Bayesian Inference II
Lab 1: Simulation

1. Show that, if $X \sim Exp(\lambda)$, then

   - $Y = X^2$ has pdf
   $$f_Y(y) = \lambda \exp\{-\lambda\sqrt{y}\}\frac{1}{2\sqrt{y}},\ y \geq 0$$
   - $Z = \sqrt{X}$ has pdf
   $$f_Z(z) = \lambda \exp\{-\lambda z^2\}2z,\ z \geq 0$$

   Write R code to simulate draws from the distributions of Y and Z.

2. A method for simulating standard normal random variables is the Box-Muller algorithm:

   - Generate independent $U_1, U2 \sim U[0,1]$
   - Let $\theta = 2\pi U_1$ and $R = \sqrt{-2\log U_2}$
   - Then $X = R\cos\theta$ and $Y = R\sin\theta$ are independent standard normal random variables.

   Show mathematically why this algorithm works. Write R code to implement this algorithm and use a sample of size $n = 1000$ from your code to show graphically that it works.

3. Show that, if $X \sim Exp(1)$, then $W = aX^{1/b} \sim Weibull$ with pdf
   $$f_W(w) = ba^{-b}w^{b-1}\exp\left\{-(w/a)^b\right\},\ w \in R^+,\ a, b \in R^+.$$
   Explain how you would simulate draws from this Weibull distribution.

4. Using the method of inversion, write R code to simulate from the Cauchy distribution. Draw 5 samples of each of the following sizes $n = 100, 500, 1000, 10000$ from the Cauchy distribution using your code and report the sample means and variances. What do you observe?

5. Construct R functions to sample from (a) the Bernoulli(p) distribution, (b) from the distribution of the discrete random variable X with probability function $P(X = 1) = 0.3$, $P(X = 2) = 0.2$, $P(X = 3) = 0.2$, $P(X = 4) = 0.3$.

6. Use rejection sampling to simulate from the logistic distribution with pdf
   $$f(x) = \frac{e^{-x}}{(1+e^{-x})^2},\ x \in R.$$
   Hint: Use the envelope $g(x) = e^{-x}$, $x \geq 0$, and choose some appropriate value of K such that $f(x) \leq Kg(x)$, for all $x \geq 0$.

7. Try different Monte Carlo methods to estimate the integral
   $$\int_0^1 [x(x^2-1)(x-2)]^{1/2}.$$
   Write R code to implement your methods and report your estimates.