(1) if no three points are collinear, the six points lie on a non-degenerate conic;
(2) if three points (say a, b, and c) are collinear either: one or more of a', b', c',
lies on the line abc or: a', b', and c' are collinear.

In interpreting these results it should be noted that the exact form of (18')
is not unique. Indeed it has 60 equivalent forms obtained from the one given by
permuting any five of the given points. The equivalence of these forms can be
shown directly by careful algebraic manipulation in the reduction of (18) to
(18') or from the fact that the equation of a conic through five points clearly is
independent of the order in which these points are chosen.

MUSIC AND TERNARY CONTINUED FRACTIONS*
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1. Introduction. The most important interval in music is the octave, ex-
pressed by the ratio, 2:1. The number of notes within the octave has varied at
different times and in different places. The pentatonic or black-key scale is very
old and is widely distributed, occurring in such countries as Scotland and China.
The heptatonic, diatonic, or white-key scale, in its various major and minor
forms, is the foundation of our classic musical system. The duodecuple or chro-
matic scale includes both the white keys and the black keys of the piano, and has
become increasingly important during the past two or three centuries.

After harmony was introduced into music during the late Middle Ages,
major and minor triads emerged as the principal chords. The major triad, as
C E G, was regarded with especial favor, because it occurs naturally in the
harmonic series, as on bugles, and can be expressed by the simple ratios, 4:5:6.
A system of tuning for the diatonic scale known today as just intonation gained
support in the 16th century, because its principal triads, C E G, F A C, and
G B D, had these just ratios. But an important minor triad, D F A, is harsh in
just intonation, and other unsatisfactory triads result when this tuning is ex-
tended to the complete chromatic scale.

To correct the worst of these pitch errors, various systems of temperament
were devised, in which the intonation of some (or all) notes was altered slightly,
so that no intervals would be unusable. In practice, temperament was often

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article.
haphazard, a rule of thumb which some musical theorists exalted into definite systems. More important are the regular systems, those in which a particular value of the perfect fifth (as $C^G$) is used as the tuning unit. The meantone temperament is a regular system in which the fifth ($3/2$) is flatted by $1/4$ syntonic comma, and the major third is pure ($5/4$). A comma is a very small interval, the difference between two larger intervals. The syntonic, or Ptolemaic, comma is the difference between two tones, $(9/8)^2$, and a pure major third, $(5/4)$. Since the difference between intervals is expressed by the quotient of their ratios, this comma has the ratio of $81/64$ to $5/4$, or $81:80$. A fifth tempered by $1/4$ syntonic comma then has the ratio $(3/2)(80/81)^{1/4} = 5^{1/4}:1$, in place of $3:2$. Other regular systems in which the fifth is tempered by an aliquot part of the comma, such as $1/3$ or $1/5$, may be thought of as varieties of the meantone temperament.

Today the only tuning system in common use is equal temperament, in which the fifth is tempered by $1/12$ ditonic comma. The ditonic, or Pythagorean, comma is the difference between six tones, $(9/8)^6$, and the pure octave, $(2/1)$; that is, the ratio $531441:524288$, which lies between $75:74$ and $74:73$. A fifth tempered by $1/12$ ditonic comma has the ratio $(3/2)(8/9)^{12}2^{1/12} = 2^{7/12}:1$. This same ratio is obtained directly from the consideration that the interval of the fifth contains 7 semitones; that is, it is $7/12$ of an octave.

Equal temperament was used on fretted instruments (lutes and viols) in the early 16th century. It was advocated for keyboard instruments before the end of the 16th century, but was not universally accepted until the middle of the 19th century. The reason for its slow acceptance was the sharpness of its major thirds. These deviations will be clearer if the intervals are expressed in a logarithmic unit known as the cent, $2^{11/200}$. The octave contains 1200 cents; the fifth of equal temperament, 700; its third, 400. The pure fifth, with ratio $3:2$, contains 1200 (log $3/2$) = log 2 = 702 cents, and thus has been slightly flatted in equal temperament; but the pure third, with ratio $5:4$, contains 386.3 cents, and thus has been sharpened by almost 14 cents. The syntonic comma contains 21.5 cents; the ditonic comma, 23.5.

The tremendous advantage of equal temperament over other regular systems of 12 notes is that it is a closed, or cyclic, system. In a cyclic tuning system the $n$th power of the value chosen for the fifth will be a higher octave of the initial note. A great many suggestions have been made from time to time for improving the quality of the thirds by increasing the number of notes in the octave. The best of these systems operate upon the principle that an equal division of the octave is much to be preferred to an extension of just intonation. In making the division, care must be taken that the improvement of the thirds does not result in too great an impairment of the fifths. An increase in the number of notes in the octave has the further advantage that such enharmonic pairs as $G#$ and $A_b$ can be differentiated.

2. **Systems of multiple division.** In the middle of the 16th century Nicola
Vicentino described his six-manual Archicembalo, in which the octave is divided into 31 equal parts. Since he wrote before the invention of logarithms, he had no way of expressing these intervals by numbers on a monochord. But he directed that the fifths are to be tempered "according to the usage and tuning common to all the keyboard instruments," that is, according to the 1/4 comma meantone temperament. It remained for Christian Huyghens to show that the two systems are almost identical. The meantone fifth contains 696.6 cents; its third 386.3. The fifth of the 31-division contains 696.8 cents; its third, 387.1.

The division of the octave into 19 equal parts was accomplished a few years after Vicentino's successful attempt at multiple division. Both Zarlino and Salinas discussed harpsichords with 19 keys to the octave, and the latter advocated a system in which the fifth is tempered by 1/3 comma. This corresponds very closely to the equal 19-division. The fifths of both systems contain 694.7 cents; their thirds, 378.9. The large distortion of both intervals would make this system inferior to the ordinary 1/4 comma meantone temperament were it not that it is a cyclic system of only a few more notes than our present equal temperament of 12 notes. There have been eloquent advocates of the 19-division even in our own day, including the Russian-American, Joseph Yasser.

The best cyclic division with fewer than 100 parts is the 53-division. It was implied by the Pythagoreans, who held that there are 4 commas in the diatonic semitone and 9 in a tone, whence the octave will contain $5 \cdot 9 + 2 \cdot 4 = 53$ commas. Since $1200 \div 53 = 22.6$ cents, this is a mean comma between the syntonic and ditonic commas. It was appreciated by a medieval Chinese theorist, King Fang, and was referred to by 17th century European musical theorists (Mersenne and Kircher); it was used by Nicholas Mercator as a "common measure" for all intervals, and applied by Bosanquet to the "generalized" keyboard of his Enharmonic Harmonium. The fifths of the 53-division are practically perfect (701.9 cents), and its thirds are slightly flat (384.9 cents). Since this is a "positive" system, with fifths larger than those of equal temperament, its third must be formed as a diminished fourth, as C Fb. (The "negative" definition of the third in terms of fifth and octave is: $T = 4F - 2O$; the "positive," $T = 5O - 8F$.) This would be a confusing feature to a performer used to thirds that are really thirds.

Of the three systems [1] of multiple division discussed above, only the 19-division would have any practical value. The others would have been too expensive to construct and too cumbersome to play. And so, as we discuss various theories of the division of the octave, it must be realized that such a scheme is useless to the musician. Of course, some of the workers in this field have tried seriously to perfect instruments for multiple division, believing they would help the cause of music thereby. Events have proved them wrong. But others, who glibly talked of dividing the octave into more than 100 parts, had nothing to say about how their theories could be applied to instruments. They may have been deceiving themselves, but probably were aware that this was nothing but mathematical speculation.
3. Theories of multiple division. Joseph Sauveur, the French geometer and acoustician, was greatly interested in multiple division. In one of his articles [2] he gave a very impressive list of octave divisions, consisting of 25 terms: 12, 17, 19, 31, 43, 50, 53, 55, 67, 74, 98, 105, 112, 117, 122, 136, 141, 153, 160, 177, 184, 189, 208, 232, 256. All of these except 12, 17 (which he calls an Oriental system), and 53 are what Bosanquet called “negative,” that is, they have fifths that are flatter than those of equal temperament, and thus form their major thirds properly.

Sauveur's theory is based on the division of the tone, as $C D$, into a diatonic semitone, as $C D\flat$, and a chromatic semitone, as $D\flat D$. In just intonation, if the ratio 16:15 or 112 cents is taken for the diatonic semitone, and the ratio 135:128 or 92 cents for the chromatic semitone, the ratio of the diatonic to the chromatic semitone will be 112:92 or 28:23. Convenient approximations to this ratio are 5:4, 6:5, and 11:9. Since the octave contains 7 diatonic and 5 chromatic semitones, we may use the formula, $O = 7d + 5c$, where $d > c$. If $d/c = 5/4$, $O = 7 \cdot 5 + 5 \cdot 4 = 55$ parts; if $d/c = 6/5$, $O = 7 \cdot 6 + 5 \cdot 5 = 67$ parts; if $d/c = 11/9$, $O = 7 \cdot 11 + 5 \cdot 9 = 122$ parts.

In Sauveur's list there are combined several independent series, each determined by the formula $d = c + i$, where $i = 1, 2, 3, 4$. If, for example, $i = 4$, $O = 7(c+4) + 5c = 12c + 28$. Using only odd values of $c$ from 7 through 19, Sauveur obtained 112, 136, 160, 184, 208, 232, 256. The ratio of the third to the fifth to the octave in this series is $(4c + 8):(14c - 5):(24c - 8)$. The limit of these ratios is 4:7:12, as in equal temperament. Therefore, as $c$ increases beyond a certain value, the fifths continue to improve, but the thirds become sharper and sharper, since equal temperament has good fifths and very sharp thirds. This will be true of each of Sauveur's series. It would seem to be a false theory by which series are generated in which the middle terms are the best. Romieu [3] and Bosanquet [4], both of whom did outstanding work in this field, had no better theories than Sauveur.

M. W. Drobisch [5] made the first notable contribution to the theory of multiple division of the octave through the use of continued fractions. Having expressed the ratio of the fifth to the octave (log 3/2: log 2) as a decimal, 0.5849625, or as a fraction, 46797/80000, he used ordinary continued fractions to find successive approximations to this ratio. He obtained for his denominators (the octaves) this series: 2, 5, 12, 41, 53, 306, 665, [15601], ···· Then, apparently not trusting these results, he found all the powers of 3/2 from the 13th to the 53rd, to ascertain which of them approach a pure octave. This should have checked closely with his previous list, to which 17 and 29 would be semiconvergents. His complete list is: 17, 19, 22, 29, 31, 41, 43, 46, 51, 53. Having eliminated all positive divisions (those with raised fifths), he still had 19, 31, and 43 to add to his previous list.

Drobisch's continued fraction expansion was the first really scientific method of dividing the octave with regard to the principal consonances, the thirds and the fifths. The difficulty with it is that there are three magnitudes to be com-
pared (third, fifth, octave), but only one ratio (third to octave, or fifth to octave) can be approximated by binary continued fractions. If we must choose a single ratio, it is better to use the ratio of fifth to octave, as Drobisch did, since the third may be expressed in terms of the fifth. But the “negative” formula, \( T = 4F - 2O \), is valid only through \( O = 12 \). The syntonic comma (81:80 or 21.5 cents) is about 1/56 octave; hence this formula will fail to give a correct number of parts for the third in any octave division greater than 28. For example, if \( O = 41 \) and \( F = 24 \), the formula gives \( T = 14 \), whereas the correct value is 13. Knowing the approximate size of the comma, we can correct the formula to read: \( T = 4F - 2O - [O/56] \). But even this would give only by accident a value for the third with as small an error as that for the fifth in the same division.

4. **Use of ternary continued fractions.** The desired solution can be obtained only by ternary continued fractions, which are a means by which the ratios of three numbers may be approximated simultaneously, just as the ratios of two numbers may be approximated by binary continued fractions. C. G. J. Jacobi [6] was the first person to explore the possibilities of ternary continued fractions. The most thorough study of the subject has been made by Oskar Perron in half a dozen articles over a period of 30 years [7]. American workers in this field include D. H. Lehmer, J. B. Coleman, and P. H. Daus, most of their articles being published in the American Journal of Mathematics. The principal questions discussed in these papers are convergence and periodicity, especially the relation between periodic ternary continued fractions and the roots of cubic equations. None of the theory developed by these men appears to have any bearing upon the present problem.

**Jacobi’s algorithm.** P. H. Daus has stated Jacobi’s algorithm for ternary continued fractions as follows [8]:

If \( u_1, v_1, w_1 \) be any three numbers, we define a ternary continued fraction expansion for them by the equations

\[
\begin{align*}
u_{n+1} &= v_n - p_n u_n; \\
v_{n+1} &= w_n - q_n u_n; \\
w_{n+1} &= u_n.
\end{align*}
\]

The numbers \((A, B, C)\), defined by the recursion formulas (3) below, form the \( n \)th convergent set to the ternary continued fraction

\[
\left( \frac{v_1}{u_1}, \frac{w_1}{u_1} \right) = (p_1, q_1; p_2, q_2; \cdots; p_n, q_n; \cdots).
\]

\[
A_n = q_n A_{n-1} + p_n A_{n-2} + A_{n-3},
\]

\[
B_n = q_n B_{n-1} + p_n B_{n-2} + B_{n-3},
\]

\[
C_n = q_n C_{n-1} + p_n C_{n-2} + C_{n-3},
\]

---

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\[
\begin{bmatrix}
A_{-2} & A_{-1} & A_0 \\
B_{-2} & B_{-1} & B_0 \\
C_{-2} & C_{-1} & C_0
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

In his earlier article [9], Daus had shown that

\[
\begin{bmatrix}
A_{n-2} & A_{n-1} & A_n \\
B_{n-2} & B_{n-1} & B_n \\
C_{n-2} & C_{n-1} & C_n
\end{bmatrix} = \prod_{1}^{n} \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & p_n \\
0 & 1 & q_n
\end{bmatrix},
\]

from which the formulas in (3) follow. In [9], Daus, following Jacobi's erroneous example, had interchanged \(p's\) and \(q's\) in the formulas for \(A_n, B_n,\) and \(C_n,\) given correctly in (3).

**Jacobi's method in the tuning problem.** For our musical problem

\[
u_1 = \log 5/4; \quad v_1 = \log 3/2; \quad w_1 = \log 2.
\]

The ratios \(A_n:B_n:C_n\) represent successive approximations to the ratios of these three logarithms. The musical interpretation is that if the octave is divided into \(C_n\) parts, \(A_n\) parts are a good approximation to the major third, \(B_n\) to the perfect fifth. In Table I are shown the results of applying Jacobi's expansion to the three logarithms.

**Table I. Jacobi's Ternary Continued Fractions, with Application to the Tuning Problem**

<table>
<thead>
<tr>
<th>(p_n)</th>
<th>(q_n)</th>
<th>(A_n)</th>
<th>(B_n)</th>
<th>(C_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
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<td>3</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>8</td>
<td>15</td>
<td>25</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>9</td>
<td>16</td>
<td>28</td>
</tr>
<tr>
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<td>1</td>
<td>10</td>
<td>18</td>
<td>31</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>28</td>
<td>51</td>
<td>87</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>478</td>
<td>263</td>
<td>817</td>
</tr>
</tbody>
</table>

There are two serious faults with these results. In the first place, the expansion converges too rapidly, and we are interested chiefly in small values, those for which \(C<100.\) In the second place, the first few terms are foreign to every other proposed solution, such as those by Sauveur and Drobisch on previous pages. Compare this table with Table IV that follows, which is a composite of Tables I–III.

**Reversed Expansion.** Jacobi's method is not the only way to obtain a ternary continued fraction expansion. What might be called the reverse of Jacobi's expansion is defined as follows:

\[
u_{n+1} = u_n - p_nv_n; \quad v_{n+1} = w_n - q_nv_n; \quad w_{n+1} = v_n.
\]
Here

\[
\begin{bmatrix}
A_{n-2} & A_{n-1} & A_n \\
B_{n-2} & B_{n-1} & B_n \\
C_{n-2} & C_{n-1} & C_n
\end{bmatrix}
= \prod_{1}^{n} \begin{bmatrix}
1 & 0 & p_n \\
0 & 0 & 1 \\
0 & 1 & q_n
\end{bmatrix}
\]

from which

\[
A_n = q_n A_{n-1} + A_{n-2} + p_n,
B_n = q_n B_{n-1} + B_{n-2},
C_n = q_n C_{n-1} + C_{n-2},
\]

with the same initial conditions as before.

**Table II. Reversed Ternary Continued Fractions, with Application to the Tuning Problem**

<table>
<thead>
<tr>
<th>(p_n)</th>
<th>(q_n)</th>
<th>(A_n)</th>
<th>(B_n)</th>
<th>(C_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>12</td>
</tr>
<tr>
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<td>3</td>
<td>13</td>
<td>24</td>
<td>41</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>17</td>
<td>31</td>
<td>53</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>98</td>
<td>179</td>
<td>306</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>214</td>
<td>389</td>
<td>665</td>
</tr>
</tbody>
</table>

Since \(v_1\) is \(\log 3/2\) and \(w_1\) is \(\log 2\), and since \(B_n\) and \(C_n\) are independent of \(p_n\), the series for the ratio of perfect fifth to octave \((B_n:C_n)\) will be the same as that obtained by Drobisch by binary continued fractions. However, the correction for the major third for octave divisions greater than 28 is no longer needed, since the correction is made directly by the formula for \(A_n\), with its extra member.

It probably is an accident that in this particular problem the solution by the reversed method yields a more familiar series than Jacobi’s expansion does. (Again compare with Table IV.) Its rate of convergence is slightly lower, but it is still somewhat high. The method is presented here chiefly as an added basis for the expansion next to be discussed.

**Defects of both expansions.** Since neither the Jacobi nor the reversed ternary continued fraction expansion converges slowly, some method must be devised for slower convergence. In binary continued fractions, about half of the semi-convergent sets that lie between two convergent sets form better approximations to the ratio of the given numbers than the former of the two convergent sets does. But semi-convergents are a doubtful expedient in ternary continued fractions. Although their peculiarities have never been completely charted, a brief study of them by the present writer shows that a careful selection of coefficients provides semi-convergents superior to the latter of the two convergent sets, something impossible in binary continued fractions. For example, between
the second and third sets of Table I may be inserted the semi-convergent sets (taking \( q_3 \) as 5 and 6, respectively) 6 11 19 and 7 13 22, both of which, as can be seen from Table IV, are superior to the next convergent set of Table I, 8 15 25.

**Mixed expansion.** It is possible, however, to obtain fairly slow convergence without recourse to semi-convergents. At any stage of the expansion there are two possible divisors, \( u_n \) and \( v_n \), the former being used in the Jacobi expansion and the latter in the reversed method. If either of these expansions is used, the rate of convergence is likely to be somewhat irregular, but approaches a mean value. But if fast convergence is desired, one should consistently divide by whichever of \( u_n \) and \( v_n \) is the smaller. In our tuning problem, slow convergence is the desideratum. Hence one should always divide by the larger of \( u_n \) and \( v_n \).

This mixed expansion, as it may be called, presents certain difficulties, since it is no longer possible to express an approximation in terms of the three immediately preceding approximations. No new definitions of \( u_{n+1} \), \( v_{n+1} \), and \( w_{n+1} \) are needed; for at any stage either the Jacobi \((J)\) or the reversed \((R)\) expansion is to be used.

Then, in terms of a matrix product, for a \( J \) stage,

\[
\begin{bmatrix}
A_{n-2} & A_{n-1} & A_n \\
B_{n-2} & B_{n-1} & B_n \\
C_{n-2} & C_{n-1} & C_n
\end{bmatrix}
= \begin{bmatrix}
A_{n-k-3} & A_{n-2} & A_{n-1} \\
B_{n-k-3} & B_{n-2} & B_{n-1} \\
C_{n-k-3} & C_{n-2} & C_{n-1}
\end{bmatrix} \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & p_n \\
0 & 1 & q_n
\end{bmatrix},
\]

where \( k \) is the number of \( R \) stages between this stage and the last previous \( J \) stage. For an \( R \) stage, substitute on the right the matrix for the reversed expansion,

\[
\begin{bmatrix}
1 & 0 & p_n \\
0 & 0 & 1 \\
0 & 1 & q_n
\end{bmatrix}.
\]

From the above we obtain,

- for an \( R \) stage: \( S_n = q_n S_{n-1} + S_{n-2} + p_n S_{n-k-3} \);
- for a \( J \) stage: \( S_n = q_n S_{n-1} + p_n S_{n-2} + S_{n-k-3} \).

It is understood that

\[ S_n = A_n, \quad B_n, \quad \text{or} \quad C_n. \]

5. **Mixed expansion in the tuning problem.** In the application of the mixed method to our tuning problem, we shall use the Jacobi \((J)\) method whenever \( u_n > v_n \), and the reversed \((R)\) method whenever \( u_n < v_n \). Hence all the \( p \)'s will be 0. In Table III this application is shown, together with a determinant in which \( A_n, B_n, \) and \( C_n \) are the coefficients of \( a, b, \) and \( c, \) respectively, in the expansion of the determinant.
Table III. Mixed Ternary Continued Fractions, with Application to the Tuning Problem

<table>
<thead>
<tr>
<th>$a$</th>
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<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>$p_n$</th>
<th>$q_n$</th>
<th>$A_n$</th>
<th>$B_n$</th>
<th>$C_n$</th>
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<td>0</td>
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<td>0</td>
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</tr>
<tr>
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<td>$q_1$</td>
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<td>0</td>
<td>0</td>
<td>J</td>
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<td>2</td>
<td></td>
</tr>
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<td>$p_2$</td>
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<td>$g_2$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>J</td>
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<td>2</td>
<td></td>
</tr>
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<td>0</td>
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<td>$g_3$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>R</td>
<td>0</td>
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<td>2</td>
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<td>1</td>
<td>$p_4$</td>
<td>$g_4$</td>
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<td>0</td>
<td>0</td>
<td>R</td>
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<td>2</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>1</td>
<td>2</td>
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<td>$g_6$</td>
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<td>0</td>
<td>0</td>
<td>R</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
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<td>0</td>
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<td>0</td>
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<td>$p_7$</td>
<td>$g_7$</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>R</td>
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<td>1</td>
<td>2</td>
<td></td>
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</tr>
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<td>0</td>
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<td>0</td>
<td>0</td>
<td>$p_8$</td>
<td>0</td>
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<td>1</td>
<td>$g_8$</td>
<td>-1</td>
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<td>2</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$p_{10}$</td>
<td>$q_{10}$</td>
<td>R</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table IV contains all the terms from the other three tables, through $C=118$, together with a few other good approximations not found in any of the previous tables. The errors in cents for the major third (in some cases to 0.1 cent) have been computed by the formula: Error $A_n=386.3-1200A_n/C_n$; the errors for the fifth by: Error $B_n=702-1200B_n/C_n$. The total error is taken as $|\text{Error } A_n| + |\text{Error } B_n|$.

Table IV. Composite of Approximations to Tuning Ratios (log 5/4: log 3/2: log 2), with Best Approximations Starred

<table>
<thead>
<tr>
<th>$A_n$</th>
<th>$B_n$</th>
<th>$C_n$</th>
<th>Error $A_n$</th>
<th>Error $B_n$</th>
<th>Total Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>*0</td>
<td>1</td>
<td>1</td>
<td>-386</td>
<td>498</td>
<td>884</td>
</tr>
<tr>
<td>*0</td>
<td>1</td>
<td>2</td>
<td>-386</td>
<td>-102</td>
<td>488</td>
</tr>
<tr>
<td>*1</td>
<td>1</td>
<td>2</td>
<td>214</td>
<td>-102</td>
<td>316</td>
</tr>
<tr>
<td>*1</td>
<td>1</td>
<td>3</td>
<td>14</td>
<td>-302</td>
<td>316</td>
</tr>
<tr>
<td>*1</td>
<td>2</td>
<td>3</td>
<td>14</td>
<td>98</td>
<td>112</td>
</tr>
<tr>
<td>*2</td>
<td>3</td>
<td>5</td>
<td>94</td>
<td>18</td>
<td>112</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>5</td>
<td>-146</td>
<td>18</td>
<td>164</td>
</tr>
<tr>
<td>*2</td>
<td>4</td>
<td>7</td>
<td>-43</td>
<td>-16</td>
<td>59</td>
</tr>
<tr>
<td>*3</td>
<td>5</td>
<td>9</td>
<td>14</td>
<td>-35</td>
<td>49</td>
</tr>
<tr>
<td>*3</td>
<td>6</td>
<td>10</td>
<td>-26</td>
<td>18</td>
<td>44</td>
</tr>
<tr>
<td>*4</td>
<td>7</td>
<td>12</td>
<td>-14</td>
<td>-2</td>
<td>16</td>
</tr>
<tr>
<td>*6</td>
<td>11</td>
<td>19</td>
<td>-7</td>
<td>-7</td>
<td>14</td>
</tr>
<tr>
<td>*7</td>
<td>13</td>
<td>22</td>
<td>-4</td>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>8</td>
<td>15</td>
<td>25</td>
<td>-2</td>
<td>18</td>
<td>20</td>
</tr>
<tr>
<td>9</td>
<td>16</td>
<td>28</td>
<td>-1</td>
<td>-16</td>
<td>17</td>
</tr>
<tr>
<td>*10</td>
<td>18</td>
<td>31</td>
<td>.8</td>
<td>5.2</td>
<td>6.0</td>
</tr>
<tr>
<td>*11</td>
<td>20</td>
<td>34</td>
<td>1.9</td>
<td>3.9</td>
<td>5.8</td>
</tr>
<tr>
<td>13</td>
<td>24</td>
<td>31</td>
<td>-6</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>*17</td>
<td>31</td>
<td>53</td>
<td>-1.4</td>
<td>-1</td>
<td>1.5</td>
</tr>
<tr>
<td>*28</td>
<td>51</td>
<td>87</td>
<td>-1.4</td>
<td>1.4</td>
<td>1.5</td>
</tr>
<tr>
<td>*38</td>
<td>69</td>
<td>118</td>
<td>.2</td>
<td>.5</td>
<td>.7</td>
</tr>
</tbody>
</table>
Of the starred best approximations in Table IV, only \( C = 9, 10, \) and 22 are missing from Table III; but none of these occurs in Tables I or II either. The only serious omission from Table III is the Hindoo division, \( C = 22, \) and it would occur as a semi-convergent, by assuming a \( J \) instead of an \( R \) for the term where 31 now stands. Note the two terms, 31 and 34, nearly the same size: it would have been impossible to include them both (even if one had been a semi-convergent) if binary or Jacobi ternary continued fractions had been used. Both 53 and 118 are present; but between them lies 87, a term superior to 65, the middle term in the binary series. The last term shown, 612, was said by Bosanquet \([4]\) to have been considered excellent by Capt. J. Herschel; it would have had no place in any other series derived by continued fractions in which 559 was also present.

6. Euler's problem. This mixed method for ternary continued fraction expansions has been shown to satisfy the need for slow convergence in our special problem. It should be equally valuable wherever slow convergence is desired. In this connection, let us return to Jacobi. His article on ternary continued fractions was preceded, both literally and logically, by an article \([10]\) in which he treated integral solutions of a linear equation in any number of variables. He made special reference, however, to Euler's problem: to solve in integers the equation \( Aa + Bb + Cc = 0, \) where the capital letters are given integers, possibly integral approximations to irrational numbers.

Euler's first example was \( 49a + 59b + 75c = 0. \) To obtain a general solution, he placed \( u \) on the right, and then replaced it by 0 for the particular solution. His method was to divide two coefficients by the remaining coefficient, always dividing by the smaller remainder in each stage. Thus his method made for rapid convergence. It is irregular, but for a purpose, as is our method of dividing by the larger remainder. In the example just given, the coefficients are so chosen that the smaller remainder is always on the left, thus agreeing with Jacobi's method, which he first used in this connection before applying it to continued fractions. Both Euler and Jacobi obtained as the smallest solution: \( a = -13, b = -7, c = 14. \)

Euler's second example was \( 1,000,000a + 1,414,214b + 1,732,051c = u, \) where the second and third coefficients will be recognized as approximations to \( \sqrt{2} \) and \( \sqrt{3}, \) with the decimal point omitted. Euler again divided by the smaller remainders, and thus got a different result from Jacobi, who divided by the left-hand remainders. With \( u = 0, \) Euler's smallest solution was: \( a = 8104, b = -6889, c = 946; \) Jacobi's, \( a = 282, b = -2377, c = 1778. \)

7. Mixed expansion in Euler's problem. The results already obtained in this paper suggest that Jacobi's results may have been better than Euler's in this second problem, not because his method was regular, but because of the slower convergence arising from a casual mixture of small and large remainders as divisors. If, then, the larger remainders are used uniformly as divisors, as in the tuning problem, better results than Jacobi's might be expected. As applied
to Euler's first problem, our mixed method gives: \( a = u - 8i + j, \ b = 3u + 13i - 11j, \ c = -3u - 5i + 8j. \) If \( u = 0, \ i = 1, \ j = 1, \) then \( a = -7, \ b = 2, \ c = 3. \) The Euler-Jacobi solution above is then obtained by letting \( i = 2, \ j = 3. \)

Again, in Euler's second problem, division by the larger remainders yields superior results. This method gives: \( a = 51u - 2403\alpha + 895\beta, \ b = 264u + 1829\alpha - 1402\beta, \ c = -245u - 106\alpha + 628\beta. \) If \( u = 0, \ \alpha = 1, \ \beta = 1, \) then \( a = -1508, \ b = 427, \ c = 522. \) Euler's solution is obtained by letting \( \alpha = -3, \ \beta = 1; \) Jacobi's, with \( \alpha = 1, \ \beta = 3. \)

8. Conclusion. Insufficient study has been given the mixed expansion for ternary continued fractions to ascertain all of its possibilities and weaknesses. Without the use of semi-convergents, the mixed method clearly provides slower convergence than Jacobi's method. It can be used to good advantage also when fast convergence is desired, by a simple reversal of the hypotheses. In this case the divisors are so selected than none of the \( p's \) will be 0. Thus the mixed expansion would seem to fulfill a definite need which Jacobi's expansion does not meet.

References

1. These three systems, as well as the three systems of 12 notes previously mentioned, are discussed, with bibliography, in the author's article, The Persistence of the Pythagorean Tuning System, Scripta Mathematica, Vol. 1 (1933), pp. 286–304.
10. C. G. J. Jacobi, Ueber die Auflösung der Gleichung \( a_1x_1 + a_2x_2 + \cdots + a_nx_n = fu, \) in Jacobi's Ges. Werke, Band 6, pp. 355–384.