5 Convergence in topological spaces

In this chapter we shall study the notion of convergence in general topological spaces, beyond metric spaces. As we have already seen (cf. Remarks 2.7 (3), 2.20 (5), and Exercise 5 of Chapter 2), the closure of a set in a topological space cannot always be described in terms of sequences (unlike in metrizable spaces).

There is, however, a generalization of the notion of a sequence that "works" (almost) equally well in arbitrary topological spaces. The basic idea is to replace the index set by a set more general than the set of natural numbers $\mathbb{N}$.

Definition 5.1. A **directed set** is a set A equipped with a binary relation $\preceq$ such that

(i) $\alpha \preceq \alpha$ for every $\alpha \in A$;

(ii) If $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$; and

(iii) For every $\alpha, \beta \in A$ there exists $\gamma \in A$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

If $\alpha \preceq \beta$, we shall also write $\beta \succeq \alpha$. We note that a set A satisfying (i) and (ii) is called a **preordered** set. Thus, a directed set is a preordered set satisfying condition (iii).

A **net** in a set X is a map $A \ni a \to x_a \in X$, where A is a directed set. We shall usually denote such a map by $(x_a)_{a \in A}$, or simply by (x_a) when the set A is clear.

Examples 5.2.

1) The set of natural numbers $\mathbb{N}$ with its usual order (and, more generally, any totally ordered set).

2) The set $\mathbb{R} \setminus \{a\}$, where $a \in \mathbb{R}$, directed by $x \preceq y \Leftrightarrow |x - a| \geq |y - a|$.

3) Let $[a, b]$ be a closed interval in $\mathbb{R}$ $(a, b \in \mathbb{R}, a < b)$. Recall that a partition of $[a, b]$ is any finite subset P of $[a, b]$ containing a and b. Let $A \equiv A[a, b]$ be the set of all partitions of $[a, b]$. The set A becomes a directed set under either of the following relations: (a) Define $P \preceq Q \Leftrightarrow \|P\| \geq \|Q\|$, where $P = \{t_0 = a < t_1 < \cdots < t_n = b\}$ and
$$\|P\| = \max\{t_k - t_{k-1} : k = 1, 2, \ldots, n\}.$$
(b) $P \preceq Q \Leftrightarrow P \subseteq Q$.

4) The family N_x of all neighbourhoods of a point x in a topological space X becomes a directed set under the relation $U \preceq V \Leftrightarrow V \subseteq U$. (We then say that N_x is directed by reverse inclusion.)

5) The Cartesian product $A \times B$ of two directed sets A and B is a directed set under the relation $(a, \beta) \preceq (a', \beta') \Leftrightarrow a \preceq a'$ and $\beta \preceq \beta'$.

(This will always be the direction on the product $A \times B$ of two directed sets.)

6) A directed set need not be partially ordered. Indeed, for the directed set $A = A[0, 1]$ of Example 3 (a), we observe that if
$$P = \left\{0, \frac{1}{3}, 1\right\} \quad \text{and} \quad Q = \left\{0, \frac{2}{3}, 1\right\},$$

then $P \preceq Q$ and $Q \preceq P$, since $\|P\| = \|Q\| = \frac{2}{3}$, but $P \neq Q$.

Note that we have already encountered Examples (1)–(3) in Analysis courses. A net with index set $\mathbb{N}$ is nothing other than a sequence. Nets as in Examples (2) and (3) also occur in the definition of the limit $\lim_{x \to a} f(x)$ of a function f of a real variable and in the definition of the Riemann integral using Riemann sums.

Definition 5.3 Let X be a topological space, $(x_a)_{a \in A}$ a net in X, and $x \in X$.

(a) We say that x is a **limit** of (x_a), or that (x_a) **converges** to x, and write

$$x_a \to x,$$

if for every neighbourhood U of x there exists $a_0 = a_0(U) \in A : a \geq a_0 \Rightarrow x_a \in U$.

(b) We say that x is a **cluster point** of (x_a) if for every neighbourhood U of x and every $a \in A$ there exists $\beta \in A$ such that $\beta \geq a$ and $x_\beta \in U$.

The following three propositions show that nets adequately replace sequences in general topological spaces.

Proposition 5.4 Let X be a topological space, $E \subseteq X$, and $x \in X$. Then:

(a) The point x is an accumulation point of E if and only if there exists a net $(x_a) \subseteq E - \{x\}$ such that $x_a \to x$.

(b) $x \in \overline{E}$ if and only if there exists a net $(x_a) \subseteq E$ such that $x_a \to x$.

Proof. (a) "$\Rightarrow$" If x is an accumulation point of E, let N_x be the family of neighbourhoods of x, directed as in Example 5.2 (4). For each $U \in N_x$, choose $x_U \in (U \setminus \{x\}) \cap E$. It is then clear that $x_U \to x$.

"$\Leftarrow$" If $(x_a) \subseteq E \setminus \{x\}$ and $x_a \to x$, then every $U \setminus \{x\}$, where $U \in N_x$, contains some x_a; hence x is an accumulation point of E.

(b) "$\Rightarrow$" By Proposition 1.11, we have $\overline{E} = E \cup E'$. Thus, if $x \in \overline{E}$, then either $x \in E$, in which case we set $x_n = x$ for every $n \in \mathbb{N}$, or $x \in E'$, in which case the result follows from assertion (a).

"$\Leftarrow$" If $(x_a) \subseteq E$ and $x_a \to x$, then every neighbourhood U of x meets E, and hence $x \in \overline{E}$ (cf. Proposition 1.9).

Proposition 5.5. Let X, Y be topological spaces and $f : X \to Y$ a function. Then f is continuous at $x \in X$ if and only if, for every net $(x_a) \subseteq X$, $x_a \to x \Rightarrow f(x_a) \to f(x)$.

Proof. Let $(x_a) \subseteq X$ be an arbitrary net such that $x_a \to x$.

If f is continuous at x and V is a neighbourhood of $f(x)$, then $f^{-1}(V)$ is a neighbourhood of x, so there exists $a_0 \in A : a \geq a_0 \Rightarrow x_a \in f^{-1}(V)$. It follows that $a \geq a_0 \Rightarrow f(x_a) \in V$. That is, $f(x_a) \to f(x)$.

Conversely, if f is not continuous at x, then there exists an open neighbourhood V of $f(x)$ such that

$$\text{for every } U \in N_x, \quad f(U) \text{ is not a subset of } V. \tag{1}$$

It follows from (1) that for every $U \in N_x$ there exists $x_U \in U$ such that $f(x_U)$ does not belong to V.

It is now clear that the net (x_U) converges to x, while the net $(f(x_U))$ does not converge to $f(x)$, a contradiction.

Definition 5.6. (a) Let A be a directed set. A subset Γ of A is said to be **cofinal** in A if for every $\alpha \in A$ there exists $\gamma \in \Gamma : \gamma \geq \alpha$.

It is easy to prove that a cofinal subset of a directed set is itself directed under the same relation.

(b) Let $(x_a)_{a\in A}$ be a net in a set X. A **subnet** of $(x_a)_{a\in A}$ is a net $(y_\beta)_{\beta\in B}$ together with a map $\varphi : \beta \in B \to \varphi(\beta) = a_\beta \in A$ such that:

(i) φ is increasing (that is, $\beta_1 \preceq \beta_2 \Rightarrow a_{\beta_1} \preceq a_{\beta_2}$).

(ii) The set $\varphi(B) = \{a_\beta : \beta \in B\}$ is a cofinal subset of A.

(iii) $y_\beta = x_{a_\beta}$ $(= x_{\varphi(\beta)})$.

Examples 5.7.

1) Let $(x_a)_{a\in A}$ be a net in a set X and B a cofinal subset of A. Consider the identity map $\varphi : \beta \in B \to \varphi(\beta) = \beta \in A$. It is then obvious that φ makes $(x_\beta)_{\beta\in B}$ a subnet of $(x_a)_{a\in A}$.

2) It should be clear that every subsequence (x_{n_k}) of a given sequence (x_n) is a special case of the preceding example (since the map $k \in \mathbb{N} \to n_k \in \mathbb{N}$ is strictly increasing). However, a subnet of a sequence need not be a subsequence. Indeed, let (x_n) be any sequence and $B = [1, +\infty)$ with its usual order. Then the map $\varphi : B \to \mathbb{N} : \varphi(t) = [t]$ (= the integer part of the real number t) is clearly increasing and $\varphi(B) = \mathbb{N}$. Thus, $y_t = x_{\varphi(t)}$, $t \in B$, is a subnet of (x_n) with an uncountable index set B, so it is not a subsequence of (x_n).

As we shall see below (and bearing the preceding examples in mind), subnets behave much like subsequences.

Proposition 5.8. If $x_a \to x$, then every subnet of $(x_a)_{a\in A}$ converges to x.

Proof. Let $y_\beta = x_{a_\beta}$, $\beta \in B$, be a subnet of (x_a). If U is a neighbourhood of x, then, since $x_a \to x$, there exists $a_0 \in A : a \geq a_0 \Rightarrow x_a \in U$. The set $\{a_\beta : \beta \in B\}$ is cofinal in A, so there exists $\beta_0 : a_{\beta_0} \geq a_0$. It follows that if $\beta \geq \beta_0$, then $a_\beta \geq a_{\beta_0} \geq a_0$. Consequently, $y_\beta = x_{a_\beta} \in U$. We conclude that $y_\beta \to x$.

Proposition 5.9. Let $(x_a)_{a\in A}$ be a net in a topological space X, and let $x \in X$. Then x is a cluster point of (x_a) if and only if (x_a) has a subnet that converges to x.

Proof. "⇒" Let x be a cluster point of (x_a). Let N_x be the family of neighbourhoods of x. The set $N_x \times A$ becomes a directed set under the relation $(U, a) \preceq (U', a')$ if and only if $U \supseteq U'$ and $a \preceq a'$ (cf. Examples 5.2 (4) and (5)).

Set $B = \{(U, a) \in N_x \times A : x_a \in U\}$; then B is a cofinal subset of $N_x \times A$. Indeed, let (U, a) be an element of $N_x \times A$, and choose $\beta \geq a : x_\beta \in U$; then $(U, a) \preceq (U, \beta)$. Thus B is a directed set, and if we define $\varphi : B \to A : \varphi(U, a) = a$, it follows easily that φ is increasing and $\varphi(B)$ is a cofinal subset of A. We claim that the net $(x_{\varphi(U,a)})_B$ converges to x. Let U be a neighbourhood of x. There exists $a_0 \in A : x_{a_0} \in U$. It follows that if $(V, a) \in B : (U, a_0) \preceq (V, a)$, then $a_0 \preceq a$ and $x_a \in V \subseteq U$, which proves the claim.

"⇐" If $y_\beta = x_{a_\beta}$ is a subnet of (x_a) that converges to x and U is a neighbourhood of x, choose $\beta_1 \in B$ such that $y_\beta \in U$ for every $\beta \geq \beta_1$. Also, given $a \in A$, choose $\beta_2 \in B$ such that $a_\beta \geq a$ for every $\beta \geq \beta_2$. Then there exists $\beta \in B$ with $\beta \geq \beta_1$ and $\beta \geq \beta_2$, and so $a_\beta \geq a$ and $x_{a_\beta} = y_\beta \in U$. Hence x is a cluster point of (x_a).

Hausdorff topological spaces can be characterized in terms of nets as follows.

Proposition 5.10. A topological space X is Hausdorff if and only if every net in X converges (if it converges at all) to at most one point.

Proof. "⇒" Suppose that X is Hausdorff. Assume that the net $(x_a)_{a \in A}$ converges to two distinct points x and y of X. Take disjoint neighbourhoods U and V of x and y, respectively. Then there exist $a_1, a_2 \in A$ such that $a \geq a_1 \Rightarrow x_a \in U$ and $a \geq a_2 \Rightarrow x_a \in V$. Since A is directed, there exists $a_0 \in A$ such that $a_0 \geq a_1$ and $a_0 \geq a_2$. Then $x_{a_0} \in U \cap V$, a contradiction.

"⇐" Suppose that every convergent net in X converges to a unique point. If X is not Hausdorff, let x and y be distinct points of X that do not have disjoint neighbourhoods. Consider the directed set $A = N_x \times N_y$, where N_x and N_y are the families of neighbourhoods of the points x and y (cf. Examples 5.2 (4) and (5)).

Define a net in X as follows. For every $(U, V) \in A$, our assumption gives $U \cap V \neq \varnothing$. Thus, using the axiom of choice, select $x_{(U,V)} \in U \cap V$.

We now claim that the net $(x_a)_{a \in A}$ satisfies

$$(1) \quad x_a \to x \quad \text{and} \quad (2) \quad x_a \to y.$$

Indeed, if U_0 is an arbitrary neighbourhood of x and we set $a_0 = (U_0, X)$, then, whenever $a = (U, V) \geq (U_0, X)$, we have $U \subseteq U_0$, and consequently

$$x_a \in U \cap V \subseteq U \subseteq U_0, \quad \text{so } x_a \in U_0.$$

Thus (1) holds. We prove (2) in the same way.

Hence the net $(x_a)_{a \in A}$ converges to two distinct limits, contrary to our assumption.

As we know from Real Analysis courses, compact metric spaces X are characterized (within the class of metric spaces) by the Bolzano–Weierstrass property: every sequence in X has a subsequence that converges to a point of X. (Cf. also Exercise 9 of Chapter 3.) A version of this property, with sequences replaced by nets, also characterizes general compact spaces.

Theorem 5.11. Let X be a topological space. The following assertions are equivalent:

(a) X is compact.

(b) Every net in X has a cluster point.

(c) Every net in X has a convergent subnet.

Proof. The equivalence of (b) and (c) follows from Proposition 5.9. It suffices to prove the equivalence (a) $\Leftrightarrow$ (b).

(a) $\Rightarrow$ (b). If X is compact and (x_a) is a net in X, let $E_a = \{x_\beta : \beta \geq a\}$. Since for every $a, \beta \in A$ there exists $\gamma \in A$ with $\gamma \geq a$ and $\gamma \geq \beta$, the family $\{\overline{E_a} : a \in A\}$ has the finite intersection property. Thus, by Remark 3.7 (b), $\bigcap_{a\in A} \overline{E_a} \neq \varnothing$. If $x \in \bigcap_{a\in A} \overline{E_a}$ and U is a neighbourhood of x, then U meets every E_a, which means that x is a cluster point of (x_a).

(b) $\Rightarrow$ (a). Suppose that X is not compact. Then there exists an open cover $\{U_\beta : \beta \in B\}$ of X with no finite subcover. Consider the set G of finite subsets of B, directed by inclusion (why?).

For every $F \in G$, let x_F be a point of the set

$$\left(\bigcup_{\beta\in F} U_\beta\right)^c = X \setminus \bigcup_{\beta\in F} U_\beta.$$

Then $(x_F)_{F\in G}$ is a net in X with no cluster point. Indeed, if $x \in X$, choose $\beta \in B$ with $x \in U_\beta$. If $F \in G$ and $F \geq \{\beta\} \Leftrightarrow \beta \in F$, then $x_F \notin U_\beta$, so x is not a cluster point of (x_F). Thus we have shown that the net (x_F) has no cluster point in X, contrary to our assumption.

Remarks 5.12.

1) Convergence in topological spaces can alternatively be developed using the notion of a filter or a filter base (cf. Definition 1.24 and the exercises in Section 1.1).

   We say that a filter base B on a topological space X converges to a point x of X, and write $B \to x$, if for every neighbourhood U of x there exists $A \in B : A \subseteq U$.

   (If B is a filter, it is clear that $B \to x \Leftrightarrow N_x \subseteq B$, where N_x is the family of neighbourhoods of x in X.)

   The two notions of convergence are equivalent in every topological space X in the following sense:

   (a) Every net $(x_a)_{a\in A}$ defines a filter base $B = B(x_a)$ such that

   $$x_a \to x \Leftrightarrow B \to x \qquad (1)$$

   (b) Every filter base B defines a net $(x_a)_{a\in A}$ such that

   $$B \to x \Leftrightarrow x_a \to x \qquad (2).$$

   For assertion (a), observe that if $(x_a)_{a\in A}$ is a net in X, then the family $B = B(x_a) = \{T_a : a \in A\}$, where $T_a = \{x_b : b \in A, a \preceq b\}$, is a filter base on X for which equivalence (1) holds.

As for assertion (b), if B is a filter base on X, define A to be the set of all ordered pairs (a, E), where $a \in E \in B$, and define on A the relation $(a, E_1) \preceq (b, E_2) \Leftrightarrow E_2 \subseteq E_1$.

Since B is a filter base, it follows easily that $(A, \preceq)$ is a directed set. The net corresponding to B for which equivalence (2) holds is defined by the map $(a, E) \to a$, that is,

$$x_{(a,E)} = a, \quad (a, E) \in A.$$

(Fill in the details in the proofs of the above assertions.)

2) Let f be a map from a set X into a topological space Y, and let B be a filter base on X. A point $y \in Y$ is said to be a limit of f with respect to the filter base B if y is a limit of the filter base $f(B) = \{f(E) : E \in B\}$ on Y (cf. Exercise 10 of Section 1.1).

We then write

$$\lim_B f = y.$$

Thus we have

$$\lim_B f = y \Leftrightarrow \text{for every neighbourhood } U \text{ of } y \text{ in } Y, \text{ there exists}$$
$$E \in B : f(E) \subseteq U.$$

Examples. (a) A sequence of points $(x_n)_{n\in\mathbb{N}}$ in a topological space X is a map $n \in \mathbb{N} \to x_n \in X$. Let B be a filter on $\mathbb{N}$. Then, according to the above definition,

$$\lim_B x_n = x \Leftrightarrow \text{for every neighbourhood } U \text{ of } x, \text{ the set}$$
$$\{n \in \mathbb{N} : x_n \in U\} \in B.$$

In particular, if B is the Fréchet filter on $\mathbb{N}$,

$$B = \{E \subseteq \mathbb{N} : \mathbb{N} \setminus E \text{ is finite}\},$$

then

$$\lim_B x_n = x \Leftrightarrow \text{for every neighbourhood } U \text{ of } x, \text{ there exists}$$
$$n_0 \in \mathbb{N} : n \geq n_0 \Rightarrow x_n \in U.$$

This means that the familiar notion of sequence convergence from classical Analysis coincides with convergence of a sequence with respect to the Fréchet filter.

(b) The definition of the Riemann integral of a bounded function $f : [a, b] \to \mathbb{R}$ can be formulated using Remark (1) (cf. also Example 5.2 (3)(a)) as follows. Let $P = \{t_0 = a < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$. Any sum of the form

$$\sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}),$$

where $x_k \in [t_{k-1}, t_k]$, $k = 1, 2, \ldots, n$, are arbitrarily chosen sample points, is called a Riemann sum and is denoted by $S(f, P, (x_k))$.

Let $R_P = \{S(f, P, (x_k)) : (x_k) \text{ is a choice of sample points for } P\}$.

Set $A_P = \bigcup\{R_Q : \|Q\| < \|P\|\}$. Then the family

$$\boldsymbol{u} = \{A_P : P \text{ is a partition of } [a, b]\}$$

is a filter base on $\mathbb{R}$, and f is (said to be) Riemann integrable if and only if there exists $I \in \mathbb{R}$ such that $\boldsymbol{u} \to I$. In this case we define

$$\int_a^b f(x)\,dx = I.$$

We can also use the direction of Example 5.2 (3)(b). Thus, define $B_P = \bigcup\{R_Q : P \subseteq Q\}$. Then $B = \{B_P : P \text{ is a partition of } [a,b]\}$ is a filter base on $\mathbb{R}$, and (it can be proved that) f is Riemann integrable if and only if there exists $I \in \mathbb{R}$ such that $B \to I$.

(Compare the definitions of the Riemann integral given in Calculus and verify that they are equivalent to the definitions above.)

Exercises

1) Let $X \neq \varnothing$. Prove that:

   (a) The set $\mathcal{P}(X)$ of all subsets of X is directed by inclusion, that is, $A \preceq B \Leftrightarrow A \subseteq B$.

   (b) Similarly, the set $F = F(X)$ of finite subsets of X is directed under the same relation.

2) Let $(x_a)_{a\in A}$ be a net in a topological space X. Prove that a point x of X is a cluster point of $(x_a) \Leftrightarrow x \in \bigcap_{a\in A} \overline{T_a}$, where $T_a = \{x_b : b \in A : a \preceq b\}$.

3) Let $(x_a)_{a\in A}$ be a net in a topological space X. Prove that $x_a \to x \Leftrightarrow$ for every cofinal $B \subseteq A$ there exists a cofinal $\Gamma \subseteq B$ such that the net $(x_\gamma)_{\gamma\in\Gamma}$ converges to x.

4) Let $X = \prod_{i\in I} X_i$ be a Cartesian product of topological spaces. Prove that:

   (a) A net $(x_a)_{a\in A}$ converges to $x \in X \Leftrightarrow$ the net $(\pi_i(x_a))_{a\in A}$ converges to $\pi_i(x)$ for every $i \in I$.

   (b) Consider the topological space $\mathbb{R}^\Gamma$ with the product topology $\mathcal{T}$.
   Prove that a net of functions $f_a : \Gamma \to \mathbb{R}$, $a \in A$, converges to the function $f : \Gamma \to \mathbb{R}$ with respect to $\mathcal{T} \Leftrightarrow \{_\dashv(\gamma) \to \{(\gamma)$ for every $\gamma \in \Gamma$. (This result also justifies the term **topology of pointwise convergence** for $\mathcal{T}$.)

5) Let $g : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, $a \in A'$ (= the set of accumulation points of A), and $b \in \mathbb{R}$. Set $F = \{A \cap (U \setminus \{a\}) : U \text{ is a neighbourhood of } a \text{ in } \mathbb{R}\}$. Prove that:

   (a) F is a filter on A.

   (b) $\lim\limits_{x\to a} g(x) = b \Leftrightarrow \lim\limits_{F} g = b$.

6) Let $X \neq \varnothing$ be a set and $f : X \to [0,+\infty)$ a function. Define the symbol $\sum_{x\in X} f(x)$ as follows:

$$\sum_{x\in X} f(x) = \sup\left\{\sum_{x\in F} f(x) : F \subseteq X \text{ is finite}\right\}.$$

   If $f : X \to [0,+\infty)$ is a given function and $A = \{x \in X : f(x) > 0\}$, prove that:

   (a) If A is uncountable, then $\sum_{x\in X} f(x) = +\infty$.

(b) If A is countably infinite, then $\sum_{x\in X} f(x) = \sum_{n=1}^{\infty} f(g(n))$, where $g : \mathbb{N} \to A$ is any one-to-one and onto function.

Hint: $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n = \{x : f(x) > \frac{1}{n}\}$. If A is uncountable, then some A_{n_0} is uncountable, and $\sum_{x\in F} f(x) > \frac{|F|}{n_0}$ for every finite $F \subseteq A_{n_0}$, and hence $\sum_{x\in X} f(x) = +\infty$.

If A is countably infinite, $g : \mathbb{N} \to A$ is one-to-one and onto, and $B_N = g(\{1, 2, \ldots, N\})$, then every finite $F \subseteq A$ is contained in some B_N. It follows that

$$\sum_{x\in F} f(x) \leq \sum_{n=1}^{N} f(g(n)) \leq \sum_{x\in X} f(x).$$

Taking the supremum first over N and then over F gives the conclusion.]

7) Let $X \neq \varnothing$ be a set and $f : X \to \mathbb{R}$ a function. For every finite subset F of X, set $Z_F = \sum_{x\in F} f(x)$, and consider the set Γ of finite subsets of X, directed by inclusion (cf. Exercise 1). Prove that the net $(Z_F)_{F\in\Gamma}$ converges to a point of $\mathbb{R}$ if and only if the set $\{x \in X; f(x) \neq 0\}$ is countable, say $\{x_n, n \geq 1\}$, and $\sum_{n=1}^{\infty} |f(x_n)| < +\infty$. In this case, $Z_F \to \sum_{n=1}^{\infty} f(x_n)$.

8) If X is a topological space, a point $x \in X$ is called a cluster point of a filter base B on X if $x \in \bigcap\{\overline{E} : E \in B\}$. Prove that:

(a) x is a cluster point of the filter base $B \Leftrightarrow$ for every neighbourhood U of x and every $E \in B$, $U \cap E \neq \varnothing$.

(b) x is a cluster point of the filter $F \Leftrightarrow$ there exists a filter F' with $F \subseteq F'$ such that $F' \to x$.

(c) An ultrafilter $\boldsymbol{u}$ on X converges to $x \in X \Leftrightarrow x$ is a cluster point of $\boldsymbol{u}$.

(d) A topological space X is compact $\Leftrightarrow$ every ultrafilter on X converges to a point of $X \Leftrightarrow$ every filter on X has a cluster point.

9) Show by an example that a subnet $y_\beta = x_{\varphi(\beta)}$, $\beta \in B$, of a sequence (x_n) need not be a subsequence of (x_n), even if B is countable and φ is one-to-one and onto $\mathbb{N}$. [*Hint.* Let B be the set of finite subsets of $\mathbb{N} \cup \{0\}$, ordered by inclusion. Consider the map $\varphi : B \to \mathbb{N} : \varphi(F) = \sum_{k\in F} 2^k$.]

10) Find an example of a partially ordered set that is not directed. [*Hint.* Let $(T, \leq)$ be a partially ordered set with a tree order. Consider two incomparable elements x and y of T.]