3

Response of Continuous- and Discrete-Time Systems

3.1 Introduction

In system theory it is important to clearly understand how inputs and initial conditions affect the response of a system. Many reasons exist for this. For example, in control theory, it is important to be able to select an input that will cause the system output to satisfy certain properties [e.g., to remain bounded (stability) or to follow a given trajectory (tracking)]. This is in stark contrast to the study of ordinary differential equations, where it is usually assumed that the forcing function (input) is given.

The goal of this chapter is to study the response of linear systems in greater detail than was done in Chapter 2. To this end, solutions of linear ordinary differential equations are reexamined, this time with an emphasis on characterizing all solutions using bases (of the solution vector space) and on determining such solutions. For convenience, certain results from Chapter 2 are repeated. We will find it convenient to treat continuous-time and discrete-time cases separately. Whereas in Chapter 2, certain fundamental issues that include input–output system descriptions, causality, linearity, and time-invariance are emphasized, here we will address in greater detail impulse (and pulse) response and transfer functions for continuous-time systems and discrete-time systems.

In Chapters 1 and 2 we addressed linear as well as nonlinear systems that may be time-varying or time-invariant. We considered this level of generality since this may be mandated during the modeling process of the systems. However, in the analysis and synthesis of such systems, simplified models involving linear time-invariant systems usually suffice. Accordingly, in the remainder of this book, we will emphasize linear, time-invariant continuous-time and discrete-time systems.

In this chapter, in Section 3.2, we further study linear systems of ordinary differential equations with constant coefficients. Specifically, in this section, we develop a general characterization of the solutions of such equations and we study the properties of the solutions by investigating the properties of fundamental matrices and state transition matrices. In Section 3.3 we address several methods of determining the state transition matrix and we study the asymptotic behavior of the solutions of such systems. In Sections 3.4 and 3.5, we further investigate the properties of the state representations and the input–output representations of continuous-time and discrete-time finite-dimensional systems. Specifically, in these sections we study equivalent representations of such systems, we investigate the properties of transfer function matrices, and for the discrete-time case we also address sampled data systems and the asymptotic behavior of the system response of time-invariant systems.

3.2 Solving $\dot{x} = Ax$ and $\dot{x} = Ax + g(t)$: The State Transition Matrix $\Phi(t, t_0)$

In this section we consider systems of linear homogeneous ordinary differential equations with constant coefficients.

$$\dot{x} = Ax \tag{3.1}$$

and linear nonhomogeneous ordinary differential equations

$$\dot{x} = Ax + g(t). \tag{3.2}$$

In Theorem 1.20 of Chapter 1 it was shown that these systems of equations, subject to initial conditions $x(t_0) = x_0$, possess unique solutions for every $(t_0, x_0) \in D$, where $D = \{(t, x) : t \in J = (a, b),\ x \in R^n\}$ and where it is assumed that $A \in R^{n\times n}$ and $g \in C(J, R^n)$. These solutions exist over the entire interval $J = (a, b)$, and they depend continuously on the initial conditions. Typically, we will assume that $J = (-\infty, \infty)$. We note that $\phi(t) \equiv 0$, for all $t \in J$, is a solution of (3.1), with $\phi(t_0) = 0$. We call this the *trivial solution*. As in Chapter 1 (refer to Section 1.8), we recall that the preceding statements are also true when $g(t)$ is piecewise continuous on J.

In the sequel, we sometimes will encounter the case where A is in Jordan canonical form that may have entries in the complex plane C. For this reason, we will allow $D = \{(t, x) : t \in J = (a, b),\ x \in R^n$ (or $x \in C^n$)$\}$ and $A \in R^{n\times n}$ [or $A \in C^{n\times n}$], as needed. For the case of real vectors, the field of scalars for the x-space will be the field of real numbers ($F = R$), whereas for the case of complex vectors, the field of scalars for the x-space will be the field of complex numbers ($F = C$). For the latter case, the theory concerning the existence and uniqueness of solutions for (3.1), as presented in Chapter 1, carries over and can be modified in the obvious way.

3.2.1 The Fundamental Matrix

Solution Space

In our first result we will make use of several facts concerning vector spaces, bases, and linear spaces, which are addressed in the appendix.

Theorem 3.1. *The set of solutions of (3.1) on the interval J forms an n-dimensional vector space.*

Proof. Let V denote the set of all solutions of (3.1) on J. Let $\alpha_1, \alpha_2 \in F$ ($F = R$ or $F = C$), and let $\phi_1, \phi_2 \in V$. Then $\alpha_1\phi_1 + \alpha_2\phi_2 \in V$ since $\frac{d}{dt}[\alpha_1\phi_1 + \alpha_2\phi_2] = \alpha_1\frac{d}{dt}\phi_1(t) + \alpha_2\frac{d}{dt}\phi_2(t) = \alpha_1 A\phi_1(t) + \alpha_2 A\phi_2(t) = A[\alpha_1\phi_1(t) + \alpha_2\phi_2(t)]$ for all $t \in J$. [Note that in this time-invariant case, it can be assumed without loss of generality, that $J = (-\infty, \infty)$.] This shows that V is a vector space.

To complete the proof of the theorem, we must show that V is of dimension n. To accomplish this, we must find n linearly independent solutions $\phi_1, \dots, \phi_n$ that span V. To this end, we choose a set of n linearly independent vectors $x_0^1, \dots, x_0^n$ in the n-dimensional x-space (i.e., in R^n or C^n). By the existence results in Chapter 1, if $t_0 \in J$, then there exist n solutions $\phi_1, \dots, \phi_n$ of (3.1) such that $\phi_1(t_0) = x_0^1, \dots, \phi_n(t_0) = x_0^n$. We first show that these solutions are linearly independent. If on the contrary, these solutions are linearly dependent, there exist scalars $\alpha_1, \dots, \alpha_n \in F$, not all zero, such that $\sum_{i=1}^n \alpha_i\phi_i(t) = 0$ for all $t \in J$. This implies in particular that $\sum_{i=1}^n \alpha_i\phi_i(t_0) = \sum_{i=1}^n \alpha_i x_0^i = 0$. But this contradicts the assumption that $\{x_0^1, \dots, x_0^n\}$ is a linearly independent set. Therefore, the solutions $\phi_1, \dots, \phi_n$ are linearly independent.

To conclude the proof, we must show that the solutions $\phi_1, \dots, \phi_n$ span V. Let ϕ be any solution of (3.1) on the interval J such that $\phi(t_0) = x_0$. Then there exist unique scalars $\alpha_1, \dots, \alpha_n \in F$ such that

$$x_0 = \sum_{i=1}^n \alpha_i x_0^i,$$

since, by assumption, the vectors $x_0^1, \dots, x_0^n$ form a basis for the x-space. Now

$$\psi = \sum_{i=1}^n \alpha_i \phi_i$$

is a solution of (3.1) on J such that $\psi(t_0) = x_0$. But by the uniqueness results of Chapter 1, we have that

$$\phi = \psi = \sum_{i=1}^n \alpha_i \phi_i.$$

Since ϕ was chosen arbitrarily, it follows that $\phi_1, \dots, \phi_n$ span V. ■

Fundamental Matrix and Properties

Theorem 3.1 enables us to make the following definition.

Definition 3.2. *A set of n linearly independent solutions of (3.1) on J, $\{\phi_1, \dots, \phi_n\}$, is called a* fundamental set of solutions *of (3.1), and the $n \times n$ matrix*

$$\Phi = [\phi_1, \phi_2, \ldots, \phi_n] = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix}$$

is called a fundamental matrix *of (3.1).* ■

We note that there are infinitely many different fundamental sets of solutions of (3.1) and, hence, infinitely many different fundamental matrices for (3.1). Clearly $[\phi_1, \phi_2, \ldots, \phi_n]$ is a basis of the solution space. We now study some of the basic properties of a fundamental matrix.

In the next result, $X = [x_{ij}]$ denotes an $n \times n$ matrix, and the derivative of X with respect to t is defined as $\dot{X} = [\dot{x}_{ij}]$. Let A be the $n \times n$ matrix given in (3.1). We call the system of n^2 equations

$$\dot{X} = AX \tag{3.3}$$

a *matrix differential equation.*

Theorem 3.3. *A fundamental matrix Φ of (3.1) satisfies the matrix equation (3.3) on the interval J.*

Proof. We have

$$\dot{\Phi} = [\dot{\phi}_1, \dot{\phi}_2, \ldots, \dot{\phi}_n] = [A\phi_1, A\phi_2, \ldots, A\phi_n] = A[\phi_1, \phi_2, \ldots, \phi_n] = A\Phi.$$

■

The next result is called *Abel's formula.*

Theorem 3.4. *If Φ is a solution of the matrix equation (3.3) on an interval J and τ is any point of J, then*

$$\det \Phi(t) = \det \Phi(\tau) \exp \left[\int_\tau^t trA ds \right]$$

for every $t \in J$. [$trA = tr[a_{ij}]$ denotes the trace of A; i.e., $trA = \sum_{j=1}^n a_{jj}$.] ■

The proof of Theorem 3.4 is omitted. We refer the reader to [1] for a proof.

Since in Theorem 3.4 τ is arbitrary, it follows that either $\det \Phi(t) \neq 0$ for all $t \in J$ or $\det \Phi(t) = 0$ for each $t \in J$. The next result provides a test on whether an $n \times n$ matrix $\Phi(t)$ is a fundamental matrix of (3.1).

Theorem 3.5. *A solution Φ of the matrix equation (3.3) is a fundamental matrix of (3.1) if and only if its determinant is nonzero for all $t \in J$.*

Proof. If $\Phi = [\phi_1, \phi_2, \ldots, \phi_n]$ is a fundamental matrix for (3.1), then the columns of Φ, $\phi_1, \ldots, \phi_n$, form a linearly independent set. Now let ϕ be a nontrivial solution of (3.1). Then by Theorem 3.1 there exist unique scalars $\alpha_1, \ldots, \alpha_n \in F$, not all zero, such that $\phi = \sum_{j=1}^{n} \alpha_j \phi_j = \Phi a$, where $a^T = (\alpha_1, \ldots, \alpha_n)$. Let $t = \tau \in J$. Then $\phi(\tau) = \Phi(\tau) a$, which is a system of n linear algebraic equations. By construction, this system of equations has a unique solution for any choice of $\phi(\tau)$. Therefore, $\det \Phi(\tau) \neq 0$. It now follows from Theorem 3.4 that $\det \Phi(t) \neq 0$ for any $t \in J$.

Conversely, let Φ be a solution of (3.3) and assume that $\det \Phi(t) \neq 0$ for all $t \in J$. Then the columns of Φ are linearly independent for all $t \in J$. Hence, Φ is a fundamental matrix of (3.1). ■

It is emphasized that a matrix may have identically zero determinant over some interval, even though its columns are linearly independent. For example, the columns of the matrix

$$\Phi(t) = \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & t \\ 0 & 0 & 0 \end{bmatrix}$$

are linearly independent, and yet $\det \Phi(t) = 0$ for all $t \in (-\infty, \infty)$. In accordance with Theorem 3.5, the above matrix cannot be a fundamental solution of the matrix equation (3.3) for any matrix A.

Theorem 3.6. *If Φ is a fundamental matrix of (3.1) and if C is any nonsingular constant $n \times n$ matrix, then ΦC is also a fundamental matrix of (3.1). Moreover, if Ψ is any other fundamental matrix of (3.1), then there exists a constant $n \times n$ nonsingular matrix P such that $\Psi = \Phi P$.*

Proof. For the matrix ΦC we have $\frac{d}{dt}(\Phi C) = \dot{\Phi} C = [A\Phi] C = A(\Phi C)$, and therefore, ΦC is a solution of the matrix equation (3.3). Furthermore, since $\det \Phi(t) \neq 0$ for $t \in J$ and $\det C \neq 0$, it follows that $\det[\Phi(t) C] = [\det \Phi(t)](\det C) \neq 0, t \in J$. By Theorem 3.5, ΦC is a fundamental matrix.

Next, let Ψ be any other fundamental matrix of (3.1) and consider the product $\Phi^{-1}(t)\Psi$. [Notice that since $\det \Phi(t) \neq 0$ for all $t \in J$, then $\Phi^{-1}(t)$ exists for all $t \in J$.] Also, consider $\Phi\Phi^{-1} = I$ where I denotes the $n \times n$ identity matrix. Differentiating both sides, we obtain $\left(\frac{d}{dt}\Phi\right)\Phi^{-1} + \Phi\left(\frac{d}{dt}\Phi^{-1}\right) = 0$ or $\frac{d}{dt}\Phi^{-1} = -\Phi^{-1}\left(\frac{d}{dt}\Phi\right)\Phi^{-1}$. Therefore, we can compute $\frac{d}{dt}(\Phi^{-1}\Psi) = \Phi^{-1}\left(\frac{d}{dt}\Psi\right) + \left(\frac{d}{dt}\Phi^{-1}\right)\Psi = \Phi^{-1}A\Psi - [\Phi^{-1}\left(\frac{d}{dt}\Phi\right)\Phi^{-1}]\Psi = \Phi^{-1}A\Psi - (\Phi^{-1}A\Phi\Phi^{-1})\Psi = \Phi^{-1}A\Psi - \Phi^{-1}A\Psi = 0$. Hence, $\Phi^{-1}\Psi = P$ or $\Psi = \Phi P$. ■

Example 3.7. It is easily verified that the system of equations

$$\begin{aligned} \dot{x}_1 &= 5x_1 - 2x_2 \\ \dot{x}_2 &= 4x_1 - x_2 \end{aligned} \tag{3.4}$$

has two linearly independent solutions given by $\phi_1(t) = (e^{3t}, e^{3t})^T$, $\phi_2(t) = (e^t, 2e^t)^T$, and therefore, the matrix

$$\Phi(t) = \begin{bmatrix} e^{3t} & e^t \\ e^{3t} & 2e^t \end{bmatrix} \tag{3.5}$$

is a fundamental matrix of (3.4).

Using Theorem 3.6 we can find the particular fundamental matrix Ψ of (3.4) that satisfies the initial condition $\Psi(0) = I$ by using $\Phi(t)$ given in (3.5). We have $\Psi(0) = I = \Phi(0)C$ or $C = \Phi^{-1}(0)$, and therefore,

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

and

$$\Psi(t) = \Phi C = \begin{bmatrix} (2e^{3t} - e^t) & (-e^{3t} + e^t) \\ (2e^{3t} - 2e^t) & (-e^{3t} + 2e^t) \end{bmatrix}.$$

3.2.2 The State Transition Matrix

In Chapter 1 we used the *method of successive approximations* (Theorem 1.15) to prove that for every $(t_0, x_0) \in J \times R^n$,

$$\dot{x} = A(t)x \tag{3.6}$$

possesses a unique solution of the form

$$\phi(t, t_0, x_0) = \Phi(t, t_0)x_0,$$

such that $\phi(t_0, t_0, x_0) = x_0$, which exists for all $t \in J$, where $\Phi(t, t_0)$ is the *state transition matrix* (see Section 1.8). We derived an expression for $\Phi(t, t_0)$ in series form, called the *Peano–Baker series* [see (1.80) of Chapter 1], and we showed that $\Phi(t, t_0)$ is the unique solution of the matrix differential equation

$$\frac{\partial}{\partial t}\Phi(t, t_0) = A(t)\Phi(t, t_0), \tag{3.7}$$

where

$$\Phi(t_0, t_0) = I \text{ for all } t \in J. \tag{3.8}$$

Of course, these results hold for (3.1) as well.

We now provide an alternative formulation of the state transition matrix, and we study some of the properties of such matrices. Even though much of the subsequent discussion applies to system (3.6), we will confine ourselves to system (3.1). In the following definition, we use the natural basis $\{e_1, e_2, \ldots, e_n\}$ (refer to Section A.2).

Definition 3.8. *A fundamental matrix Φ of (3.1) whose columns are determined by the linearly independent solutions $\phi_1, \ldots, \phi_n$ with*

$$\phi_1(t_0) = e_1, \ldots, \phi_n(t_0) = e_n, \quad t_0 \in J,$$

is called the state transition matrix Φ *for (3.1). Equivalently, if Ψ is* any *fundamental matrix of (3.1), then the matrix Φ determined by*

$$\Phi(t, t_0) \triangleq \Psi(t)\Psi^{-1}(t_0) \quad \textit{for all} \quad t, t_0 \in J, \tag{3.9}$$

is said to be the state *transition matrix of (3.1).* ■

We note that the state transition matrix of (3.1) is *uniquely* determined by the matrix A and is *independent* of the particular choice of the fundamental matrix. To show this, let Ψ_1 and Ψ_2 be two different fundamental matrices of (3.1). Then by Theorem 3.6 there exists a constant $n \times n$ nonsingular matrix P such that $\Psi_2 = \Psi_1 P$. Now by the definition of state transition matrix, we have $\Phi(t, t_0) = \Psi_2(t)[\Psi_2(t_0)]^{-1} = \Psi_1(t)PP^{-1}[\Psi_1(t_0)]^{-1} = \Psi_1(t)[\Psi_1(t_0)]^{-1}$. This shows that $\Phi(t, t_0)$ is independent of the fundamental matrix chosen.

Properties of the State Transition Matrix

In the following discussion, we summarize some of the properties of state transition matrix.

Theorem 3.9. *Let $t_0 \in J$, let $\phi(t_0) = x_0$, and let $\Phi(t, t_0)$ denote the state transition matrix for (3.1) for all $t \in J$. Then the following statements are true:*

(i) *$\Phi(t, t_0)$ is the unique solution of the matrix equation $\frac{\partial}{\partial t}\Phi(t, t_0) = A\Phi(t, t_0)$ with $\Phi(t_0, t_0) = I$, the $n \times n$ identity matrix.*
(ii) *$\Phi(t, t_0)$ is nonsingular for all $t \in J$.*
(iii) *For any $t, \sigma, \tau \in J$, we have $\Phi(t, \tau) = \Phi(t, \sigma)\Phi(\sigma, \tau)$ (semigroup property).*
(iv) *$[\Phi(t, t_0)]^{-1} \triangleq \Phi^{-1}(t, t_0) = \Phi(t_0, t)$ for all $t, t_0 \in J$.*
(v) *The unique solution $\phi(t, t_0, x_0)$ of (3.1), with $\phi(t_0, t_0, x_0) = x_0$ specified, is given by*

$$\phi(t, t_0, x_0) = \Phi(t, t_0)x_0 \ \textit{for all} \ t \in J. \tag{3.10}$$

Proof. (i) For any fundamental matrix of (3.1), say Ψ, we have, by definition, $\Phi(t, t_0) = \Psi(t)\Psi^{-1}(t_0)$, independent of the choice of Ψ. Therefore, $\frac{\partial}{\partial t}\Phi(t, t_0) = \dot{\Psi}(t)\Psi^{-1}(t_0) = A\Psi(t)\Psi^{-1}(t_0) = A\Phi(t, t_0)$. Furthermore, $\Phi(t_0, t_0) = \Psi(t_0)\Psi^{-1}(t_0) = I$.

(ii) For any fundamental matrix of (3.1) we have that $\det \Psi(t) \neq 0$ for all $t \in J$. Therefore, $\det \Phi(t, t_0) = \det[\Psi(t)\Psi^{-1}(t_0)] = \det \Psi(t) \det \Psi^{-1}(t_0) \neq 0$ for all $t, t_0 \in J$.

(iii) For any fundamental matrix Ψ of (3.1) and for the state transition matrix Φ of (3.1), we have $\Phi(t,\tau) = \Psi(t)\Psi^{-1}(\tau) = \Psi(t)\Psi^{-1}(\sigma)\Psi(\sigma)\Psi^{-1}(\tau) = \Phi(t,\sigma)\Phi(\sigma,\tau)$ for any $t,\sigma,\tau \in J$.

(iv) Let Ψ be any fundamental matrix of (3.1), and let Φ be the state transition matrix of (3.1). Then $[\Phi(t,t_0)]^{-1} = [\Psi(t)\Psi(t_0)^{-1}]^{-1} = \Psi(t_0)\Psi^{-1}(t) = \Phi(t_0,t)$ for any $t,t_0 \in J$.

(v) By the results established in Chapter 1, we know that for every $(t_0,x_0) \in D$, (3.1) has a unique solution $\phi(t)$ for all $t \in J$ with $\phi(t_0) = x_0$. To verify (3.10), we note that $\dot{\phi}(t) = \frac{\partial \Phi}{\partial t}(t,t_0)x_0 = A\Phi(t,t_0)x_0 = A\phi(t)$. ■

In Chapter 1 we pointed out that the state transition matrix $\Phi(t,t_0)$ maps the solution (state) of (3.1) at time t_0 to the solution (state) of (3.1) at time t. Since there is no restriction on t relative to t_0 (i.e., we may have $t < t_0$, $t = t_0$, or $t > t_0$), we can "move forward or backward" in time. Indeed, given the solution (state) of (3.1) at time t, we can solve the solution (state) of (3.1) at time t_0. Thus, $x(t_0) = x_0 = [\Phi(t,t_0)]^{-1}\phi(t,t_0,x_0) = \Phi(t_0,t)\phi(t,t_0,x_0)$. This *reversibility in time* is possible because $\Phi^{-1}(t,t_0)$ always exists. [In the case of discrete-time systems described by difference equations, this reversibility in time does in general not exist (refer to Section 3.5).]

3.2.3 Nonhomogeneous Equations

In Section 1.8, we proved the following result [refer to (1.87) to (1.89)].

Theorem 3.10. *Let $t_0 \in J$, let $(t_0,x_0) \in D$, and let $\Phi(t,t_0)$ denote the state transition matrix for (3.1) for all $t \in J$. Then the unique solution $\phi(t,t_0,x_0)$ of (3.2) satisfying $\phi(t_0,t_0,x_0) = x_0$ is given by*

$$\phi(t,t_0,x_0) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\eta)g(\eta)d\eta. \tag{3.11}$$

■

As pointed out in Section 1.8, when $x_0 = 0$, (3.11) reduces to the *zero state response*

$$\phi(t,t_0,0) \triangleq \phi_p(t) = \int_{t_0}^{t} \Phi(t,s)g(s)ds, \tag{3.12}$$

and when $x_0 \neq 0$, but $g(t) \equiv 0$, (3.11) reduces to the *zero input response*

$$\phi(t,t_0,x_0) \triangleq \phi_h(t) = \Phi(t,t_0)x_0 \tag{3.13}$$

and the solution of (3.2) may be viewed as consisting of a component that is due to the initial data x_0 and another component that is due to the forcing term $g(t)$. We recall that ϕ_p is called a *particular solution* of the nonhomogeneous system (3.2), whereas ϕ_h is called the *homogeneous solution*.

3.3 The Matrix Exponential e^{At}, Modes, and Asymptotic Behavior of $\dot{x} = Ax$

In the time-invariant case $\dot{x} = Ax$, the state transition matrix $\Phi(t, t_0)$ equals the matrix exponential $e^{A(t-t_0)}$, which is studied in the following discussion.

Let $D = \{(t, x) : t \in R,\ x \in R^n\}$. In view of the results of Section 1.8, it follows that for every $(t_0, x_0) \in D$, the unique solution of (3.1) with $x(0) = x_0$ specified is given by

$$\begin{aligned}\phi(t, t_0, x_0) &= \left(I + \sum_{k=1}^{\infty} \frac{A^k (t-t_0)^k}{k!} \right) x_0 \\ &= \Phi(t, t_0)x_0 \triangleq \Phi(t - t_0)x_0 \triangleq e^{A(t-t_0)}x_0, \end{aligned} \tag{3.14}$$

where $\Phi(t - t_0) = e^{A(t-t_0)}$ denotes the state transition matrix for (3.1). [By writing $\Phi(t, t_0) = \Phi(t - t_0)$, we are using a slight abuse of notation.]

In arriving at (3.14) we invoked Theorem 1.15 of Chapter 1 in Section 1.5, to show that the sequence $\{\phi_m\}$, where

$$\phi_m(t, t_0, x_0) = \left(I + \sum_{k=1}^{m} \frac{A^k (t-t_0)^k}{k!} \right) x_0 \triangleq S_m(t - t_0)x_0, \tag{3.15}$$

converges uniformly and absolutely as $m \to \infty$ to the unique solution $\phi(t, t_0, x_0)$ of (3.1) given by (3.14) on compact subsets of R. In the process of arriving at this result, we also proved the following results.

Theorem 3.11. *Let A be a constant $n \times n$ matrix (which may be real or complex), and let $S_m(t)$ denote the partial sum of matrices defined by*

$$S_m(t) = I + \sum_{k=1}^{m} \frac{t^k}{k!} A^k. \tag{3.16}$$

Then each element of the matrix $S_m(t)$ converges absolutely and uniformly on any finite t interval $(-a, a), a > 0$, as $m \to \infty$. Furthermore, $\dot{S}_m(t) = AS_{m-1}(t) = S_{m-1}(t)A$, and thus, the limit of $S_m(t)$ as $t \to \infty$ is a C^1 function on R. Moreover, this limit commutes with A. ∎

3.3.1 Properties of e^{At}

In view of Theorem 3.11, the following definition makes sense (see also Section 1.8).

Definition 3.12. *Let A be a constant $n \times n$ matrix (which may be real or complex). We define e^{At} to be the matrix*

$$e^{At} = I + \sum_{k=1}^{\infty} \frac{t^k}{k!} A^k \tag{3.17}$$

for any $-\infty < t < \infty$, and we call e^{At} a matrix exponential. ∎

We are now in a position to provide the following characterizations of e^{At}.

Theorem 3.13. *Let $J = R, t_0 \in J$, and let A be a given constant matrix for (3.1). Then*

(i) $\Phi(t) \triangleq e^{At}$ *is a fundamental matrix for all* $t \in J$.
(ii) The state transition matrix for (3.1) is given by $\Phi(t, t_0) = e^{A(t-t_0)} \triangleq \Phi(t - t_0)$, $t \in J$.
(iii) $e^{At_1}e^{At_2} = e^{A(t_1+t_2)}$ *for all* $t_1, t_2 \in J$.
(iv) $Ae^{At} = e^{At}A$ *for all* $t \in J$.
(v) $(e^{At})^{-1} = e^{-At}$ *for all* $t \in J$.

Proof. By (3.17) and Theorem 3.11 we have that $\frac{d}{dt}[e^{At}] = \lim_{m\to\infty} AS_m(t) = \lim_{m\to\infty} S_m(t)A = Ae^{At} = e^{At}A$. Therefore, $\Phi(t) = e^{At}$ is a solution of the matrix equation $\dot{\Phi} = A\Phi$. Next, observe that $\Phi(0) = I$. It follows from Theorem 3.4 that $\det[e^{At}] = e^{trace(At)} \neq 0$ for all $t \in R$. Therefore, by Theorem 3.5 $\Phi(t) = e^{At}$ is a fundamental matrix for $\dot{x} = Ax$. We have proved parts (i) and (iv).

To prove (iii), we note that in view of Theorem 3.9(iii), we have for any $t_1, t_2 \in R$ that $\Phi(t_1, t_2) = \Phi(t_1, 0)\Phi(0, t_2)$. By Theorem 3.9(i) we see that $\Phi(t, t_0)$ solves (3.1) with $\Phi(t_0, t_0) = I$. It was just proved that $\Psi(t) \triangleq e^{A(t-t_0)}$ is also a solution. By uniqueness, it follows that $\Phi(t, t_0) = e^{A(t-t_0)}$. For $t = t_1, t_0 = -t_2$, we therefore obtain $e^{A(t_1+t_2)} = \Phi(t_1, -t_2) = \Phi(t_1)\Phi(-t_2)^{-1}$, and for $t = t_1, t_0 = 0$, we have $\Phi(t_1, 0) = e^{At_1} = \Phi(t_1)$. Also, for $t = 0, t_0 = -t_2$, we obtain $\Phi(0, -t_2) = e^{t_2 A} = \Phi(-t_2)^{-1}$. Therefore, $e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$ for all $t_1, t_2 \in R$.

Finally, to prove (ii), we note that by (iii) we have $\Phi(t, t_0) \triangleq e^{A(t-t_0)} = I + \sum_{k=1}^{\infty} \frac{(t-t_0)^k}{k!}A^k = \Phi(t - t_0)$ is a fundamental matrix for $\dot{x} = Ax$ with $\Phi(t_0, t_0) = I$. Therefore, it is its state transition matrix. ■

We conclude this section by stating the solution of $\dot{x} = Ax + g(t)$,

$$\begin{aligned} \phi(t, t_0, x_0) &= \Phi(t - t_0)x_0 + \int_{t_0}^{t} \Phi(t - s)g(s)ds \\ &= e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}g(s)ds \\ &= e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-As}g(s)ds, \end{aligned} \tag{3.18}$$

for all $t \in R$. In arriving at (3.18), we have used expression (1.87) of Chapter 1 and the fact that in this case, $\Phi(t, t_0) = e^{A(t-t_0)}$.

3.3.2 How to Determine e^{At}

We begin by considering the specific case

$$A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}. \tag{3.19}$$

From (3.17) it follows immediately that

$$e^{At} = I + tA = \begin{bmatrix} 1 & \alpha t \\ 0 & 1 \end{bmatrix}. \tag{3.20}$$

As another example, we consider

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{3.21}$$

where $\lambda_1, \lambda_2 \in R$. Again, from (3.17) it follows that

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \lambda_1^k & 0 \\ 0 & 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \lambda_2^k \end{bmatrix} \\ &= \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}. \end{aligned} \tag{3.22}$$

Unfortunately, in general it is much more difficult to evaluate the matrix exponential than the preceding examples suggest. In the following discussion, we consider several methods of evaluating e^{At}.

The Infinite Series Method

In this case we evaluate the partial sum $S_m(t)$ (see Theorem 3.11)

$$S_m(t) = I + \sum_{k=1}^{m} \frac{t^k}{k!} A^k$$

for some fixed t, say, t_1, and for $m = 1, 2, \ldots$ until no significant changes occur in succeeding sums. This yields the matrix e^{At_1}. This method works reasonably well if the smallest and largest real parts of the eigenvalues of A are not widely separated.

In the same spirit as above, we could use any of the vector differential solvers to solve $\dot{x} = Ax$, using the natural basis for R^n as n linearly independent initial conditions [i.e., using as initial conditions the vectors $e_1 = (1, 0, \ldots, 0)^T$, $e_2 = (0, 1, 0, \ldots, 0)^T, \ldots, e_n = (0, \ldots, 0, 1)^T$] and observing that in view of (3.14), the resulting solutions are the columns of e^{At} (with $t_0 = 0$).

Example 3.14. There are cases when the definition of e^{At} (in series form) directly produces a closed-form expression. This occurs for example when $A^k = 0$ for some k. In particular, if all the eigenvalues of A are at the origin, then $A^k = 0$ for some $k \leq n$. In this case, only a finite number of terms in (3.17) will be nonzero and e^{At} can be evaluated in closed form. This was precisely the case in (3.19).

The Similarity Transformation Method

Let us consider the initial-value problem

$$\dot{x} = Ax, \quad x(t_0) = x_0; \tag{3.23}$$

let P be a real $n \times n$ nonsingular matrix, and consider the transformation $x = Py$, or equivalently, $y = P^{-1}x$. Differentiating both sides with respect to t, we obtain $\dot{y} = P^{-1}\dot{x} = P^{-1}APy = Jy, y(t_0) = y_0 = P^{-1}x_0$. The solution of the above equation is given by

$$\psi(t, t_0, y_0) = e^{J(t-t_0)}P^{-1}x_0. \tag{3.24}$$

Using (3.24) and $x = Py$, we obtain for the solution of (3.23),

$$\phi(t, t_0, x_0) = Pe^{J(t-t_0)}P^{-1}x_0. \tag{3.25}$$

Now suppose that the similarity transformation P given above has been chosen in such a manner that

$$J = P^{-1}AP \tag{3.26}$$

is in Jordan canonical form (refer to Section A.6). We first consider the case when A has n linearly independent eigenvectors, say, v_i, that correspond to the eigenvalues λ_i (not necessarily distinct), $i = 1, \dots, n$. (Necessary and sufficient conditions for this to be the case are given in Sections A.5 and A.6. A sufficient condition for the eigenvectors v_i, $i = 1, \dots, n$, to be linearly independent is that the eigenvalues of $A, \lambda_1, \dots, \lambda_n$, be distinct.) Then P can be chosen so that $P = [v_1, \dots, v_n]$ and the matrix $J = P^{-1}AP$ assumes the form

$$J = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}. \tag{3.27}$$

Using the power series representation

$$e^{Jt} = I + \sum_{k=1}^{\infty} \frac{t^k J^k}{k!}, \tag{3.28}$$

we immediately obtain the expression

$$e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}. \tag{3.29}$$

Accordingly, the solution of the initial-value problem (3.23) is now given by

$$\phi(t, t_0, x_0) = P \begin{bmatrix} e^{\lambda_1(t-t_0)} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n(t-t_0)} \end{bmatrix} P^{-1} x_0. \tag{3.30}$$

In the general case when A has repeated eigenvalues, it is no longer possible to diagonalize A (see Section A.6). However, we can generate n linearly independent vectors $v_1, \dots, v_n$ and an $n \times n$ similarity transformation $P = [v_1, \dots, v_n]$ that takes A into the Jordan canonical form $J = P^{-1}AP$. Here J is in the block diagonal form given by

$$J = \begin{bmatrix} J_0 & & & 0 \\ & J_1 & & \\ & & \ddots & \\ 0 & & & J_s \end{bmatrix}, \tag{3.31}$$

where J_0 is a diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_k$ (not necessarily distinct), and each $J_i, i \geq 1$, is an $n_i \times n_i$ matrix of the form

$$J_i = \begin{bmatrix} \lambda_{k+i} & 1 & 0 & \cdots & 0 \\ 0 & \lambda_{k+i} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_{k+i} \end{bmatrix}, \tag{3.32}$$

where λ_{k+i} need not be different from λ_{k+j} if $i \neq j$, and where $k + n_1 + \cdots + n_s = n$.

Now since for any square block diagonal matrix

$$C = \begin{bmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_l \end{bmatrix}$$

with $C_i, i = 1, \dots, l$, square, we have that

$$C^k = \begin{bmatrix} C_1^k & & 0 \\ & \ddots & \\ 0 & & C_l^k \end{bmatrix},$$

it follows from the power series representation of e^{Jt} that

$$e^{Jt} = \begin{bmatrix} e^{J_0 t} & & & 0 \\ & e^{J_1 t} & & \\ & & \ddots & \\ 0 & & & e^{J_s t} \end{bmatrix}, \tag{3.33}$$

$t \in R$. As shown earlier, we have

$$e^{J_0 t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_k t} \end{bmatrix}. \tag{3.34}$$

For J_i, $i = 1, \dots, s$, we have

$$J_i = \lambda_{k+i} I_i + N_i, \tag{3.35}$$

where I_i denotes the $n_i \times n_i$ identity matrix and N_i is the $n_i \times n_i$ nilpotent matrix given by

$$N_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}. \tag{3.36}$$

Since $\lambda_{k+i} I_i$ and N_i commute, we have that

$$e^{J_i t} = e^{\lambda_{k+i} t} e^{N_i t}. \tag{3.37}$$

Repeated multiplication of N_i by itself results in $N_i^k = 0$ for all $k \geq n_i$. Therefore, the series defining e^{tN_i} terminates, resulting in

$$e^{tJ_i} = e^{\lambda_{k+i} t} \begin{bmatrix} 1 & t & \cdots & t^{n_i - 1}/(n_i - 1)! \\ 0 & 1 & \cdots & t^{n_i - 2}/(n_i - 2)! \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad i = 1, \dots, s. \tag{3.38}$$

It now follows that the solution of (3.23) is given by

$$\phi(t, t_0, x_0) = P \begin{bmatrix} e^{J_0(t-t_0)} & 0 & \cdots & 0 \\ 0 & e^{J_1(t-t_0)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & e^{J_s(t-t_0)} \end{bmatrix} P^{-1} x_0. \tag{3.39}$$

Example 3.15. In system (3.23), let $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 1$, and corresponding eigenvectors for A are given by $v_1 = (1, 0)^T$ and $v_2 = (1, 1)^T$, respectively. Then $P = [v_1, v_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, and $J = P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, as expected. We obtain $e^{At} = P e^{Jt} P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & e^t - e^{-t} \\ 0 & e^t \end{bmatrix}$.

Suppose next that in (3.23) the matrix A is either in *companion form* or that it has been transformed into this form via some suitable similarity transformation P, so that $A = A_c$, where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}. \tag{3.40}$$

Since in this case we have $x_{i+1} = \dot{x}_i$, $i = 1, \ldots, n-1$, it should be clear that in the calculation of e^{At} we need to determine, via some method, only the first row of e^{At}. We demonstrate this by means of a specific example.

Example 3.16. In system (3.23), assume that $A = A_c = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, which is in companion form. To demonstrate the above observation, let us compute e^{At} by some other method, say diagonalization. The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -2$, and a set of corresponding eigenvectors is given by $v_1 = (1, -1)^T$ and $v_2 = (1, -2)^T$. We obtain $P = [v_1, v_2] = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ and $J = P^{-1} A_c P = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, $e^{At} = P e^{Jt} P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (e^{-t} - e^{-2t}) \\ (-2e^{-t} + 2e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix}$. Note that the second row of the above matrix is the derivative of the first row, as expected.

The Cayley–Hamilton Theorem Method

If $\alpha(\lambda) = \det(\lambda I - A)$ is the characteristic polynomial of an $n \times n$ matrix A, we have that $\alpha(A) = 0$, in view of the Cayley–Hamilton Theorem; i.e., every $n \times n$ matrix satisfies its characteristic equation (refer to Sections A.5 and A.6). Using this result, along with the series definition of the matrix exponential e^{At}, it is easily shown that

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i(t) A^i \tag{3.41}$$

[Refer to Sections A.5 and A.6 for the details on how to determine the terms $\alpha_i(t)$.]

The Laplace Transform Method

We assume that the reader is familiar with the basics of the (one-sided) Laplace transform. If $f(t) = [f_1(t), \ldots, f_n(t)]^T$, where $f_i : [0, \infty) \to R$, $i = 1, \ldots, n$, and if each f_i is Laplace transformable, then we define the Laplace transform of the vector f component-wise; i.e., $\hat{f}(s) = [\hat{f}_1(s), \ldots, \hat{f}_n(s)]^T$, where $\hat{f}_i(s) = \mathcal{L}[f_i(t)] \triangleq \int_0^\infty f_i(t)e^{-st}dt$.

We define the Laplace transform of a matrix $C(t) = [c_{ij}(t)]$ similarly. Thus, if each $c_{ij} : [0, \infty) \to R$ and if each c_{ij} is Laplace transformable, then the Laplace transform of $C(t)$ is defined as $\widehat{C}(s) = \mathcal{L}[c_{ij}(t)] = [\mathcal{L}c_{ij}(t)] = [\hat{c}_{ij}(s)]$.

Laplace transforms of some of the common time signals are enumerated in Table 3.1. Also, in Table 3.2 we summarize some of the more important properties of the Laplace transform. In Table 3.1, $\delta(t)$ denotes the *Dirac delta distribution* (see Subsection 2.4.3) and $p(t)$ represents the *unit step function.*

Table 3.1. Laplace transforms

$f(t)(t \geq 0)$	$\hat{f}(s) = \mathcal{L}[f(t)]$
$\delta(t)$	1
$p(t)$	$1/s$
$t^k/k!$	$1/s^{k+1}$
e^{-at}	$1/(s+a)$
$t^k e^{-at}$	$k!/(s+a)^{k+1}$
$e^{-at}\sin bt$	$b/[(s+a)^2+b^2]$
$e^{-at}\cos bt$	$(s+a)/[(s+a)^2+b^2]$

Table 3.2. Laplace transform properties

Time differentiation	$df(t)/dt$	$s\hat{f}(s) - f(0)$
	$d^k f(t)/dt^k$	$s^k\hat{f}(s) - [s^{k-1}f(0) + \cdots + f^{(k-1)}(0)]$
Frequency shift	$e^{-at}f(t)$	$\hat{f}(s+a)$
Time shift	$f(t-a)p(t-a), a > 0$	$e^{-as}\hat{f}(s)$
Scaling	$f(t/\alpha), \alpha > 0$	$\alpha\hat{f}(\alpha s)$
Convolution	$\int_0^t f(\tau)g(t-\tau)d\tau = f(t) * g(t)$	$\hat{f}(s)\hat{g}(s)$
Initial value	$\lim_{t\to 0^+} f(t) = f(0^+)$	$\lim_{s\to\infty} s\hat{f}(s)^\dagger$
Final value	$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} s\hat{f}(s)^\ddagger$

† If the limit exists.

‡ If $s\hat{f}(s)$ has no singularities on the imaginary axis or in the right half s plane.

Now consider once more the initial-value problem (3.23), letting $t_0 = 0$; i.e.,

$$\dot{x} = Ax, \quad x(0) = x_0. \tag{3.42}$$

Taking the Laplace transform of both sides of $\dot{x} = Ax$, and taking into account the initial condition $x(0) = x_0$, we obtain $s\hat{x}(s) - x_0 = A\hat{x}(s)$, $(sI - A)\hat{x}(s) = x_0$, or

$$\hat{x}(s) = (sI - A)^{-1}x_0. \tag{3.43}$$

It can be shown by analytic continuation that $(sI-A)^{-1}$ exists for all s, except at the eigenvalues of A. Taking the inverse Laplace transform of (3.43), we obtain the solution

$$\phi(t) = \mathcal{L}^{-1}[(sI - A)^{-1}]x_0 = \Phi(t, 0)x_0 = e^{At}x_0. \tag{3.44}$$

It follows from (3.42) and (3.44) that $\hat{\Phi}(s) = (sI - A)^{-1}$ and that

$$\Phi(t, 0) \triangleq \Phi(t - 0) = \Phi(t) = \mathcal{L}^{-1}[(sI - A)^{-1}] = e^{At}. \tag{3.45}$$

Finally, note that when $t_0 \neq 0$, we can immediately compute $\Phi(t, t_0) = \Phi(t - t_0) = e^{A(t-t_0)}$.

Example 3.17. In (3.42), let $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$. Then

$$(sI - A)^{-1} = \begin{bmatrix} s+1 & -2 \\ 0 & s-1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{(s+1)(s-1)} \\ 0 & \frac{1}{s-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \left(\frac{1}{s-1} - \frac{1}{s+1}\right) \\ 0 & \frac{1}{s-1} \end{bmatrix}.$$

Using Table 3.1, we obtain $\mathcal{L}^{-1}[(sI - A)^{-1}] = e^{At} = \begin{bmatrix} e^{-t} & (e^t - e^{-t}) \\ 0 & e^t \end{bmatrix}$.

Before concluding this subsection, we briefly consider initial-value problems described by

$$\dot{x} = Ax + g(t), \quad x(t_0) = x_0. \tag{3.46}$$

We wish to apply the Laplace transform method discussed above in solving (3.46). To this end we assume $t_0 = 0$ and we take the Laplace transform of both sides of (3.46) to obtain $s\hat{x}(s) - x_0 = A\hat{x}(s) + \hat{g}(s)$, $(sI - A)\hat{x}(s) = x_0 + \hat{g}(s)$, or

$$\begin{aligned} \hat{x}(s) &= (sI - A)^{-1}x_0 + (sI - A)^{-1}\hat{g}(s) \\ &= \hat{\Phi}(s)x_0 + \hat{\Phi}(s)\hat{g}(s) \\ &\triangleq \hat{\phi}_h(s) + \hat{\phi}_p(s). \end{aligned} \tag{3.47}$$

Taking the inverse Laplace transform of both sides of (3.47) and using (3.18) with $t_0 = 0$, we obtain $\phi(t) = \phi_h(t) + \phi_p(t) = \mathcal{L}^{-1}[(sI - A)^{-1}]x_0 + \mathcal{L}^{-1}[(sI - A)^{-1}\hat{g}(s)] = \Phi(t)x_0 + \int_0^t \Phi(t - \eta)g(\eta)d\eta$, where ϕ_h denotes the homogeneous solution and ϕ_p is the particular solution, as expected.

Example 3.18. Consider the initial-value problem given by

$$\dot{x}_1 = -x_1 + x_2,$$
$$\dot{x}_2 = -2x_2 + u(t),$$

with $x_1(0) = -1$, $x_2(0) = 0$, and

$$u(t) = \begin{cases} 1 & \text{for } t > 0, \\ 0 & \text{for } t \leq 0. \end{cases}$$

It is easily verified that in this case

$$\hat{\Phi}(s) = \begin{bmatrix} \frac{1}{s+1} & \left(\frac{1}{s+1} - \frac{1}{s+2}\right) \\ 0 & \frac{1}{s+2} \end{bmatrix},$$

$$\Phi(t) = \begin{bmatrix} e^{-t} & (e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix},$$

$$\phi_h(t) = \begin{bmatrix} e^{-t} & (e^{-t} - e^{-2t}) \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ 0 \end{bmatrix},$$

$$\hat{\phi}_p(s) = \begin{bmatrix} \frac{1}{s+1} & \left(\frac{1}{s+1} - \frac{1}{s+2}\right) \\ 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\frac{1}{s}\right) + \frac{1}{2}\left(\frac{1}{s+2}\right) - \frac{1}{s+1} \\ \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{s+2}\right) \end{bmatrix},$$

$$\phi_p(t) = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} \\ \frac{1}{2} - \frac{1}{2}e^{-2t} \end{bmatrix},$$

and

$$\phi(t) = \phi_h(t) + \phi_p(t) = \begin{bmatrix} \frac{1}{2} - 2e^{-t} + \frac{1}{2}e^{-2t} \\ \frac{1}{2} - \frac{1}{2}e^{-2t} \end{bmatrix}.$$

3.3.3 Modes, Asymptotic Behavior, and Stability

In this subsection we study the qualitative behavior of the solutions of $\dot{x} = Ax$ by means of the modes of such systems, to be introduced shortly. Although we will not address the stability of systems in detail until Chapter 4, the results here will enable us to give some general stability characterizations for such systems.

Modes: General Case

We begin by recalling that the unique solution of

$$\dot{x} = Ax, \tag{3.48}$$

satisfying $x(0) = x_0$, is given by

$$\phi(t, 0, x_0) = \Phi(t, 0)x(0) = \Phi(t, 0)x_0 = e^{At}x_0. \tag{3.49}$$

We also recall that $\det(sI - A) = \prod_{i=1}^{\sigma}(s - \lambda_i)^{n_i}$, where $\lambda_1, \dots, \lambda_\sigma$ denote the σ distinct eigenvalues of A, where λ_i with $i = 1, \dots, \sigma$, is assumed to be repeated n_i times (i.e., n_i is the algebraic multiplicity of λ_i), and $\Sigma_{i=1}^{\sigma} n_i = n$.

To introduce the modes for (3.48), we must show that

$$\begin{aligned} e^{At} &= \sum_{i=1}^{\sigma}\sum_{k=0}^{n_i-1} A_{ik}t^k e^{\lambda_i t} \\ &= \sum_{i=1}^{\sigma}[A_{i0}e^{\lambda_i t} + A_{i1}te^{\lambda_i t} + \cdots + A_{i(n_i-1)}t^{n_i-1}e^{\lambda_i t}], \end{aligned} \tag{3.50}$$

where

$$A_{ik} = \frac{1}{k!}\,\frac{1}{(n_i - 1 - k)!}\lim_{s\to\lambda_i}[[(s - \lambda_i)^{n_i}(sI - A)^{-1}]^{(n_i-1-k)}]. \tag{3.51}$$

In (3.51), $[\cdot]^{(l)}$ denotes the lth derivative with respect to s.

Equation (3.50) shows that e^{At} can be expressed as the sum of terms of the form $A_{ik}t^k e^{\lambda_i t}$, where $A_{ik} \in R^{n\times n}$. We call $A_{ik}t^k e^{\lambda_i t}$ *a mode of system (3.48)*. If an eigenvalue λ_i is repeated n_i times, there are n_i modes, $A_{ik}t^k e^{\lambda_i t}$, $k = 0, 1, \dots, n_i - 1$, in e^{At} associated with λ_i. Accordingly, the solution (3.49) of (3.48) is determined by the n modes of (3.48) corresponding to the n eigenvalues of A and by the initial condition $x(0)$. We note that by selecting $x(0)$ appropriately, modes can be combined or eliminated $[A_{ik}x(0) = 0]$, thus affecting the behavior of $\phi(t, 0, x_0)$.

To verify (3.50) we recall that $e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$ and we make use of the partial fraction expansion method to determine the inverse Laplace transform. As in the scalar case, it can be shown that

$$(sI - A)^{-1} = \sum_{i=1}^{\sigma}\sum_{k=0}^{n_i-1}(k!A_{ik})(s - \lambda_i)^{-(k+1)}, \tag{3.52}$$

where the $(k!A_{ik})$ are the coefficients of the partial fractions ($k!$ is for scaling). It is known that these coefficients can be evaluated for each i by multiplying both sides of (3.52) by $(s - \lambda_i)^{n_i}$, differentiating $(n_i - 1 - k)$ times with respect to s, and then evaluating the resulting expression at $s = \lambda_i$. This yields (3.51). Taking the inverse Laplace transform of (3.52) and using the fact that $\mathcal{L}[t^k e^{\lambda_i t}] = k!(s - \lambda_i)^{-(k+1)}$ (refer to Table 3.1) results in (3.50).

When all n eigenvalues λ_i of A are distinct, then $\sigma = n, n_i = 1, i = 1, \dots, n$, and (3.50) reduces to the expression

$$e^{At} = \sum_{i=1}^{n} A_i e^{\lambda_i t}, \tag{3.53}$$

where

$$A_i = \lim_{s \to \lambda_i} [(s - \lambda_i)(sI - A)^{-1}]. \tag{3.54}$$

Expression (3.54) can also be derived directly, using a partial fraction expansion of $(sI - A)^{-1}$ given in (3.52).

Example 3.19. For (3.48) we let $A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$, for which the eigenvalue $\lambda_1 = -2$ is repeated twice; i.e., $n_1 = 2$. Applying (3.50) and (3.51), we obtain

$$e^{At} = A_{10}e^{\lambda_1 t} + A_{11}te^{\lambda_1 t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} te^{-2t}.$$

Example 3.20. For (3.48) we let $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, for which the eigenvalues are given by (the complex conjugate pair) $\lambda_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}, \lambda_2 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$. Applying (3.53) and (3.54), we obtain

$$A_1 = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 + 1 & 1 \\ -1 & \lambda_1 \end{bmatrix} = \frac{1}{j\sqrt{3}} \begin{bmatrix} \frac{1}{2} + j\frac{\sqrt{3}}{2} & 1 \\ -1 & -\frac{1}{2} + j\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$A_2 = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 + 1 & 1 \\ -1 & \lambda_2 \end{bmatrix} = \frac{1}{-j\sqrt{3}} \begin{bmatrix} \frac{1}{2} - j\frac{\sqrt{3}}{2} & 1 \\ -1 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} \end{bmatrix}$$

[i.e., $A_1 = A_2^*$, where $(\cdot)^*$ denotes the complex conjugate of $(\cdot)$], and

$$\begin{aligned} e^{At} &= A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} = A_1 e^{\lambda_1 t} + A_1^* e^{\lambda_1^* t} \\ &= 2(Re\, A_1)(Re\, e^{\lambda_1 t}) - 2(Im\, A_1)(Im\, e^{\lambda_1 t}) \\ &= 2e^{-\frac{1}{2}t} \left[\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \cos\frac{\sqrt{3}}{2}t - \begin{bmatrix} -\frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{bmatrix} \sin\left(\frac{\sqrt{3}}{2}t\right) \right]. \end{aligned}$$

The last expression involves only real numbers, as expected, since A and e^{At} are real matrices.

Example 3.21. For (3.48) we let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, for which the eigenvalue $\lambda_1 = 1$ is repeated twice; i.e., $n_1 = 2$. Applying (3.50) and (3.51), we obtain

$$e^{At} = A_{10}e^{\lambda_1 t} + A_{11}te^{\lambda_1 t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e^{t} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} te^{t} = Ie^{t}.$$

This example shows that not all modes of the system are necessarily present in e^{At}. What is present depends in fact on the number and dimensions of the individual blocks of the Jordan canonical form of A corresponding to identical eigenvalues. To illustrate this further, we let for (3.48), $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, where the two repeated eigenvalues $\lambda_1 = 1$ belong to the same Jordan block. Then $e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e^t + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} te^t$.

Stability of an Equilibrium

In Chapter 4 we will study the *qualitative properties* of linear dynamical systems, including systems described by (3.48). This will be accomplished by studying the *stability properties* of such systems or, more specifically, the *stability properties* of an *equilibrium* of such systems.

If $\phi(t, 0, x_e)$ denotes the solution of system (3.48) with $x(0) = x_e$, then x_e is said to be an *equilibrium* of (3.48) if $\phi(t, 0, x_e) = x_e$ for all $t \geq 0$. Clearly, $x_e = 0$ is an equilibrium of (3.48). In discussing the qualitative properties, it is often customary to speak, somewhat loosely, of the *stability properties of system (3.48)*, rather than the stability properties of the equilibrium $x_e = 0$ of system (3.48).

We will show in Chapter 4 that the following qualitative characterizations of system (3.48) are actually *equivalent* to more fundamental qualitative characterizations of the equilibrium $x_e = 0$ of system (3.48):

1. The system (3.48) is said to be *stable* if all solutions of (3.48) are bounded for all $t \geq 0$ [i.e., for any $\phi(t, 0, x_0) = (\phi_1(t, 0, x_0), \ldots, \phi_n(t, 0, x_0))^T$ of (3.48), there exist constants M_i, $i = 1, \ldots, n$ (which in general will depend on the solution on hand) such that $|\phi_i(t, 0, x_0)| < M_i$ for all $t \geq 0$].
2. The system (3.48) is said to be *asymptotically stable* if it is stable and if all solutions of (3.48) tend to the origin as t tends to infinity [i.e., for any solution $\phi(t, 0, x_0) = (\phi_1(t, 0, x_0), \ldots, \phi_n(t, 0, x_0))^T$ of (3.48), we have $\lim_{t \to \infty} \phi_i(t, 0, x_0) = 0$, $i = 1, \ldots, n$].
3. The system (3.48) is said to be *unstable* if it is not stable.

By inspecting the modes of (3.48) given by (3.50), (3.51) and (3.53), (3.54), the following stability criteria for system (3.48) are now evident:

1. The system (3.48) is *asymptotically stable* if and only if all eigenvalues of A have negative real parts (i.e., $Re\lambda_j < 0$, $j = 1, \ldots, n$).
2. The system (3.48) is *stable* if and only if $Re\lambda_j \leq 0$, $j = 1, \ldots, n$, and for all eigenvalues with $Re\lambda_j = 0$ having multiplicity $n_j > 1$, it is true that

$$\lim_{s \to \lambda_j} [(s - \lambda_j)^{n_j} (sI - A)^{-1}]^{(n_j - 1 - k)} = 0, \quad k = 1, \ldots, n_j - 1. \tag{3.55}$$

3. System (3.48) is *unstable* if and only if (2) is not true.

We note in particular that if $Re\lambda_j = 0$ and $n_j > 1$, then there will be modes $A_{jk}t^k$, $k = 0, \ldots, n_j - 1$ that will yield terms in (3.50) whose norm will tend to infinity as $t \to \infty$, unless their coefficients are zero. This shows why the necessary and sufficient conditions for stability of (3.48) include condition (3.55).

Example 3.22. The systems in Examples 3.19 and 3.20 are asymptotically stable. A system (3.48) with $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ is stable, since the eigenvalues of A above are $\lambda_1 = 0$, $\lambda_2 = -1$. A system (3.48) with $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is unstable since the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = -1$. The system of Example 3.21 is also unstable.

Modes: Distinct Eigenvalue Case

When the eigenvalues λ_i of A are distinct, there is an alternative way to (3.54) of computing the matrix coefficients A_i, expressed in terms of the corresponding right and left eigenvectors of A. This method offers great insight into questions concerning the presence or absence of modes in the response of a system. Specifically, if A has n distinct eigenvalues λ_i, then

$$e^{At} = \sum_{i=1}^{n} A_i e^{\lambda_i t}, \tag{3.56}$$

where

$$A_i = v_i \tilde{v}_i, \tag{3.57}$$

where $v_i \in R^n$ and $(\tilde{v}_i)^T \in R^n$ are right and left eigenvectors of A corresponding to the eigenvalue λ_i, respectively.

To prove the above assertions, we recall that $(\lambda_i I - A)v_i = 0$ and $\tilde{v}_i(\lambda_i I - A) = 0$. If $Q \triangleq [v_1, \ldots, v_n]$, then the $\tilde{v}_i$ are the rows of

$$P = Q^{-1} = \begin{bmatrix} \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \end{bmatrix}.$$

The matrix Q is of course nonsingular, since the eigenvalues λ_i, $i = 1, \ldots, n$, are by assumption distinct and since the corresponding eigenvectors are linearly independent. Notice that $Q \operatorname{diag}[\lambda_1, \ldots, \lambda_n] = AQ$ and that $\operatorname{diag}[\lambda_1, \ldots, \lambda_n]P = PA$. Also, notice that $\tilde{v}_i v_j = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j\,. \end{cases}$$

In view of this, we now have $(sI - A)^{-1} = [sI - Q\,\mathrm{diag}[\lambda_1, \ldots, \lambda_n]Q^{-1}]^{-1} = Q[sI - \mathrm{diag}[\lambda_1, \cdots, \lambda_n]]^{-1}Q^{-1} = Q\,\mathrm{diag}[(s-\lambda_1)^{-1}, \ldots, (s-\lambda_n)^{-1}]Q^{-1} = \sum_{i=1}^{n} v_i\tilde{v}_i(s-\lambda_i)^{-1}$. If we take the inverse Laplace transform of the above expression, we obtain (3.56).

If we choose the initial value $x(0)$ for (3.48) to be colinear with an eigenvector v_j of A [i.e., $x(0) = \alpha v_j$ for some real $\alpha \neq 0$], then $e^{\lambda_j t}$ is the only mode that will appear in the solution ϕ of (3.48). This can easily be seen from our preceding discussion. In particular if $x(0) = \alpha v_j$, then (3.56) and (3.57) yield

$$\phi(t, 0, x(0)) = e^{At}x(0) = v_1\tilde{v}_1 x(0)e^{\lambda_1 t} + \cdots + v_n\tilde{v}_n x(0)e^{\lambda_n t} = \alpha v_j e^{\lambda_j t} \quad (3.58)$$

since $\tilde{v}_i v_j = 1$ when $i = j$, and $\tilde{v}_i v_j = 0$ otherwise.

Example 3.23. In (3.48) we let $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$. The eigenvalues of A are given by $\lambda_1 = -1$ and $\lambda_2 = 1$ and $Q = [v_1, v_2] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $Q^{-1} = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$. Then $e^{At} = v_1\tilde{v}_1 e^{\lambda_1 t} + v_2\tilde{v}_2 e^{\lambda_2 t} = \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 & 1/2 \\ 0 & 1 \end{bmatrix} e^{t}$. If in particular we choose $x(0) = \alpha v_1 = (\alpha, 0)^T$, then $\phi(t, 0, x(0)) = e^{At}x(0) = \alpha(1, 0)^T e^{-t}$, which contains only the mode corresponding to the eigenvalue $\lambda_1 = -1$. Thus, for this particular choice of initial vector, the unstable behavior of the system is suppressed.

Remark

We conclude our discussion of modes and asymptotic behavior by briefly considering systems of linear, nonhomogeneous, ordinary differential equations $\dot{x} = Ax + g(t)$ in (3.2) for the special case where $g(t) = Bu(t)$,

$$\dot{x} = Ax + Bu, \quad (3.59)$$

where $B \in R^{n \times m}, u : R \to R^m$, and where it is assumed that the Laplace transform of u exists. Taking the Laplace transform of both sides of (3.59) and rearranging yields

$$\hat{x}(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}B\hat{u}(s). \quad (3.60)$$

By taking the inverse Laplace transform of (3.60), we see that the solution ϕ is the sum of modes that correspond to the singularities or poles of $(sI - A)^{-1}x(0)$ and of $(sI - A)^{-1}B\hat{u}(s)$. If in particular (3.48) is asymptotically stable (i.e., for $\dot{x} = Ax, Re\lambda_i < 0$, $i = 1, \ldots, n$) and if u in (3.59) is bounded (i.e., there is an M such that $|u_i(t)| < M$ for all $t \geq 0$, $i = 1, \ldots, m$), then it is easily seen that the solutions of (3.59) are bounded as well. Thus, the fact that the system (3.48) is asymptotically stable has repercussions on the asymptotic behavior of the solution of (3.59). Issues of this type will be addressed in greater detail in Chapter 4.

3.4 State Equation and Input–Output Description of Continuous-Time Systems

This section consists of three subsections. We first study the response of linear continuous-time systems. Next, we examine transfer functions of linear time-invariant systems, given the state equations of such systems. Finally, we explore the equivalence of internal representations of systems.

3.4.1 Response of Linear Continuous-Time Systems

We consider once more systems described by linear equations of the form

$$\dot{x} = Ax + Bu, \tag{3.61a}$$

$$y = Cx + Du, \tag{3.61b}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}, D \in R^{p\times m}$, and $u : R \to R^m$ is assumed to be continuous or piecewise continuous. We recall that in (3.61), x denotes the state vector, u denotes the system input, and y denotes the system output. From Section 2.2 we recall that for given initial conditions $t_0 \in R, x(t_0) = x_0 \in R^n$ and for a given input u, the unique solution of (3.61a) is given by

$$\phi(t, t_0, x_0) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)Bu(s)ds \tag{3.62}$$

for $t \in R$, where Φ denotes the state transition matrix of A. Furthermore, by substituting (3.62) into (3.61b), we obtain, for all $t \in R$, the *total system response* given by

$$y(t) = C\Phi(t, t_0)x_0 + C\int_{t_0}^{t} \Phi(t, s)Bu(s)ds + Du(t). \tag{3.63}$$

Recall that the total response (3.63) may be viewed as consisting of the sum of two components, the *zero-input response* given by the term

$$\psi(t, t_0, x_0, 0) = C\Phi(t, t_0)x_0 \tag{3.64}$$

and the *zero-state response* given by the term

$$\rho(t, t_0, 0, u) = C\int_{t_0}^{t} \Phi(t, s)Bu(s)ds + Du(t). \tag{3.65}$$

The cause of the former is the initial condition x_0 [and can be obtained from (3.63) by letting $u(t) \equiv 0$], whereas for the latter the cause is the input u [and can be obtained by setting $x_0 = 0$ in (3.63)].

The zero-state response can be used to introduce the *impulse response* of the system (3.61). We recall from Subsection 2.4.3 that by using the Dirac delta distribution δ, we can rewrite (3.63) with $x_0 = 0$ as

$$\begin{aligned} y(t) &= \int_{t_0}^{t} [C\Phi(t,\tau)B + D\delta(t-\tau)]u(\tau)d\tau \\ &= \int_{t_0}^{t} H(t,\tau)u(\tau)d\tau, \end{aligned} \tag{3.66}$$

where $H(t,\tau)$ denotes the impulse response matrix of system (3.61) given by

$$H(t,\tau) = \begin{cases} C\Phi(t,\tau)B + D\delta(t-\tau), & t \geq \tau, \\ 0, & t < \tau. \end{cases} \tag{3.67}$$

Now recall that

$$\Phi(t,t_0) = e^{A(t-t_0)}. \tag{3.68}$$

The solution of (3.61a) is thus given by

$$\phi(t,t_0,x_0) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}Bu(s)ds, \tag{3.69}$$

the *total response* of system (3.61) is given by

$$y(t) = Ce^{A(t-t_0)}x_0 + C\int_{t_0}^{t} e^{A(t-s)}Bu(s)ds + Du(t) \tag{3.70}$$

and the *zero-state response* of (3.61), is given by $y(t) = \int_{t_0}^{t}[Ce^{A(t-\tau)}B + D\delta(t-\tau)]u(\tau)d\tau = \int_{t_0}^{t} H(t,\tau)u(\tau)d\tau = \int_{t_0}^{t} H(t-\tau)u(\tau)d\tau$, where the *impulse response matrix* H of system (3.61) is given by

$$H(t-\tau) = \begin{cases} Ce^{A(t-\tau)}B + D\delta(t-\tau), & t \geq \tau, \\ 0, & t < \tau, \end{cases} \tag{3.71}$$

or, as is more commonly written,

$$H(t) = \begin{cases} Ce^{At}B + D\delta(t), & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{3.72}$$

At this point it may be worthwhile to consider some specific cases.

Example 3.24. In (3.61), let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [0,1], D = 0$ and consider the case when $t_0 = 0, x(0) = (1,-1)^T$, u is the unit step, and $t \geq 0$. We can easily compute the solution of (3.61a) as

$$\phi(t, t_0, x_0) = \phi_h(t, t_0, x_0) + \phi_p(t, t_0, x_0) = \begin{bmatrix} 1-t \\ -1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}t^2 \\ t \end{bmatrix}$$

with $t_0 = 0$ and for $t \geq 0$. The total system response $y(t) = Cx(t)$ is given by the sum of the zero-input response and the zero-state response, $y(t, t_0, x_0, u) = \psi(t, t_0, x_0, 0) + \rho(t, t_0, 0, u) = -1 + t$, $t \geq 0$.

Example 3.25. Consider the time-invariant system given above in Example 3.24. It is easily verified that in the present case

$$\Phi(t) = e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Then $H(t, \tau) = Ce^{A(t-\tau)}B = 1$ for $t \geq \tau$ and $H(t, \tau) = 0$ for $t < \tau$. Thus, the response of this system to an impulse input for zero initial conditions is the unit step.

As one might expect, external descriptions of finite-dimensional linear systems are not as complete as internal descriptions of such systems. Indeed, the utility of impulse responses is found in the fact that they represent the input–output relations of a system quite well, assuming that the system is at rest. To describe other dynamic behavior, one needs in general additional information [e.g., the initial state vector (or perhaps the history of the system input since the last time instant when the system was at rest) as well as the internal structure of the system].

Internal descriptions, such as state-space representations, constitute more complete descriptions than external descriptions. However, the latter are simpler to apply than the former. Both types of representations are useful. It is quite straightforward to obtain external descriptions of systems from internal descriptions, as was demonstrated in this section. The reverse process, however, is not quite as straightforward. The process of determining an internal system description from an external description is called *realization* and will be addressed in Chapter 8. The principal issue in system realization is to obtain minimal order internal descriptions that model a given system, avoiding the generation of unnecessary dynamics.

3.4.2 Transfer Functions

Next, if as in [(2.95) in Chapter 2], we take the Laplace transform of (3.71), we obtain the input–output relation

$$\hat{y}(s) = \widehat{H}(s)\hat{u}(s). \tag{3.73}$$

We recall from Section 2.4 that $\widehat{H}(s)$ is called the *transfer function matrix* of system (3.61). We can evaluate this matrix in a straightforward manner by first taking the Laplace transform of both sides of (3.61a) and (3.61b) to obtain

$$s\hat{x}(s) - x(0) = A\hat{x}(s) + B\hat{u}(s), \tag{3.74}$$
$$\hat{y}(s) = C\hat{x}(s) + D\hat{u}(s). \tag{3.75}$$

Using (3.74) to solve for $\hat{x}(s)$, we obtain

$$\hat{x}(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}B\hat{u}(s). \tag{3.76}$$

Substituting (3.76) into (3.75) yields

$$\hat{y}(s) = C(sI - A)^{-1}x(0) + C(sI - A)^{-1}B\hat{u}(s) + D\hat{u}(s) \tag{3.77}$$

and

$$y(t) = \mathcal{L}^{-1}\hat{y}(s) = Ce^{At}x(0) + C\int_0^t e^{A(t-s)}Bu(s)ds + Du(t), \tag{3.78}$$

as expected.

If in (3.77) we let $x(0) = 0$, we obtain the Laplace transform of the zero-state response given by

$$\begin{aligned}\hat{y}(s) &= [C(sI - A)^{-1}B + D]\hat{u}(s)\\ &= \widehat{H}(s)\hat{u}(s),\end{aligned} \tag{3.79}$$

where $\widehat{H}(s)$ denotes the transfer function of system (3.61), given by

$$\widehat{H}(s) = C(sI - A)^{-1}B + D. \tag{3.80}$$

Recalling that $\mathcal{L}[e^{At}] = \Phi(s) = (sI - A)^{-1}$ [refer to (3.45)], we could of course have obtained (3.80) directly by taking the Laplace transform of $H(t)$ given in (3.73).

Example 3.26. In Example 3.24, let $t_0 = 0$ and $x(0) = 0$. Then

$$\begin{aligned}\widehat{H}(s) &= C(sI - A)^{-1}B + D = [0, 1]\begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\\ &= [0, 1]\begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1/s\end{aligned}$$

and $H(t) = \mathcal{L}^{-1}\widehat{H}(s) = 1$ for $t \geq 0$, as expected (see Example 3.24).

Next, as in Example 3.24, let $x(0) = (1, -1)^T$ and let u be the unit step. Then $\hat{y}(s) = C(sI - A)^{-1}x(0) + \widehat{H}(s)\hat{u}(s) = [0, 1/s](1, -1)^T + (1/s)(1/s) = -1/s + 1/s^2$ and $y(t) = \mathcal{L}^{-1}[\hat{y}(s)] = -1 + t$ for $t \geq 0$, as expected (see Example 3.24).

We note that the eigenvalues of the matrix A in Example 3.26 are the roots of the equation $\det(sI - A) = s^2 = 0$, and are given by $s_1 = 0, s_2 = 0$, whereas the transfer function $\hat{H}(s)$ in this example has only one pole (the zero of its denominator polynomial), located at the origin. It will be shown in Chapter 8 (on realization) that the *poles of the transfer function* $\hat{H}(s)$ *(of a SISO system)* are in general a subset of the eigenvalues of A. In Chapter 5 we will introduce and study two important system theoretic concepts, called *controllability* and *observability*. We will show in Chapter 8 that the eigenvalues of A are precisely the poles of the transfer function $\hat{H}(s) = C(sI - A)^{-1}B + D$ if and only if the system (3.61) is observable and controllable. This is demonstrated in the next example.

Example 3.27. In (3.61), let $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [-3, 3], D = 0$. The eigenvalues of A are the roots of the equation $\det(sI - A) = s^2 + 2s + 1 = (s+1)^2 = 0$ given by $s_1 = -1, s_2 = -1$, and the transfer function of this SISO system is given by

$$\begin{aligned} \hat{H}(s) = C(sI - A)^{-1}B + D &= [-3, 3] \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= 3[-1, 1] \frac{1}{(s+1)^2} \begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{3(s-1)}{(s+1)^2}, \end{aligned}$$

with poles (the zeros of the denominator polynomial) also given by $s_1 = -1, s_2 = -1$.

If in Example 3.27 we replace $B = [0, 1]^T$ and $D = 0$ by $B = \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix}$ and $D = [0, 0]$, then we have a multi-input system whose transfer function is given by

$$\hat{H}(s) = \left[\frac{3(s-1)}{(s+1)^2}, \frac{3}{(s+1)} \right].$$

The concepts of poles and zeros for MIMO systems (also called multivariable systems) will be introduced in Chapter 7. The determination of the poles of such systems is not as straightforward as in the case of SISO systems. It turns out that in the present case the poles of $\hat{H}(s)$ are $s_1 = -1, s_2 = -1$, the same as the eigenvalues of A.

Before proceeding to our next topic, the equivalence of internal representations, an observation concerning the transfer function $\hat{H}(s)$ of system (3.61), given by (3.80), $\hat{H}(s) = C(sI - A)^{-1}B + D$ is in order. Since the numerator matrix polynomial of $(sI - A)^{-1}$ is of degree $(n-1)$, while its denominator polynomial, the characteristic polynomial $\alpha(s)$ of A, is of degree n, it is clear that

$$\lim_{s\to\infty} \widehat{H}(s) = D,$$

a real-valued $m \times n$ matrix, and in particular, when the *direct link matrix* D in the output equation (3.61b) is zero, then

$$\lim_{s\to\infty} \widehat{H}(s) = 0,$$

the $m \times n$ matrix with zeros as its entries. In the former case (when $D \neq 0$), $\widehat{H}(s)$ is said to be a *proper transfer function*, whereas in the latter case (when $D = 0$), $\widehat{H}(s)$ is said to be a *strictly proper transfer function*.

When discussing the realization of transfer functions by state-space descriptions (in Chapter 8), we will study the properties of transfer functions in greater detail. In this connection, there are also systems that can be described by models corresponding to transfer functions $\widehat{H}(s)$ that are *not proper*. The differential equation representation of a differentiator (or an inductor) given by $y(t) = (d/dt)u(t)$ is one such example. Indeed, in this case the system cannot be represented by (3.61) and the transfer function, given by $\widehat{H}(s) = s$ is not proper. Such systems will be discussed in Chapter 10.

3.4.3 Equivalence of State-Space Representations

In Subsection 3.3.2 it was shown that when a linear, autonomous, homogeneous system of first-order ordinary differential equations $\dot{x} = Ax$ is subjected to an appropriately chosen similarity transformation, the resulting set of equations may be considerably easier to use and may exhibit latent properties of the system of equations. It is therefore natural that we consider a similar course of action in the case of the linear systems (3.61).

We begin by letting

$$\tilde{x} = Px, \tag{3.81}$$

where P is a real, nonsingular matrix (i.e., P is a similarity transformation). Consistent with what has been said thus far, we see that such transformations bring about a *change of basis* for the state space of system (3.61). Application of (3.81) to this system will result, as will be seen, in a system description of the same form as (3.61), but involving different state variables. We will say that the system (3.61), and the system obtained by subjecting (3.61) to the transformation (3.81), constitute *equivalent internal representations* of an underlying system. We will show that equivalent internal representations (of the same system) possess identical external descriptions, as one would expect, by showing that they have identical impulse responses and transfer function matrices. In connection with this discussion, two important notions called *zero-input equivalence* and *zero-state equivalence* of a system will arise in a natural manner.

If we differentiate both sides of (3.81), and if we apply $x = P^{-1}\tilde{x}$ to (3.61), we obtain the equivalent internal representation of (3.61) given by

$$\dot{\tilde{x}} = \widetilde{A}\tilde{x} + \widetilde{B}u, \tag{3.82a}$$

$$y = \widetilde{C}\tilde{x} + \widetilde{D}u, \tag{3.82b}$$

where

$$\widetilde{A} = PAP^{-1}, \quad \widetilde{B} = PB, \quad \widetilde{C} = CP^{-1}, \quad \widetilde{D} = D \tag{3.83}$$

and where $\tilde{x}$ is given by (3.81). It is now easily verified that the system (3.61) and the system (3.82) have the same external representation. Recall that for (3.61) and for (3.82), we have for the impulse response

$$H(t,\tau) \triangleq H(t-\tau,0) = \begin{cases} Ce^{A(t-\tau)}B + D\delta(t-\tau), & t \geq \tau, \\ 0, & t < \tau, \end{cases} \tag{3.84}$$

and

$$\widetilde{H}(t,\tau) \triangleq \widetilde{H}(t-\tau,0) = \begin{cases} \widetilde{C}e^{\widetilde{A}(t-\tau)}\widetilde{B} + \widetilde{D}\delta(t-\tau), & t \geq \tau, \\ 0, & t < \tau. \end{cases} \tag{3.85}$$

Recalling from Subsection 3.3.2 [see (3.25)] that

$$e^{\widetilde{A}(t-\tau)} = Pe^{A(t-\tau)}P^{-1}, \tag{3.86}$$

we obtain from (3.83)–(3.85) that $\widetilde{C}e^{\widetilde{A}(t-\tau)}\widetilde{B}+\widetilde{D}\delta(t-\tau)=CP^{-1}Pe^{A(t-\tau)}P^{-1}PB+D\delta(t-\tau)=Ce^{A(t-\tau)}B+D\delta(t-\tau)$, which proves, in view of (3.84) and (3.85), that

$$\widetilde{H}(t,\tau) = H(t,\tau), \tag{3.87}$$

and this in turn shows that

$$\widehat{\widetilde{H}}(s) = \widehat{H}(s). \tag{3.88}$$

This last relationship can also be verified by observing that $\widehat{\widetilde{H}}(s) = \widetilde{C}(sI - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D} = CP^{-1}(sI - PAP^{-1})^{-1}PB + D = CP^{-1}P(sI - A)^{-1}P^{-1}PB + D = C(sI - A)^{-1}B + D = \widehat{H}(s)$.

Next, recall that in view of (3.70) we have for (3.61) that

$$\begin{aligned} y(t) &= Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} H(t-\tau,0)u(\tau)d\tau \\ &= \psi(t,t_0,x_0,0) + \rho(t,t_0,0,u) \end{aligned} \tag{3.89}$$

and for (3.82) that

$$\begin{aligned} y(t) &= \widetilde{C}e^{\widetilde{A}(t-t_0)}\tilde{x}_0 + \int_{t_0}^{t} \widetilde{H}(t-\tau,0)u(\tau)d\tau \\ &= \widetilde{\psi}(t,t_0,\tilde{x}_0,0) + \tilde{\rho}(t,t_0,0,u) \end{aligned} \tag{3.90}$$

where ψ and $\widetilde{\psi}$ denote the zero-input response of (3.61) and (3.82), respectively, whereas ρ and $\tilde{\rho}$ denote the zero-state response of (3.61) and (3.82), respectively. The relations (3.89) and (3.90) give rise to the following concepts: Two state-space representations are *zero-state equivalent* if they give rise to the same impulse response (the same external description). Also, two state-space representations are *zero-input equivalent* if for any initial state vector for one representation there exists an initial state vector for the second representation such that the zero-input responses for the two representations are identical.

The following result is now clear: *If two state-space representations are equivalent, then they are both zero-state and zero-input equivalent.* They are clearly zero-state equivalent since $H(t,\tau) = \widetilde{H}(t,\tau)$. Also, in view of (3.89) and (3.90), we have $\widetilde{C}e^{\widetilde{A}(t-t_0)}\tilde{x}_0 = (CP^{-1})[Pe^{A(t-t_0)}P^{-1}]\tilde{x}_0 = Ce^{A(t-t_0)}x_0$, where (3.86) was used. Therefore, the two state representations are also zero-input equivalent.

The converse to the above result is in general not true, since there are representations that are both zero-state and zero-input equivalent, yet not equivalent. In Chapter 8, which deals with state-space realizations of transfer functions, we will consider this topic further.

Example 3.28. System (3.61) with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [-1, -5], \quad D = 1$$

has the transfer function

$$H(s) = C(sI - A)^{-1}B + D = \frac{-5s - 1}{s^2 + 3s + 2} + 1 = \frac{(s-1)^2}{(s+1)(s+2)}.$$

Using the similarity transformation

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

yields the equivalent representation of the system given by

$$\widetilde{A} = PAP^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \widetilde{B} = PB = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \widetilde{C} = CP^{-1} = [4, 9]$$

and $\widetilde{D} = D = 1$. Note that the columns of P^{-1}, given by $[1, -1]^T$ and $[1, -2]^T$, are eigenvectors of A corresponding to the eigenvalues $\lambda_1 = -1, \lambda_2 = -2$ of A; that is, P was chosen to diagonalize A. Notice that A (which is in companion form) has characteristic polynomial $s^2+3s+2 = (s+1)(s+2)$. Notice also that the eigenvectors given above are of the form $[1, \lambda_i]^T$, $i = 1, 2$. The transfer function of the equivalent representation of the system is now given by

$$\hat{\tilde{H}}(s) = \widetilde{C}(sI - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D} = [4, 0]\begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1$$
$$= \frac{-5s - 1}{(s+1)(s+2)} + 1 = H(s).$$

Finally, it is easily verified that $e^{\widetilde{A}t} = Pe^{At}P^{-1}$.

From the above discussion it should be clear that systems [of the form (3.61)] described by equivalent representations have identical behavior to the *outside world*, since both their zero-input and zero-state responses are the same. Their states, however, are in general not identical, but are related by the transformation $\tilde{x}(t) = Px(t)$.

3.5 State Equation and Input–Output Description of Discrete-Time Systems

In this section, which consists of five subsections, we address the state equation and input–output description of linear discrete-time systems. In the first subsection we study the response of linear time-invariant systems described by the difference equations (2.15) [or (1.8)]. In the second subsection we consider transfer functions for linear time-invariant systems, whereas in the third subsection we address the equivalence of the internal representations of time-invariant linear discrete-time systems [described by (2.15)]. Some of the most important classes of discrete-time systems include linear sampled-data systems that we develop in the fourth subsection. In the final part of this section, we address the modes and asymptotic behavior of linear time-invariant discrete-time systems.

3.5.1 Response of Linear Discrete-Time Systems

We consider once again systems described by linear time-invariant equations of the form

$$x(k+1) = Ax(k) + Bu(k), \tag{3.91a}$$
$$y(k) = Cx(k) + Du(k), \tag{3.91b}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, and $D \in R^{p\times m}$. We recall that in (3.91), x denotes the state vector, u denotes the system input, and y denotes the system output. For given initial conditions $k_0 \in Z, x(k_0) = x_{k_0} \in R^n$ and for a given input u, equation (3.91a) possesses a unique solution $x(k)$, which is defined for all $k \geq k_0$, and thus, the response $y(k)$ for (3.91b) is also defined for all $k \geq k_0$.

Associated with (3.91a) is the linear autonomous, homogeneous system of equations given by

$$x(k+1) = Ax(k). \tag{3.92}$$

We recall from Section 2.3 that the solution of the initial-value problem

$$x(k+1) = Ax(k), \quad x(k_0) = x_{k_0} \tag{3.93}$$

is given by

$$x(k) = \Phi(k, k_0)x_{k_0} = A^{k-k_0}x_{k_0}, \quad k > k_0, \tag{3.94}$$

where $\Phi(k, k_0)$ denotes the state transition matrix of (3.92) with

$$\Phi(k, k) = I \tag{3.95}$$

[refer to (2.31) to (2.34) in Chapter 2].

Common properties of the state transition matrix $\Phi(k, l)$, such as for example the *semigroup property* (forward in time) given by

$$\Phi(k, l) = \Phi(k, m)\Phi(m, l), \quad k \geq m \geq l,$$

can quite easily be derived from (3.94), (3.95). We caution the reader, however, that not all of the properties of the state transition matrix $\Phi(t, \tau)$ for continuous-time systems $\dot{x} = Ax$ carry over to the discrete-time case (3.92). In particular we recall that if for the continuous-time case we have $t > \tau$, then future values of the state ϕ at time t can be obtained from past values of the state ϕ at time τ, and vice versa, from the relationships $\phi(t) = \Phi(t, \tau)\phi(\tau)$ and $\phi(\tau) = \Phi^{-1}(t, \tau)\phi(t) = \Phi(\tau, t)\phi(t)$, i.e., for continuous-time systems a principle of *time reversibility exists*. This principle is in general not true for system (3.92), unless $A^{-1}(k)$ exists. The reason for this lies in the fact that $\Phi(k, l)$ will not be nonsingular if A is not nonsingular.

Example 3.29. In (3.94), let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $x(0) = \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$, $\alpha \in R$. The initial state $x(0)$ at $k_0 = 0$ for *any* $\alpha \in R$ will map into the state $x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Accordingly, in this case, time reversibility will not apply.

Example 3.30. In (3.93), let $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$. In view of (3.94) we have that $A^{(k-k_0)} = \begin{bmatrix} (-1)^{(k-k_0)} & 1-(-1)^{(k-k_0)} \\ 0 & 1 \end{bmatrix}$, $k \geq k_0$; i.e., $A^{(k-k_0)} = A$ when $(k - k_0)$ is odd, and $A^{(k-k_0)} = I$ when $(k - k_0)$ is even. Therefore, given $k_0 = 0$ and $x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then $x(k) = Ax(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $k = 1, 3, 5, \ldots$, and

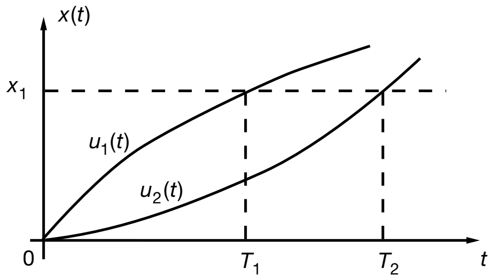

Figure 3.1. Plots of states for Example 3.30

$x(k) = Ix(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $k = 2, 4, 6, \ldots$. A plot of the states $x(k) = [x_1(k), x_2(k)]^T$ is given in Figure 3.1.

Continuing, we recall that the solutions of initial-value problems determined by linear nonhomogeneous systems (2.35) are given by expression (2.36). Utilizing (2.36), the solution of (3.91a) for given $x(k_0)$ and $u(k)$ is given as

$$x(k) = \Phi(k, k_0)x(k_0) + \sum_{j=k_0}^{k-1} \Phi(k, j+1)Bu(j), \quad k > k_0. \tag{3.96}$$

This expression in turn can be used to determine the system response for system (3.91) as

$$\begin{aligned} y(k) &= C\Phi(k, k_0)x(k_0) \\ &\quad + \sum_{j=k_0}^{k-1} C\Phi(k, j+1)Bu(j) + Du(k), \quad k > k_0, \\ y(k_0) &= Cx(k_0) + Du(k_0), \end{aligned} \tag{3.97}$$

or

$$\begin{aligned} y(k) &= CA^{(k-k_0)}x(k_0) + \sum_{j=k_0}^{k-1} CA^{k-(j+1)}Bu(j) + Du(k), \quad k > k_0, \\ y(k_0) &= Cx(k_0) + Du(k_0). \end{aligned} \tag{3.98}$$

Since the system (3.91) is time-invariant, we can let $k_0 = 0$ without loss of generality to obtain from (3.98) the expression

$$y(k) = CA^k x(0) + \sum_{j=0}^{k-1} CA^{k-(j+1)}Bu(j) + Du(k), \quad k > 0. \tag{3.99}$$

As in the continuous-time case, the *total system response* (3.97) may be viewed as consisting of two components, the *zero-input response*, given by

$$\psi(k) = C\Phi(k, k_0)x(k_0), \quad k > k_0,$$

and the *zero-state response*, given by

$$\left.\begin{aligned} \rho(k) &= \sum_{j=k_0}^{k-1} C\Phi(k, j+1)Bu(j) + Du(k), && k > k_0, \\ \rho(k_0) &= Du(k_0), && k = k_0. \end{aligned}\right\} \tag{3.100}$$

Finally, in view of (2.67), we recall that the (discrete-time) unit impulse response matrix of system (3.91) is given by

$$H(k, l) = \begin{cases} CA^{k-(l+1)}B, & k > l, \\ D, & k = l, \\ 0, & k < l, \end{cases} \tag{3.101}$$

and in particular, when $l = 0$ (i.e., when the pulse is applied at time $l = 0$),

$$H(k, 0) = \begin{cases} CA^{k-1}B, & k > 0, \\ D, & k = 0, \\ 0, & k < 0. \end{cases} \tag{3.102}$$

Example 3.31. In (3.91), let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad D = 0.$$

We first determine A^k by using the Cayley–Hamilton Theorem (refer to Section A.5). To this end we compute the eigenvalues of A as $\lambda_1 = 0, \lambda_2 = -1$, we let $A^k = f(A)$, where $f(s) = s^k$, and we let $g(s) = \alpha_1 s + \alpha_0$. Then $f(\lambda_1) = g(\lambda_1)$, $\alpha_0 = 0$ and $f(\lambda_2) = g(\lambda_2)$, or $(-1)^k = -\alpha_1 + \alpha_0$. Therefore, $A^k = \alpha_1 A + \alpha_0 I = -(-1)^k \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & (-1)^{k-1} \\ 0 & (-1)^k \end{bmatrix}$, $k = 1, 2, \ldots$, or $A^k = \begin{bmatrix} \delta(k) & (-1)^{k-1}p(k-1) \\ 0 & (-1)^k p(k) \end{bmatrix}$, $k = 0, 1, 2, \ldots$, where $A^0 = I$, and where $p(k)$ denotes the unit step given by

$$p(k) = \begin{cases} 1, & k \geq 0, \\ 0, & k < 0. \end{cases}$$

The above expression for A^k is now substituted into (3.98) to determine the response $y(k)$ for $k > 0$ for a given initial condition $x(0)$ and a given input $u(k), k \geq 0$. To determine the unit impulse response, we note that $H(k, 0) = 0$ for $k < 0$ and $k = 0$. When $k > 0, H(k, 0) = CA^{k-1}B = (-1)^{k-2}p(k-2)$ for $k > 0$ or $H(k, 0) = 0$ for $k = 1$ and $H(k, 0) = (-1)^{k-2}$ for $k = 2, 3, \ldots$.

3.5.2 The Transfer Function and the z-Transform

We assume that the reader is familiar with the concept and properties of the *one-sided z-transform* of a real-valued sequence $\{f(k)\}$, given by

$$\mathcal{Z}\{f(k)\} = \hat{f}(z) = \sum_{j=0}^{\infty} z^{-j} f(j). \tag{3.103}$$

An important property of this transform, which is useful in solving difference equations, is given by the relation

$$\begin{aligned}\mathcal{Z}\{f(k+1)\} &= \sum_{j=0}^{\infty} z^{-j} f(j+1) = \sum_{j=1}^{\infty} z^{-(j-1)} f(j) \\ &= z\left[\sum_{j=0}^{\infty} z^{-j} f(j) - f(0)\right] \\ &= z\left[\mathcal{Z}\{f(k)\} - f(0)\right] = z\hat{f}(z) - zf(0). \end{aligned} \tag{3.104}$$

If we take the z-transform of both sides of (3.91a), we obtain, in view of (3.104), $z\hat{x}(z) - zx(0) = A\hat{x}(z) + B\hat{u}(z)$ or

$$\hat{x}(z) = (zI - A)^{-1} zx(0) + (zI - A)^{-1} B\hat{u}(z). \tag{3.105}$$

Next, by taking the z-transform of both sides of (3.91b), and by substituting (3.105) into the resulting expression, we obtain

$$\hat{y}(z) = C(zI - A)^{-1} zx(0) + [C(zI - A)^{-1} B + D]\hat{u}(z). \tag{3.106}$$

The time sequence $\{y(k)\}$ can be recovered from its one-sided z-transform $\hat{y}(z)$ by applying the *inverse z-transform*, denoted by $\mathcal{Z}^{-1}[\hat{y}(z)]$.

In Table 3.3 we provide the one-sided z-transforms of some of the commonly used sequences, and in Table 3.4 we enumerate some of the more frequently encountered properties of the one-sided z-transform.

The *transfer function matrix* $\widehat{H}(z)$ of system (3.91) relates the z-transform of the output y to the z-transform of the input u under the assumption that $x(0) = 0$. We have

$$\hat{y}(z) = \widehat{H}(z)\hat{u}(z), \tag{3.107}$$

where

$$\widehat{H}(z) = C(zI - A)^{-1} B + D. \tag{3.108}$$

To relate $\widehat{H}(z)$ to the impulse response matrix $H(k, l)$, we notice that $\mathcal{Z}\{\delta(k - l)\} = z^{-l}$, where δ denotes the *discrete-time impulse* (or *unit pulse* or *unit sample*) defined in (2.51); i.e.,

$$\delta(k - l) = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases} \tag{3.109}$$

Table 3.3. Some commonly used z-transforms

$\{f(k)\}, \quad k \geq 0$	$\hat{f}(z) = \mathcal{Z}\{f(k)\}$
$\delta(k)$	1
$p(k)$	$1/(1 - z^{-1})$
k	$z^{-1}/(1 - z^{-1})^2$
k^2	$[z^{-1}(1 + z^{-1})]/(1 - z^{-1})^3$
a^k	$1/(1 - az^{-1})$
$(k + 1)a^k$	$1/(1 - az^{-1})^2$
$[(1/l!)(k + 1) \cdots (k + l)]a^k \quad l \geq 1$	$1/(1 - az^{-1})^{l+1}$
$a \cos \alpha k + b \sin \alpha k$	$\dfrac{a + z^{-1}(b \sin \alpha - a \cos \alpha)}{1 - 2z^{-1} \cos \alpha + z^{-2}}$

Table 3.4. Some properties of z-transforms

	$\{f(k)\}, k \geq 0$	$f(z)$
Time shift	$f(k + 1)$	$z\hat{f}(z) - zf(0)$
-Advance	$f(k + l) \quad l \geq 1$	$z^l \hat{f}(z) - z \sum_{i=1}^{l} z^{l-i} f(i - 1)$
Time shift	$f(k - 1)$	$z^{-1}\hat{f}(z) + f(-1)$
-Delay	$f(k - l) \quad l \geq 1$	$z^{-l}\hat{f}(z) + \sum_{i=1}^{l} z^{-l+i} f(-i)$
Scaling	$a^k f(k)$	$\hat{f}(z/a)$
	$kf(k)$	$-z(d/dz)\hat{f}(z)$
Convolution	$\sum_{l=0}^{\infty} f(l)g(k - l) = f(k) * g(k)$	$\hat{f}(z)\hat{g}(z)$
Initial value	$f(l)$ with $f(k) = 0, \quad k < l$	$\lim_{z\to\infty} z^l \hat{f}(z)^{\dagger}$
Final value	$\lim_{k\to\infty} f(k)$	$\lim_{z\to 1}(1 - z^{-1})\hat{f}(z)^{\ddagger}$

† If the limit exists.

‡ If $(1 - z^{-1})\hat{f}(z)$ has no singularities on or outside the unit circle.

This implies that the z-transform of a unit pulse applied at time zero is $\mathcal{Z}\{\delta(k)\} = 1$. It is not difficult to see now that $\{H(k, 0)\} = \mathcal{Z}^{-1}[\hat{y}(z)]$, where $\hat{y}(z) = \widehat{H}(z)\hat{u}(z)$ with $\hat{u}(z) = 1$. This shows that

$$\mathcal{Z}^{-1}[\widehat{H}(z)] = \mathcal{Z}^{-1}[C(zI - A)^{-1}B + D] = \{H(k, 0)\}, \tag{3.110}$$

where the unit impulse response matrix $H(k, 0)$ is given by (3.102).

The above result can also be derived directly by taking the z-transform of $\{H(k, 0)\}$ given in (3.102) (prove this). In particular, notice that the z-transform of $\{A^{k-1}\}$, $k = 1, 2, \ldots$ is $(zI - A)^{-1}$ since

$$\begin{aligned}
\mathcal{Z}\{0, A^{k-1}\} &= \sum_{j=1}^{\infty} z^{-j} A^{j-1} = z^{-1} \sum_{j=0}^{\infty} z^{-j} A^j \\
&= z^{-1}(I + z^{-1}A + z^{-2}A^2 + \ldots) \\
&= z^{-1}(I - z^{-1}A)^{-1} = (zI - A)^{-1}.
\end{aligned} \tag{3.111}$$

Above, the matrix determined by the expression $(1-\lambda)^{-1} = 1+\lambda+\lambda^2+\cdots$ was used. It is easily shown that the corresponding series involving A converges. Notice also that $\mathcal{Z}\{A^k\}, k = 0, 1, 2, \ldots$ is $z(zI - A)^{-1}$. This fact can be used to show that the inverse z-transform of (3.106) yields the time response (3.99), as expected.

We conclude this subsection with a specific example.

Example 3.32. In system (3.91), we let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1\ 0], \quad D = 0.$$

To verify that $\mathcal{Z}^{-1}[z(zI-A)^{-1}] = A^k$, we compute $z(zI-A)^{-1} = z\left[\begin{smallmatrix} z & -1 \\ 0 & z+1 \end{smallmatrix}\right]^{-1}$ $= z\begin{bmatrix} \frac{1}{z} & \frac{1}{z(z+1)} \\ 0 & \frac{1}{z+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{z+1} \\ 0 & \frac{z}{z+1} \end{bmatrix}$ and

$$\mathcal{Z}^{-1}[z(zI - A)^{-1}] = \begin{bmatrix} \delta(k) & (-1)^{k-1}p(k-1) \\ 0 & (-1)^k p(k) \end{bmatrix}$$

or

$$A^k = \begin{cases} \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], & \text{when } k = 0, \\ \left[\begin{smallmatrix} 0 & (-1)^{k-1} \\ 0 & (-1)^k \end{smallmatrix}\right], & \text{when } k = 1, 2, \ldots, \end{cases}$$

as expected from Example 3.31.

Notice that

$$\begin{aligned} \mathcal{Z}^{-1}[(zI - A)^{-1}] &= \mathcal{Z}^{-1}\left[\begin{bmatrix} 1/z & 1/[z(z+1)] \\ 0 & 1/(z+1) \end{bmatrix}\right] \\ &= \begin{bmatrix} \delta(k-1)p(k-1) & \delta(k-1)p(k-1) - (-1)^{k-1}p(k-1) \\ 0 & (-1)^{k-1}p(k-1) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ for } k = 0, \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for } k = 1, \end{aligned}$$

and

$$\mathcal{Z}^{-1}[(zI - A)^{-1}] = \begin{bmatrix} 0 & -(-1)^{k-1} \\ 0 & (-1)^{k-1} \end{bmatrix} \text{ for } k = 2, 3, \ldots,$$

which is equal to $A^k, k \geq 0$, delayed by one unit; i.e., it is equal to $A^{k-1}, k = 1, 2, \ldots$, as expected.

Next, we consider the system response with $x(0) = 0$ and $u(k) = p(k)$. We have

$$\begin{aligned} y(k) &= \mathcal{Z}^{-1}[\hat{y}(z)] = \mathcal{Z}^{-1}[C(zI - A)^{-1}B \cdot \hat{u}(z)] \\ &= \mathcal{Z}^{-1}\left[\frac{1}{(z+1)(z-1)}\right] = \mathcal{Z}^{-1}\left[\tfrac{1/2}{z-1} - \tfrac{1/2}{z+1}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}[(1)^{k-1} - (-1)^{k-1}]p(k-1) \\
&= \begin{cases} 0, & k = 0, \\ \frac{1}{2}(1 - (-1)^{k-1}), & k = 1, 2, \ldots, \end{cases} \\
&= \begin{cases} 0, & k = 0, \\ 0, & k = 1, 3, 5, \ldots, \\ 1, & k = 2, 4, 6, \ldots. \end{cases}
\end{aligned}
$$

Note that if $x(0) = 0$ and $u(k) = \delta(k)$, then

$$
\begin{aligned}
y(k) &= \mathcal{Z}^{-1}[C(zI - A)^{-1}B] = \mathcal{Z}^{-1}\left[\frac{1}{z(z+1)}\right] \\
&= \delta(k-1)p(k-1) - (-1)^{k-1}p(k-1) \\
&= \begin{cases} 0, & k = 0, 1, \\ (-1)^{k-2}, & k = 2, 3, \ldots, \end{cases}
\end{aligned}
$$

which is the unit impulse response of the system (refer to Example 3.31).

3.5.3 Equivalence of State-Space Representations

Equivalent representations of linear discrete-time systems are defined in a manner analogous to the continuous-time case. For systems (3.91), we let P denote a real nonsingular $n \times n$ matrix and we define

$$\tilde{x}(k) = Px(k). \tag{3.112}$$

Substituting (3.112) into (3.91) yields the equivalent system representation

$$\tilde{x}(k+1) = \widetilde{A}\tilde{x}(k) + \widetilde{B}u(k), \tag{3.113a}$$

$$y(k) = \widetilde{C}\tilde{x}(k) + \widetilde{D}u(k), \tag{3.113b}$$

where

$$\widetilde{A} = P^{-1}AP, \quad \widetilde{B} = PB, \quad \widetilde{C} = CP^{-1}, \quad \widetilde{D} = D. \tag{3.114}$$

We note that the terms in (3.114) are identical to corresponding terms obtained for the case of linear continuous-time systems.

We conclude by noting that if $\widehat{H}(z)$ and $\widehat{\widetilde{H}}(z)$ denote the transfer functions of the unit impulse response matrices of system (3.91) and system (3.113), respectively, then it is easily verified that $\widehat{H}(z) = \widehat{\widetilde{H}}(z)$.

3.5.4 Sampled-Data Systems

Discrete-time dynamical systems arise in a variety of ways in the modeling process. There are systems that are inherently defined only at discrete points in time, and there are representations of continuous-time systems at discrete points in time. Examples of the former include digital computers and devices (e.g., digital filters) where the behavior of interest of a system is adequately described by values of variables at discrete-time instants (and what happens between the discrete instants of time is quite irrelevant to the problem on hand); inventory systems where only the inventory status at the end of each day (or month) is of interest; economic systems, such as banking, where, e.g., interests are calculated and added to savings accounts at discrete-time intervals only; and so forth. Examples of the latter include simulations of continuous-time processes by means of digital computers, making use of difference equations that approximate the differential equations describing the process in question; feedback control systems that employ digital controllers and give rise to sampled-data systems (as discussed further in the following); and so forth.

In providing a short discussion of sampled-data systems, we make use of the specific class of linear feedback control systems depicted in Figure 3.2. This system may be viewed as an interconnection of a subsystem S_1, called the *plant* (the object to be controlled), and a subsystem S_2, called the *digital controller*. The plant is described by the equations

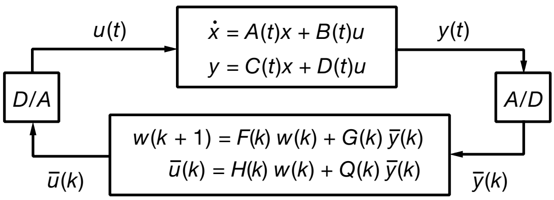

Figure 3.2. Digital control system

$$\dot{x} = A(t)x + B(t)u, \tag{3.115a}$$

$$y = C(t)x + D(t)u, \tag{3.115b}$$

where all symbols in (3.115) are defined as in (2.3) and where we assume that $t \geq t_0 \geq 0$.

Since our presentation pertains equally to the time-varying and time-invariant cases, we will first address the more general time-varying case. Next, we specialize our results to the time-invariant case.

The subsystem S_2 accepts the continuous-time signal $y(t)$ as its input, and it produces the piecewise continuous-time signal $u(t)$ as its output, where $t \geq t_0$. The continuous-time signal y is converted into a discrete-time signal $\{\bar{y}(k)\}$,

$k \geq k_0 \geq 0$, $k, k_0 \in Z$, by means of an analog-to-digital (A/D) converter and is processed according to a control algorithm given by the difference equations

$$w(k+1) = F(k)w(k) + G(k)\bar{y}(k), \tag{3.116a}$$
$$\bar{u}(k) = H(k)w(k) + Q(k)\bar{y}(k), \tag{3.116b}$$

where the $w(k), \bar{y}(k), \bar{u}(k)$ are real vectors and the $F(k), G(k), H(k)$, and $Q(k)$ are real, time-varying matrices with a consistent set of dimensions. Finally, the discrete-time signal $\{\bar{u}(k)\}$, $k \geq k_0 \geq 0$, is converted into the continuous-time signal u by means of a digital-to-analog (D/A) converter. To simplify our discussion, we assume in the following that $t_0 = k_0$.

An (ideal) A/D converter is a device that has as input a continuous-time signal, in our case y, and as output a sequence of real numbers, in our case $\{\bar{y}(k)\}, k = k_0, k_0 + 1, \ldots$, determined by the relation

$$\bar{y}(k) = y(t_k). \tag{3.117}$$

In other words, the (ideal) A/D converter is a device that *samples* an input signal, in our case $y(t)$, at times $t_0, t_1, \ldots$ producing the corresponding sequence $\{y(t_0), y(t_1), \ldots\}$.

A *D/A converter* is a device that has as input a discrete-time signal, in our case the sequence $\{\bar{u}(k)\}, k = k_0, k_0 + 1, \ldots$, and as output a continuous-time signal, in our case u, determined by the relation

$$u(t) = \bar{u}(k), t_k \leq t < t_{k+1}, \quad k = k_0, k_0 + 1, \ldots. \tag{3.118}$$

In other words, the D/A converter is a device that keeps its output constant at the last value of the sequence entered. We also call such a device a *zero-order hold*.

The system of Figure 3.2, as described above, is an example of a *sampled-data system*, since it involves truly *sampled data* (i.e., sampled signals), making use of an *ideal A/D converter*. In practice the digital controller S_2 uses *digital signals* as variables. In the scalar case, such signals are represented by real-valued sequences whose numbers belong to a subset of R consisting of a discrete set of points. (In the vector case, the previous statement applies to the components of the vector.) Specifically, in the present case, after the signal $y(t)$ has been sampled, it must be *quantized* (or *digitized*) to yield a *digital signal*, since only such signals are representable in a digital computer. If a computer uses, e.g., 8-bit words, then we can represent $2^8 = 256$ distinct levels for a variable, which determine the signal quantization. By way of a specific example, if we expect in the representation of a function a signal that varies from 9 to 25 volts, we may choose a 0.1-volt quantization step. Then 2.3 and 2.4 volts are represented by two different numbers (quantization levels); however, 2.315, 2.308, and 2.3 are all represented by the bit combination corresponding to 2.3. Quantization is an approximation and for short wordlengths may lead to significant errors. Problems associated with *quantization effects* will not be addressed in this book.

In addition to being a sampled-data system, the system represented by (3.115) to (3.118) constitutes a *hybrid system* as well, since it involves descriptions given by ordinary differential equations and ordinary difference equations. The analysis and synthesis of such systems can be simplified appreciably by replacing the description of subsystem S_1 (the plant) by a set of ordinary difference equations, valid only at discrete points in time $t_k, k = 0, 1, 2, \ldots$. [In terms of the blocks of Figure 3.2, this corresponds to considering the plant S_1, together with the D/A and A/D devices, to obtain a system with input $\bar{u}(k)$ and output $\bar{y}(k)$, as shown in Figure 3.3.] To accomplish this, we apply the variation of constants formula to (3.115a) to obtain

$$x(t) = \Phi(t, t_k)x(t_k) + \int_{t_k}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau, \tag{3.119}$$

where the notation $\phi(t, t_k, x(t_k)) = x(t)$ has been used. Since the input $u(t)$

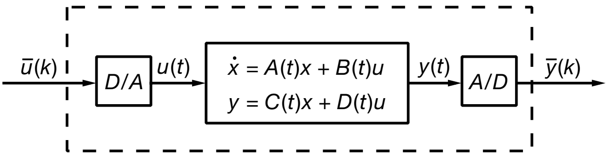

Figure 3.3. System described by (3.121) and (3.124)

is the output of the zero-order hold device (the D/A converter), given by (3.118), we obtain from (3.119) the expression

$$x(t_{k+1}) = \Phi(t_{t+1}, t_k)x(t_k) + \left[\int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau)B(\tau)d\tau\right] u(t_k). \tag{3.120}$$

Since $\bar{x}(k) \triangleq x(t_k)$ and $\bar{u}(k) \triangleq u(t_k)$, we obtain a discrete-time version of the state equation for the plant, given by

$$\bar{x}(k+1) = \bar{A}(k)\bar{x}(k) + \bar{B}(k)\bar{u}(k), \tag{3.121}$$

where

$$\left.\begin{aligned} \bar{A}(k) &\triangleq \Phi(t_{k+1}, t_k), \\ \bar{B}(k) &\triangleq \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau)B(\tau)d\tau. \end{aligned}\right\} \tag{3.122}$$

Next, we assume that the output of the plant is sampled at instants t'_k that do not necessarily coincide with the instants t_k at which the input to the plant is adjusted, and we assume that $t_k \leq t'_k < t_{k+1}$. Then (3.115) and (3.119) yield

$$y(t'_k) = C(t'_k)\Phi(t'_k, t_k)x(t_k) + \left[C(t'_k)\int_{t_k}^{t'_k}\Phi(t'_k, \tau)B(\tau)d\tau\right]u(t_k) + D(t'_k)u(t_k). \tag{3.123}$$

Defining $\bar{y}(k) \triangleq y(t'_k)$, we obtain from (3.123),

$$\bar{y}(k) = \bar{C}(k)\bar{x}(k) + \bar{D}(k)\bar{u}(k), \tag{3.124}$$

where

$$\left.\begin{aligned} \bar{C}(k) &\triangleq C(t'_k)\Phi(t'_k, t_k), \\ \bar{D}(k) &\triangleq C(t'_k)\int_{t_k}^{t'_k}\Phi(t'_k, \tau)B(\tau)d\tau + D(t'_k). \end{aligned}\right\} \tag{3.125}$$

Summarizing, (3.121) and (3.124) constitute a state-space representation, valid at discrete points in time, of the plant [given by (3.115a)] and including the A/D and D/A devices [given by (3.117) and (3.118), see Figure 3.3]. Furthermore, the entire hybrid system of Figure 3.2, valid at discrete points in time, can now be represented by (3.121), (3.124), and (3.116).

Time-Invariant System With Constant Sampling Rate

We now turn to the case of the time-invariant plant, where $A(t) \equiv A, B(t) \equiv B, C(t) \equiv C$, and $D(t) \equiv D$, and we assume that $t_{k+1} - t_k = T$ and $t'_k - t_k = \alpha$ for all $k = 0, 1, 2, \ldots$. Then the expressions given in (3.121), (3.122), (3.124), and (3.125) assume the form

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k), \tag{3.126a}$$

$$\bar{y}(k) = \bar{C}\bar{x}(k) + \bar{D}\bar{u}(k), \tag{3.126b}$$

where

$$\left.\begin{aligned} \bar{A} &= e^{AT}, & \bar{B} &= \left(\int_0^T e^{A\tau}d\tau\right)B, \\ \bar{C} &= Ce^{A\alpha}, & \bar{D} &= C\left(\int_0^\alpha e^{A\tau}d\tau\right)B + D. \end{aligned}\right\} \tag{3.127}$$

If $t'_k = t_k$, or $\alpha = 0$, then $\bar{C} = C$ and $\bar{D} = D$.

In the preceding, T is called the *sampling period* and $1/T$ is called the *sampling rate*. Sampled-data systems are treated in great detail in texts dealing with digital control systems and with digital signal processing.

Example 3.33. In the control system of Figure 3.2, let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1, 0], \quad D = 0,$$

let T denote the sampling period, and assume that $\alpha = 0$. The discrete-time state-space representation of the plant, preceded by a zero-order hold

(D/A converter) and followed by a sampler [an (ideal) A/D converter], both sampling synchronously at a rate of 1/T, is given by $\bar{x}(k+1) = \bar{A}\bar{x}(k)+\bar{B}\bar{u}(k)$, $\bar{y}(k) = \bar{C}x(k)$, where

$$\bar{A} = e^{AT} = \sum_{j=1}^{\infty}(T^j/j!)A^j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} T = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix},$$

$$\bar{B} = \left(\int_0^T e^{A\tau} d\tau\right) B = \left(\int_0^T \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} d\tau\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} T & T^2/2 \\ 0 & T \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix},$$

$$\bar{C} = C = [1\ 0].$$

The transfer function (relating $\bar{y}$ to $\bar{u}$) is given by

$$\begin{aligned} \widehat{H}(z) &= \bar{C}(zI - \bar{A})^{-1}\bar{B} \\ &= [1\ 0] \begin{bmatrix} z-1 & -T \\ 0 & z-1 \end{bmatrix}^{-1} \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \\ &= [1\ 0] \begin{bmatrix} 1/(z-1) & T/(z-1)^2 \\ 0 & 1/(z-1) \end{bmatrix} \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \\ &= \frac{T^2}{2} \frac{(z+1)}{(z-1)^2}. \end{aligned}$$

The transfer function of the continuous-time system (continuous-time description of the plant) is determined to be $\widehat{H}(s) = C(sI-A)^{-1}B = 1/s^2$, the double integrator.

The behavior of the system between the discrete instants, $t, t_k \le t < t_{k+1}$, can be determined by using (3.119), letting $x(t_k) = x(k)$ and $u(t_k) = u(k)$.

An interesting observation, useful when calculating $\bar{A}$ and $\bar{B}$, is that both can be expressed in terms of a single series. In particular, $\bar{A} = e^{AT} = I+TA+(T^2/2!)A^2+\cdots = I+TA\Psi(T)$, where $\Psi(T) = I+(T/2!)A+(T^2/3!)A^2+\cdots = \sum_{j=0}^{\infty}(T^j/(j+1)!)A^j$. Then $\bar{B} = (\int_0^T e^{A\tau}d\tau)\ B = (\sum_{j=0}^{\infty}(T^{j+1}/(j+1)!)A^j)$ $B = T\Psi(T)B$. If $\Psi(T)$ is determined first, and then both $\bar{A}$ and $\bar{B}$ can easily be calculated.

Example 3.34. In Example 3.33, $\Psi(T) = I + TA = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$. Therefore, $\bar{A} = I + TA\Psi(T) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$ and $\bar{B} = T\Psi(T)B = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$, as expected.

3.5.5 Modes, Asymptotic Behavior, and Stability

As in the case of continuous-time systems, we study in this subsection the qualitative behavior of the solutions of linear, autonomous, homogeneous ordinary difference equations

$$x(k+1) = Ax(k) \tag{3.128}$$

in terms of the modes of such systems, where $A \in R^{n\times n}$ and $x(k) \in R^n$ for every $k \in Z^+$. From before, the unique solution of (3.128) satisfying $x(0) = x_0$ is given by

$$\phi(k, 0, x_0) = A^k x_0. \tag{3.129}$$

Let $\lambda_1, \ldots, \lambda_\sigma$, denote the σ distinct eigenvalues of A, where λ_i with $i = 1, \ldots, \sigma$, is assumed to be repeated n_i times so that $\sum_{i=1}^{\sigma} n_i = n$. Then

$$\det(zI - A) = \prod_{i=1}^{\sigma}(z - \lambda_i)^{n_i}. \tag{3.130}$$

To introduce the modes for (3.128), we first derive the expressions

$$\begin{aligned} A^k &= \sum_{i=1}^{\sigma}[A_{i0}\lambda_i^k p(k) + \sum_{l=1}^{n_i-1} A_{il}k(k-1)\cdots(k-l+1)\lambda_i^{k-l}p(k-l)] \\ &= \sum_{i=1}^{\sigma}[A_{i0}\lambda_i^k p(k) + A_{i1}k\lambda_i^{k-1}p(k-1) + \cdots \\ &\quad + A_{i(n_i-1)}k(k-1)\cdots(k-n_i+2)\lambda_i^{k-(n_i-1)}p(k-n_i+1)], \end{aligned} \tag{3.131}$$

where

$$A_{il} = \frac{1}{l!}\,\frac{1}{(n_i-1-l)!}\lim_{z\to\lambda_i}\{[(z-\lambda_i)^{n_i}(zI-A)^{-1}]^{(n_i-1-l)}\}. \tag{3.132}$$

In (3.132), $[\cdot]^{(q)}$ denotes the qth derivative with respect to z, and in (3.131), $p(k)$ denotes the unit step [i.e., $p(k) = 0$ for $k < 0$ and $p(k) = 1$ for $k \geq 0$]. Note that if an eigenvalue λ_i of A is zero, then (3.131) must be modified. In this case,

$$\sum_{i=0}^{n_i-1} A_{il}l!\delta(k-l) \tag{3.133}$$

are the terms in (3.131) corresponding to the zero eigenvalue.

To prove (3.131), (3.132), we proceed as in the proof of (3.50), (3.51). We recall that $\{A^k\} = \mathcal{Z}^{-1}[z(zI-A)^{-1}]$ and we use the partial fraction expansion method to determine the z-transform. In particular, as in the proof of (3.50), (3.51), we can readily verify that

$$z(zI - A)^{-1} = z \sum_{i=1}^{\sigma} \sum_{l=0}^{n_i - 1} (l! A_{il})(z - \lambda_i)^{-(l+1)}, \tag{3.134}$$

where the A_{il} are given in (3.132). We now take the inverse z-transform of both sides of (3.134). We first notice that

$$\begin{aligned} \mathcal{Z}^{-1}[z(z-\lambda_i)^{-(l+1)}] &= \mathcal{Z}^{-1}[z^{-l} z^{l+1} (z - \lambda_i)^{-(l+1)}] \\ &= \mathcal{Z}^{-1}[z^{-l}(1 - \lambda_i z^{-1})^{-(l+1)}] = f(k-l)p(k-l) \\ &= \begin{cases} f(k-l), & \text{for } k \geq l, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Referring to Tables 3.3 and 3.4 we note that $f(k)p(k) = \mathcal{Z}^{-1}[(1-\lambda_i z^{-1})^{-(l+1)}] = [\frac{1}{l!}(k+1)\cdots(k+l)]\lambda_i^k$ for $\lambda_i \neq 0$ and $l \geq 1$. Therefore, $\mathcal{Z}^{-1}[l! z(z-\lambda_i)^{-(l+1)}] = l! f(k-l)p(k-l) = k(k-1)\cdots(k-l+1)\lambda_i^{k-l}, l \geq 1$. For $l = 0$, we have $\mathcal{Z}^{-1}[(1 - \lambda_i z^{-1})^{-1}] = \lambda_i^k$. This shows that (3.131) is true when $\lambda_i \neq 0$. Finally, if $\lambda_i = 0$, we note that $\mathcal{Z}^{-1}[l! z^{-l}] = l! \delta(k-l)$, which implies (3.133).

Note that one can derive several alternative but equivalent expressions for (3.131) that correspond to different ways of determining the inverse z-transform of $z(zI - A)^{-1}$ or of determining A^k via some other methods.

In complete analogy with the continuous-time case, we call the terms $A_{il} k(k-1)\cdots(k-l+1)\lambda_i^{k-l}$ the *modes of the system* (3.128). There are n_i modes corresponding to the eigenvalues $\lambda_i, l = 0, \ldots, n_i - 1$, and the system (3.128) has a total of n modes.

It is particularly interesting to study the matrix $A^k, k = 0, 1, 2, \ldots$ using the Jordan canonical form of A, i.e., $J = P^{-1}AP$, where the similarity transformation P is constructed by using the generalized eigenvectors of A. We recall once more that $J = \text{diag}[J_1, \ldots, J_\sigma] \triangleq \text{diag}[J_i]$ where each $n_i \times n_i$ block J_i corresponds to the eigenvalue λ_i and where, in turn, $J_i = \text{diag}[J_{i1}, \ldots, J_{il_i}]$ with J_{ij} being smaller square blocks, the dimensions of which depend on the length of the chains of generalized eigenvectors corresponding to J_i (refer to Subsection 3.3.2). Let J_{ij} denote a typical Jordan canonical form block. We shall investigate the matrix J_{ij}^k, since $A^k = P^{-1}J^kP = P^{-1}\text{diag}[J_{ij}^k]P$.

Let

$$J_{ij} = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & \lambda_i \end{bmatrix} = \lambda_i I + N_i, \tag{3.135}$$

where

$$N_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

and where we assume that J_{ij} is a $t \times t$ matrix. Then

$$\begin{aligned}(J_{ij})^k &= (\lambda_i I + N_i)^k \\ &= \lambda_i^k I + k\lambda_i^{k-1} N_i + \frac{k(k-1)}{2!}\lambda_i^{k-2} N_i^2 + \cdots + k\lambda_i N_i^{k-1} + N_i^k. \quad (3.136)\end{aligned}$$

Now since $N_i^k = 0$ for $k \geq t$, a typical $t \times t$ Jordan block J_{ij} will generate terms that involve only the scalars $\lambda_i^k, \lambda_i^{k-1}, \ldots, \lambda_i^{k-(t-1)}$. Since the largest possible block associated with the eigenvalue λ_i is of dimension $n_i \times n_i$, the expression of A^k in (3.131) should involve at most the terms $\lambda_i^k, \lambda_i^{k-1}, \ldots, \lambda_i^{k-(n_i-1)}$, which it does.

The above enables us to prove the following useful fact: Given $A \in R^{n\times n}$, there exists an integer $k \geq 0$ such that

$$A^k = 0 \quad (3.137)$$

if and only if all the eigenvalues λ_i of A are at the origin. Furthermore, the smallest k for which (3.137) holds is equal to the dimension of the largest block J_{ij} of the Jordan canonical form of A.

The second part of the above assertion follows readily from (3.136). We ask the reader to prove the first part of the assertion.

We conclude by observing that when all n eigenvalues λ_i of A are distinct, then

$$A^k = \sum_{i=1}^{n} A_i \lambda_i^k, \quad k \geq 0, \quad (3.138)$$

where

$$A_i = \lim_{z \to \lambda_i} [(z - \lambda_i)(zI - A)^{-1}]. \quad (3.139)$$

If $\lambda_i = 0$, we use $\delta(k)$, the unit pulse, in place of λ_i^k in (3.138). This result is straightforward, in view of (3.131), (3.132).

Example 3.35. In (3.128) we let $A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & 1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = \lambda_2 = \frac{1}{2}$, and therefore, $n_1 = 2$ and $\sigma = 1$. Applying (3.131), (3.132), we obtain

$$\begin{aligned}A^k &= A_{10}\lambda_1^k p(k) + A_{11} k \lambda_1^{k-1} p(k-1) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\frac{1}{2}\right)^k p(k) + \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} (k) \left(\frac{1}{2}\right)^{k-1} p(k-1).\end{aligned}$$

Example 3.36. In (3.128) we let $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = -1, \lambda_2 = 1$ (so that $\sigma = 2$). Applying (3.138), (3.139), we obtain

$$A^k = A_{10}\lambda_1^k + A_{20}\lambda_2^k = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} (-1)^k + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad k \geq 0.$$

Note that this same result was obtained by an entirely different method in Example 3.30.

Example 3.37. In (3.128) we let $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = -1$, and $\sigma = 2$. Applying (3.138), (3.139), we obtain

$$A_0 = \lim_{z \to 0} [z(zI - A)^{-1}] = \frac{1}{z+1} \begin{bmatrix} z+1 & 1 \\ 0 & z \end{bmatrix} |_{z=0} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A_1 = \lim_{z \to -1} \left[(z+1) \frac{1}{z(z+1)} \begin{bmatrix} z+1 & 1 \\ 0 & z \end{bmatrix} \right] = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

and

$$A^k = A_0\delta(k) + A_1(-1)^k = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \delta(k) + \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} (-1)^k, \quad k \geq 0.$$

As in the case of continuous-time systems described by (3.48), various notions of stability of an equilibrium for discrete-time systems described by linear, autonomous, homogeneous ordinary difference equations (3.128) will be studied in detail in Chapter 4. If $\phi(k, 0, x_e)$ denotes the solution of system (3.128) with $x(0) = x_e$, then x_e is said to be an *equilibrium* of (3.128) if $\phi(k, 0, x_e) = x_e$ for all $k \geq 0$. Clearly, $x_e = 0$ is an equilibrium of (3.128). In discussing the qualitative properties, it is customary to speak, somewhat informally, of the stability properties of (3.128), rather than the stability properties of the equilibrium $x_e = 0$ of system (3.128).

The concepts of *stability*, *asymptotic stability*, and *instability* of system (3.128) are now defined in an identical manner as in Subsection 3.3.3 for system (3.48), except that in this case continuous-time t ($t \in R^+$) is replaced by discrete-time k ($k \in Z^+$).

By inspecting the modes of system (3.128) [given by (3.131) and (3.132)], we can readily establish the following stability criteria:

1. The system (3.128) is *asymptotically stable* if and only if all eigenvalues of A are within the unit circle of the complex plane (i.e., $|\lambda_j| < 1$, $j = 1, \ldots, n$).

2. The system (3.128) is *stable* if and only if $|\lambda_j| \leq 1$, $j = 1, \ldots, n$, and for all eigenvalues with $|\lambda_j| = 1$ having multiplicity $n_j > 1$, it is true that

$$\lim_{z \to \lambda_j} [[(z - \lambda_j)^{n_j}(zI - A)^{-1}]^{(n_j - 1 - l)}] = 0 \text{ for } l = 1, \ldots, n_j - 1. \quad (3.140)$$

3. The system (3.128) is *unstable* if and only if (2) is not true.

Example 3.38. The system given in Example 3.35 is asymptotically stable. The system given in Example 3.36 is stable. In particular, note that the solution $\phi(k, 0, x(0)) = A^k x(0)$ for Example 3.36 is bounded.

When the eigenvalues λ_i of A are distinct, then as in the continuous-time case [refer to (3.56), (3.57)], we can readily show that

$$A^k = \sum_{j=1}^{n} A_j \lambda_j^k, \quad A_j = v_j \tilde{v}_j, \quad k \geq 0, \quad (3.141)$$

where the v_j and $\tilde{v}_j$ are right and left eigenvectors of A corresponding to λ_j, respectively. If $\lambda_j = 0$, we use $\delta(k)$, the unit pulse, in place of λ_j^k in (3.141).

In proving (3.141), we use the same approach as in the proof of (3.56), (3.57). We have $A^k = Q \operatorname{diag}[\lambda_1^k, \ldots, \lambda_n^k] Q^{-1}$, where the columns of Q are the n right eigenvectors and the rows of Q^{-1} are the n left eigenvectors of A.

As in the continuous-time case [system (3.48)], the initial condition $x(0)$ for system (3.128) can be selected to be colinear with the eigenvector v_i to eliminate from the solution of (3.128) all modes except the ones involving λ_i^k.

Example 3.39. As in Example 3.36, we let $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$. Corresponding to the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 1$, we have the right and left eigenvectors $v_1 = (1, 0)^T, v_2 = (1, 1)^T, \tilde{v}_1 = (1, -1)$, and $\tilde{v}_2 = (0, 1)$. Then

$$\begin{aligned} A^k &= [v_1 \ \tilde{v}_1]\lambda_1^k + [v_2 \ \tilde{v}_2]\lambda_2^k \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} (-1)^k + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (1)^k, \quad k \geq 0. \end{aligned}$$

Choose $x(0) = \alpha(1, 0)^T = \alpha v_1$ with $\alpha \neq 0$. Then

$$\phi(k, 0, x(0)) = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} (-1)^k,$$

which contains only the mode associated with $\lambda_1 = -1$.

We conclude our discussion of modes and asymptotic behavior by briefly considering the state equation

$$x(k+1) = Ax(k) + Bu(k), \tag{3.142}$$

where x, u, A, and B are as defined in (3.91a). Taking the $\mathcal{Z}$-transform of both sides of (3.142) and rearranging yields

$$\tilde{x}(z) = z(zI - A)^{-1}x(0) + (zI - A)^{-1}B\tilde{u}(z). \tag{3.143}$$

By taking the inverse $\mathcal{Z}$-transform of (3.143), we see that the solution ϕ of (3.142) is the sum of modes that correspond to the singularities or poles of $z(zI - A)^{-1}x(0)$ and of $(zI - A)^{-1}B\tilde{u}(z)$. If in particular, system (3.128) is asymptotically stable [i.e., for $x(k+1) = Ax(k)$, all eigenvalues λ_j of A are such that $|\lambda_j| < 1$, $j = 1, \dots, n$] and if $u(k)$ in (3.142) is bounded [i.e., there is an M such that $|u_i(k)| < M$ for all $k \geq 0, i = 1, \dots, m$], then it is easily seen that the solutions of (3.142) are bounded as well.

3.6 An Important Comment on Notation

Chapters 1–3 are primarily concerned with the basic (qualitative) properties of systems of first-order ordinary differential equations, such as, e.g., the system of equations given by

$$\dot{x} = Ax, \tag{3.144}$$

where $x \in R^n$ and $A \in R^{n\times n}$. In the arguments and proofs to establish various properties for such systems, we highlighted the solutions by using the ϕ-notation. Thus, the unique solution of (3.144) for a given set of initial data (t_0, x_0) was written as $\phi(t, t_0, x_0)$ with $\phi(t_0, t_0, x_0) = x_0$. A similar notation was used in the case of the equation given by

$$\dot{x} = f(t, x) \tag{3.145}$$

and the equations given by

$$\dot{x} = Ax + Bu, \tag{3.146a}$$

$$y = Cx + Du, \tag{3.146b}$$

where in (3.145) and in (3.146) all symbols are defined as in (1.11) (see Chapter 1) and as in (3.61) of this chapter, respectively.

In the study of control systems such as system (3.61), the center of attention is usually the control input u and the resulting evolution of the system state in the state space and the system output. In the development of control systems theory, the x-notation has been adopted to express the solutions of systems. Thus, the solution of (3.61a) is denoted by $x(t)$ [or $x(t, t_0, x_0)$ when t_0 and x_0 are to be emphasized] and the evolution of the system output y

in (3.61b) is denoted by $y(t)$. In all subsequent chapters, except Chapter 4, we will also follow this practice, employing the usual notation utilized in the control systems literature. In Chapter 4, which is concerned with the stability properties of systems, we will use the ϕ-notation when studying the Lyapunov stability of an equilibrium [such as system (3.144)] and the x-notation when investigating the input–output properties of control systems [such as system (3.61)].

3.7 Summary and Highlights

Continuous-Time Systems

- *The state transition matrix* $\Phi(t, t_0)$ of $\dot{x} = Ax$

$$\Phi(t, t_0) \triangleq \Psi(t)\Psi^{-1}(t_0), \tag{3.9}$$

  where $\Psi(t)$ is any fundamental matrix of $\dot{x} = Ax$. See Definitions 3.8 and 3.2 and Theorem 3.9 for properties of $\Phi(t, t_0)$. In the present time-invariant case

$$\Phi(t, t_0) = e^{A(t-t_0)},$$

  where

$$e^{At} = I + \sum_{k=1}^{\infty} \frac{t^k A^k}{k!} \tag{3.17}$$

  is the matrix exponential. See Theorem 3.13 for properties.
- *Methods to evaluate* e^{At}. Via infinite series (3.17) and via similarity transformation

$$e^{At} = P^{-1} e^{Jt} P$$

  with $J = P^{-1}AP$ [see (3.25)] where J is diagonal or in Jordan canonical form; via the Cayley–Hamilton Theorem [see (3.41)] and via the Laplace transform, where

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}], \tag{3.45}$$

  or via the system modes [see (3.50)], which simplify to

$$e^{At} = \sum_{i=1}^{n} A_i e^{\lambda_i t} \tag{3.53}$$

  when the n eigenvalues of A, λ_i, are distinct. See also (3.56), (3.57).
- *Modes of the system.* e^{At} is expressed in terms of the modes $A_{ik} t^k e^{\lambda_i t}$ in (3.50). The distinct eigenvalue case is found in (3.53), (3.54) and in (3.56), (3.57).
- *The stability of an equilibrium* of $\dot{x} = Ax$ is defined and related to the eigenvalues of A using the expression for e^{At} in terms of the modes.

- Given $\dot{x} = Ax + Bu$,

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)s$$

is its solution, the *variation of constants formula.*
If in addition $y = Cx + Du$, then the *total response of the system* is

$$y(t) = Ce^{A(t-t_0)}x_0 + C\int_{t_0}^t e^{A(t-s)}Bu(s)ds + Du(t). \tag{3.70}$$

The *impulse response* is

$$H(t) = \begin{cases} Ce^{At}B + D\delta(t), & t \geq \tau, \\ 0, & t < 0, \end{cases} \tag{3.72}$$

and the *transfer function* is

$$\widehat{H}(s) = C(sI - A)^{-1}B + D. \tag{3.80}$$

Note that $H(s) = \mathcal{L}(H(t,0))$.
- *Equivalent representations*

$$\begin{aligned} \dot{\tilde{x}} &= \widetilde{A}\tilde{x} + \widetilde{B}u, \\ y &= \widetilde{C}\tilde{x} + \widetilde{D}u, \end{aligned} \tag{3.82}$$

where

$$\widetilde{A} = PAP^{-1}, \quad \widetilde{B} = PB, \quad \widetilde{C} = CP^{-1}, \quad \widetilde{D} = D \tag{3.83}$$

is equivalent to $\dot{x} = Ax + Bu$, $y = Cx + Du$.

Discrete-Time Systems

- Consider the *discrete-time system*

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) + Du(k). \tag{3.91}$$

Then

$$y(k) = CA^k x(0) + \sum_{j=0}^{k-1} CA^{k-(j+1)}Bu(j) + Du(k), \quad k > 0. \tag{3.99}$$

The *discrete-time unit impulse response* is

$$H(k,0) = \begin{cases} CA^{k-1}B & k \geq 0, \\ D & k = 0, \\ 0 & k < 0, \end{cases} \tag{3.102}$$

and the *transfer function* is

$$\widehat{H}(z) = C(zI - A)^{-1}B + D. \tag{3.108}$$

Note that $\widehat{H}(z) = \mathcal{Z}\{H(k,0)\}$.

- $A^k = \mathcal{Z}^{-1}(z(zI - A)^{-1})$. A^k may also be calculated using the Cayley–Hamilton theorem. Note that when all n eigenvalues of A, λ_i, are distinct then

$$A^k = \sum_{j=0}^{n} A_i \lambda_i^k, \quad k \geq 0, \tag{3.138}$$

$A_i \lambda_i^k$ are the modes of the system.
- The *stability of an equilibrium* of $x(k+1) = Ax(k)$ is defined and related to the eigenvalues of A using the expressions of A^k in terms of the modes.

Sampled Data Systems

- When $\dot{x} = Ax + Bu$, $y = Cx + Du$ is the system in Figure 3.3, the discrete-time description is

$$\begin{aligned} \bar{x}(k+1) &= \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k), \\ \bar{y}(k) &= \bar{C}\bar{x}(k) + \bar{D}\bar{u}(k), \end{aligned} \tag{3.126}$$

with

$$\begin{aligned} &\bar{A} = e^{AT}, \quad \bar{B} = \left[\int_0^T e^{A\tau} d\tau\right] B, \\ &\bar{C} = C, \qquad \bar{D} = D, \end{aligned} \tag{3.127}$$

where T is the sampling period.

3.8 Notes

Our treatment of basic aspects of linear ordinary differential equations in Sections 3.2 and 3.3 follows along lines similar to the development of this subject given in Miller and Michel [8].

State-space and input–output representations of continuous-time systems and discrete-time systems, addressed in Sections 3.4 and 3.5, respectively, are addressed in a variety of textbooks, including Kailath [7], Chen [4], Brockett [3], DeCarlo [5], Rugh [11], and others. For further material on sampled-data systems, refer to Aström and Wittenmark [2] and to the early works on this subject that include Jury [6] and Ragazzini and Franklin [9].

Detailed treatments of the Laplace transform and the z-transform, discussed briefly in Sections 3.3 and 3.5, respectively, can be found in numerous texts on signals and linear systems, control systems, and signal processing.

In the presentation of the material in all the sections of this chapter, we have relied principally on Antsaklis and Michel [1].

The state representation of systems received wide acceptance in systems theory beginning in the late 1950s. This was primarily due to the work of R.

E. Kalman and others in filtering theory and quadratic control theory and due to the work of applied mathematicians concerned with the stability theory of dynamical systems. For comments and extensive references on some of the early contributions in these areas, refer to Kailath [7] and Sontag [12]. Of course, differential equations have been used to describe the dynamical behavior of artificial systems for many years. For example, in 1868 J. C. Maxwell presented a complete treatment of the behavior of devices that regulate the steam pressure in steam engines called flyball governors (Watt governors) to explain certain phenomena.

The use of state-space representations in the systems and control area opened the way for the systematic study of systems with multi-inputs and multi-outputs. Since the 1960s an alternative description is also being used to characterize time-invariant MIMO control systems that involves usage of polynomial matrices or differential operators. Some of the original references on this approach include Rosenbrock [10] and Wolovich [13]. This method, which corresponds to system descriptions by means of higher order ordinary differential equations (rather than systems of first-order ordinary differential equations, as is the case in the state-space description), is addressed in Sections 7.5 and 8.5 and in Chapter 10.

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Exercises

For the first 12 exercises, the reader may want to refer to the appendix, which contains appropriate material on matrices and linear algebra.

3.1. (a) Let $(V, F) = (R^3, R)$. Determine the representation of $v = (1, 4, 0)^T$ with respect to the basis $v^1 = (1, -1, 0)^T, v^2 = (1, 0, -1)^T$, and $v^3 = (0, 1, 0)^T$.

(b) Let $V = F^3$, and let F be the field of rational functions. Determine the representation of $\tilde{v} = (s+2, 1/s, -2)^T$ with respect to the basis $\{v^1, v^2, v^3\}$ given in (a).

3.2. Find the relationship between the two bases $\{v^1, v^2, v^3\}$ and $\{\bar{v}^1, \bar{v}^2, \bar{v}^3\}$ (i.e., find the matrix of $\{\bar{v}^1, \bar{v}^2, \bar{v}^3\}$ with respect to $\{v^1, v^2, v^3\}$) where $v^1 = (2, 1, 0)^T, v^2 = (1, 0, -1)^T, v^3 = (1, 0, 0)^T, \bar{v}^1 = (1, 0, 0)^T, \bar{v}^2 = (0, 1, -1)$, and $\bar{v}^3 = (0, 1, 1)$. Determine the representation of the vector $e_2 = (0, 1, 0)^T$ with respect to both of the above bases.

3.3. Let $\alpha \in R$ be fixed. Show that the set of all vectors $(x, \alpha x)^T, x \in R$, determines a vector space of dimension one over $F = R$, where vector addition and multiplication of vectors by scalars is defined in the usual manner. Determine a basis for this space.

3.4. Show that the set of all real $n \times n$ matrices with the usual operation of matrix addition and the usual operation of multiplication of matrices by scalars constitutes a vector space over the reals [denoted by $(R^{n \times n}, R)$]. Determine the dimension and a basis for this space. Is the above statement still true if $R^{n \times n}$ is replaced by $R^{m \times n}$, the set of real $m \times n$ matrices? Is the above statement still true if $R^{n \times n}$ is replaced by the set of nonsingular matrices? Justify your answers.

3.5. Let $v^1 = (s^2, s)^T$ and $v^2 = (1, 1/s)^T$. Is the set $\{v^1, v^2\}$ linearly independent over the field of rational functions? Is it linearly independent over the field of real numbers?

3.6. Determine the rank of the following matrices, carefully specifying the field:

$$\text{(a)} \begin{bmatrix} j \\ 3j \\ -1 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 1 & 4 & -5 \\ 7 & 0 & 2 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} (s+4) & -2 \\ (s^2-1) & 6 \\ 0 & 2s+3 \\ s & -s+4 \end{bmatrix}, \quad \text{(d)} \left(\frac{s+1}{s^2} \right),$$

where $j = \sqrt{-1}$.

3.7. (a) Determine bases for the range and null space of the matrices

$$A_1 = [1\,0\,1], \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad A_3 = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}.$$

(b) Characterize all solutions of $A_1 x = 1$ (see Subsection A.3.1).

3.8. Show that $e^{(A_1+A_2)t} = e^{A_1 t} e^{A_2 t}$ if $A_1 A_2 = A_2 A_1$.

3.9. Show that there exists a similarity transformation matrix P such that

$$PAP^{-1} = A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix}$$

if and only if there exists a vector $b \in R^n$ such that the rank of $[b, Ab, \dots, A^{n-1}b]$ is n; i.e., $\rho[b, Ab, \dots, A^{n-1}b] = n$.

3.10. Show that if λ_i is an eigenvalue of the companion matrix A_c given in Exercise 3.9, then a corresponding eigenvector is $v^i = (1, \lambda_i, \dots, \lambda_i^{n-1})^T$.

3.11. Let λ_i be an eigenvalue of a matrix A, and let v^i be a corresponding eigenvector. Let $f(\lambda) = \sum_{k=0}^{l} \alpha_k \lambda^k$ be a polynomial with real coefficients. Show that $f(\lambda_i)$ is an eigenvalue of the matrix function $f(A) = \sum_{k=0}^{l} \alpha_k A^k$. Determine an eigenvector corresponding to $f(\lambda_i)$.

3.12. For the matrices

$$A_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

determine the matrices $A_1^{100}, A_2^{100}, e^{A_1 t}$, and $e^{A_2 t}, t \in R$.

3.13. For the system

$$\dot{x} = Ax + Bu, \tag{3.147}$$

where all symbols are as defined in (3.61a), derive the *variation of constants formula* (3.11), using the change of variables $z(t) = \Phi(t_0, t)x(t)$.

3.14. Show that $\frac{\partial}{\partial \tau}\Phi(t, \tau) = -\Phi(t, \tau)A$ for all $t, \tau \in R$.

3.15. The *adjoint equation* of (3.1) is given by

$$\dot{z} = -A^T z. \tag{3.148}$$

Let $\Phi(t, t_0)$ and $\Phi_a(t, t_0)$ denote the state transition matrices of (3.1) and its adjoint equation, respectively. Show that $\Phi_a(t, t_0) = [\Phi(t_0, t)]^T$.

3.16. Consider the system described by

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{3.149}$$

where all symbols are as in (3.61) with $D = 0$, and consider the *adjoint equation* of (3.149), given by

$$\dot{z} = -A^T z + C^T v, \quad w = B^T z. \tag{3.150}$$

(a) Let $H(t,\tau)$ and $H_a(t,\tau)$ denote the impulse response matrices of (3.149) and (3.150), respectively. Show that at the times when the impulse responses are nonzero, they satisfy $H(t,\tau) = H_a(\tau,t)^T$.
(b) Show that $H(s) = -H_a(-s)^T$, where $H(s)$ and $H_a(s)$ are the transfer matrices of (3.149) and (3.150), respectively.

3.17. Compute e^{At} for

$$A = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

3.18. Given is the matrix

$$A = \begin{bmatrix} 1/2 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

(a) Determine e^{At}, using the different methods covered in this text. Discuss the advantages and disadvantages of these methods.
(b) For system (3.1) let A be as given. Plot the components of the solution $\phi(t, t_0, x_0)$ when $x_0 = x(0) = (1,1,1)^T$ and $x_0 = x(0) = (2/3, 1, 0)^T$. Discuss the differences in these plots, if any.

3.19. Show that for $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, we have $e^{At} = e^{at}\begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}$.

3.20. Given is the system of equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

with $x(0) = (1,0)^T$ and

$$u(t) = p(t) = \begin{cases} 1, & t \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Plot the components of the solution of ϕ. For different initial conditions $x(0) = (a,b)^T$, investigate the changes in the asymptotic behavior of the solutions.

3.21. The system (3.1) with $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is called the *harmonic oscillator* (refer to Chapter 1) because it has periodic solutions $\phi(t) = (\phi_1(t), \phi_2(t))^T$. Simultaneously, for the same values of t, plot $\phi_1(t)$ along the horizontal axis and $\phi_2(t)$ along the vertical axis in the x_1-x_2 plane to obtain a *trajectory* for this system for the specific initial condition $x(0) = x_0 = (x_1(0), x_2(0))^T = (1, 1)^T$. In plotting such trajectories, time t is viewed as a parameter, and arrows are used to indicate increasing time. When the horizontal axis corresponds to position and the vertical axis corresponds to velocity, the x_1-x_2 plane is called the *phase plane* and ϕ_1, ϕ_2 (resp. x_1, x_2) are called *phase variables*.

3.22. First, determine the solution ϕ of $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with $x(0) = (1, 1)^T$. Next, determine the solution ϕ of the above system for $x(0) = \alpha(1, -1)^T, \alpha \in R, \alpha \neq 0$, and discuss the properties of the two solutions.

3.23. In Subsection 3.3.3 it is shown that when the n eigenvalues λ_i of a real $n \times n$ matrix A are distinct, then $e^{At} = \sum_{i=1}^{n} A_i e^{\lambda_i t}$ where $A_i = \lim_{s \to \lambda_i} [(s - \lambda_i)(sI - A)^{-1}] = v_i \tilde{v}_i$ [refer to (3.53), (3.54), and (3.57)], where $v_i, \tilde{v}_i$ are the right and left eigenvectors of A, respectively, corresponding to the eigenvalue λ_i. Show that (a) $\sum_{i=1}^{n} A_i = I$, where I denotes the $n \times n$ identity matrix, (b) $AA_i = \lambda_i A_i$, (c) $A_i A = \lambda_i A_i$, (d) $A_i A_j = \delta_{ij} A_i$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ when $i \neq j$.

3.24. Consider the system

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{3.151}$$

where all symbols are defined as in (3.61) with $D = 0$. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = [1,\ 0,\ 1,\ 0]. \tag{3.152}$$

(a) Find equivalent representations for system (3.151), (3.152), given by

$$\dot{\tilde{x}} = \widetilde{A}\tilde{x} + \widetilde{B}u, \quad y = \widetilde{C}\tilde{x}, \tag{3.153}$$

where $\tilde{x} = Px$, when $\widetilde{A}$ is in (i) the Jordan canonical (or diagonal) form and (ii) the companion form.

(b) Determine the transfer function matrix for this system.

3.25. Consider the system (3.61) with $B = 0$.

(a) Let

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad C = [1,\ 1,\ 1].$$

If possible, select $x(0)$ in such a manner so that $y(t) = te^{-t}, t \geq 0$.

(b) Determine conditions under which it is possible to specify $y(t), t \geq 0$, using only the initial data $x(0)$.

3.26. Consider the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1/2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} u, \quad y = [1,\ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a) Determine $x(0)$ so that for $u(t) = e^{-4t}, y(t) = ke^{-4t}$, where k is a real constant. Determine k for the present case. Notice that $y(t)$ does not have any transient components.
(b) Let $u(t) = e^{\alpha t}$. Determine $x(0)$ that will result in $y(t) = ke^{\alpha t}$. Determine the conditions on α for this to be true. What is k in this case?

3.27. Consider the system (3.61) with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 3 & 0 & -3 & 1 \\ -1 & 1 & 4 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) For $x(0) = [1, 1, 1, 1]^T$ and $u(t) = [1, 1]^T, t \geq 0$, determine the solution $\phi(t, 0, x(0))$ and the output $y(t)$ for this system and plot the components $\phi_i(t, 0, x(0)), i = 1, 2, 3, 4$ and $y_i(t), i = 1, 2$.
(b) Determine the transfer function matrix $H(s)$ for this system.

3.28. Consider the system

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k), \tag{3.154}$$

where all symbols are defined as in (3.91) with $D = 0$. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad C = [1\ 1],$$

and let $x(0) = 0$ and $u(k) = 1, k \geq 0$.

(a) Determine $\{y(k)\}, k \geq 0$, by working in the (i) time domain and (ii) z-transform domain, using the transfer function $H(z)$.
(b) If it is known that when $u(k) = 0$, then $y(0) = y(1) = 1$, can $x(0)$ be uniquely determined? If your answer is affirmative, determine $x(0)$.

3.29. Consider $\hat{y}(z) = H(z)\hat{u}(z)$ with transfer function $H(z) = 1/(z + 0.5)$.

(a) Determine and plot the unit pulse response $\{h(k)\}$.
(b) Determine and plot the unit step response.

(c) If

$$u(k) = \begin{cases} 1, & k = 1, 2, \\ 0, & \text{elsewhere}, \end{cases}$$

determine $\{y(k)\}$ for $k = 0, 1, 2, 3$, and 4 via (i) convolution and (ii) the z-transform. Plot your answer.

(d) For $u(k)$ given in (c), determine $y(k)$ as $k \to \infty$.

3.30. Consider the system (3.91) with $x(0) = x_0$ and $k \geq 0$. Determine conditions under which there exists a sequence of inputs so that the state remains at x_0, i.e., so that $x(k) = x_0$ for all $k \geq 0$. How is this input sequence determined? Apply your method to the specific case

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

3.31. For system (3.92) with $x(0) = x_0$ and $k \geq 0$, it is desired to have the state go to the zero state for any initial condition x_0 in at most n steps; i.e., we desire that $x(k) = 0$ for any $x_0 = x(0)$ and for all $k \geq n$.

(a) Derive conditions in terms of the eigenvalues of A under which the above is true. Determine the minimum number of steps under which the above behavior will be true.

(b) For part (a), consider the specific cases

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hint: Use the Jordan canonical form for A. Results of this type are important in *deadbeat control*, where it is desired that a system variable attains some desired value and settles at that value in a finite number of time steps.

3.32. Consider a continuous-time system described by the transfer function $H(s) = 4/(s^2 + 2s + 2)$; i.e., $\hat{y}(s) = H(s)\hat{u}(s)$.

(a) Assume that the system is at rest, and assume a unit step input; i.e., $u(t) = 1, t \geq 0, u(t) = 0, t < 0$. Determine and plot $y(t)$ for $t \geq 0$.

(b) Obtain a discrete-time approximation for the above system by following these steps: (i) Determine a *realization* of the form (3.61) of $H(s)$ (see Exercise 3.33); (ii) assuming a sampler and a zero-order hold with sampling period T, use (3.151) to obtain a discrete-time system representation

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k), \quad \bar{y}(k) = \bar{C}\bar{x}(k) + \bar{D}\bar{u}(k) \tag{3.155}$$

and determine $\bar{A}, \bar{B}$, and $\bar{C}$ in terms of T.

(c) For the unit step input, $u(k) = 1$ for $k \geq 0$ and $u(k) = 0$ for $k < 0$, determine and plot $\bar{y}(k), k \geq 0$, for different values of T, assuming the system is at rest. Compare $\bar{y}(k)$ with $y(t)$ obtained in part (a).

(d) Determine for (3.155) the transfer function $\bar{H}(z)$ in terms of T. Note that $\bar{H}(z) = \bar{C}(zI - \bar{A})^{-1}\bar{B} + \bar{D}$. It can be shown that $\bar{H}(z) = (1 - z^{-1})\mathcal{Z}\{\mathcal{L}^{-1}[H(s)/s]_{t=kT}\}$. Verify this for the given $H(s)$.

3.33. Given a proper rational transfer function matrix $H(s)$, the state-space representation $\{A, B, C, D\}$ is called a *realization of* $H(s)$ if $H(s) = C(sI - A)^{-1}B + D$. Thus, the system (3.61) is a realizations of $H(s)$ if its transfer function matrix is equal to $H(s)$. Realizations of $H(s)$ are studied at length in Chapter 8. When $H(s)$ is scalar, it is straightforward to derive certain realizations, and in the following, we consider one such realization.

Given a proper rational scalar transfer function $H(s)$, let $D \triangleq \lim_{s\to\infty} H(s)$ and let

$$H_{sp}(s) \triangleq H(s) - D = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0},$$

a strictly proper rational function.

(a) Let

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{3.156}$$
$$C = [b_0\ b_1 \cdots b_{n-1}],$$

and show that $\{A, B, C, D\}$ is indeed a realization of $H(s)$. Also, show that $\{\widetilde{A} = A^T, \widetilde{B} = C^T, \widetilde{C} = B^T, \widetilde{D} = D\}$ is a realization of $H(s)$ as well. These two state-space representations are said to be in *controller (companion) form* and in *observer (companion) form*, respectively (refer to Chapter 6).

(b) In particular find realizations in controller and observer form for (i) $H(s) = 1/s^2$, (ii) $H(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, and (iii) $H(s) = (s+1)^2/(s-1)^2$.

3.34. Assume that $H(s)$ is a $p \times m$ proper rational transfer function matrix. Expand $H(s)$ in a Laurent series about the origin to obtain

$$H(s) = H_0 + H_1 s^{-1} + \cdots + H_k s^{-k} + \cdots = \sum_{k=0}^{\infty} H_k s^{-k}. \tag{3.157}$$

The elements of the sequence $\{H_0, H_1, \ldots, H_k, \ldots\}$ are called the *Markov parameters* of the system. These parameters provide an alternative representation of the transfer function matrix $H(s)$, and they are useful in Realization Theory (refer to Chapter 8).

(a) Show that the impulse response $H(t,0)$ can be expressed as

$$H(t,0) = H_0\delta(t) + \sum_{k=1}^{\infty} H_k(t^{k-1}/(k-1)!). \tag{3.158}$$

In the following discussion, we assume that the system in question is described by (3.61).

(b) Show that

$$H(s) = D + C(sI-A)^{-1}B = D + \sum_{k=1}^{\infty}[CA^{k-1}B]s^{-k}, \tag{3.159}$$

which shows that the elements of the sequence $\{D, CB, CAB, \ldots, CA^{k-1}B, \ldots\}$ are the Markov parameters of the system; i.e., $H_0 = D$ and $H_k = CA^{k-1}B$, $k = 1, 2, \ldots$.

(c) Show that

$$H(s) = D + \frac{1}{\alpha(s)}C[R_{n-1}s^{n-1} + \cdots + R_1 s + R_0]B, \tag{3.160}$$

where $\alpha(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = \det(sI - A)$, the characteristic polynomial of A, and $R_{n-1} = I, R_{n-2} = AR_{n-1} + a_{n-1}I = A + a_{n-1}I, \ldots, R_0 = A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I$.
Hint: Write $(sI-A)^{-1} = \frac{1}{\alpha(s)}[\text{adjoint}(sI-A)] = \frac{1}{\alpha(s)}[R_{n-1}s^{n-1} + \cdots + R_1 s + R_0]$, and equate the coefficients of equal powers of s in the expression

$$\alpha(s)I = (sI-A)[R_{n-1}s^{n-1} + \cdots + R_1 s + R_0]. \tag{3.161}$$

3.35. The *frequency response matrix* of a system described by its $p \times m$ transfer function matrix evaluated at $s = j\omega$,

$$H(\omega) \triangleq \hat{H}(s)|_{s=j\omega},$$

is a very useful means of characterizing a system, since typically it can be determined experimentally, and since control system specifications are frequently expressed in terms of the frequency responses of transfer functions. When the poles of $\hat{H}(s)$ have negative real parts, the system turns out to be bounded-input/bounded-output (BIBO) stable (refer to Chapter 4). Under these conditions, the frequency response $H(\omega)$ has a clear physical meaning, and this fact can be used to determine $H(\omega)$ experimentally.

(a) Consider a stable SISO system given by $\hat{y}(s) = \hat{H}(s)\hat{u}(s)$. Show that if $u(t) = k\sin(\omega_0 t + \phi)$ with k constant, then $y(t)$ at steady-state (i.e., after all transients have died out) is given by

$$y_{ss}(t) = k|H(\omega_0)|\sin(\omega_0 t + \phi + \theta(\omega_0)),$$

where $|H(\omega)|$ denotes the magnitude of $H(\omega)$ and $\theta(\omega) = \arg H(\omega)$ is the argument or phase of the complex quantity $H(\omega)$.
From the above it follows that $H(\omega)$ completely characterizes the system response at steady state (of a stable system) to a sinusoidal input. Since $u(t)$ can be expressed in terms of a series of sinusoidal terms via a Fourier series (recall that $u(t)$ is piecewise continuous), $H(\omega)$ characterizes the steady-state response of a stable system to any bounded input $u(t)$. This physical interpretation does not apply when the system is not stable.

(b) For the $p \times m$ transfer function matrix $\hat{H}(s)$, consider the frequency response matrix $H(\omega)$ and extend the discussion of part (a) above to MIMO systems to give a physical interpretation of $H(\omega)$.

3.36. (*Double integrator*)

(a) Plot the response of the double integrator of Example 3.33 to a unit step input.

(b) Consider the discrete-time state-space representation of the double integrator of Example 3.33 for $T = 0.5$, 1, 5 sec and plot the unit step responses. Compare with your results in (a).

3.37. *(Spring mass system)* Consider the spring mass system of Example 1.1. For $M_1 = 1$ kg, $M_2 = 1$ kg, $K = 0.091$ N/m, $K_1 = 0.1$ N/m, $K_2 = 0.1$ N/m, $B = 0.0036$ N sec/m, $B_1 = 0.05$ N sec/m, and $B_2 = 0.05$ N sec/m, the state-space representation of the system in (1.27) assumes the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.1910 & -0.0536 & 0.0910 & 0.0036 \\ 0 & 0 & 0 & 1 \\ 0.0910 & 0.0036 & -0.1910 & -0.0536 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

where $x_1 \triangleq y_1$, $x_2 \triangleq \dot{y}_1$, $x_3 \triangleq y_2$, and $x_4 \triangleq \dot{y}_2$.

(a) Determine the eigenvalues and eigenvectors of the matrix A of the system and express $x(t)$ in terms of the modes and the initial conditions $x(0)$ of the system, assuming that $f_1 = f_2 = 0$.

(b) For $x(0) = [1, 0, -0.5, 0]^T$ and $f_1 = f_2 = 0$, plot the states for $t \geq 0$.

(c) Let $y = Cx$ with $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ denote the output of the system. Determine the transfer function between y and $u \triangleq [f_1, f_2]^T$.

(d) For zero initial conditions, $f_1(t) = \delta(t)$ (the unit impulse), and $f_2(t) = 0$, plot the states for $t \geq 0$ and comment on your results.

(e) It is desirable to explore what happens when the mass ratio M_2/M_1 takes on different values. For this, let $M_2 = \alpha M_1$ with $M_1 = 1$ kg and $\alpha = 0.1$, 0.5, 2, 5. All other parameter values remain the same. Repeat (a) to (d) for the different values of α and discuss your results.

3.38. *(Automobile suspension system)* [M.L. James, G.M. Smith, and J.C. Wolford, *Applied Numerical Methods for Digital Computation*, Harper and Row, 1985, p. 667.] Consider the spring mass system in Figure 3.4, which describes part of the suspension system of an automobile. The data for this system are given as

$$\begin{aligned}
m_1 &= \tfrac{1}{4} \times (\text{mass of the automobile}) = 375 \text{ kg},\\
m_2 &= \text{ mass of one wheel} = 30 \text{ kg},\\
k_1 &= \text{ spring constant} = 1500 \text{ N/m},\\
k_2 &= \text{ linear spring constant of tire} = 6500 \text{ N/m},\\
c &= \text{ damping constant of dashpot} = 0,\ 375,\ 750, \text{ and } 1125 \text{ N sec/m},\\
x_1 &= \text{ displacement of automobile body from equilibrium position m},\\
x_3 &= \text{ displacement of wheel from equilibrium position m},\\
v &= \text{ velocity of car} = 9,\ 18,\ 27, \text{ or } 36 \text{ m/sec}.
\end{aligned}$$

A linear model $\dot{x} = Ax + Bu$ for this system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{c}{m_1} & \frac{k_1}{m_1} & \frac{c}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{c}{m_2} & -\frac{k_1+k_2}{m_2} & -\frac{c}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_2}{m_2} \end{bmatrix} u(t),$$

where $u(t) = \frac{1}{6} \sin \frac{2\pi vt}{20}$ describes the profile of the roadway.

(a) Determine the eigenvalues of A for all of the above cases.
(b) Plot the states for $t \geq 0$ when the input $u(t) = \frac{1}{6} \sin \frac{2\pi vt}{20}$ and $x(0) = [0, 0, 0, 0]^T$ for all the above cases. Comment on your results.

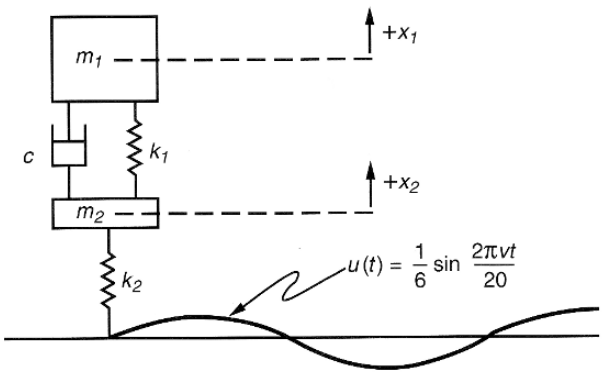

Figure 3.4. Model of an automobile suspension system