1.3 Relative topology and subspaces

Definition 1.37. Let $(X, \mathcal{T})$ be a topological space. If $Y \subseteq X$, then the family

$$\mathcal{T}|\,Y \stackrel{\text{def}}{=} \{Y \cap U : U \in \mathcal{T}\}$$

is a topology on the set Y, called the **relative** (or **induced**) topology on Y. The space $(Y, \mathcal{T}|\,Y)$ is called a **subspace** of X.

It is easy to verify that $\mathcal{T}|\,Y$ is a topology. It contains $\varnothing$ and Y, since

$$\varnothing = Y \cap \varnothing \quad \text{and} \quad Y = Y \cap X.$$

The fact that it is closed under arbitrary unions and finite intersections follows from the equalities

$$\bigcup_{a\in J}(U_a \cap Y) = \left(\bigcup_{a\in J} U_a\right) \cap Y \quad \text{and}$$
$$(U_1 \cap Y) \cap \cdots \cap (U_n \cap Y) = (U_1 \cap \cdots \cap U_n) \cap Y.$$

The members of $\mathcal{T}|\,Y$ are said to be **open in** Y or **relatively open** in Y.

Examples 1.38.

1) Let $X = \mathbb{R}$ with the usual topology and $Y = [0, 1) \cup \{2\}$. The following sets are open in Y:

   (a) $\{2\}$;

   (b) all open intervals in $\mathbb{R}$ contained in $[0, 1)$;

   (c) all intervals of the form $[0, a)$, $0 < a < 1$.

   The sets $[0, 1)$ and $\{2\}$ are both open and closed in Y. Observe that a set that is open (or closed) in Y need not be open (or closed) in $\mathbb{R}$.

2) We adjoin two points, $+\infty$ and $-\infty$, to the set of real numbers $\mathbb{R}$, and consider the topology $\mathcal{T}_e$ on the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ with subbasis consisting of the sets of the form

$$[-\infty, a) \stackrel{\text{def}}{=} \{-\infty\} \cup \{x \in \mathbb{R} : x < a\} \quad \text{and}$$

$$(a, +\infty] \stackrel{\text{def}}{=} \{+\infty\} \cup \{x \in \mathbb{R} : x > a\} \quad \text{for } a \in \mathbb{R}.$$

   The topological space $(\overline{\mathbb{R}}, \mathcal{T}_e)$ is called the **extended real line**.

   It is easy to check that the relative topology on $\mathbb{R}$, viewed as a subspace of $(\overline{\mathbb{R}}, \mathcal{T}_e)$, coincides with the usual topology on $\mathbb{R}$.

   Note: If we define $-\infty < x < +\infty$ for every $x \in \mathbb{R}$, then $\overline{\mathbb{R}}$ is a totally ordered set, and for every $\varnothing \neq A \subseteq \overline{\mathbb{R}}$, both $\sup A$ and $\inf A$ always exist (and may be equal to $\pm\infty$). The sets (intervals) $[-\infty, a)$ and $(a, +\infty]$ are neighbourhoods of $-\infty$ and $+\infty$, respectively.

3) Let (X, d) be a metric space and $Y \subseteq X$. Then

$$\mathcal{T}_d | Y = \mathcal{T}\big(d | Y \times Y\big).$$

In other words, the relative topology on Y coincides with the topology induced on Y by the restriction of the metric from $X \times X$ to $Y \times Y$. First observe that the open balls $B_Y(x, \varepsilon)$ in the metric space $(Y, d | Y \times Y)$ have the form $B_Y(x, \varepsilon) = Y \cap B(x, \varepsilon)$, where $x \in Y$ (why?).

The proof of the claim is now easy. Let $V \in \mathcal{T}_d | Y$. Then $V = Y \cap U$ for some $U \in \mathcal{T}_d$. If $x \in V$, then $x \in U$, so there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$, and thus

$$x \in Y \cap B(x, \varepsilon) \subseteq Y \cap U = V \quad \text{or} \quad x \in B_Y(x, \varepsilon) \subseteq V.$$

It follows that V can be written as a union of open balls in $(Y, d | Y \times Y)$, and hence $V \in \mathcal{T}(d | Y \times Y)$. Conversely, if $V \in \mathcal{T}(d | Y \times Y)$, then for each $x \in V$ there exists $\varepsilon_x > 0$ such that $B_Y(x, \varepsilon_x) \subseteq V$. That is, $x \in V \Rightarrow Y \cap B(x, \varepsilon_x) \subseteq V$, and consequently

$$V = \left[\bigcup_{x \in V} B(x, \varepsilon_x)\right] \cap Y \in \mathcal{T}_d | Y.$$

4) A subspace of a subspace of a topological space X is a subspace of X. (The proof is straightforward.)

Proposition 1.39. Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. Then the following hold:

(a) $V \subseteq Y$ is open in Y if and only if $V = U \cap Y$ for some open set $U \subseteq X$.

(b) $A \subseteq Y$ is closed in Y if and only if $A = F \cap Y$ for some closed set $F \subseteq X$.

(c) If $A \subseteq Y$, then $\mathrm{cl}_Y A = \mathrm{cl}_X A \cap Y$, $A'_Y = A' \cap Y$, $Y \cap \mathrm{int}(A) \subseteq \mathrm{int}_Y(A)$, and $\mathrm{Bd}_Y(A) \subseteq Y \cap \mathrm{Bd}(A)$.

(d) If $\{U_a : a \in J\}$ is a basis (respectively, a subbasis) for $\mathcal{T}$, then the family $\{Y \cap U_a : a \in J\}$ is a basis (respectively, a subbasis) for $\mathcal{T} | Y$.

Proof (a) This follows from the definition of the relative topology.

(b) A is closed in $Y \Leftrightarrow Y \setminus A$ is open in $Y \Leftrightarrow Y \setminus A = U \cap Y$ for some $U \in \mathcal{T}$ $\Leftrightarrow A = Y \cap (X \setminus U)$, where $X \setminus U = F$ is closed in X.

(c) Using the characterisation of the closure of a set and the fact that $A \subseteq Y$, we have, for $y \in Y$,

$$\begin{aligned} y \in \mathrm{cl}_X A \cap Y &\Leftrightarrow U \cap A \neq \varnothing \text{ for every neighbourhood } U \text{ of } y \text{ in } X \\ &\Leftrightarrow (U \cap Y) \cap A \neq \varnothing \text{ for every neighbourhood } U \text{ of } y \text{ in } X \\ &\Leftrightarrow y \in \mathrm{cl}_Y A. \end{aligned}$$

The second equality is proved similarly (where A'_Y denotes the derived set of A in the space $(Y, \mathcal{T} | Y)$). The other two relations are easily checked and are therefore left as an exercise.

(d) This claim is also straightforward and is left as an exercise.

Remarks 1.40. Let Y be a subspace of a topological space X.

1) If Y is an open subset of X, then $U \subseteq Y$ is open in $Y \Leftrightarrow U$ is open in X. The corresponding result holds if Y is a closed subset of X. (Exercise.)

2) If $p \in Y$, then $\mathcal{N}_p^Y = \{U \cap Y : U \in \mathcal{N}_p\}$ is the neighbourhood system of p in the space Y. Also, if B_p is a neighbourhood basis at p in X, then $B_p^Y = \{U \cap Y : U \in B_p\}$ is a neighbourhood basis at p in Y. (Exercise.)

Examples 1.41.

1) Let X be a topological space, Y a subspace of X, and $A \subseteq Y$. Then $Y \cap \text{int}(A) \subseteq \text{int}_Y(A)$ (Proposition 1.39(c)). Equality does not hold in general. Indeed, consider the Euclidean plane $\mathbb{R}^2$, let $X = \{(x, 0) : x \in \mathbb{R}\}$ be the x-axis, and let $A = \{(x, 0) : x \in (0, 1)\}$. It is then clear that $\text{int}(A) = \varnothing$, and hence $X \cap \text{int}(A) = \varnothing$, whereas $\text{int}_X(A) = A \neq \varnothing$. Thus equality does not hold in general.

2) In the Euclidean plane $\mathbb{R}^2$, the real line is identified with the subset (linear subspace) $\mathbb{R} \times \{0\}$ of $\mathbb{R}^2$. It is easy to verify that the relative topology on $\mathbb{R} \times \{0\}$ coincides with the usual topology on $\mathbb{R}$. (Fill in the details.)

Exercises

1) Let $(X, \mathcal{T})$ be a topological space, (x_n) a sequence in X, and $x \in X$. State the "correct" definition of convergence of (x_n) to x.

   [Hint: $x_n \xrightarrow{\mathcal{T}} x \Leftrightarrow$ for every neighbourhood U of x, there exists $\nu_0 \in \mathbb{N}$ such that $n \geq \nu_0 \Rightarrow x_n \in U$.]

2) Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces, and let B_1 and B_2 be bases for $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively.

   (a) Prove that the family

   $$\Sigma = \{X_1 \times V : V \in B_2\} \cup \{U \times X_2 : U \in \mathcal{T}_1\}$$

   is a subbasis and that $B = \{U_1 \times U_2 : U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2\}$ is a basis for the product topology $\mathcal{T}$ on $X_1 \times X_2$. Also prove that if (x_n, y_n), $n \geq 1$, is a sequence in $X_1 \times X_2$ and $(x, y) \in X_1 \times X_2$, then

   $$(x_n, y_n) \xrightarrow{\mathcal{T}} (x, y) \Leftrightarrow x_n \xrightarrow{\mathcal{T}_1} x \text{ and } y_n \xrightarrow{\mathcal{T}_2} y.$$

   (b) Let $(X_1, \mathcal{T}_1), \ldots, (X_n, \mathcal{T}_n)$, $(n \geq 2)$, be a finite sequence of topological spaces. Define the product topology on the Cartesian product $X = X_1 \times \cdots \times X_n$ and prove a result analogous to (a).

   (c) Let $X_1 = X_2 = \cdots = X_n = \mathbb{R}$, with the usual topology. Prove that the Euclidean topology on $\mathbb{R}^n$ coincides with the product topology on it.

3) Consider the topological space $(\mathbb{R}_s, \mathcal{T}_s)$ (the Sorgenfrey line). Prove that:

   (a) The family
   $$\Sigma = \{(-\infty, b) : b \in \mathbb{Q}\} \cup \{[a, +\infty) : a \in \mathbb{R}\}$$
   is a subbasis and $B = \{[a, b) : a, b \in \mathbb{R},\ b \in \mathbb{Q} \text{ and } a < b\}$ is a basis for $\mathcal{T}_s$. (Here $\mathbb{Q}$ denotes the rational numbers.)

   (b) The family $B_1 = \{[a, b) : a, b \in \mathbb{Q},\ a < b\}$ is a basis for a topology on $\mathbb{R}$ different from both the usual topology and $\mathcal{T}_S$.

   (c) $(\mathbb{R}_s, \mathcal{T}_s)$ has a countable neighbourhood basis at every point.

   (d) Every interval $[a, b)$, $a, b \in \mathbb{R}$, $a < b$, is both an open and a closed subset of $(\mathbb{R}_s, \mathcal{T}_s)$.

   (e) The Euclidean topology $\mathcal{T}$ is strictly coarser than $\mathcal{T}_S$.

   (f) Characterise the convergent sequences (x_n) in the topological space $\mathbb{R}_S$, and prove that they include all decreasing bounded sequences of real numbers. Does the sequence $x_n = 1 - \frac{1}{n}$, $n \geq 1$, converge in $\mathbb{R}_S$?

4) Let $(X, \leq)$ be a totally ordered set (for every $a, b \in X$ with $a \neq b$, either $a < b$ or $b < a$) with at least two elements.

   For $a, b \in X$ with $a < b$, define the open interval (a, b) by $(a, b) = \{x \in X : a < x < b\}$. Also, put $(-\infty, a) = \{x \in X : x < a\}$ and $(a, +\infty) = \{x \in X : a < x\}$ for $a \in X$.

   The intervals $[a, b]$, $[a, b)$, and $(a, b]$ are defined similarly. Prove that:

   (a) The family B of all intervals of the form $(-\infty, a)$ and $(b, +\infty)$, $a, b \in X$, is a subbasis for a topology $\mathcal{T}_0(X)$ on X, called the **order topology**.

   (b) The usual topology on $\mathbb{R}$ coincides with the order topology associated with the usual order on $\mathbb{R}$.

   (c) For which of the following subsets $X \subseteq \mathbb{R}$ does the topology $\mathcal{T}_0(X)$ (where $X \subseteq \mathbb{R}$ is totally ordered by the order inherited from $\mathbb{R}$) coincide with the relative topology inherited from $\mathbb{R}$ with its usual topology: $\mathbb{N}$, $[0, 1) \cup \{2\}$, and $\{0\} \cup \{x \in \mathbb{R} : |x| > 1\}$?

5) Consider the extended real line $(\overline{\mathbb{R}}, \mathcal{T}_e)$. Prove that:

   (a) $(\overline{\mathbb{R}}, \mathcal{T}_e)$ has a countable basis.

   (b) The topology $\mathcal{T}_e$ coincides with the topology $\mathcal{T}_0(\overline{\mathbb{R}})$ (cf. Exercise 4(c)).

   (c) Let $(x_n) \subseteq \mathbb{R}$. Then $x_n \xrightarrow{\mathcal{T}_e} +\infty \Leftrightarrow$ for every $\varepsilon > 0$ there exists $\nu_0 \in \mathbb{N}$ such that $n \geq \nu_0 \Rightarrow x_n > \varepsilon$. Prove the corresponding result for $x_n \xrightarrow{\mathcal{T}_e} -\infty$.

   (d) If $\varnothing \neq A \subseteq \overline{\mathbb{R}}$, then $\sup A \in \overline{A}$.

6) Let B be a basis for a topology $\mathcal{T}$ on a set X. Prove that $\mathcal{T}$ is equal to the intersection of all topologies on X that contain B. Prove the same result for a subbasis.

7) Prove that the rational numbers $\mathbb{Q}$, as a subspace of $\mathbb{R}$ with its usual topology, do not have the discrete topology.

8) Describe the relative topology on the unit circle $S_1 = \{z \in \mathbb{R}^2 : \|z\|_2 = 1\}$ in the Euclidean plane.