1.4. Continuous functions

The notion of a continuous function is fundamental and is first studied for functions of one variable, and then of two or more variables, in Calculus courses. In Real Analysis, this notion is extended to spaces more general than Euclidean spaces, namely metric spaces.

Here we shall give a definition of continuity for functions defined on topological spaces that includes the previous definitions as special cases. What allows us to extend the classical definition is the notion of an **open** set and the related notion of a **neighbourhood** of a point. The latter expresses the "closeness" of two points: the set of points that are "sufficiently close" to a point a is a neighbourhood of a. In metric spaces, balls are neighbourhoods and "closeness" becomes quantitative, since a point x is "sufficiently close" to a if x belongs to some ball $B(a,\varepsilon)$ (that is, $d(x,a)<\varepsilon$).

Let us begin by recalling the definition of continuity in metric spaces.

Let (X,d), (Y,ρ) be metric spaces and $f:X\to Y$ a function. We say that f is continuous at $a\in X$ if for every $\varepsilon>0$ there exists $\delta=\delta(\varepsilon,a)>0$ such that

$$x\in X \text{ and } d(x,a)<\delta \;\Rightarrow\; \rho\big(f(x),f(a)\big)<\varepsilon.$$

Using balls, this definition can also be formulated as follows: for every ball $B(f(a),\varepsilon)$ in Y, there exists a ball $B(a,\delta)$ in X such that

$$f\big(B(a,\delta)\big)\subseteq B\big(f(a),\varepsilon\big).$$

It is this last form of the definition that we shall generalize to topological spaces.

Definition 1.42. Let X, Y be topological spaces, $f:X\to Y$ a function, and $a\in X$. We say that f is **continuous at** a if for every neighbourhood V of $f(a)$ in Y there exists a neighbourhood U of a in X such that $f(U)\subseteq V$.

We say that f is continuous on X if and only if f is continuous at every point of X. (If there is no ambiguity about the domain of f, we shall simply say that f is continuous.)

The basic theorem about continuous functions is the following:

Theorem 1.43. Let X and Y be topological spaces and $f:X\to Y$ a function. The following assertions are equivalent:

(a) f is continuous.

(b) For every open subset V (or: for every member V of a subbase) of Y, $f^{-1}(V)$ is open in X.

(c) For every closed subset F of Y, $f^{-1}(F)$ is closed in X.

(d) $f\big(\overline{A}\big)\subseteq\overline{f(A)}$ for every $A\subseteq X$.

Proof: Before proceeding with the proof of the theorem, recall the following:

If X, Y are sets and $f : X \to Y$ is a function, then the map

$$A \in P(Y) \longmapsto f^{-1}(A) \in P(X)$$

preserves all set operations (union, intersection, and difference). That is,

$$f^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f^{-1}(A_i), \quad f^{-1}\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f^{-1}(A_i),$$
$$f^{-1}(Y \setminus A) = X \setminus f^{-1}(A),$$

for every $A \subseteq Y$ and every family $(A_i)_{i\in I}$ of subsets of Y.

(a) $\Rightarrow$ (b) Let V be open in Y and $x \in f^{-1}(V)$. Since f is continuous at x and V is a neighbourhood of $f(x)$ (being open), there exists a neighbourhood U of x such that $f(U) \subseteq V$. It follows that

$$U \subseteq f^{-1}\big(f(U)\big) \subseteq f^{-1}(V),$$

and so $f^{-1}(V)$ is open, since for every $x \in f^{-1}(V)$ there exists a neighbourhood U of x such that $U \subseteq f^{-1}(V)$.

(b) $\Rightarrow$ (a) Let $x \in X$ and V be a neighbourhood of $f(x)$. Then there exists an open set $W \subseteq Y$ such that $f(x) \in W \subseteq V$. Thus $x \in f^{-1}(W)$ and $U = f^{-1}(W)$ is an open neighbourhood of x. It follows that $f(U) \subseteq W \subseteq V$, and hence f is continuous at x. (Let $\mathcal{C}$ be a subbase for the topology $\mathcal{T}_Y$ of Y, and let $\mathcal{T}_X$ be the topology of X. Suppose that $V \in \mathcal{C} \Rightarrow f^{-1}(V) \in \mathcal{T}_X$. The same then holds for finite intersections of members of $\mathcal{C}$, and consequently for arbitrary unions of such intersections. Thus $V \in \mathcal{T}_Y \Rightarrow f^{-1}(V) \in \mathcal{T}_X$.)

(b) $\Leftrightarrow$ (c). The proof of this equivalence is based on the observation that

$$f^{-1}(Y \setminus F) = X \setminus f^{-1}(F) \tag{1}$$

for every $F \subseteq Y$. Since $F \subseteq Y$ is closed $\Leftrightarrow$ $V = Y \setminus F$ is open, it follows from (1) that

$$U = X \setminus f^{-1}(F) = f^{-1}(V) \subseteq X$$

is open $\Leftrightarrow$ $f^{-1}(F)$ is closed in X.

Thus assertions (a), (b), and (c) are equivalent, and it suffices to prove (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (c).

(c) $\Rightarrow$ (d) Let $A \subseteq X$. Set $F = \overline{f(A)}$.

Then F is closed in Y and, by our assumption, $f^{-1}(F)$ is closed in X. Observe that $f^{-1}(F)$ contains A. Indeed,

$$f(A) \subseteq F \Rightarrow A \subseteq f^{-1}\big(f(A)\big) \subseteq f^{-1}(F).$$

It follows that

$$\overline{A} \subseteq f^{-1}(F) \Rightarrow f\big(\overline{A}\big) \subseteq f\big(f^{-1}(F)\big) \subseteq F = \overline{f(A)}.$$

(d) $\Rightarrow$ (c). Let $F \subseteq Y$ be closed and $A = f^{-1}(F)$. To prove that A is closed, it suffices to show that $\overline{A} \subseteq A$. Observe that $f(A) \subseteq F$. Thus, if $x \in \overline{A}$, then

$$f(x) \in f\big(\overline{A}\big) \subseteq \overline{f(A)} = \overline{F} = F.$$

Hence $x \in f^{-1}(F) = A$, and therefore $\overline{A} \subseteq A$.

Note. Throughout the rest of these notes, whenever we refer to the topological space $\mathbb{R}^n$, $n \geq 1$ (or to the space $\mathbb{R}^n$), we shall always mean that it is equipped with the Euclidean topology (and its subsets with the subspace topology). Otherwise, we shall explicitly specify the topology under consideration.

We next prove some elementary properties of continuous functions.

Definition 1.44. Let X, Y be sets, $A \subseteq X$, and $f : X \to Y$ a function. The function

$$f|_A : A \to Y : \quad (f|_A)(x) = f(x) \quad \text{for every } x \in A$$

is called the **restriction** of f to A.

Proposition 1.45. Let X, Y, Z be topological spaces.

(a) **Composition:** If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then $g \circ f : X \to Z$ is also continuous.

(b) **Restricting the domain:** If $f : X \to Y$ is continuous and $A \subseteq X$ is equipped with the subspace topology, then the restriction $f|_A : A \to Y$ is continuous.

(c) **Restricting the codomain:** If $f : X \to Y \subseteq Z$ is a function and Y is a subspace of the topological space Z, then

$$f : X \to Z \text{ is continuous } \Leftrightarrow \ \widetilde{f} : X \to Y \text{ is continuous.}$$

(Here $\widetilde{f}$ denotes the same function with codomain the subspace Y of Z.)

Proof. (a) Observe that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Let $V \subseteq Z$ be open. Then, since g is continuous, $g^{-1}(V)$ is open in Y, and, since f is continuous,

$$f^{-1}\big(g^{-1}(V)\big) = (g \circ f)^{-1}(V)$$

is open in X. It follows from Theorem 1.43 that $g \circ f$ is continuous.

(b) Let $V \subseteq Y$ be open. Then $(f|_A)^{-1}(V) = f^{-1}(V) \cap A$. Since f is continuous, $f^{-1}(V)$ is open, so $f^{-1}(V) \cap A$ is open in A.

(c) Observe that the map $I : Y \to Z : I(y) = y$ for $y \in Y$ is continuous. (If $U \subseteq Z$ is open, then $I^{-1}(U) = U \cap Y$ is open in Y.)

($\Rightarrow$) Suppose that f is continuous. If $V \subseteq Y$ is open in Y, then there exists a set U open in Z such that $V = U \cap Y$.

Then

$$\begin{aligned}\widetilde{f}^{-1}(V) &= f^{-1}(Y \cap U) = f^{-1}(Y) \cap f^{-1}(U) \\ &= X \cap f^{-1}(U) = f^{-1}(U)\end{aligned}$$

is open in X. Thus $\widetilde{f}$ is continuous.

"$\Leftarrow$" Observe that $f = I \circ \widetilde{f}$. Thus, if $\widetilde{f}$ is continuous, then, since I is also continuous, assertion (a) of the proposition implies that f is continuous as a composition of continuous functions.

Examples 1.46.

1) Let (X, d), (Y, ρ) be metric spaces, $a \in X$, and $f : X \to Y$ a function continuous at a. Then, of course (according to the discussion preceding Definition 1.4), f is continuous at a in the familiar sense of Real Analysis.

2) If X and Y are topological spaces, then every constant function $f : X \to Y$ is continuous. (The inverse image of every open set $U \subseteq Y$ is either $\varnothing$ or X.)

3) Let X be a set with two topologies $\mathcal{T}_1$, $\mathcal{T}_2$. The identity map

$$I : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2) : I(x) = x,$$

where $x \in X$, is continuous if and only if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

4) Let X, Y, Z be topological spaces and $f : X \to Y$, $g : X \to Z$ functions. Define

$$\varphi : X \to Y \times Z : \varphi(x) = \big(f(x), g(x)\big), \quad x \in X.$$

Then φ is continuous (where $Y \times Z$ is equipped with the product topology; cf. Example 1.33 (1)) if and only if f and g are continuous. Indeed, if $U \subseteq Y$ and $V \subseteq Z$ are open, it is easy to verify that

$$\varphi^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V).$$

It follows that if $V = Z$, then $\varphi^{-1}(U \times Z) = f^{-1}(U)$, and if $U = Y$, then $\varphi^{-1}(Y \times V) = g^{-1}(V)$. Since the sets of the form $U \times Z$ and $Y \times V$, where $U \subseteq Y$ and $V \subseteq Z$ are open, form a subbase for $Y \times Z$, the conclusion follows from Theorem 1.43 (b).

It is easy to check that this example generalizes to a finite sequence of functions $f_k : X \to Y_k$, $k = 1, 2, \ldots, n$. (Fill in the details.)

As special cases of the above example, we mention the continuous functions

$$f : \mathbb{R} \to \mathbb{R}^2 : f(t) = (\cos t, \sin t)$$

and

$$F : \mathbb{R}^2 \to \mathbb{R}^3 : F(x, y) = (x + y^2, x^3, e^x).$$

5) Let X, Y be topological spaces. Consider the projections

$$\pi_1 : X \times Y \to X : \pi_1(x, y) = x$$

and

$$\pi_2 : X \times Y \to Y : \pi_2(x, y) = y,$$

onto the first and second coordinates, respectively. The maps π_1, π_2 are continuous (when $X \times Y$ is equipped with the product topology). Indeed, $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$, and these sets are open if U and V are open.

Proposition 1.47. Let X, Y be topological spaces and $\{A_a : a \in A\}$ a family of subsets of X such that $\bigcup_{a\in A} A_a = X$. For each $a \in A$, let $f_a : A_a \to Y$ be a continuous function such that

$$f_a|_{A_a \cap A_\beta} = f_\beta|_{A_a \cap A_\beta}, \quad \text{for every } a, \beta \in A.$$

Suppose that either:

(1) The sets A_a are open,

or (2) The sets A_a are closed and the family is finite, say $\{A_1, \ldots, A_n\}$.

Then there exists a (unique) continuous function $f : X \to Y$ extending every f_a; that is, for every $a \in A$, $f|_{A_a} = f_a$.

Proof. For each $x \in X$, define $f(x) = f_a(x)$, where a is an index for which $x \in A_a$. It is easy to prove that this does indeed define a function $f : X \to Y$ such that $f|_{A_a} = f_a$ for every $a \in A$.

(1) Let $V \subseteq Y$ be open. Then $f^{-1}(V) \cap A_a = f_a^{-1}(V)$. By the continuity of f_a, $f_a^{-1}(V)$ is open in A_a, and hence, since A_a is an open subset of X, it is open in X. It follows that

$$f^{-1}(V) = \bigcup_{a\in A} \big(f^{-1}(V) \cap A_a\big) = \bigcup_{a\in A} f_a^{-1}(V)$$

is open in X. Thus f is continuous.

(2) Let $F \subseteq Y$ be closed. Then $f^{-1}(F) \cap A_k = f_k^{-1}(F)$. By the continuity of f_k, $f_k^{-1}(F)$ is closed in A_k, and hence, since A_k is a closed subset of X, it is closed in X.

It follows that

$$f^{-1}(F) = \bigcup_{k=1}^{n} \big(f^{-1}(F) \cap A_k\big) = \bigcup_{k=1}^{n} f_k^{-1}(F).$$

Thus $f^{-1}(F)$ is closed as a finite union of closed sets, and f is continuous.

Remarks 1.48.

1) Let X be a set and $\{A_a : a \in A\}$ a family of subsets of X such that $\bigcup_{a\in A} A_a = X$. Such a family is called a **cover** or a **covering** of X.

2) Assertion (1) of Proposition 1.47 expresses the fact that continuity of a function is a local property.

3) As an application of Proposition 1.47, observe that the following functions are continuous:

$$f(x) = \begin{cases} x^2, & x \geq 0, \\ \dfrac{x}{3}, & x \leq 0, \end{cases} \qquad g(x) = \begin{cases} \sin x, & x \geq 0, \\ 0, & x \in [-1, 0], \\ -x - 1, & x \leq -1. \end{cases}$$

Definition 1.49. Let X, Y be topological spaces and $f : X \to Y$ a one-to-one and onto function. We say that f is a **homeomorphism** if both f and its inverse $f^{-1} : Y \to X$ are continuous. In this case, X and Y are said to be **homeomorphic**, and we write $X \cong Y$.

If X and Y are topological spaces and $f : X \to Y$ is a function, we say that f is a **topological embedding** if it is one-to-one and continuous and $f^{-1} : f(X) \to X$ is continuous (where $f(X)$ is regarded as a subspace of Y).

Let $f : X \to Y$ be a homeomorphism. It is easy to verify that $U \subseteq X$ is open if and only if $f(U) \subseteq Y$ is open. Thus f induces a one-to-one correspondence between the topologies $\mathcal{T}(X)$ and $\mathcal{T}(Y)$ of X and Y. Consequently, every property of X that can be expressed in terms of its topology (that is, its open sets) also holds for Y. Such a property is called a **topological property** of X.

More generally, a property of topological spaces is called a **topological invariant** if, whenever it holds for a space X, it also holds for every space homeomorphic to X.

Observe that homeomorphism is an equivalence relation on the class of topological spaces:

$$\text{(a) } X \cong X, \qquad \text{(b) } X \cong Y \Leftrightarrow Y \cong X,$$
$$\text{and (c) } X \cong Y \text{ and } Y \cong Z \Rightarrow X \cong Z.$$

(Fill in the details.)

Examples 1.50.

1) Let $f : X \to Y$ be a homeomorphism. If A is a subspace of X, then $A \cong f(A)$. Indeed, it is easy to verify that the map $f|_A : A \to f(A) \subseteq Y$ is a homeomorphism.

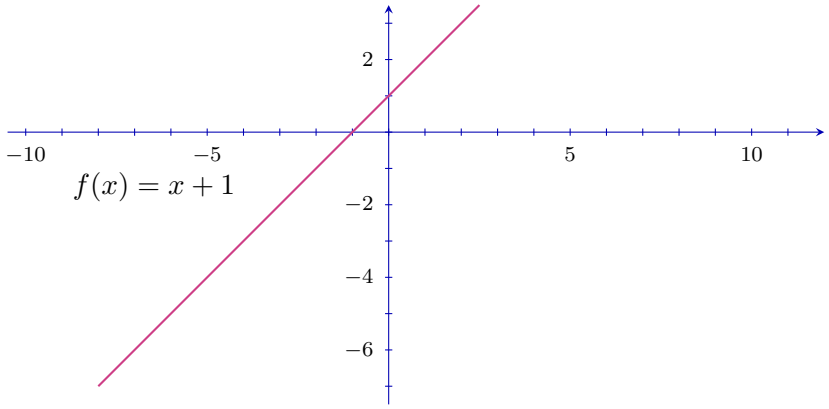
2) Let $c, d \in \mathbb{R}$ with $c \neq 0$. Then the map $f : \mathbb{R} \to \mathbb{R} : f(x) = cx + d$, $x \in \mathbb{R}$, is a homeomorphism, since both f and its inverse

$$f^{-1}(y) = \frac{1}{c}y - \frac{d}{c}, \quad y \in \mathbb{R},$$

are polynomials of degree one. In particular, if $a, b \in \mathbb{R}$ with $-\infty < a < b < +\infty$, then

$$[a, b] \cong [0, 1], \quad (a, b) \cong (0, 1), \quad [a, b) \cong [0, 1) \quad \text{and} \quad (a, b] \cong (0, 1].$$

Indeed, the map $f(x) = (b - a)x + a$, $x \in \mathbb{R}$, is a homeomorphism, $f(0) = a$, $f(1) = b$, and $f([0, 1]) = [a, b]$.



3) The function

$$f : \mathbb{R} \to (-1, 1) : f(x) = \frac{x}{1 + |x|}, \quad x \in \mathbb{R},$$

is a homeomorphism between $\mathbb{R}$ and $(-1, 1)$. Indeed, both f and its inverse

$$f^{-1} : (-1, 1) \to \mathbb{R} : f^{-1}(y) = \frac{y}{1 - |y|}$$

are continuous as compositions of continuous functions.

In particular,

$$\mathbb{R} \cong (-1, 1), \quad [0, +\infty) \cong [0, 1), \quad (-\infty, 0] \cong (-1, 0].$$

Combining Examples (2) and (3), we conclude that if $-\infty \leq a < b \leq +\infty$, then $(a, b) \cong (0, 1)$. (Why?)

4) Let $(\overline{\mathbb{R}}, \mathcal{T}_e)$ be the extended real line (cf. Example 1.38 (2)). Then the map

$$F : \overline{\mathbb{R}} \to [-1, 1] : F(x) = \frac{x}{1 + |x|}, \quad x \in \mathbb{R},$$

with $F(\pm\infty) = \pm 1$, is a homeomorphism. Consequently, the extended real line $\overline{\mathbb{R}}$ is homeomorphic to the interval $[-1, 1]$. (Fill in the details.)

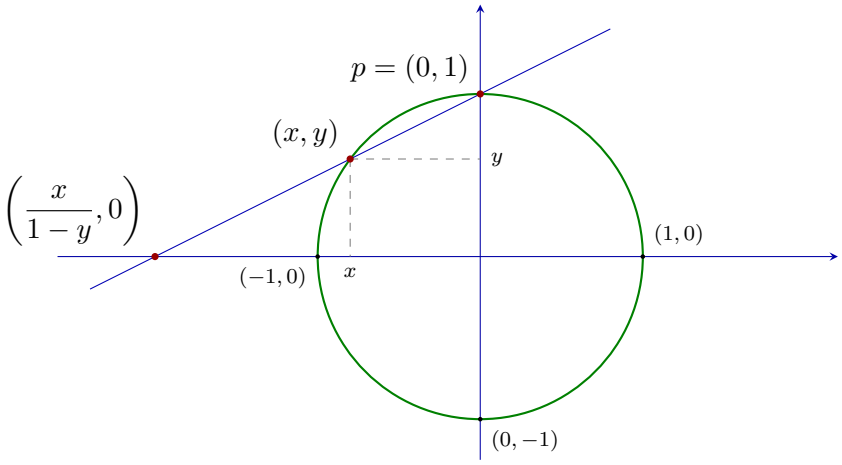
5) Consider the point $p = (0, 1)$ on the unit circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

in the Euclidean plane. Then $S^1 \setminus \{p\} \cong \mathbb{R}$. Indeed, the map

$$\varphi : (x, y) \in S^1 \setminus \{p\} \longmapsto \left(\frac{x}{1 - y}, 0\right) \in \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$$

is, as is easy to verify (also geometrically), a homeomorphism between $S^1 \setminus \{p\}$ and the subspace $\mathbb{R} \times \{0\}$ of $\mathbb{R}^2$. Since $\mathbb{R}$ can be identified with $\mathbb{R} \times \{0\}$ through the topological embedding $x \in \mathbb{R} \longmapsto (x, 0) \in \mathbb{R}^2$, the conclusion follows.



6) A continuous function $f : X \to Y$ that is also one-to-one and onto need not be a homeomorphism. One such function is the identity $I : (\mathbb{R}, \mathcal{T}) \to \mathbb{R}$, where $\mathcal{T}$ is the discrete topology on $\mathbb{R}$.

Another example is the identity $I : \mathbb{R}_s \to \mathbb{R}$. (Cf. Example 1.33 (2).)

Exercises

1) Let X be a set and $A \subseteq X$. The characteristic function X_A of A is defined by $\mathrm{X}_A(x) = 1$ if $x \in A$ and $\mathrm{X}_A(x) = 0$ if $x \in X \setminus A$. Suppose that X is a topological space and $A \subseteq X$. Prove that:

(a) The characteristic function X_A is continuous if and only if A is both open and closed in X.

(b) Prove that in the space $\mathbb{R}_s$ (= the Sorgenfrey line), the interval $[a, b)$ is both open and closed for every $a, b \in \mathbb{R}$ with $a < b$.

2) Let X be a topological space and $f, g : X \to \mathbb{R}$ continuous functions.

Prove that:

(a) The functions $f \pm g$, λf ($\lambda \in \mathbb{R}$), $f \cdot g$, $\max(f, g)$, and $\min(f, g)$ are continuous. In particular, the set $C(X)$ of all continuous functions $f : X \to \mathbb{R}$ is a vector space over $\mathbb{R}$ (with operations defined pointwise).

(b) If $f(x) \neq 0$, then the function $\dfrac{1}{f}$ is continuous on its domain.

(c) If X is a topological space and $f, g : X \to \mathbb{C}$ are continuous functions, then $f \pm g$, λf ($\lambda \in \mathbb{C}$), and $f \cdot g$ are continuous. If $f \neq 0$, then $\dfrac{1}{f}$ is continuous on its domain.

[*Hint.* For (b):

$$f \cdot g = \frac{1}{4}\left[|f + g|^2 - |f - g|^2\right],$$

$$\max(f,g) = \frac{1}{2}\left[f+g+|f-g|\right], \qquad \min(f,g) = \frac{1}{2}\left[f+g-|f-g|\right].$$

For $\max(f,g)$ and $\min(f,g)$, you may also use Proposition 1.47 (2). For (c), observe that the set of complex numbers $\mathbb{C}$ can be identified topologically with $\mathbb{R}^2$, and that if $f : X \to \mathbb{C} \cong \mathbb{R}^2$ is a function, then $f = (\operatorname{Re} f, \operatorname{Im} f)$.]

3) Prove that the functions $(x,y) \in \mathbb{R}^2 \longmapsto x+y \in \mathbb{R}$ and $(x,y) \in \mathbb{R}^2 \longmapsto x \cdot y \in \mathbb{R}$ are continuous.

[*Hint.* Let $(a,b) \in \mathbb{R}^2$ and $\varepsilon > 0$. For addition, observe that

$$|x-a| < \frac{\varepsilon}{2} \text{ and } |y-b| < \frac{\varepsilon}{2}$$
$$\Rightarrow |(x+y)-(a+b)| \leq |x-a| + |y-b| < \varepsilon.$$

For multiplication, choose $0 < \delta < 1$ such that $\delta(|a|+|b|+1) < \varepsilon$. Then the inequalities $|x-a| < \delta$ and $|y-b| < \delta$, together with the identity

$$xy - ab = a(y-b) + (x-a)b + (x-a)(y-b),$$

imply that $|xy-ab| < \varepsilon$.]

4) (a) Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is "right-continuous" (that is, $\lim\limits_{\substack{x \to a \\ x > a}} f(x) = f(a)$ for every $a \in \mathbb{R}$). Prove that f is continuous when regarded as a function from $\mathbb{R}_S$ to $\mathbb{R}$.

(b) What kinds of functions $f : \mathbb{R} \to \mathbb{R}$ are continuous as maps from $\mathbb{R}$ to $\mathbb{R}_S$, or from $\mathbb{R}_S$ to $\mathbb{R}_S$?

5) Find a function $f : \mathbb{R} \to \mathbb{R}$ with exactly one point of continuity.

6) Let X be a topological space, $f_n : X \to \mathbb{R}$, $n \geq 1$, a sequence of continuous functions, and $f : X \to \mathbb{R}$ a function. Prove that if $f_n \to f$ uniformly on X, then f is also continuous.

7) Let $(X, \|\cdot\|)$ be a normed space. Prove that the map

$$f : X \to B(0,1) \left(= \{x \in X : \|x\| < 1\}\right) :$$
$$f(x) = \frac{x}{1+\|x\|}, \quad x \in X$$

is a homeomorphism.

8) Adjoin a point, denoted by ∞, to the set $\mathbb{R}$, and consider the topology $\mathcal{T}$ on $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with subbase consisting of the sets of the form

$$\{x \in \mathbb{R} : x < a\}, \qquad \{x \in \mathbb{R} : x > a\},$$

together with the sets

$$\{\infty\} \cup \{x \in \mathbb{R} : x > a\} \cup \{x \in \mathbb{R} : x < a\}, \quad a \in \mathbb{R}.$$

Prove that:

(a) The subspace topology on $\mathbb{R}$, regarded as a subspace of $(\widetilde{\mathbb{R}}, \mathcal{T})$, is the Euclidean topology.

(b) The map $F : S^1 \to \widetilde{\mathbb{R}}$ defined by

$$F(x,y) = \frac{x}{1-y}, \quad \text{if } (x,y) \neq (0,1), \quad \text{and} \quad F(0,1) = \infty,$$

is a homeomorphism. (Cf. Example 1.50 (5).) Give a geometric interpretation of F.

9) Let $f : \mathbb{R} \to S^1 : f(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$. Prove that:

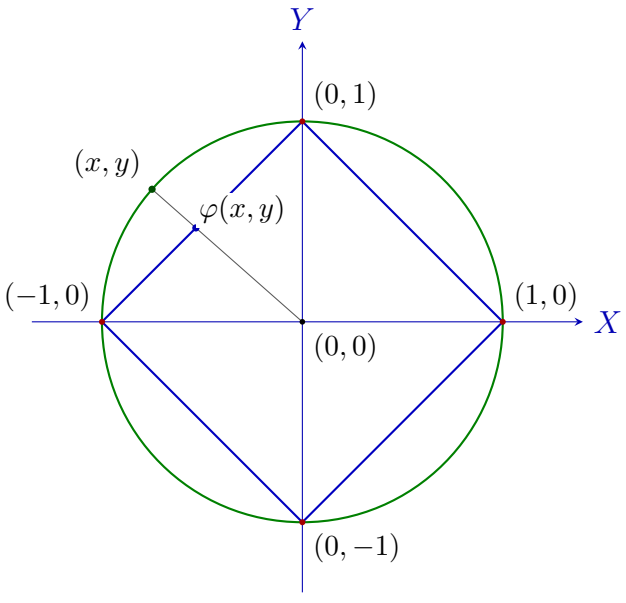
(a) f is continuous and maps onto S^1.

(b) $f|_{[0,2\pi)}$ is one-to-one and maps onto S^1, but is not a homeomorphism.

(c) If we identify the points 0 and 2π of $[0, 2\pi]$, what topology $\mathcal{T}$ on $[0, 2\pi)$ can make $f|_{[0,2\pi)}$ a homeomorphism?

[*Hint.* For (c): take as a neighbourhood base at 0 all sets of the form $[0, \varepsilon) \cup (2\pi - \varepsilon, 2\pi)$, $0 < \varepsilon < 2\pi$. Relate this "construction" to the space $(\widetilde{\mathbb{R}}, \mathcal{T})$ of Exercise (8).]

10) Explain geometrically why the map φ in the figure is a homeomorphism between the unit circle S^1 and the square $S = \{(x,y) : |x| + |y| = 1\}$ with vertices $(0,1)$, $(-1,0)$, $(0,-1)$, and $(1,0)$.



Then prove that φ extends to a homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(D) = \Omega$, where D and Ω are the interiors of S^1 and S.

[*Hint.* For each $\vec{x} = (x,y) \in \mathbb{R}^2$, set

$$\|\vec{x}\|_1 = |x| + |y| \quad \text{and} \quad \|\vec{x}\|_2 = \sqrt{x^2 + y^2}.$$

The required map φ is

$$\varphi(\vec{x}) = \frac{\|\vec{x}\|_2}{\|\vec{x}\|_1} \cdot \vec{x}, \quad \vec{x} \neq (0,0) \quad \text{and} \quad \varphi(0,0) = (0,0),$$

and its inverse is

$$\varphi^{-1}(\vec{y}) = \frac{\|\vec{y}\|_1}{\|\vec{y}\|_2} \cdot \vec{y}, \quad \vec{y} \neq (0,0) \quad \text{and} \quad \varphi^{-1}(0,0) = (0,0).$$

Observe that if $(\vec{x}_n) \subseteq \mathbb{R}^2$, then $\|\vec{x}_n\|_2 \to 0 \Leftrightarrow \|\vec{x}_n\|_1 \to 0$.]

11) Let $p = (0,0,1)$ be the "north" pole of the sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

in $\mathbb{R}^3$. Prove that the map

$$\varphi : (x,y,z) \in S^2 \setminus \{p\} \longmapsto \left(\frac{x}{1-z}, \frac{y}{1-z}, 0 \right) \in \mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^3$$

is a homeomorphism between $S^2 \setminus \{p\}$ and the subspace $\mathbb{R}^2 \times \{0\} \cong \mathbb{R}^2$ of $\mathbb{R}^3$. This map is called the **stereographic projection**. Similarly, one can prove that

$$S^n \setminus \{(0,0,\ldots,1)\} \cong \mathbb{R}^n \times \{0\} \cong \mathbb{R}^n,$$

where

$$S^n = \{(x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$