

Μέθοδοι Εσωτερικού Σημείου για ΓΠ.

R. Vanderbei, Linear Programming (2020)

① Συμβολισμός

Primal

$$\begin{aligned} z_p &= \max C^T x \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

Dual

$$\begin{aligned} z_D &= \min b^T y \\ A^T y &\geq C \\ y &\geq 0 \end{aligned}$$

(KM)

$$\begin{aligned} \max C^T x \\ Ax + w &= b \\ x, w &\geq 0 \end{aligned}$$

$$\begin{aligned} \min b^T y \\ A^T y - z &= C \\ y, z &\geq 0 \end{aligned}$$

$$A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, C \in \mathbb{R}^n, x, z \in \mathbb{R}^n, y, w \in \mathbb{R}^m$$

Από θεωρήμα συμπληρωματικότητας (σε HKM)

Αν x^*, y^* είναι βέλτεστη λύση των

P, D αντίστοιχα

$$y_i^* (b_i - a_i^T x^*) = 0 \quad i = 1, \dots, m$$

$$x_j^* (A_j^T y - c_j) = 0 \quad j = 1, \dots, n$$

Ισοδύναμα σε κανονική μορφή

$$x_j^* z_j^* = 0 \quad j = 1, \dots, n$$

$$y_i^* w_i^* = 0 \quad i = 1, \dots, m$$

Έστω για $x \in \mathbb{R}^n$ $X = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}_{n \times n}$

αντιστοίχα $Y = \begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_m \end{pmatrix}_{m \times m}, W, Z$

$$e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$Xe = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$Xe = x, \quad Ye = y, \dots$$

$$X^{-1} = \begin{pmatrix} 1/x_1 & & \\ & \ddots & \\ & & 1/x_n \end{pmatrix} \quad (\text{όπου } x_j \neq 0 \forall j)$$

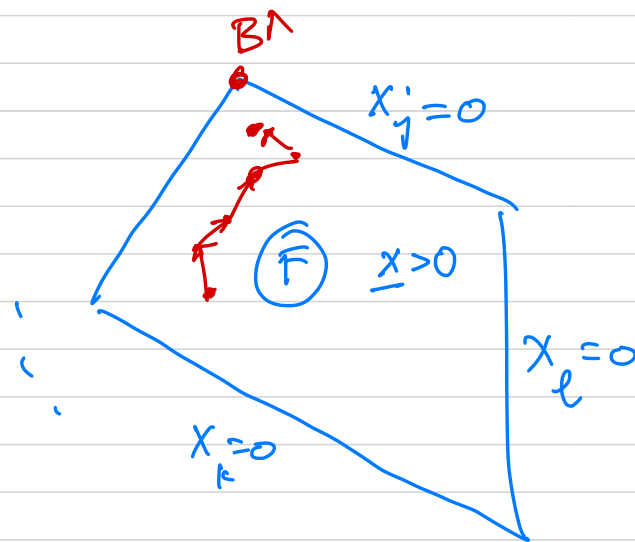
$$X^{-1}e = \begin{pmatrix} 1/x_1 \\ \vdots \\ 1/x_n \end{pmatrix}$$

2

$$z_p = \max C^T x$$

$$Ax + y = b$$

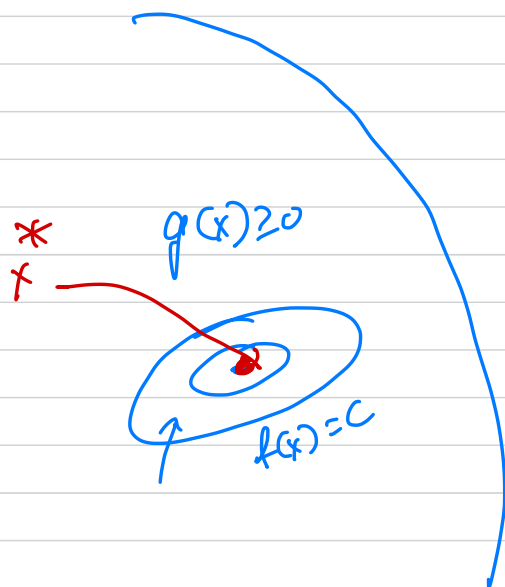
$$x \geq 0$$



Η μέθοδος που θα δοίτε δοκιμάτε

σε συνδυασμό { Lagrangean optimization
+ penalty barrier method.

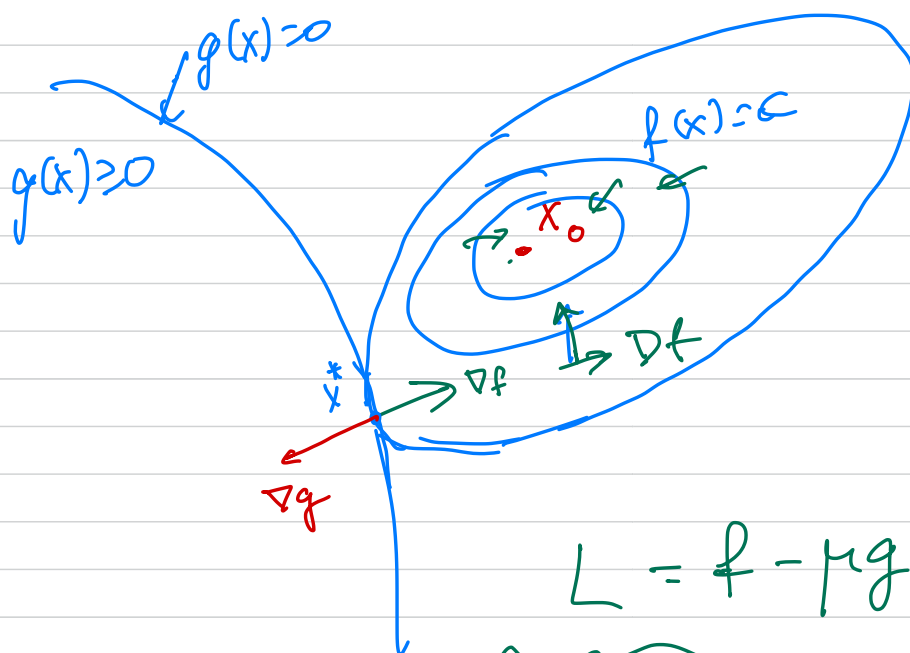
Εστω το πρόβλημα $\max f(x)$
 $g(x) \geq 0$



$$\nabla f(x) = 0 \Rightarrow x = x^*$$

$g(x^*) > 0$
 με Slater's condition

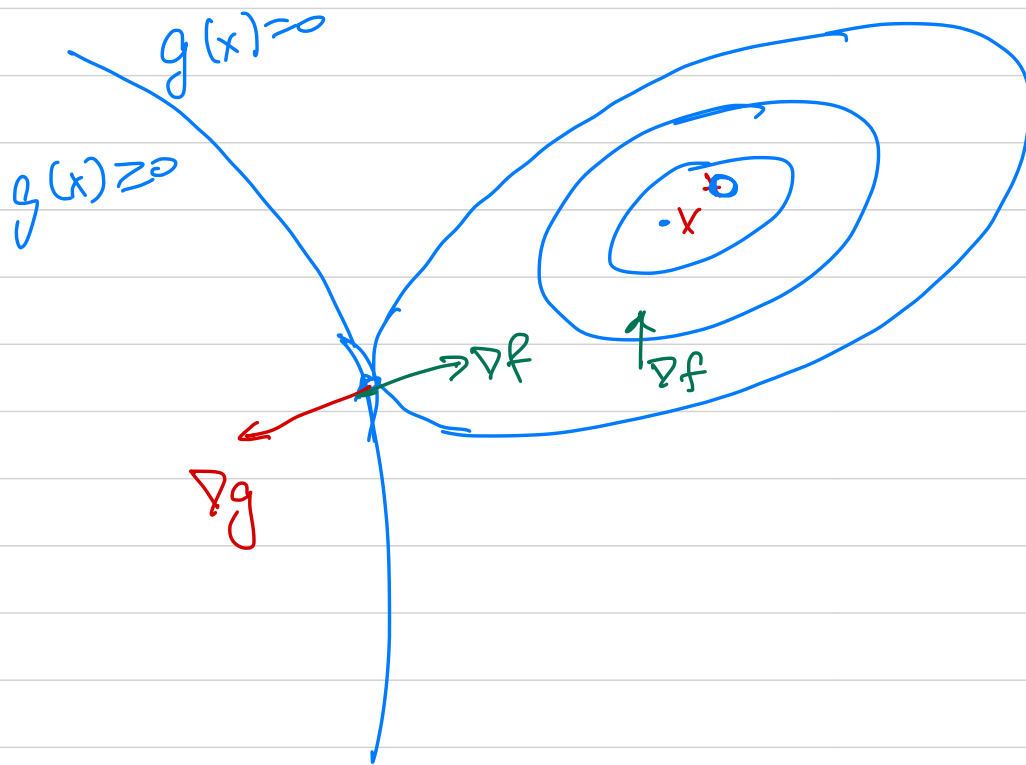
Εστω ότι το $x^0 : \nabla f(x^0) = 0$ είναι $g(x^0) < 0$



$$L = f - \mu g$$

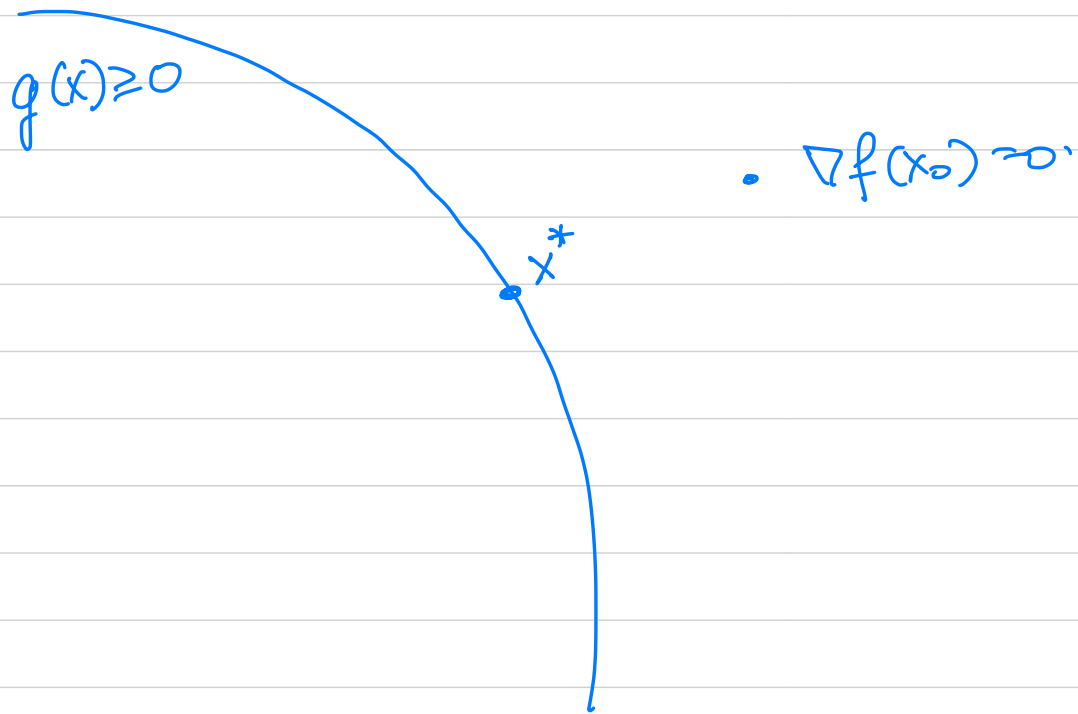
(Lagrangian
 eben)

$$\nabla f = \mu \nabla g \Rightarrow \nabla (f - \mu g) = 0$$



$$\nabla f = \mu \nabla g$$

Penalty barrier method



Barrier function

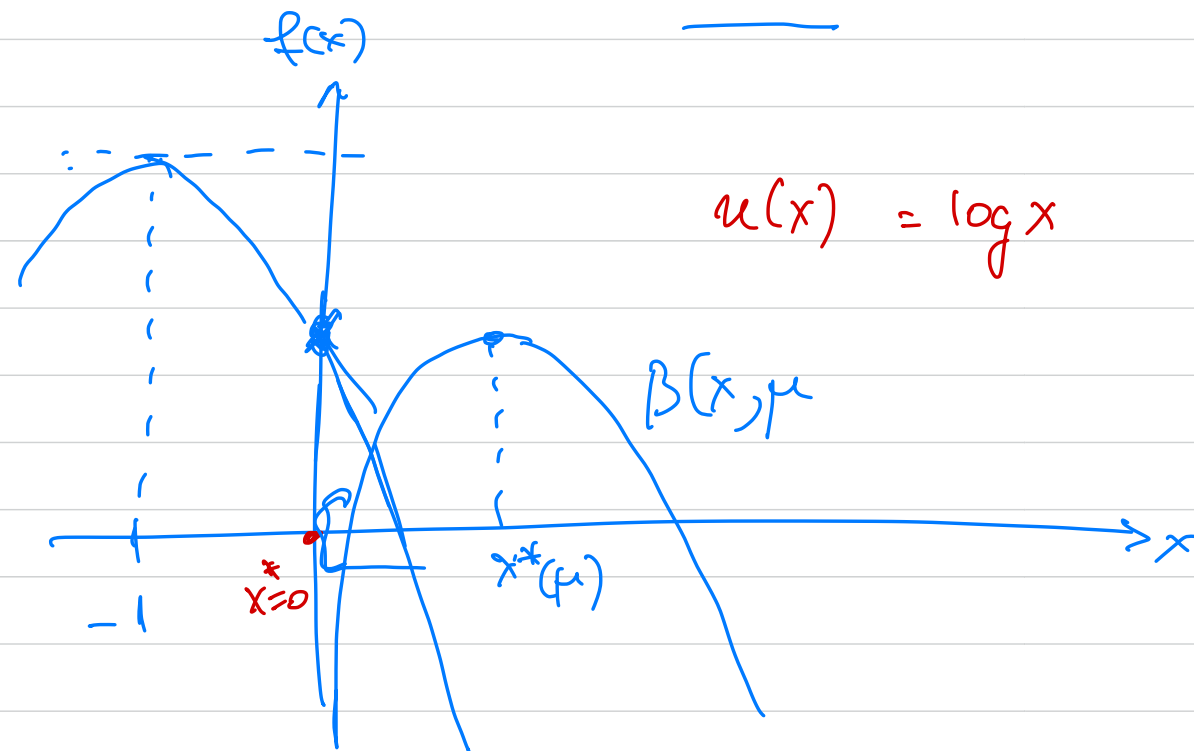
$$B(x, \mu) = f(x) + \mu u(x)$$

$u(x)$: penalty function : $\lim_{g(x) \rightarrow 0} (u(x)) = -\infty$

Given $x^*(\mu) : \max_x B(x, \mu)$

Then $\mu \rightarrow 0 \Rightarrow x^*(\mu) \rightarrow x^*$

1. x. $\max_{\underline{x \geq 0}} f(x) = \underline{2 - (x+1)^2}$



$$B(x, \mu) = 2 - (x+1)^2 + \mu \log x$$

$$\frac{\partial B}{\partial x} = -2(x+1) + \frac{\mu}{x}$$

$$\frac{\partial^2 B}{\partial x^2} = -2 - \frac{\mu}{x^2} < 0$$

$$\frac{\partial B}{\partial x} = 0 \Rightarrow \frac{\mu}{x} - 2x - 2 = 0 \Rightarrow -2x^2 - 2x + \mu = 0$$

$$\Rightarrow x^*(\mu) = \sqrt{1+2\mu} - 1$$

$$B x^* = 1 + \mu \left[\log(\sqrt{1+2\mu} - 1) - 2 \right] \quad . \quad ?$$

Primal Problem (KM)

$$z_p = \max C^T x$$

$$Ax + w = b \leftrightarrow \text{Lagrangian}$$

$$x, w \geq 0 \leftrightarrow \text{penalty barrier}$$

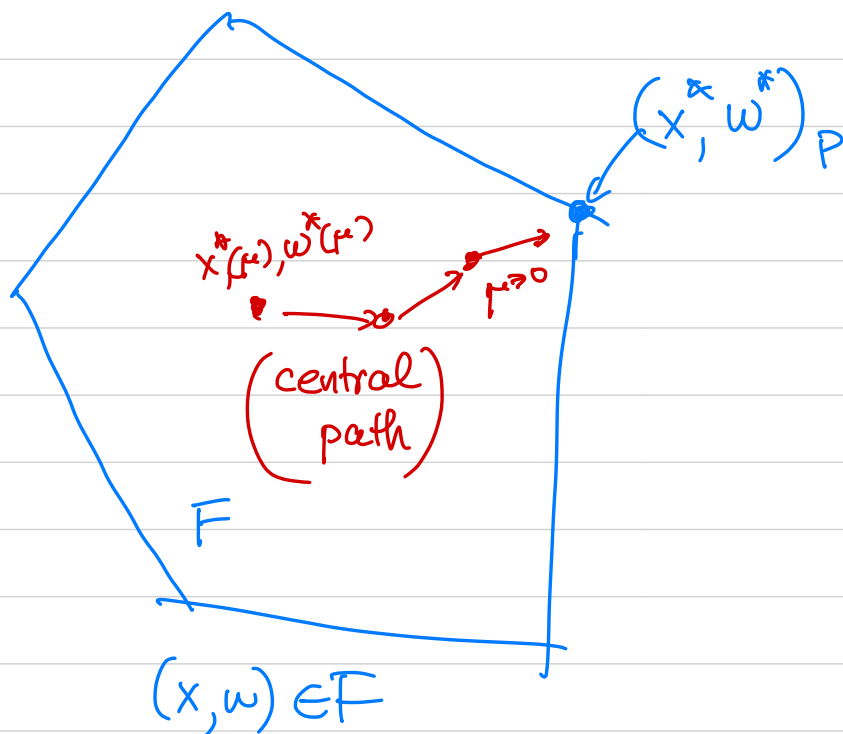
↓ penalty barrier (log)

$$z^*(\mu) = \max_{Ax + w = b} C^T x + \mu \sum_j \log x_j + \mu \sum_i \log w_i$$

$$\lim_{\mu \rightarrow 0^+} z^*(\mu) = z_p$$

$$x^*(\mu) \rightarrow x^*, \quad w^*(\mu) \rightarrow w^*$$

$$\forall \mu > 0 : x^*(\mu), w^*(\mu) \in \text{interior}(F)$$



To $z(\mu)$: μ is now Lagrangean function

$$L(x, w, y) = c^T x + \mu \sum_j \log x_j + \underbrace{\mu \sum_i \log w_i}_{\substack{\uparrow \\ \text{Lagrange} \\ \text{multipliers}}} + y^T (b - Ax - w)$$

$$\frac{\partial L}{\partial x_j} = c_j + \frac{\mu}{x_j} - A_j^T \cdot y \quad j = 1, \dots, n \quad (1)$$

$$\begin{aligned} y^T (b - Ax - w) &= \sum_{i=1}^m y_i (b_i - a_i^T x - w_i) \\ &= \sum_i y_i \left(b_i - \sum_{j=1}^n a_{ij} x_j - w_i \right) \end{aligned}$$

$$\frac{\partial}{\partial x_j} [y^T (b - Ax - w)] = - \sum_i y_i a_{ij}$$

$$\begin{aligned} A &= \begin{pmatrix} & a_{1j} & & \\ \cdot & \cdot & a_{2j} & \cdot & \cdot & \cdot \\ & \vdots & & & \\ & a_{mj} & & & \end{pmatrix} = -y^T A_j \\ &= -A_j^T y \end{aligned}$$

$$\frac{\partial L}{\partial w_i} = \frac{\mu}{w_i} - y_i \quad i=1, \dots, m \quad (2)$$

$$\frac{\partial L}{\partial y_i} = b_i - a_i^T x - w_i \quad i=1, \dots, m \quad (3)$$

$$\nabla L = 0$$

$$A_j^T y - \frac{\mu}{x_j} = c_j \quad j=1, \dots, n$$

$$y_i = \frac{\mu}{w_i} \quad i=1, \dots, m$$

$$Ax + w = b$$

} $\Rightarrow$

$$\Leftrightarrow Ax + w = b$$

$$A^T y - \mu X^T e = c$$

$$y = \mu W^T e$$

$$\text{Ερω} \quad z = \mu X^T e$$

$\Leftrightarrow$

$$z(\mu) : \nabla L(x, w, y) = 0 \Leftrightarrow$$

$$Ax + w = b$$

$$A^T y - z = c$$

$$z = \mu x^T e$$

$$y = \mu w^T e$$

$$\Leftrightarrow$$

$$\Leftrightarrow z e = \mu x^T e$$

$$\begin{aligned} Ax + w &= b \\ A^T y - z &= c \\ x z e &= \mu e \\ y w e &= \mu e \end{aligned}$$

$$\Leftrightarrow x_j z_j = \mu \quad \forall j$$

$$\Leftrightarrow y_i w_i = \mu \quad \forall i$$

$$z(\mu) = \max B(x, w)$$

$$Ax + w = b$$

Τίτλοι συνδίκου για max : $H_B = \begin{pmatrix} \frac{\partial^2 B}{\partial x_i \partial x_j} \end{pmatrix}$ αρνητ. ημιοριστός

$$\frac{\partial B}{\partial x_j} = c_j + \frac{\mu}{x_j}$$

$$\frac{\partial B}{\partial w_i} = \frac{\mu}{w_i}$$

$$\frac{\partial^2 B}{\partial x_j^2} = -\frac{\mu}{x_j^2} < 0$$

$$\frac{\partial^2 B}{\partial w_i^2} = -\frac{\mu}{w_i^2} < 0$$

$$\frac{\partial^2 B}{\partial x_i \partial x_j} = \frac{\partial^2 B}{\partial w_i \partial w_j} = \frac{\partial^2 B}{\partial x_i \partial w_j} = 0$$

$$H_B = \begin{pmatrix} -\mu/x_1^2 & & & \\ & -\mu/x_2^2 & & \\ & & \ddots & \\ & & & -\mu/x_n^2 & \\ & & & & -\mu/w_1^2 & \\ & & & & & \ddots & \\ & & & & & & -\mu/w_m^2 \end{pmatrix}$$

apvrtas a nepozitišas $\Rightarrow$ max