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This technique can also be used to prove the following theorem about Kronecker products. (See, e.g., [**1**], pp. 235–243.)

THEOREM. *If $A = (a_{ij})$ and $B = (b_{ij})$ are positive semidefinite, then the Kronecker product $A \otimes B = (a_{ij}B)$ is positive semidefinite.*

As a final demonstration of these ideas, we show that the symmetric $n \times n$ matrices

$$A = \begin{bmatrix} n & n-1 & n-2 & \cdots & 1 \\ & 2(n-1) & 2(n-2) & \cdots & 2 \\ & & 3(n-2) & \cdots & 3 \\ & \vdots & \vdots & \vdots & \\ & & & & n \end{bmatrix},$$

$$B = \begin{bmatrix} a_1^2 & a_1^2 & a_1^2 & \cdots & a_1^2 \\ & a_1^2 + a_2^2 & a^2 + a_2^2 & \cdots & a_1^2 + a_2^2 \\ & & a_1^2 + a_2^2 + a_3^2 & \cdots & a_1^2 + a_2^2 + a_3^2 \\ & & \cdots & \cdots & \\ & & \cdots & \cdots & \\ & & & & a_1^2 + \cdots + a_n^2 \end{bmatrix},$$

are positive semidefinite by showing that they are covariance matrices.

(i) If $X_1, \ldots, X_n$ are independent random variables from a uniform distribution on $[0,1]$ and $X_{(1)} \leqslant \cdots \leqslant X_{(n)}$ are the order statistics, then, except for a constant, the matrix A is the covariance matrix of $(X_{(1)}, \ldots, X_{(n)})$.

(ii) The matrix B is the covariance matrix of random variables

$$Z_1 = X_1,\ Z_2 = X_1 + X_2, \ldots, Z_n = X_1 + \cdots + X_n,$$

where $X_1, \ldots, X_n$ are independent random variables with zero means and variances $a_1^2, \ldots, a_n^2$, respectively.

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ENUMERATION IN MUSIC THEORY

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The calculation of the number of chords and tone rows using Pólya's Theorem and Burnside's Lemma can add a little variety to the applications of these theorems usually given in combinatorics and algebra texts.

The *n-scale* is taken to be equal (well) tempered and consists of the integers from 0 to $n - 1$. We equate octave notes, so our scale is mathematically Z_n with addition. For example, ordinary Western music has $n = 12$, Debussy used a whole tone scale with $n = 6$, and Ives used a quarter tone scale with $n = 24$. (Of course, in terms of frequencies of pitches, the physical scale is a multiplicative group, but this makes no difference here.)

A *k-chord* in the n-scale is an equivalence class (defined in Section 1) of subsets of k elements each of Z_n, and an *n-tone row* is an equivalence class (defined in Section 2) of permutations in S_n.

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1. Chords. The equivalence relation is induced by one of two permutation groups. We can use the transpositions (not to be confused with permutations which exchange two elements) $T^i: Z_n \to Z_n : a \mapsto i + a (\mathrm{mod}\, n)$; a jazz guitarist generally uses chords which can be transposed easily so not so many fingerings must be memorized. Or we can use the larger group of transpositions (as above) and the inversion $I: a \to -a(\mathrm{mod}\, n)$. A typical group element looks like $T^i I: a \mapsto i - a(\mathrm{mod}\, n)$. This use of the word "inversion" is standard in serial music and must be distinguished from the usual meaning of octave transposition of one or more notes of a chord. For example, under the larger group, a C major chord is equivalent to a C minor chord: $\{C, E, G\} = \{0,4,7\} \sim \{0,3,7\} = \{C, E\flat, G\}$, the $\sim$ being via $T^7 I$.

It is easy to see that these groups are the cyclic groups C_n and the dihedral groups D_n, respectively, and we are thus dealing with what is known as the two-color necklace problems, either one-sided or two-sided [**3**, p. 162]. That is, we have a circular necklace of notes to choose from, and we choose the ones in the chord by coloring them one color and the rest another color.

Pólya's Theorem is the appropriate tool for this situation: a group acting on a domain set D, inducing a group action on the set of functions into some range set R. It suffices to let G be a group of permutations of D. G then acts on the set of all functions R^D in a natural way, inducing an equivalence relation.

To keep track of what's important in such a problem, we use enumerators, polynomials or formal power series whose coefficients are the numbers of objects of specified types. The enumerator of the range here can be whatever we like, say $\Sigma_{r \in R} w(r)$, where w, the "weight," is any function from R to a commutative ring containing the rationals. The weight of a function from D to R is then defined to be the product of the weights of its image values, and the enumerator or inventory of a set of functions is the sum of the weights of the individual functions. Usually this set of functions is a set of equivalence class representatives when we have an equivalence relation on R^D.

In our situation of chords, $R = \{0,1\}$, $w(0) = 1$, $w(1) = x$, and the weight of a function from D to this R is x^k, where k is just the size of the subset of D mapped to 1. This is how we enumerate subsets or chords.

The importance of Pólya's theorem is that it relates the inventory of equivalence classes of functions to the "store enumerator," the enumerator of the range, via the cycle index P_G of the group G acting on D, which is defined by

$$P_G(t_1, t_2, \dots) = \frac{1}{|G|} \sum_{g \in G} t_1^{a_1} t_2^{a_2} \cdots,$$

where a_i = the number of i-cycles in g.

Pólya's Enumeration Theorem [**2**, p. 148] states that the enumerator or inventory of equivalence classes under G of functions in R^D is

$$P_G\left(\sum w(r), \sum w(r)^2, \dots, \sum w(r)^k, \dots\right).$$

The cycle indices of C_n and D_n are well known [**3**, pp. 149–150]:

$$P_{C_n}(t_1, t_2, \dots, t_n) = \frac{1}{n} \sum_{j|n} \phi(j)\, t_j^{n/j} = \frac{1}{n}\Phi,$$

and

$$P_{D_n}(t_1, t_2, \dots, t_n) \begin{cases} = \dfrac{1}{2n}\left[\Phi + n t_1 t_2^{(n-1)/2}\right], & \text{if } n \text{ is odd,} \\ = \dfrac{1}{2n}\left[\Phi + \dfrac{n}{2} t_1^{(n/2)-1}\left(t_1^2 + t_2\right)\right], & \text{if } n \text{ is even.} \end{cases}$$

Here we use $\phi(j)$, which is the Euler phi-function, the number of positive integers less than j which are relatively prime to j, and Φ, which is short for the sum in the first equation.

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Our enumerator is just $1 + x$, and $t_j = 1 + x^j$. Hence the number of k-chords is the coefficient of x^k in $P(1 + x, 1 + x^2, \ldots, 1 + x^n)$. For C_n, a little rearrangement produces

$$\#\text{ of }k\text{-chords} = \frac{1}{n}\sum_{j|(n,k)} \phi(j)\binom{n/j}{k/j} = \frac{1}{n}\Phi_n(k),$$

where $\Phi_n(k)$ is short for the summation. For D_n, the additional term gives

$$\#\text{ of }k\text{-chords} = \begin{cases} \dfrac{1}{2n}\left[\Phi_n(k) + n\dbinom{(n-1)/2}{[k/2]}\right], & \text{if } n \text{ is odd,} \\ \dfrac{1}{2n}\left[\Phi_n(k) + n\dbinom{n/2}{k/2}\right], & \text{if } n \text{ is even and } k \text{ is even,} \\ \dfrac{1}{2n}\left[\Phi_n(k) + n\dbinom{(n/2)-1}{[k/2]}\right], & \text{if } n \text{ is even and } k \text{ is odd.} \end{cases}$$

The situation in twelve-tone music is $n = 12$ with dihedral symmetry, and we obtain the following table.

$k =$	0	1	2	3	4	5	6	7	8	9	10	11	12
number	1	1	6	12	29	38	50	38	29	12	6	1	1

Thus, there are 50 hexachords in twelve-tone music and only 6 intervals, a fact well known in music theory, [**1**].

2. Tone rows. Here our equivalence classes are induced by the group generated by

transposition $T: S_n \to S_n : (a_1, \ldots, a_n) \mapsto (a_1 + 1, \ldots, a_n + 1) \pmod n$;
inversion $I: (a_1, a_2, \ldots, a_n) \to (a_1, 2a_1 - a_2, \ldots, 2a_1 - a_n) \pmod n$;
retrogradation $R: (a_1, \ldots, a_n) \to (a_n, \ldots, a_1)$.

This group is not well known; however the entire structure of the group is not really needed. We can take a cue from music theorists, who regard transposition as such a basic transformation that they don't work with all permutations, but work with equivalence classes of permutations under transposition, those beginning with 0 being regarded as class representatives. Hence our set is now the set of $(n-1)!$ permutations of $\{1, \ldots, n-1\}$ with a prefix of 0, and our group is generated by R and I. On this new set, $RI = IR$, so we have the Klein four group V acting.

Burnside's Lemma is the appropriate tool in this setting.

Burnside's Lemma [**2**, p. 136] states that for a group G acting on D, the number of equivalence classes is

$$\frac{1}{|G|}\sum_{g \in G} (\#\ \textit{of elements of D fixed by g}).$$

The computation proceeds as follows (see the appendix for details). In order to avoid triviality, we assume that $n \geqslant 3$.

Group Element	Number Fixed: n odd	Number Fixed: n even
e	$(n-1)!$	$(n-1)!$
I	0	0
R	0	$(n-2)(n-4)\cdots(2)$
IR	$(n-1)(n-3)\cdots(2)$	$(n/2)(n-2)(n-4)\cdots(2)$.

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Thus the number of n-tone rows is

$$\tfrac{1}{4}[(n-1)!+(n-1)(n-3)\cdots(2)] \qquad \text{if } n \text{ is odd;}$$
$$\tfrac{1}{4}[(n-1)!+(n-2)(n-4)\cdots(2)(1+n/2)] \qquad \text{if } n \text{ is even.}$$

For example, there are 9985920 twelve tone rows, a fact which does not seem to be in the literature.

Appendix. Computation of numbers of fixed elements.

1. $I(0,a_2,\ldots,a_n)=(0,-a_2,\ldots,-a_n)\sim(0,a_2,\ldots,a_n)$ implies $a_i\equiv -a_i \pmod n$ for $i=2,\ldots,n$ since no transposition is allowed because the first element 0 is fixed. There is at most one nonzero solution to $x\equiv -x \pmod n$, but that is not enough to fill out the permutation.

2. $R(a_1,\ldots,a_n)=(a_n,\ldots,a_1)\sim(a_1,\ldots,a_n)$ implies that a t exists such that $a_1\equiv a_n+t$, $a_2\equiv a_{n-1}+t,\ldots \pmod n$. If n is odd, the middle element is fixed and no transposition is allowed: $t=0$. But then $a_n=a_1$, a contradiction. If n is even, the first and last congruences imply that $2t\equiv 0$; hence, $t=0$ or $t=n/2$. The first is impossible just as when n is odd, but the other gives fixed permutations. Since $a_1=0$, $a_n=n/2$. For a_2, we can choose any of $n-2$ elements, and this determines a_{n-1}. For a_3, we have $n-4$ choices, etc.

3. $IR(a_1,\ldots,a_n)=(-a_n,\ldots,-a_1)\sim(a_1,\ldots,a_n)$ implies that a t exists such that $a_1+a_n\equiv t$, $a_2+a_{n-1}\equiv t,\ldots$. The last congruence for n odd is $2a_{(n+1)/2}\equiv t \pmod n$. Clearly it is not important that we fix the first element as 0; we could fix $a_{(n+1)/2}$ as 0 and obtain the same count. Thus we may assume that $t=0$, $a_{(n+1)/2}=0$. This allows $n-1$ choices for a_1, with a_n thus determined, $n-3$ choices for a_2, etc. For n even, we fix $a_1=0$, $a_n\neq t\neq 0$. A little thought shows that t must be odd in order for us to complete the permutation. If $t=2k$, then there is no mate for k in the permutation. There are thus $n/2$ choices for $t=a_n$. For a_2, there are $n-2$ choices, with a_{n-1} determined thereby, etc.

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BIJECTING EULER'S PARTITIONS-RECURRENCE

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A partition of an integer n is a nonincreasing sequence of positive integers $\lambda(1)\geqslant\lambda(2)\geqslant\cdots\geqslant\lambda(t)>0$, such that $\lambda(1)+\cdots+\lambda(t)=n$. The set of partitions of n is denoted $\mathrm{Par}(n)$ and its cardinality $|\mathrm{Par}(n)|$ is written $p(n)$. For example,

$$\mathrm{Par}(5)=\{5;4,1;3,2;3,1,1;2,2,1;2,1,1,1;1,1,1,1,1\} \quad\text{and}\quad p(5)=7.$$

There is no closed form formula for $p(n)$ but Euler ([1], p. 12) gave a very efficient way for compiling a table of $p(n)$ by proving the recurrence

$$(1) \qquad \sum_{j \text{ even}} p(n-a(j)) = \sum_{j \text{ odd}} p(n-a(j)), \quad \text{where } a(j)=(3j^2+j)/2.$$

Euler used generating functions to prove this formula. Garsia and Milne [2] gave a very nice

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