1.2 Bases and subbases

The task of defining a topology becomes easier if we can describe enough open sets to generate all the open sets.

Definition 1.29. Let (X, τ) be a topological space. A family $\mathcal{B} \subseteq \tau$ is called a *base* for τ if every nonempty open set (that is, every element of $\tau \setminus \{\varnothing\}$) is a union of members of $\mathcal{B}$.

We should note the distinction between a base and a neighbourhood base in a topological space. The former is a *global* concept, whereas the latter is *local*. Moreover, the members of a base are open sets, whereas those of a neighbourhood base need not be open, as we have already seen (cf. Examples 1.23). The following proposition characterises the families $\mathcal{B} \subseteq \tau$ that form a base for τ.

Proposition 1.30. Let $\mathcal{B} \subseteq \tau$. The following statements are equivalent.

(a) $\mathcal{B}$ is a base for τ.

(b) For every $U \in \tau$ and every $x \in U$, there exists $V \in \mathcal{B}$ such that $x \in V \subseteq U$.

(c) Set $\mathcal{B}_x = \{V \in \mathcal{B} : x \in V\}$ for $x \in X$. Then $\mathcal{B}_x$ is a neighbourhood base at x for every $x \in X$.

Proof. (a) $\Rightarrow$ (b). Let $U \in \tau$ and $x \in U$. Since $\mathcal{B}$ is a base for τ,

$$U = \bigcup \{V_i : i \in I\},$$

where $V_i \in \mathcal{B}$ for every $i \in I$. Hence there is at least one $V_i \in \mathcal{B}$ such that $x \in V_i \subseteq U$.

(b) $\Rightarrow$ (c). Let $x \in X$ and let W be a neighbourhood of x. Then there exists $U \in \tau$ such that $x \in U \subseteq W$. By our assumption, there exists $V \in \mathcal{B}$ such that $x \in V \subseteq U$. The conclusion follows from the definition of a neighbourhood base.

(c) $\Rightarrow$ (a). Let $U \in \tau$. If $x \in U$, our assumption gives a set $V \in \mathcal{B}_x$, and hence $V \in \mathcal{B}$, such that $x \in V \subseteq U$. It follows that U can be expressed as a union of members of $\mathcal{B}$, so $\mathcal{B}$ is a base for τ.

Examples 1.31.

1) Let (X, τ) be a topological space. Clearly, τ itself is a base for τ. If $\tau = \mathcal{P}(X)$ is the discrete topology on X, then $\mathcal{B} = \{\{x\} : x \in X\}$ is a base for τ. We already see that a topology can have different bases, as the following examples will also illustrate.

2) Let (X, d) be a metric space. Set

$$\mathcal{B}_1 = \{B(x, \varepsilon) : x \in X \text{ and } \varepsilon > 0\}$$

and

$$\mathcal{B}_2 = \left\{ B\left(x, \frac{1}{n}\right) : x \in X,\ n \geq 1 \right\}.$$

Since an open set in a metric space is a union of open balls, the families $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for the metric topology τ_d on X. Note that if $D \subseteq X$ is dense (that is, if $\overline{D} = X$), then the family

$$\mathcal{B} = \{B(x, q) : x \in D \text{ and } q \in \mathbb{Q},\ q > 0\}$$

is also a base for τ_d.

(Indeed, let $U \subseteq X$ be open and let $x \in U$. There exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. Choose a rational number q with $0 < q < \varepsilon$. If $y \in D$ and $d(x, y) < q/3$, then

$$x \in B\left(y, \frac{q}{3}\right) \subseteq B(x, q) \subseteq B(x, \varepsilon) \subseteq U.$$

The conclusion follows from Proposition 1.30.)

In particular, if D is countable and dense in X (that is, if X is separable), then X has a countable base. In this case we say that X is *second countable*.

3) Let $X = \mathbb{R}$ with its usual topology. The family of open intervals in $\mathbb{R}$,

$$\mathcal{B} = \{(a, b) : a < b,\ a, b \in \mathbb{R}\},$$

is a base for the topology on $\mathbb{R}$. This also follows from the previous example, since

$$(a, b) = B(x, \varepsilon) = (x - \varepsilon, x + \varepsilon), \quad \text{where } x = \frac{a+b}{2} \text{ and } \varepsilon = \frac{b-a}{2}.$$

Since the rational numbers are dense in $\mathbb{R}$, it follows from example (2) that the family

$$\mathcal{B}' = \{(a, b) : a, b \in \mathbb{Q},\ a < b\}$$

is also a countable base for $\mathbb{R}$.

4) Let $X = \mathbb{R}^2$ be the Euclidean plane. Observe that $D = \{(p, q) : p, q \in \mathbb{Q}\}$ is a countable dense subset of $\mathbb{R}^2$ (why?).

Thus, by example (2), the family $\mathcal{B}$ of all open discs $B(x, r)$ in $\mathbb{R}^2$ with centres $x \in D$ and positive rational radii r is a countable base for $\mathbb{R}^2$.

A different base $\mathcal{B}'$ for $\mathbb{R}^2$ consists of all open rectangles $V = (a, b) \times (c, d)$ (why?). This example generalises to every Euclidean space $\mathbb{R}^n$. (Fill in the details.)

We next examine the conditions under which an arbitrary family $\mathcal{B}$ of subsets of a set X can be a base for some topology on X.

Proposition 1.32. Let X be a set and let $\mathcal{B} \subseteq \mathcal{P}(X)$. Then $\mathcal{B}$ is a base for a unique topology on X if and only if it satisfies the following conditions.

(a) For every $x \in X$, there exists $U \in \mathcal{B}$ such that $x \in U$ (that is, $X = \bigcup\{U : U \in \mathcal{B}\}$).

(b) For every $U_1, U_2 \in \mathcal{B}$ and every $x \in U_1 \cap U_2$, there exists $U_3 \in \mathcal{B}$ such that $x \in U_3 \subseteq U_1 \cap U_2$.

Proof. "⇒". Suppose that $\mathcal{B}$ is a base for a topology τ on X. Since X is open, condition (a) follows from the definition of a base. If $U_1, U_2 \in \mathcal{B}$ and $x \in U_1 \cap U_2$, then $U_1 \cap U_2$ is open, so condition (b) follows from Proposition 1.30.

"⇐". Set

$$\tau = \left\{ V \subseteq X : \exists \{U_i : i \in I\} \subseteq \mathcal{B} \text{ such that } V = \bigcup_{i \in I} U_i \right\}.$$

We shall show that τ satisfies the definition of a topology. Condition (a) gives $X \in \tau$, and, taking $I = \varnothing$, we obtain

$$\varnothing = \bigcup_{i \in \varnothing} U_i \in \tau.$$

The family τ is closed under arbitrary unions. Indeed, let $\{U_j : j \in J\} \subseteq \tau$. By the definition of τ, for every $j \in J$ we can write $U_j = \bigcup_{i \in I_j} B_{j,i}$, where each $B_{j,i} \in \mathcal{B}$. Consequently,

$$\bigcup_{j \in J} U_j = \bigcup_{j \in J} \left(\bigcup_{i \in I_j} B_{j,i} \right),$$

which is a union of members of $\mathcal{B}$ and therefore belongs to τ.

The family τ is closed under finite intersections. It is enough to prove this for intersections of two sets (why?). Let $U_1, U_2 \in \tau$, and write $U_1 = \bigcup_{i \in I_1} B_i$ and $U_2 = \bigcup_{j \in I_2} B'_j$. Then

$$U_1 \cap U_2 = \bigcup_{(i,j) \in I_1 \times I_2} (B_i \cap B'_j). \tag{1}$$

By condition (b), if $B_i, B'_j \in \mathcal{B}$ and $x \in B_i \cap B'_j$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq B_i \cap B'_j$.

It follows that $B_i \cap B'_j$ is a union of members of $\mathcal{B}$, so by (1), $U_1 \cap U_2$ is also a union of members of $\mathcal{B}$ and therefore belongs to τ. The uniqueness of τ is immediate from its definition and the definition of a base.

Examples 1.33.

1) Let (X_1, τ_1) and (X_2, τ_2) be topological spaces. Consider the family $\mathcal{B}$ of all "open rectangles" in the Cartesian product $X_1 \times X_2$, that is, all sets of the form $U \times V$, where $U \in \tau_1$ and $V \in \tau_2$. The family $\mathcal{B}$ satisfies conditions (a) and (b) of Proposition 1.32. Indeed, (a) is obvious. For (b), observe that if $U_1 \times V_1$ and $U_2 \times V_2$ belong to $\mathcal{B}$, then

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2).$$

Thus $\mathcal{B}$ satisfies the stronger property of being closed under intersections of two sets, and hence under finite intersections.

The topology τ on $X_1 \times X_2$ with base $\mathcal{B}$ is important: it is called the *product topology* and will be studied in detail in the next chapter. For now, generalise the example to a finite collection of spaces $X_1, X_2, \ldots, X_n$.

2) Let $\mathbb{R}$ be the set of real numbers. Consider the family $\mathcal{B}$ of all half-open intervals of the form $[x, r)$, where $x, r \in \mathbb{R}$ and $x < r$.

It is easy to show that $\mathcal{B}$ satisfies conditions (a) and (b) of Proposition 1.32.

This defines a topology on $\mathbb{R}$ with base $\mathcal{B}$. The set $\mathbb{R}$ equipped with this topology, denoted by τ_s, is called the *Sorgenfrey line* and is denoted by $\mathbb{R}_S$. It is easy to verify that the usual (Euclidean) topology on $\mathbb{R}$ is strictly coarser than τ_s, and that the rational numbers form a dense subset of $\mathbb{R}_S$ (so $\mathbb{R}_S$ is separable).

We shall prove that $\mathbb{R}_S$ is not metrisable. Since it is separable, it is enough to show that it has no countable base (cf. Example 1.31 (2)).

Let $\mathcal{B}_1 = \{U_n : n \geq 1\}$ be a countable family of open subsets of $\mathbb{R}_S$. We may assume that each U_n is nonempty. Set $a_n = \inf U_n$ for $n \geq 1$, allowing $a_n = -\infty$ when U_n is unbounded below. Clearly, there exists $x_0 \in \mathbb{R}$ such that $x_0 \neq a_n$ for every $n \geq 1$. Observe that the interval $[x_0, x_0 + 1)$, which is open in $\mathbb{R}_S$, cannot be expressed as a union of a subfamily of $\mathcal{B}_1$ (why?). Thus $\mathcal{B}_1$ cannot be a base for $\mathbb{R}_S$.

The topological space $\mathbb{R}_S$ is an important example that will be useful later.

At this point, we may ask the following question. What happens if we start with an arbitrary family of subsets of a set X, adjoin all finite intersections of its members, and then take all possible unions of the resulting sets? This leads us to the notion of a subbase for a topology.

Definition 1.34. Let (X, τ) be a topological space and let $\Sigma \subseteq \tau$. The family Σ is called a *subbase* for τ if and only if the family

$$\mathcal{B} = \left\{ \bigcap_{k=1}^{n} C_k : n \geq 1,\ C_1, C_2, \ldots, C_n \in \Sigma \right\}$$

is a base for τ.

Proposition 1.35. Let $X \neq \varnothing$ be a set and let $\Sigma \subseteq \mathcal{P}(X)$ satisfy $X = \bigcup\{C : C \in \Sigma\}$. Then Σ is a subbase for a unique topology on X.

Proof. Set

$$\mathcal{B} = \left\{ \bigcap_{k=1}^{n} C_k : n \geq 1,\ C_1, \ldots, C_n \in \Sigma \right\}.$$

Clearly, $\Sigma \subseteq \mathcal{B}$. Observe that $\mathcal{B}$ satisfies conditions (a) and (b) of Proposition 1.32, and hence is a base for a topology τ on X. Indeed, (a) is immediate, since $\Sigma \subseteq \mathcal{B}$ and $X = \bigcup\{C : C \in \Sigma\}$. If $B_1, B_2 \in \mathcal{B}$, then

$$B_1 = \bigcap_{i=1}^{n} C_i \qquad \text{and} \qquad B_2 = \bigcap_{j=1}^{m} C'_j.$$

Consequently, $B_1 \cap B_2$ is a finite intersection of members of Σ. Thus $\mathcal{B}$ is closed under finite intersections, and condition (b) holds.

Examples 1.36.

1) Let (X, τ) be a topological space. Then the topology τ itself is a subbase for τ.

2) Let $X = \mathbb{R}$ with its usual topology τ. Then all intervals of the form $(-\infty, a)$ and $(b, +\infty)$, where $a, b \in \mathbb{R}$, form a subbase for τ. This is easy to verify.

3) In $\mathbb{R}_S$ (the Sorgenfrey line), the intervals of the form $[a, +\infty)$ and $(-\infty, b)$, where $a, b \in \mathbb{R}$, form a subbase for the topology of the space. Again, this is easy to check.

4) Let (X_1, τ_1) and (X_2, τ_2) be topological spaces. Then the product topology τ on $X_1 \times X_2$ (cf. Example 1.33 (1)) has the family

$$\Sigma = \{X_1 \times V : V \in \tau_2\} \cup \{U \times X_2 : U \in \tau_1\}$$

as a subbase.