# 1 Topological spaces

## 1.1 Elementary concepts of topology

In this section we introduce the basic concepts of topology, namely open and closed sets, the closure and interior of a set, and the neighbourhood of a point, and discuss the relationships between them.

**Definition 1.1.** Let $X \neq \varnothing$ be a set. A *topology* $\mathcal{T}$ on $X$ is a subset $\mathcal{T} \subseteq \mathcal{P}(X)$ such that:

(i) $\varnothing, X \in \mathcal{T}$.

(ii) The union of the members of any family of elements of $\mathcal{T}$ belongs to $\mathcal{T}$. (If $\{U_i : i \in I\} \subseteq \mathcal{T}$, where $I$ is an arbitrary index set, then $\bigcup_{i\in I} U_i \in \mathcal{T}$.)

(iii) The intersection of the members of any finite family of elements of $\mathcal{T}$ belongs to $\mathcal{T}$. (If $n \in \mathbb{N}$ and $U_1, \ldots, U_n \in \mathcal{T}$, then $\bigcap_{k=1}^{n} U_k \in \mathcal{T}$.)

The pair $(X, \mathcal{T})$ is called a *topological space*, and the members of $\mathcal{T}$ are called *open subsets* of $X$.

When there is no risk of confusion, we usually write "the topological space $X$" and use the abbreviation t.s. (= topological space).

**Examples 1.2.**

1) Let $(X, d)$ be a metric space. Recall that a nonempty subset $V \subseteq X$ is called open if for every $x \in V$ there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq V$. It is easy to prove that

$$\mathcal{T}_d = \{V \subseteq X : V \neq \varnothing \text{ and } V \text{ is open}\} \cup \{\varnothing\}$$

is a topology on $X$.

**Terminology.** A topological space $(X, \mathcal{T})$ is called *metrizable* if there is a metric $d$ on $X$ such that $\mathcal{T} = \mathcal{T}_d$.

2) Let $X \neq \varnothing$ be a set. Put $\mathcal{T} = \mathcal{P}(X)$. Clearly, $\mathcal{T}$ is a topology on $X$, called the *discrete topology* on $X$. Observe that $\tau = \mathcal{T}_d$, where $d$ is the discrete metric on $X$. (Proof: If $x \in X$, then $B(x, \varepsilon) = \{x\}$ for $0 < \varepsilon < 1$.)

3) Let $X \neq \varnothing$ be a set. Put $\mathcal{T} = \{\varnothing, X\}$. Clearly, $\mathcal{T}$ is a topology on $X$, called the *trivial topology* on $X$.

Observe that:

a) If $|X| = 1$, then $(X, \mathcal{T})$ is a metrizable topological space. Indeed, $\mathcal{P}(X) = \{\varnothing, X\} = \mathcal{T}$, and the conclusion follows from Example 2.

b) If $|X| \geq 2$, then $(X, \mathcal{T})$ is not metrizable. Indeed, suppose that $d$ is a metric on $X$ such that $\mathcal{T} = \mathcal{T}_d$. If $a, b \in X$, $a \neq b$, and $\varepsilon = d(a, b) > 0$, then the open balls $B(a, \varepsilon/2)$ and $B(b, \varepsilon/2)$ are disjoint, nonempty proper subsets of $X$. Thus they belong to $\mathcal{T}$, so $|\mathcal{T}| \geq 3$, a contradiction.

4) Let $X = \{a, b\}$ with $a \neq b$ and $\mathcal{T} = \{\varnothing, X, \{a\}\}$. As before, one can prove that $(X, \mathcal{T})$ is not metrizable. The space $(X, \mathcal{T})$ is called the *Sierpiński space*.

5) Let $X \neq \varnothing$ be a set. Put

$$\mathcal{T} = \{A \subseteq X : X \setminus A \text{ is finite}\} \cup \{\varnothing\}.$$

It is easy to prove that $\mathcal{T}$ is a topology on $X$. Note that a set $A \subseteq X$ is called *cofinite* if $X \setminus A$ is finite. Thus the members of $\mathcal{T}$ are the cofinite subsets of $X$ and $\varnothing$. The topology $\mathcal{T}$ is called the *cofinite topology*.

**Definition 1.3.** Let $X$ be a topological space. A subset $A \subseteq X$ is called *closed* if $X \setminus A$ is open.

**Proposition 1.4.** Let $X$ be a topological space and let $\mathcal{F}$ be the family of closed subsets of $X$. Then

(a) $\varnothing, X \in \mathcal{F}$.

(b) The intersection of any family of members of $\mathcal{F}$ is closed.

(c) The union of any finite family of members of $\mathcal{F}$ is closed.

**Proof.** Part (a) is obvious. If $A_i$, $i \in I$, are closed, then the sets $X \setminus A_i$ are open, so by De Morgan's laws

$$X \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \setminus A_i)$$

is open. Hence $\bigcap_{i \in I} A_i$ is closed. Similarly, for closed sets $A_1, \ldots, A_n$,

$$X \setminus \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X \setminus A_i)$$

is open, so $\bigcup_{i=1}^{n} A_i$ is closed.

**Definition 1.5.** Let $X$ be a topological space and $A \subseteq X$. The *closure* (or *closed hull*) of $A$ is the set

$$\overline{A} = \bigcap \{B : B \text{ is closed and } A \subseteq B\}.$$

The set $\overline{A}$ is also denoted by $\mathrm{cl}_X A$ or $\mathrm{cl}\, A$. Observe that $\overline{A}$, being an intersection of closed sets, is closed; in fact, it is the smallest closed set containing $A$, with respect to inclusion.

**Proposition 1.6.** Let $X$ be a topological space. The closure operation $\mathcal{P}(X) \ni A \mapsto \overline{A} \in \mathcal{P}(X)$ satisfies the following properties:

(a) $\overline{\varnothing} = \varnothing$.

(b) $A \subseteq \overline{A}$.

(c) $\overline{\overline{A}} = \overline{A}$.

(d) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(e) $A$ is closed $\iff A = \overline{A}$.

**Proof.** Parts (a) and (b) are obvious. For (c), $\overline{A}$ is closed, so $\overline{\overline{A}} = \overline{A}$. For (d), observe that if $C \subseteq D$, then $\overline{C} \subseteq \overline{D}$. Since $A, B \subseteq A \cup B$, we obtain $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Also, $A \cup B \subseteq \overline{A} \cup \overline{B}$ and the latter set is closed, so $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Part (e) follows from the definition, since $\overline{A}$ is closed.

**Examples 1.7.**

1) If $X$ has the discrete topology $\mathcal{T} = \mathcal{P}(X)$, every $A \subseteq X$ is closed.

2) Let $(X, d)$ be a metric space. If $x \in X$ and $\varepsilon > 0$, the open ball $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ is an open set, and the closed ball $\widehat{B}(x, \varepsilon) = \{y \in X : d(x, y) \leq \varepsilon\}$ is a closed set.

3) Let $X = \mathbb{R}$ with the usual metric. Every closed interval $[a, b]$, $a < b$, is closed. The sets $(-\infty, a]$ and $[b, +\infty)$ are also closed. The set $[0, 1)$ is neither open nor closed (why?).

4) Every line in the Euclidean plane $X = \mathbb{R}^2$ is a closed set (why?).

5) Let $X = \mathbb{Q}$ with the usual metric. The set $V = [\sqrt{2}, \sqrt{3}] \cap \mathbb{Q} = (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ is both an open and a closed subset of $\mathbb{Q}$, with $\varnothing \neq V \neq \mathbb{Q}$.

**Remarks 1.8.**

1) For every topological space $X$ and $A, B \subseteq X$, we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$. The corresponding identity does not hold in general for intersections. Indeed, for $X = \mathbb{R}$, $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$, we have $\overline{A \cap B} = \varnothing$, whereas $\overline{A} \cap \overline{B} = \mathbb{R}$. In general, $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ (why?).

2) If $(X, d)$ is a metric space, $x \in X$ and $\varepsilon > 0$, then $\overline{B(x, \varepsilon)} \subseteq \widehat{B}(x, \varepsilon)$. Equality does not hold in general. For the discrete metric on $X$, $\widehat{B}(x, 1) = X$, whereas $\overline{B(x, 1)} = B(x, 1) = \{x\}$. The equality $\overline{B(x, \varepsilon)} = \widehat{B}(x, \varepsilon)$ holds in Euclidean spaces and, more generally, in normed spaces. If $(X, \|\cdot\|)$ is a normed space, the metric is $d(x, y) = \|x - y\|$.

The next proposition gives an equivalent description of the closure of a set.

**Proposition 1.9.** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The following are equivalent:

(a) $x \in \overline{A}$.

(b) For every open set $U \subseteq X$ with $x \in U$, we have $U \cap A \neq \varnothing$.

That is, $\overline{A} = \{x \in X : U \cap A \neq \varnothing \text{ for every } U \in \mathcal{T} \text{ with } x \in U\}$.

**Proof.** (a)⇒(b): If there were an open set $U \ni x$ with $U \cap A = \varnothing$, then $X \setminus U$ would be closed and would contain $A$. Thus $\overline{A} \subseteq X \setminus U$, contradicting $x \in \overline{A} \cap U$.

(b)⇒(a): If $x \notin \overline{A}$, then the open set $X \setminus \overline{A}$ contains $x$ and does not meet $A$, a contradiction.

**Definition 1.10.** Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$ and $x \in X$. The point $x$ is called an *accumulation point* (a.p.) of $A$ if $U \cap (A \setminus \{x\}) \neq \varnothing$ for every open set $U \ni x$. The set of accumulation points of $A$ is called the *derived set* and is denoted by $A'$. Observe that $A' \subseteq \overline{A}$. A point $x \in A$ is called an *isolated point* of $A$ if there exists $U \in \mathcal{T}$ with $U \cap A = \{x\}$.

**Proposition 1.11.** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then $\overline{A} = A \cup A'$.

**Proof.** Let $x \in \overline{A}$. If $x \in A$, then $x \in A \cup A'$. If $x \notin A$, then for every open set $U \ni x$ we have $U \cap (A \setminus \{x\}) = U \cap A \neq \varnothing$, so $x \in A'$. Conversely, $A \subseteq \overline{A}$ and, by the definition of an accumulation point and Proposition 1.9, $A' \subseteq \overline{A}$.

**Corollary 1.12.** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The set $A$ is closed if and only if $A' \subseteq A$. In particular, if $A' = \varnothing$, then $A$ is closed.

**Proof.** If $A$ is closed, then $\overline{A} = A$, so $A' \subseteq \overline{A} = A$. If $A' \subseteq A$, then $\overline{A} = A \cup A' = A$, so $A$ is closed.

**Remarks and examples 1.13.** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$.

(a) If $x \in A$, then $x$ is an a.p. of $A$ if and only if it is not an isolated point of $A$ (why?).

(b) For $A = \{1\} \cup [2, 3] \subseteq \mathbb{R}$, the point 1 is isolated and hence is not an a.p. of $A$. Also, every point of $B = [0, 1]$ is an a.p. of both $[0, 1]$ and $(0, 1)$. Thus an a.p. may or may not belong to the set.

(c) If $A \subseteq \mathbb{R}$ (or $A \subseteq \mathbb{R}^n$) is open, then $A \subseteq A' = \overline{A}$ (why?). Moreover, an infinite set may have no a.p.: for example, $\mathbb{Z} \subseteq \mathbb{R}$.

We now introduce the interior of a set in a topological space, which is the concept dual to closure.

**Definition 1.14.** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The *interior* of $A$ is defined to be the set

$$A^\circ = \bigcup \{B : B \subseteq A \text{ and } B \text{ is open}\}.$$

It is also denoted by $\operatorname{int}(A)$, $\operatorname{int}_X(A)$ or $\operatorname{int}_{\mathcal{T}} A$. It is open, being a union of open sets, and is the largest open set contained in $A$. The points of $A^\circ$ are called *interior points* of $A$.

**Examples 1.15.**

1) In $\mathbb{R}$, the interior of $A = [0, 1)$ is $(0, 1)$.

2) In the Euclidean plane $\mathbb{R}^2$, the interior of the closed disc $\widehat{B}(x, \varepsilon)$ is the open disc $B(x, \varepsilon)$.

3) An infinite set may have empty interior: in $\mathbb{R}$, $\mathbb{Q}^\circ = \varnothing$. Similarly, $\mathbb{Q}^n$ and $\mathbb{Z}^n$ have empty interior in $\mathbb{R}^n$ (give the details).

The duality between closure and interior is described in the following proposition.

**Proposition 1.16.** Let $X$ be a topological space and $A \subseteq X$. Then

$$\text{(a)} \quad X \setminus \overline{A} = (X \setminus A)^\circ, \qquad \text{(b)} \quad X \setminus A^\circ = \overline{X \setminus A}.$$

**Proof.** For (a), $x \in X \setminus \overline{A}$ if and only if there exists an open set $U \ni x$ with $U \cap A = \varnothing$, if and only if $x \in (X \setminus A)^\circ$. Part (b) follows by applying (a) to $X \setminus A$.

**Proposition 1.17.** Let $X$ be a topological space. The interior operation $\mathcal{P}(X) \ni A \mapsto A^\circ \in \mathcal{P}(X)$ satisfies:

(a) $A^\circ \subseteq A$.

(b) $(A^\circ)^\circ = A^\circ$.

(c) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

(d) $A$ is open $\iff A = A^\circ$.

**Proof.** Parts (a) and (b) are obvious. If $C \subseteq D$, then $C^\circ \subseteq D^\circ$, so $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$. Conversely, $A^\circ \cap B^\circ$ is open and contained in $A \cap B$, so it is contained in $(A \cap B)^\circ$. Part (d) follows from the definition.

**Remark 1.18.** The equality $(A \cup B)^\circ = A^\circ \cup B^\circ$ does not hold in general. For $X = \mathbb{R}$, $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$, we have $(A \cup B)^\circ = \mathbb{R}$, whereas $A^\circ = B^\circ = \varnothing$. In general, $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ (why?).

**Definition 1.19.** Let $X$ be a topological space and $A \subseteq X$. The *boundary* $\mathrm{Bd}(A)$ (or $\mathrm{Bd}_X(A)$ or $\partial A$ or $\partial_X A$) is defined by

$$\mathrm{Bd}(A) = \overline{A} \cap \overline{X \setminus A}.$$

By Proposition 1.16, $\overline{X \setminus A} = X \setminus A^\circ$, so $\mathrm{Bd}(A) = \overline{A} \setminus A^\circ$. The points of $\mathrm{Bd}(A)$ are called *boundary points*. A point $x \in X$ is a boundary point if and only if every open set $U \ni x$ meets both $A$ and $X \setminus A$. The set $A$ and its complement have the same boundary.

**Proposition 1.20.** Let $X$ be a topological space and $A \subseteq X$. Then:

(a) $\overline{A} = A \cup \mathrm{Bd}(A) = A^\circ \cup \mathrm{Bd}(A)$ and $A^\circ \cap \mathrm{Bd}(A) = \varnothing$.

(b) $X = A^\circ \cup (X \setminus A)^\circ \cup \mathrm{Bd}(A)$, where the three sets are pairwise disjoint.

**Proof.** We have $A^\circ \cup \mathrm{Bd}(A) = A^\circ \cup (\overline{A} \setminus A^\circ) = \overline{A}$, and this union is disjoint. Since $A^\circ \subseteq A \subseteq \overline{A}$, we also obtain $\overline{A} = A \cup \mathrm{Bd}(A)$. For (b), the two interiors are disjoint, and each is disjoint from the boundary. Finally, $(X \setminus A)^\circ = X \setminus \overline{A}$, so the union of the three sets is $\overline{A} \cup (X \setminus \overline{A}) = X$.

**Remarks and examples 1.21.** In what follows, $\mathbb{R}^n$ is equipped with the Euclidean metric

$$d(x, y) = \|x - y\|_2 = \left( \sum_{k=1}^{n} (x_k - y_k)^2 \right)^{1/2}.$$

1) The boundary of $I = [0,1] \subseteq \mathbb{R}$ is $\mathrm{Bd}(I) = I \setminus I^\circ = \{0,1\}$.

2) The boundary of the upper half-plane $A = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ is the line $\{(x,0) : x \in \mathbb{R}\}$.

3) The boundary of the open disc $B(x,\varepsilon)$ in $\mathbb{R}^2$ is the circle $S(x,\varepsilon) = \{y \in \mathbb{R}^2 : \|x-y\|_2 = \varepsilon\}$.

4) More generally, the boundary of the open ball $B(x,\varepsilon)$ in $\mathbb{R}^n$ is $S(x,\varepsilon) = \{y \in \mathbb{R}^n : \|x-y\|_2 = \varepsilon\}$. It suffices to first prove that $\overline{B(x,\varepsilon)} = \widehat{B}(x,\varepsilon)$ (exercise).

5) If $A \subseteq \mathbb{R}^n$ is open, then $x \in \mathbb{R}^n$ is a boundary point of $A$ if and only if it is an accumulation point of $A$ and $x \notin A$ (why?).

**Definition 1.22.** Let $X$ be a topological space and $x \in X$.

(a) A *neighbourhood* of $x$ is a set $U \subseteq X$ such that $x \in U^\circ$. The set of neighbourhoods of $x$ is denoted by $\mathcal{N}_x$ and is called the *neighbourhood system* of $x$.

(b) A *neighbourhood base* $\mathcal{B}_x$ at $x$ is a family of subsets of $X$ such that $\mathcal{B}_x \subseteq \mathcal{N}_x$ and for every $U \in \mathcal{N}_x$ there exists $V \in \mathcal{B}_x$ with $V \subseteq U$.

The family $\mathcal{N}_x$ is itself a neighbourhood base. In general, there are many neighbourhood bases at the same point. The members of $\mathcal{B}_x$ are called *basic neighbourhoods* of $x$.

**Examples 1.23.**

1) For a topological space $(X,\mathcal{T})$ and $x \in X$, the family $\mathcal{B}_x = \{U \in \mathcal{T} : x \in U\}$ is a neighbourhood base at $x$.

2) In the discrete topology, $\mathcal{B}_x = \{\{x\}\}$ is a neighbourhood base and $\mathcal{N}_x = \{A \subseteq X : x \in A\}$.

3) In a metric space $(X,d)$, for $x \in X$, the families

$$\mathcal{B}_x^1 = \{B(x,\varepsilon) : \varepsilon > 0\}, \qquad \mathcal{B}_x^2 = \{B(x,1/n) : n \geq 1\}$$

are neighbourhood bases. The second is countable.

4) In $\mathbb{R}^2$, for $p = (a,b)$, the families $\mathcal{B}_p^1 = \{B(p,\varepsilon) : \varepsilon > 0\}$ and $\mathcal{B}_p^2 = \{U_n : n \geq 1\}$ are neighbourhood bases, where

$$U_n = (a - 1/n, a + 1/n) \times (b - 1/n, b + 1/n).$$

The second is countable, and $\mathcal{B}_p^1 \cap \mathcal{B}_p^2 = \varnothing$. A third neighbourhood base is the family of triangles in the plane that contain $p$ in their interiors (why?).

**Definition 1.24.** Let $S \neq \varnothing$ be a set.

(a) A nonempty family $\mathcal{F}$ of subsets of $S$ is a *filter* on $S$ if:

$$\text{(F1) } \varnothing \notin \mathcal{F}; \qquad \text{(F2) } F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F};$$

$$\text{(F3) } F \in \mathcal{F},\ F \subseteq G \subseteq S \Rightarrow G \in \mathcal{F}.$$

(b) A nonempty family $\mathcal{B}$ of subsets of $S$ is a *filter base* if:

$$\text{(B1) } \varnothing \notin \mathcal{B}; \qquad \text{(B2) } A, B \in \mathcal{B} \Rightarrow \exists C \in \mathcal{B} : \ C \subseteq A \cap B.$$

**Remarks 1.25.**

1) If $\mathcal{B}$ is a filter base on $S$, the family $\mathcal{F}(\mathcal{B})$ of all supersets in $S$ of members of $\mathcal{B}$ is a filter, called the *filter generated* by $\mathcal{B}$. Every filter is a filter base.

2) If $\varnothing \neq C \subseteq S$, then $\mathcal{F}(C) = \{A \subseteq S : C \subseteq A\}$ is the *principal filter* generated by $C$. A filter $\mathcal{F}$ is called *free* if $\bigcap_{F \in \mathcal{F}} F = \varnothing$. The *Fréchet filter* on an infinite set $S$ consists of all cofinite subsets of $S$. It is free, hence not principal, and is contained in every free filter on $S$ (why?).

3) For a topological space $(X, \mathcal{T})$ and $x \in X$, the neighbourhood system $\mathcal{N}_x$ is a filter, and every neighbourhood base $\mathcal{B}_x$ is a filter base (complete the details).

**Definition 1.26.** Let $X$ be a set and let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on $X$. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_2$ is said to be *finer* (or *larger*) than $\mathcal{T}_1$, and $\mathcal{T}_1$ is said to be *coarser* (or *smaller*) than $\mathcal{T}_2$.

**Proposition 1.27 (Hausdorff's criterion).** Let $X$ be a set and let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on $X$. The following are equivalent:

(a) $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

(b) For every $x \in X$ and every neighbourhood $U$ of $x$ in $\mathcal{T}_1$, there exists a neighbourhood $V$ of $x$ in $\mathcal{T}_2$ such that $x \in V \subseteq U$.

**Proof.** (a)⇒(b): For a neighbourhood $U$ of $x$ in $\mathcal{T}_1$, take $V = \operatorname{int}_{\mathcal{T}_1} U \in \mathcal{T}_1 \subseteq \mathcal{T}_2$, with $x \in V \subseteq U$.

(b)⇒(a): Let $U \in \mathcal{T}_1$. For every $x \in U$, the hypothesis gives a $\mathcal{T}_2$-neighbourhood $V$ of $x$ with $x \in V \subseteq U$. Thus $x \in \operatorname{int}_{\mathcal{T}_2} V \subseteq U$. Hence $U$ is a union of $\mathcal{T}_2$-open sets and belongs to $\mathcal{T}_2$.

**Remark 1.28.** For two metrics $d_1, d_2$ on $X$ and the corresponding topologies $\mathcal{T}_{d_1}, \mathcal{T}_{d_2}$, the criterion takes the form

$$\mathcal{T}_{d_1} \subseteq \mathcal{T}_{d_2} \quad \Longleftrightarrow \quad \forall x \in X \ \forall \varepsilon > 0 \ \exists \delta > 0 : \ B_2(x, \delta) \subseteq B_1(x, \varepsilon),$$

where $B_i$ denotes the open ball with respect to $d_i$.

# Exercises

1) Let $X \neq \varnothing$ and let $\mathcal{T}$ be the cofinite topology on $X$. Prove that (a) if $d$ is a metric on $X$, then $\mathcal{T} \subseteq \mathcal{T}_d$; (b) if $X$ is finite, then $\mathcal{T} = \mathcal{P}(X)$; (c) if $X$ is infinite, then $(X, \mathcal{T})$ is not metrizable.

2) Prove that each of the following families is a topology on $\mathbb{N}$:

(a) The sets $\varnothing$, $\mathbb{N}$ and all initial segments $I_n = \{1, 2, \ldots, n\}$, $n \geq 1$ (initial segment topology).

(b) The sets $\varnothing$, $\mathbb{N}$ and all final segments $T_n = \{n, n+1, \ldots\}$, $n \geq 1$ (final segment topology).

3) Find all topologies on a set $X$ with $|X| = 2$ or $|X| = 3$. [Hint: There are 4 and 29, respectively.]

4) Let $X$ be an infinite set and let $\mathcal{T}$ be a topology on $X$ such that $X$ is the only infinite open set. Is $\mathcal{T}$ trivial?

5) Let $X$ be an infinite set and let $\mathcal{T}$ be a topology on $X$ such that every infinite subset of $X$ belongs to $\mathcal{T}$. Prove that $\mathcal{T}$ is the discrete topology.

6) Let $(X, \|\cdot\|)$ be a normed space, $x \in X$ and $\varepsilon > 0$. Prove that

(a) $\overline{B(x,\varepsilon)} = \widehat{B}(x,\varepsilon)$,

(b) $\operatorname{int}(\widehat{B}(x,\varepsilon)) = B(x,\varepsilon)$,

(c) $\operatorname{Bd}(B(x,\varepsilon)) = \operatorname{Bd}(\widehat{B}(x,\varepsilon)) = S(x,\varepsilon)$,

where $S(x,\varepsilon) = \{y \in X : \|x - y\| = \varepsilon\}$.

7) Find $\operatorname{int}(A)$, $\overline{A}$ and $\operatorname{Bd}(A)$ for the following subsets of $\mathbb{R}^2$:

(a) $\{(x,y) \in \mathbb{R}^2 : 0 < y < x + 1,\ x > -1\}$.

(b) $\{(r\cos\theta, r\sin\theta) : 0 < r < 1,\ 0 < \theta < 2\pi\}$.

(c) $\{x_n : n \geq 1\} \cup \{x\}$, where $(x_n) \subseteq \mathbb{R}^2$, $x \in \mathbb{R}^2$, $x_n \to x$.

(d) $\{(x,y) \in \mathbb{R}^2 :$ either $x$ or $y$ is irrational$\}$.

8) Find the accumulation points of $A$ in each of the following cases:

(a) $A = \{(-1)^n n/(n+1) : n \in \mathbb{N}\} \subseteq \mathbb{R}$.

(b) $A = \{(\cos(2\pi/n), \sin(2\pi/n)) : n \in \mathbb{N}\} \subseteq \mathbb{R}^2$.

(c) $A = \{(x,y) \in \mathbb{R}^2 : (x^2 + y^2)(y^2 - x^2 + 1) \leq 0\}$.

The spaces $\mathbb{R}$ and $\mathbb{R}^2$ are equipped with the usual Euclidean topology.

9) Let $S$ be an infinite set. Prove that (a) the Fréchet filter is indeed a filter and is free; (b) every free filter on $S$ contains it.

10) Let $X, Y$ be sets, $f : X \to Y$ a function, and $\mathcal{B}$ a filter base on $X$. Prove that the family $f(\mathcal{B}) = \{f(B) : B \in \mathcal{B}\}$ is a filter base on $Y$. In particular, if $\mathcal{F}$ is a filter on $X$, then the family $f(\mathcal{F}) = \{f(E) : E \in \mathcal{F}\}$ is a filter base on $Y$.

11) If $A \subseteq \mathbb{N}$, we say that $A$ has a density if the limit

$$\lim_{n \to +\infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n}$$

exists. Prove that the family of subsets of $\mathbb{N}$ with density 1 is a free filter on $\mathbb{N}$.

12) Let $S \neq \varnothing$ and let $\mathcal{U}$ be a filter on $S$. The filter $\mathcal{U}$ is called an *ultrafilter* if it is not properly contained in any other filter. Prove that:

(a) If $x \in S$, then the principal filter $\mathcal{F}(\{x\}) = \{A \subseteq S : x \in A\}$ is an ultrafilter.

(b) If $S$ is finite, then every ultrafilter on $S$ has the form $\mathcal{F}(\{x\})$ for some $x \in S$.

(c) Every filter $\mathcal{F}$ on $S$ is contained in an ultrafilter. [Hint: Order the class of all filters containing $\mathcal{F}$ by inclusion ($\subseteq$) and apply Zorn's lemma.]

(d) Let $\mathcal{F}$ be a filter on a set $S$. Then $\mathcal{F}$ is an ultrafilter if and only if, for every $A \subseteq S$, we have $A \in \mathcal{F}$ or $S \setminus A \in \mathcal{F}$.

(e) If $S$ is an infinite set, $\mathcal{F}$ is a free filter and $\mathcal{F} \subseteq \mathcal{F}_1$, where $\mathcal{F}_1$ is a filter, then $\mathcal{F}_1$ is also free. Consequently, there exist free ultrafilters on $S$.

13) Prove by induction that, if a finite set $X$ has $n$ elements, there are at least $(n-1)!$ distinct topologies on $X$.

[Hint: Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{x_1, \ldots, x_n, x_{n+1}\}$. If $\mathcal{T}$ is a topology on $X$, take $i \leq n$ and define a topology $\mathcal{T}_i$ on $Y$ as follows: For every $U \in \mathcal{T}$ with $x_i \in U$, put $U_i = (U \setminus \{x_i\}) \cup \{x_{n+1}\}$. The topology $\mathcal{T}_i$ consists of all the sets $U_i$, together with $Y \setminus \{x_i\}$ and $Y$.]