$, and its equivalent representation $\dot{\hat{x}} = \widehat{A}\hat{x} + \widehat{B}u$, $y = \widehat{C}\hat{x} + \widehat{D}u$, where $\widehat{A} = PAP^{-1}$, $\widehat{B} = PB$, $\widehat{C} = CP^{-1}$, and $\widehat{D} = D$. Let W_r and W_o denote the reachability and observability Gramians, respectively.

(a) Show that $\widehat{W_r} = PW_rP^*$ and $\widehat{W_o} = (P^{-1})^* W_o P^{-1}$, where P^* denotes the complex conjugate transpose of P. Note that $P^* = P^T$ when only real coefficients in the system equations are involved.
   Using singular-value decomposition (refer to Section A.9), write

$$W_r = U_r \Sigma_r V_r^* \quad \text{and} \quad W_o = U_o \Sigma_o V_o^*,$$

   where $U^*U = I$, $VV^* = I$, and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ with σ_i the singular values of W. Define

$$H = (\Sigma_o^{1/2})^* U_o^* U_r (\Sigma_r^{1/2}),$$

   and using singular-value decomposition, write

$$H = U_H \Sigma_H V_H,$$

   where $U_H^* U_H = I$, $V_H V_H^* = I$. Prove the following:

(b) If $P = P_{\text{in}} \triangleq V_H(\sum_r^{1/2})^{-1}V_r^*$, then $\widehat{W}_r = I$, $\widehat{W}_o = \Sigma_H^2$.
(c) If $P = P_{\text{out}} \triangleq U_H^*(\sum_o^{1/2})^* V_o^*$, then $\widehat{W}_r = \Sigma_H^2$, $\widehat{W}_o = I$.
(d) If $P = P_{ib} = P_{\text{in}} \sum_H^{1/2} = \sum_H^{1/2} P_{\text{out}}$, then $\widehat{W}_r = \widehat{W}_o = \Sigma_H$. Note that the equivalent representations $\{\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}\}$ in (b), (c), and (d) are called, respectively, *input-normal, output-normal, and internally balanced representations.*

8.10. Consider a system described by

$$\begin{bmatrix} \hat{y}_1(s) \\ \hat{y}_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{2}{s^2} \\ 0 & \frac{s+1}{s} \end{bmatrix} \begin{bmatrix} \hat{u}_1(s) \\ \hat{u}_2(s) \end{bmatrix}.$$

(a) What is the order of a controllable and observable realization of this system?
(b) If we consider such a realization, is the resulting system controllable from the input u_2? Is it observable from the output y_1? Explain your answers.

8.11. Consider the system described by $H(s) = \frac{1}{s-(1+\epsilon)}(\hat{y}(s) = H(s)\hat{u}(s))$ and $C(s) = \frac{s-1}{s+2}(\hat{u}(s) = C(s)\hat{r}(s))$ connected in series ($\epsilon \in R$).

(a) Derive minimal state-space realizations for $H(s)$ and $C(s)$, and determine a (second order) state-space description for the system $\hat{y}(s) = H(s)C(s)\hat{r}(s)$.
(b) Let $\epsilon = 0$, and discuss the implications regarding the overall transfer function and your state-space representations in (a). Is the overall system now controllable, observable, asymptotically stable? Are the poles of the overall transfer function stable? [That is, is the overall system BIBO stable? (See Chapter 4.)] Plot the states and the output for some nonzero initial condition and a unit step input, and comment on your results.
(c) In practice, if $H(s)$ is a given system to be controlled and $C(s)$ is a controller, it is unlikely that ϵ will be exactly equal to zero and therefore the situation in (a), rather than (b), will arise. In view of this, comment on whether open-loop stabilization can be used in practice. Carefully explain your reasoning.

8.12. Consider the transfer function $H(s) = \begin{bmatrix} 1 & \frac{1}{s} & \frac{s-1}{s} \\ 0 & \frac{s+1}{s^2} & 0 \end{bmatrix}$. Determine a minimal realization in

(a) Polynomial Matrix Fractional Description (PMFD) form,
(b) State-Space Description (SSD) form.