# 2 Operations on topological spaces

## 2.1 The product topology

In an earlier section we defined the product topology on the Cartesian product $X \times Y$ of two topological spaces $X, Y$ (Example 1.33 (1)). In this section we shall define the *product topology*, or *Cartesian topology*, for an arbitrary family of topological spaces $\{X_i : i \in I\}$. First, we must extend the definition of the Cartesian product to an arbitrary family of sets. This extension is based on the observation that, if $X_1, \ldots, X_n$ are sets, then the elements of the Cartesian product $X_1 \times \cdots \times X_n$, that is, the $n$-tuples $(x_1, \ldots, x_n)$ with $x_k \in X_k$, $1 \leq k \leq n$, can be viewed as precisely those functions $f : \{1, 2, \ldots, n\} \to \bigcup_{k=1}^{n} X_k$ such that $f(k) \in X_k$ for every $k = 1, 2, \ldots, n$.

**Definition 2.1.** Let $\{X_i : i \in I\}$ be a family of sets. The *Cartesian product* $\prod_{i\in I} X_i$ is the set of all functions $f : I \to \bigcup\{X_i : i \in I\}$ such that $f(i) \in X_i$ for every $i \in I$.

If $X_i = X$ for every $i \in I$, then the Cartesian product $\prod\{X_i : i \in I\}$ is denoted by $X^I$ (= the set of all functions $f : I \to X$).

The notations $\prod_{i\in I} X_i$ and $\prod\{X_i : i \in I\}$ are equivalent. An element $f \in \prod_{i\in I} X_i$ is usually written as $(x_i)_{i\in I}$ or $(x_i)$, meaning that $f(i) = x_i$ for every $i \in I$.

With this notation, $x_j \in X_j$ is called the *$j$-th coordinate* of the element $(x_i)$. The set $X_j$ is called the *$j$-th factor* of $\prod_{i\in I} X_i$. For every $j \in I$, the map $\pi_j : \prod_{i\in I} X_i \to X_j$, $\pi_j((x_i)) = x_j$ (that is, the map $f \mapsto f(j)$), is called the *projection onto the $j$-th coordinate*.

We now state the *axiom of choice*.

**Proposition 2.2.** Let $\{X_i : i \in I\}$ be a nonempty family of nonempty sets. Then $\prod_{i\in I} X_i \neq \varnothing$. That is,

$$I \neq \varnothing \text{ and } X_i \neq \varnothing \text{ for every } i \in I \quad \Longrightarrow \quad \prod_{i\in I} X_i \neq \varnothing.$$

**Remark 2.3.** If $I = \varnothing$ or $X_i = \varnothing$ for some $i \in I$, then $\prod_{i\in I} X_i = \varnothing$. Thus, in what follows we shall always consider nonempty families of nonempty sets or topological spaces, without explicitly mentioning this.

**Examples 2.4.**

1) The set of all sequences of real numbers, which is a vector space under pointwise operations,

$$\mathbb{R}^{\mathbb{N}} = \{(x_n) : x_n \in \mathbb{R},\ \forall n \in \mathbb{N}\}.$$

Similarly, the vector space of all sequences of complex numbers,

$$\mathbb{C}^{\mathbb{N}} = \{(z_n) : z_n \in \mathbb{C},\ \forall n \in \mathbb{N}\}.$$

2) More generally, if $\Gamma \neq \varnothing$ is a set, then $\mathbb{R}^{\Gamma}$ is the vector space of all functions $f : \Gamma \to \mathbb{R}$. (The space $\mathbb{C}^{\Gamma}$ is defined similarly.)

3) The sets $\mathbb{N}^{\mathbb{N}}$, $\{0, 1\}^{\mathbb{N}}$ and $\prod_{n=1}^{\infty}[-n, n]$ are, respectively, the set of all sequences with values in $\mathbb{N}$, the set of all sequences with values in $\{0, 1\}$ and the set of all sequences of real numbers $(x_n)$ such that $|x_n| \leq n$ for every $n \in \mathbb{N}$.

The next proposition summarizes some simple and useful properties of the Cartesian product and of the projections $\pi_i$, $i \in I$.

**Proposition 2.5.** Let $\{X_i : i \in I\}$ be a family of sets. For every $i \in I$, let $A_i, B_i$ be subsets of $X_i$. Then

(a) If $A_i \subseteq B_i$ for every $i \in I$, then $\prod_{i\in I} A_i \subseteq \prod_{i\in I} B_i$.

(b) $\displaystyle\prod_{i\in I} A_i \cap \prod_{i\in I} B_i = \prod_{i\in I}(A_i \cap B_i)$.

(c) $\displaystyle\prod_{i\in I} A_i \cup \prod_{i\in I} B_i \subseteq \prod_{i\in I}(A_i \cup B_i)$.

(d) If $F \subseteq I$, we define $\prod_{i\in F} A_i \times \prod_{i\in I\setminus F} X_i$ to be the set of all $(x_i) \in \prod_{i\in I} X_i$ such that $x_i \in A_i$ for every $i \in F$. Then

$$\prod_{i\in F} A_i \times \prod_{i\in I\setminus F} X_i = \bigcap_{i\in F} \pi_i^{-1}(A_i).$$

In particular, $A_j \times \prod_{i\neq j} X_i = \pi_j^{-1}(A_j)$ for every $j \in I$.

*Proof.* Since statements (a), (b) and (c) are straightforward, we prove only (d).

$$\begin{aligned} x = (x_i) \in \prod_{i\in F} A_i \times \prod_{i\in I\setminus F} X_i &\iff x = (x_i) \text{ with } x_i \in A_i \text{ for every } i \in F \\ &\iff x = (x_i) \text{ with } \pi_i(x) \in A_i \text{ for every } i \in F \\ &\iff x = (x_i) \text{ with } x \in \pi_i^{-1}(A_i) \text{ for every } i \in F \\ &\iff x \in \bigcap_{i\in F} \pi_i^{-1}(A_i). \end{aligned}$$

□

**Definition 2.6.** Let $(X_i, \tau_i)$, $i \in I$, be a family of topological spaces. The product topology, or Cartesian topology, $\tau$ on the Cartesian product $X = \prod_{i\in I} X_i$ is the topology having the family

$$\begin{aligned} B &= \Big\{\prod_{i\in I} U_i : U_i \in \tau_i \text{ for every } i \in I \text{ and } \{i \in I : U_i \neq X_i\} \text{ is finite}\Big\} \\ &= \Big\{\prod_{i\in F} U_i \times \prod_{i\in I\setminus F} X_i : U_i \in \tau_i \text{ for every } i \in F \text{ and } F \subseteq I \text{ is finite}\Big\} \end{aligned}$$

as a basis. It is clear, using Proposition 2.5, that $B$ satisfies the conditions of Proposition 1.32 and is therefore indeed a basis for a unique topology on $X$.

**Remarks 2.7.**

1) If $B_i$ is a basis for the topological space $X_i$ for every $i \in I$, then the family

$$B_1 = \Big\{\prod_{i\in I} U_i : U_i \in B_i \cup \{X_i\} \text{ for every } i \in I \text{ and } \{i \in I : U_i \neq X_i\} \text{ is finite}\Big\}$$

is also a basis for $\tau$ (why?).

2) The set $\Sigma = \{\pi_i^{-1}(U_i) : i \in I,\ U_i \in \tau_i\}$ is a subbasis for the topology $\tau$. (By Proposition 2.5 (d), each member of the basis $B$ for $\tau$ is a finite intersection of members of $\Sigma$.)

3) If we set $B' = \big\{\prod_{i\in I} U_i : U_i \in \tau_i \text{ for every } i \in I\big\}$, then $B \subseteq B'$, and it is easy to show that $B'$ is a basis for a topology on $\prod_{i\in I} X_i$. This topology is in general finer than the product topology. If $I$ is finite, it clearly coincides with the product topology. It is called the *box topology*. The box topology is less important than the product topology and will not be studied in detail in these notes. From now on, whenever we consider a Cartesian product of topological spaces, we shall always assume that it is equipped with the product topology, unless explicitly stated otherwise.

**Definition 2.8.** Let $X, Y$ be topological spaces and let $f : X \to Y$ be a function. We say that $f$ is *open* (respectively, *closed*) if, for every open set $U \subseteq X$ (respectively, closed set $F \subseteq X$), the set $f(U)$ (respectively, $f(F)$) is open (respectively, closed) in $Y$.

**Proposition 2.9.** The projection $\pi_j : \prod_{i\in I} X_i \to X_j$ is a continuous open map for every $j \in I$. In general, it is not closed.

*Proof.* Let $U \subseteq X_j$ be open. Then $\pi_j^{-1}(U) = U \times \prod_{i\neq j} X_i$ is open in $\prod_{i\in I} X_i$, so $\pi_j$ is continuous.

Let $U \subseteq \prod_{i\in I} X_i$ be open. Then $U$ can be written as a union of basic open sets, say

$$U = \bigcup_{a\in A} V_a \quad \Longrightarrow \quad \pi_j(U) = \bigcup_{a\in A} \pi_j(V_a).$$

Thus, it suffices to observe that, if $V \subseteq \prod_{i\in I} X_i$ is a basic open set, then $\pi_j(V)$ is open in $X_j$. Hence $\pi_j$ is open.

A projection need not be a closed map. Indeed, the set $F = \{(x, 1/x) : x > 0\}$ is closed in $\mathbb{R}^2$ (why?), whereas $\pi_1(F) = (0, +\infty)$ is an open subset of $\mathbb{R}$. □

**Proposition 2.10.**

(a) The product topology $\tau$ is the smallest topology for which every projection $\pi_j$ is continuous.

(b) If $X$ is a topological space, then a function $f : X \to \prod_{i\in I} X_i$ is continuous if and only if $\pi_j \circ f$ is continuous for every $j \in I$.

(If we set $f_j = \pi_j \circ f$, $j \in I$, then $f$ is given by $f(x) = \big(f_j(x)\big)_{j\in I}$.)

*Proof.*

(a) Let $\tau_1$ be a topology on $\prod_{i\in I} X_i$ such that every projection $\pi_j : \prod_{i\in I} X_i \to X_j$ is continuous. Then, for every $j \in I$ and every open set $U \subseteq X_j$, $\pi_j^{-1}(U)$ is open in $\big(\prod_{i\in I} X_i, \tau_1\big)$. Thus, $\tau_1$ contains all members of the subbasis $\Sigma$ for the product topology described in Remark 2.7 (2), and hence $\tau \subseteq \tau_1$.

(b) $\Rightarrow$ If $f$ is continuous, the conclusion follows from Proposition 2.9, since $\pi_j \circ f$ is a composition of continuous functions.

$\Leftarrow$ Let $U \times \prod_{i\neq j} X_i$ be an arbitrary subbasic open set, where $U \subseteq X_j$ is open. Then

$$f^{-1}\Big(U \times \prod_{i\neq j} X_i\Big) = f^{-1}\big(\pi_j^{-1}(U)\big) = (\pi_j \circ f)^{-1}(U),$$

which is open in $X$ by our assumption on the functions $\pi_j \circ f$, $j \in I$. Thus, the inverse image under $f$ of each subbasic open set in $\prod_{i\in I} X_i$ is open in $X$, so $f$ is continuous. □

**Remarks 2.11.**

1) Let $\{X_i : i \in I\}$ be a family of topological spaces and, for every $i \in I$, let $A_i$ be a subspace of $X_i$. Then the product topology on $\prod_{i\in I} A_i$ coincides with the subspace topology on $\prod_{i\in I} A_i$ induced by the product topology on $\prod_{i\in I} X_i$. (Why?)

2) Recall that, if $X$ is a topological space, $(x_n)$ is a sequence in $X$ and $x \in X$, then

$$x_n \to x \iff \begin{array}{c}\text{for every neighbourhood } U \text{ of } x \text{ there exists } n_0 \in \mathbb{N} \\ \text{such that } n \geq n_0 \Rightarrow x_n \in U.\end{array}$$

If $X, Y$ are topological spaces and $f : X \to Y$ is continuous, then, for every sequence $(x_n) \subseteq X$, $x_n \to x \Rightarrow f(x_n) \to f(x)$. (Why?)

**Proposition 2.12.** Let $\{X_i : i \in I\}$ be a family of topological spaces, let $(x_n)$ be a sequence in $\prod_{i\in I} X_i$, and let $x \in \prod_{i\in I} X_i$. Write $x_n = (x_i^n)_{i\in I}$ and $x = (x_i)_{i\in I}$. Then $x_n \to x$ if and only if $x_i^n \to x_i$ for every $i \in I$. In other words, $x_n \to x$ if and only if $(x_n)$ converges coordinatewise to $x$.

*Proof.* $\Rightarrow$ The projections $\pi_i$ are continuous, so the conclusion follows from Proposition 2.10 together with Remark 2.11 (2).

$\Leftarrow$ Let $U = \prod_{i\in F} V_i \times \prod_{i\in I\setminus F} X_i$ be a basic neighbourhood of $x$. Then $x_i \in V_i$ for every $i \in F$.

By our assumption, for every $i \in F$ there exists $\nu_i \in \mathbb{N}$ such that $n \geq \nu_i \Rightarrow x_i^n \in V_i$. Since $F$ is finite, $\nu_0 = \max\{\nu_i : i \in F\}$ exists. Thus, $n \geq \nu_0$ implies $x_n \in U$. Consequently, $x_n \to x$. □

**Examples 2.13.**

1) A basis for the product topology on the vector space $\mathbb{R}^{\mathbb{N}}$ consists of all sets of the form

$$U_1 \times U_2 \times \cdots \times U_n \times \mathbb{R}^{\mathbb{N}\setminus\{1,2,\ldots,n\}},$$

where $n \in \mathbb{N}$ and $U_1, \ldots, U_n$ are open subsets of $\mathbb{R}$.

Another basis for the same topology on $\mathbb{R}^{\mathbb{N}}$ consists of the sets of the form

$$(a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R}^{\mathbb{N}\setminus\{1,2,\ldots,n\}},$$

where $n \in \mathbb{N}$ and $a_k < b_k$ for every $k = 1, 2, \ldots, n$.

2) A more general version of the preceding example is the vector space $\mathbb{R}^{\Gamma}$, where $\Gamma \neq \varnothing$ is a set, of all functions $f : \Gamma \to \mathbb{R}$. A basis for the product topology $\tau$ on $\mathbb{R}^{\Gamma}$ consists of the sets of the form

$$U_{\gamma_1} \times \cdots \times U_{\gamma_n} \times \mathbb{R}^{\Gamma\setminus\{\gamma_1,\ldots,\gamma_n\}},$$

where $n \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $U_{\gamma_1}, \ldots, U_{\gamma_n}$ are open in $\mathbb{R}$. As before, the sets $U_\gamma$ may be replaced by open intervals in $\mathbb{R}$. Analogous statements hold for the product topology on the space $\mathbb{C}^{\Gamma}$ of complex-valued functions $f : \Gamma \to \mathbb{C}$.

If $f_n : \Gamma \to \mathbb{R}$, $n \geq 1$, is a sequence of functions and $f : \Gamma \to \mathbb{R}$ is a function, then Proposition 2.12 gives

$$f_n \xrightarrow{\tau} f \iff \text{for every } \gamma \in \Gamma, \quad f_n(\gamma) \to f(\gamma).$$

That is, $f_n \to f$ pointwise on $\Gamma$.

3) Important subspaces of $\mathbb{R}^{\mathbb{N}}$ are the Hilbert cube $[0,1]^{\mathbb{N}}$, the Baire space $\mathbb{N}^{\mathbb{N}}$ and the Cantor set $\{0,1\}^{\mathbb{N}}$. Important subspaces of $\mathbb{R}^{\Gamma}$, with $\Gamma$ uncountable, are the cube $[0,1]^{\Gamma}$ and the generalized Cantor set $\{0,1\}^{\Gamma}$. If $\Gamma$ is finite and $|\Gamma| = n$, then $\mathbb{R}^{\Gamma}$ can be identified with $\mathbb{R}^n$, and the product topology on $\mathbb{R}^n$ is equal to the Euclidean topology. The fact that these two topologies coincide is geometrically clear when $n = 2$ or $3$ (cf. Example 1.31 (4)). For the general case, we refer to Remark 2.20 (6) and Exercise 6.

4) Let $g_n : \mathbb{R} \to \mathbb{R}$, $n \geq 1$, be a sequence of continuous functions. It follows immediately from Proposition 2.10 (b) that the function

$$g : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}, \qquad g(t) = \big(g_1(t), g_2(t), \ldots, g_n(t), \ldots\big), \quad t \in \mathbb{R},$$

is continuous. For example, the functions

$$g(t) = (t, t, \ldots, t, \ldots), \qquad f(t) = (t, t^2, \ldots, t^n, \ldots),$$
$$h(t) = \big(t, \tfrac{1}{2}t, \tfrac{1}{3}t, \ldots\big), \qquad \varphi(t) = (1, 2, e^t, \cos t, 1, 1, \ldots, 1, \ldots)$$

are continuous.