

25-5-2026

Vanderbei

17.2

$0 < \theta \leq \pi/2$

$$\begin{cases} \sin \theta \geq 0 \\ \cos \theta \geq 0 \end{cases}$$

dual

$$\begin{aligned} \max (\cos \theta) x_1 + (\sin \theta) x_2 \\ x_1 \leq 1 \\ x_2 \leq 1 \\ x_1, x_2 \geq 0 \end{aligned}$$

$$(x^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, z^* = \cos \theta + \sin \theta)$$

$$\begin{aligned} \min y_1 + y_2 \\ y_1 \geq \cos \theta \\ y_2 \geq \sin \theta \\ y_1, y_2 \geq 0 \end{aligned}$$

Beitrag zu central path or analyzing problem

$$(x, w, y, z)$$

$$\begin{aligned} \max (\cos \theta) x_1 + \sin \theta x_2 \\ x_1 + w_1 = 1 \\ x_2 + w_2 = 1 \\ x_1, x_2, w_1, w_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \min y_1 + y_2 \\ y_1 - z_1 = \cos \theta \\ y_2 - z_2 = \sin \theta \\ y_1, y_2, z_1, z_2 \geq 0 \end{aligned}$$

Egrows Central Path

$$Ax + w = b : \quad x_1 + w_1 = 1 \quad (\Rightarrow 0 < x_1 < 1) \\ x_2 + w_2 = 1$$

$$xz = \mu c : \quad x_1 z_1 = \mu \\ x_2 z_2 = \mu$$

$$A^T y - z = c : \quad y_1 - z_1 = \cos \theta \\ y_2 - z_2 = \sin \theta$$

$$y w = \mu e : \quad y_1 w_1 = \mu \\ y_2 w_2 = \mu$$

$$(x, y, w, z) \geq 0$$

$$\begin{aligned}
 w_1 &= 1 - x_1 \\
 y_1 w_1 &= \mu \Rightarrow y_1 (1 - x_1) = \mu \Rightarrow y_1 = \frac{\mu}{1 - x_1} \\
 x_1 z_1 &= \mu \Rightarrow z_1 = \frac{\mu}{x_1} \\
 y_1 - z_1 &= \cos \theta
 \end{aligned}
 \left. \vphantom{\begin{aligned} y_1 w_1 &= \mu \\ x_1 z_1 &= \mu \\ y_1 - z_1 &= \cos \theta \end{aligned}} \right\} \Rightarrow$$

$$\frac{\mu}{1 - x_1} - \frac{\mu}{x_1} = \cos \theta \Rightarrow \dots \Rightarrow \cos \theta x_1^2 + (2\mu - \cos \theta) x_1 - \mu = 0$$

$$\Delta = 4\mu^2 + \cos^2 \theta > 0 \quad x_1^{(1)} x_1^{(2)} = -\frac{\mu}{\cos \theta} < 0$$

$$\begin{aligned}
 x_1 &= \frac{\cos \theta - 2\mu + \sqrt{4\mu^2 + \cos^2 \theta}}{2 \cos \theta} \Rightarrow \begin{cases} z_1 = \frac{\mu}{x_1} \\ w_1 = 1 - x_1 \\ y_1 = \frac{\mu}{x_1} + \frac{\mu}{1 - x_1} \end{cases}
 \end{aligned}$$

Óµora pa za x_2, w_2, z_2, y_2

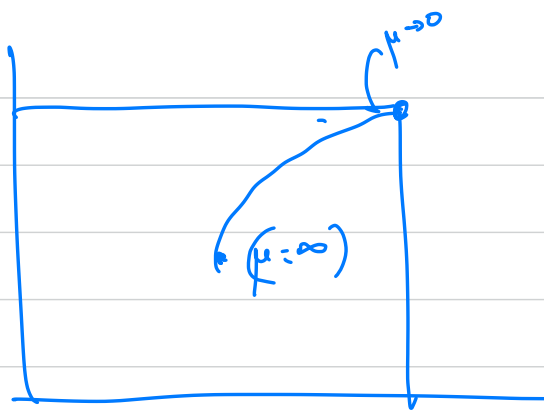
$$\lim_{\mu \rightarrow \infty} x_{1\mu} \quad x_1 = \cos \theta - 2\mu + 2\mu \sqrt{1 + \frac{\cos^2 \theta}{4\mu^2}}$$

$$\lim_{\mu \rightarrow \infty} x_{1\mu} = \frac{1}{2}$$

$$\lim_{\mu \rightarrow \infty} x_{2\mu} = \frac{1}{2}$$

$$\lim_{\mu \rightarrow 0} x_{1\mu} = 1$$

$$\lim_{\mu \rightarrow 0} x_{2\mu} = 1$$



17.4

Εστω το πρόβλημα

$$\max c_1 x_1 + \dots + c_n x_n$$

$$\left(\text{υπόδ: } \begin{array}{l} c_j > 0 \\ a_j > 0 \end{array} \right)$$

$$a_1 x_1 + \dots + a_n x_n = b$$

$$x_j \geq 0$$

$$x_j^* = \begin{cases} \frac{b}{a_k} & , j=k \\ 0 & , j \neq k \end{cases}$$

$$\text{όπου } \frac{c_k}{a_k} = \max \left\{ \frac{c_j}{a_j} , j=1, \dots, n \right\}$$

Αλλάζει μεταβλητή $x_j = \xi_j^2 \quad (\xi_j \in \mathbb{R})$

$$\Rightarrow \max \sum c_j \xi_j^2$$

$$\sum a_j \xi_j^2 = b$$

$$\text{Εστω } V = \{ x \in \mathbb{R}^n : \sum a_j x_j = b, x_j \geq 0 \}$$

$$W = \{ \xi \in \mathbb{R}^n : \sum a_j \xi_j^2 = b \}$$

$$\textcircled{1} \text{ Δ.Ο. } V \subseteq W$$

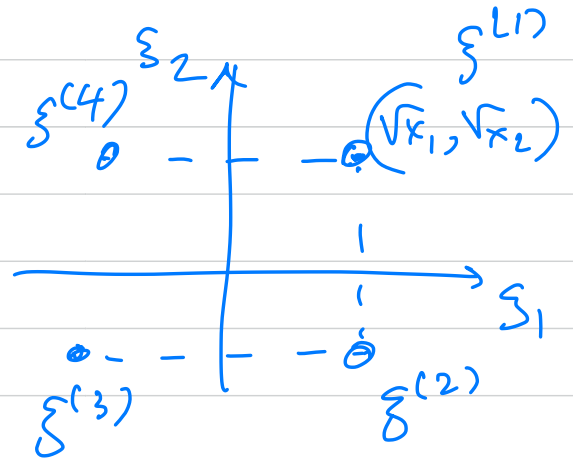
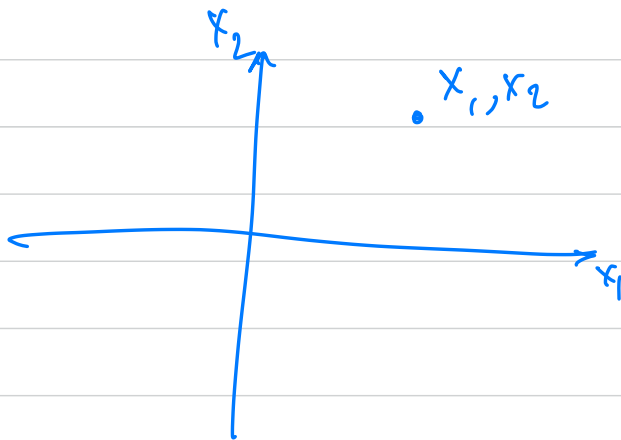
$$\forall x \in V \Rightarrow \sum a_j x_j = b \quad \Rightarrow \text{θετίζοντας } \xi_j = \sqrt{x_j} \Rightarrow$$

$$x_j \geq 0$$

$$=, \sum a_j \xi_j^2 = b \Rightarrow \xi \in W$$

$$\Rightarrow V \subseteq W$$

$$\text{or } x \in V \Rightarrow \xi_j = -\sqrt{x_j} = \pm \begin{pmatrix} \sqrt{x_1} \\ \vdots \\ \sqrt{x_n} \end{pmatrix} \in W$$



$$\begin{array}{l} \max \sum c_j \xi_j^2 \\ \sum a_j \xi_j^2 = b \end{array} \quad \left| \quad L(\xi, \lambda) = \sum c_j \xi_j^2 + \lambda (b - \sum a_j \xi_j^2) \right.$$

$$\frac{\partial L}{\partial \xi_j} = 2c_j \xi_j - 2\lambda a_j \xi_j = 2\xi_j (c_j - \lambda a_j) = 0 \quad \forall j$$

$$\frac{\partial L}{\partial \lambda} = \sum a_j \xi_j^2 = b$$

$$\forall j \text{ or } c_j - \lambda a_j \neq 0 \Rightarrow \xi_j = 0$$

$$\text{or } \lambda \neq \frac{c_j}{a_j} \Rightarrow \xi_j = 0$$

$$\text{Υποθέτουμε } \frac{c_j}{a_j} \text{ διαφορετικά μεταξύ τους } \frac{c_j}{a_j} \neq \frac{c_i}{a_i} \quad \forall i \neq j$$

$$\eta = \frac{c_k}{a_k} \text{ για το οποίο είναι } k=1, \dots, n$$

$$\text{δεν μπορεί } \eta \neq \frac{c_k}{a_k} \forall k \text{ γιατί τότε } \sum_k = 0 \forall k$$

$$\text{ε' επιπλέον } \underline{\sum_k a_k \xi_k^2 = 0 \neq b}$$

$$\text{Επιπλέον } \eta = \frac{c_k}{a_k} \text{ για κάποιο } k. \quad \underline{(a_k \xi_k^2 = b)}$$

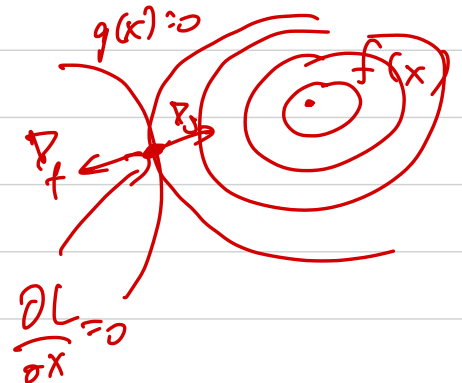
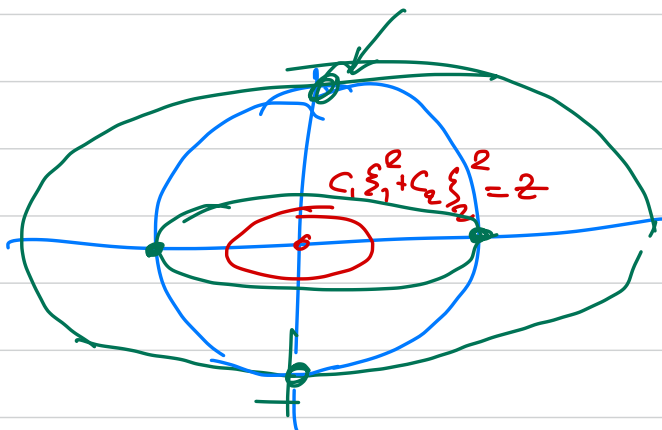
$$\Rightarrow \xi_j = \begin{cases} \pm \sqrt{\frac{b}{a_k}} & , j=k \\ 0 & , j \neq k \end{cases} \Rightarrow \text{ζωνικά} \\ \text{ακρόζωλα στο } L$$

$$\text{αρκ. } f(\xi) = c_k \left(\sqrt{\frac{b}{a_k}} \right)^2 = \frac{c_k b}{a_k}$$

$$f^* = b \max_k \left(\frac{c_k}{a_k} \right) \rightarrow \text{dvm } k' \text{ του } \Pi \cap \Pi.$$

$$\max c_1 \xi_1^2 + c_2 \xi_2^2$$

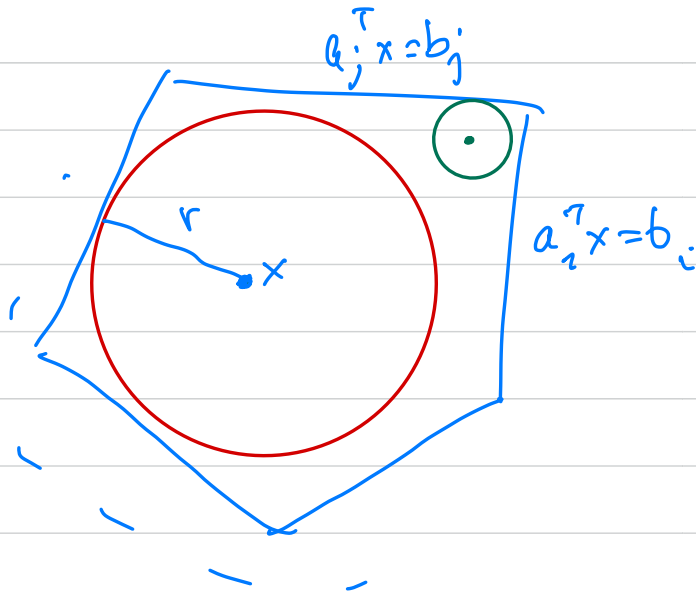
$$\text{πχ. αν } a_1 = a_2 = 1 \quad \xi_1^2 + \xi_2^2 = b$$



Άσκηση

P : κυρτό πολέδρο

$$P = \{x \in \mathbb{R}^n : a_i^T x \leq b_i \quad i=1, \dots, m\}$$



Θέλω εκεινο το x^* έτσι ώστε ο κύκλος να έχει τη μέγιστη δυνατή ακτίνα

Μορφοποιώ μέσω ΠΓΠ.

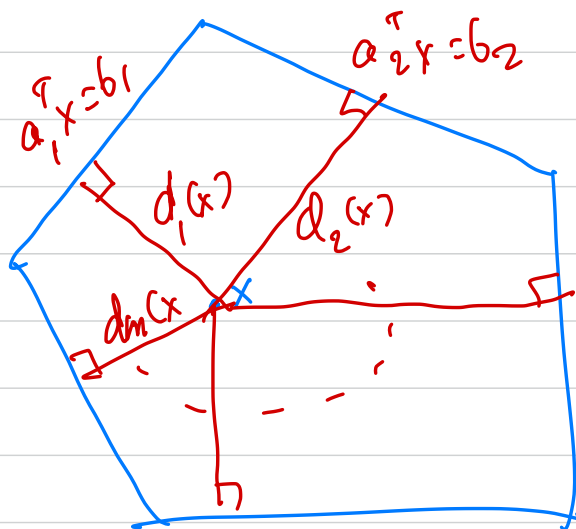
1) $\forall x \in P$ έχω $r^*(x)$: μέγιστη δυνατή ακτίνα κύκλου με κέντρο x που ανήκει σε φορέα του

σε P

$$\text{Θέλουμε } z = \max_{x \in P} r^*(x)$$

$$2) \quad \forall x \in P \quad : \quad r^*(x) = \min_{i=1, \dots, m} d_i(x)$$

oder $d_i(x) = \text{abstand von } x \text{ auf}$
 $\text{zu } a_i^T x = b$



Ergebnis
$$z = \max_{x \in P} \min_{i=1, \dots, m} d_i(x)$$

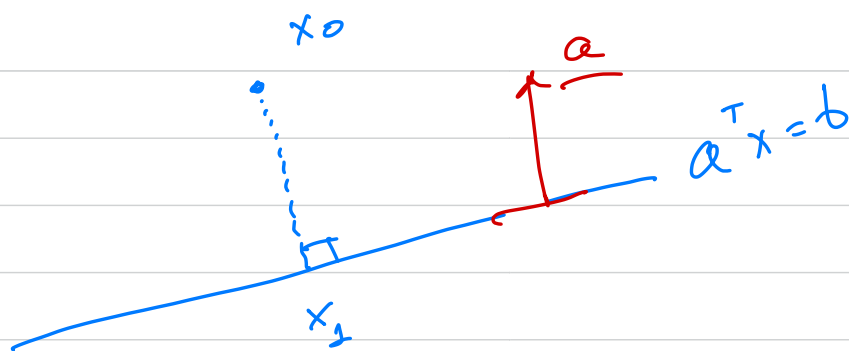
$$= \max \left\{ \min d_i(x) : x \in \mathbb{R}^n, Ax \leq b \right\}$$

$$d_i(x) = ?$$

$$d_i(x) = \frac{|a_i^T x - b|}{\|a_i\|_2} = \frac{|a_i^T x - b|}{\sqrt{a_1^2 + \dots + a_n^2}}$$

$$\theta_i |a_i^T x - b| \quad \theta_i > 0$$

(nüs bräivej)



$$d(x_0) = \|x_0 - x_1\| = \sqrt{\min \{ \|x_0 - x\|^2 : a^T x = b \}}$$

$$\min \left\{ \|x - x_0\|^2 : a^T x = b \right\} \quad \text{Lagrange} \Rightarrow$$

$$\Rightarrow \dots \quad x_1 - x_0 = \lambda a \quad \lambda \in \mathbb{R}$$

$$\Rightarrow \left. \begin{array}{l} x_1 = x_0 + \lambda a \\ a^T x = b \end{array} \right\} \Rightarrow a^T x_0 + \lambda a^T a = b \Rightarrow$$

$$\Rightarrow \lambda \|a\|^2 = b - a^T x_0 \Rightarrow \lambda = \frac{b - a^T x_0}{\|a\|^2}$$

$$\dots \quad x_1 - x_0 = \frac{(b - a^T x_0)}{\|a\|^2} \cdot \underline{a}$$

$$\Rightarrow \|x_1 - x_0\| = \frac{|b - a^T x_0|}{\|a\|} \cdot \cancel{\|a\|}$$

Exercises

$$z = \max \min_{i=1, \dots, m} \theta_i |a_i^T x - b_i|$$

v.n. $Ax \leq b$

$$\begin{aligned} \forall x : Ax \leq b &\Rightarrow a_i^T x - b_i \leq 0 \Rightarrow b_i - a_i^T x \geq 0 \\ (\forall x \in P) &\Rightarrow |a_i^T x - b_i| = b_i - a_i^T x \end{aligned}$$

$$\begin{aligned} \Rightarrow z^* &= \max_x \min_{i=1, \dots, m} (b_i - a_i^T x) \theta_i \\ &\quad a_i^T x \leq b_i, \quad i=1, \dots, m \end{aligned}$$

$\Downarrow$ pw linear concave

$$\begin{aligned} = \max \quad & z \\ & z \leq (b_i - a_i^T x) \theta_i \quad i=1, \dots, m \\ & Ax \leq b \end{aligned}$$

Ασκηση

$$\max C^T x$$

$$Ax = b$$

$$x \geq 0$$

$$A = \begin{pmatrix} 3 & 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 18 \\ 7 \end{pmatrix} \quad c = \begin{pmatrix} 3 \\ 4 \\ 1 \\ 5 \\ 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

- 1) Δ.Ο. η βασική λύση του B είναι ΒΕΛ
- 2) Ξεράστε αν είναι βέλτου.