Στοχαστικά Μοντέλα

Σειρά Ασκήσεων 1 - Υποδείξεις - σύντομες απαντήσεις

Άσκηση 1

(a) Θεώρημα Διπλής Μέσης Τιμής

(b) $Var(X) = E(X^2) - (EX)^2$

$$E(X^2) = E_Y\left[E(X^2|Y)\right] = E_Y\left[Var(X|Y) + \left(E(X|Y)\right)^2\right]$$

$$= E_Y\left(\sigma^2_{X|Y}(Y)\right) + E_Y\left(m_{X|Y}(Y)^2\right)$$

$$(EX)^2 = \left[E_Y(E(X|Y))\right]^2 = \left(E_Y(m_{X|Y}(Y))\right)^2$$

$$\Rightarrow Var(X) = E_Y\left(\sigma^2_{X|Y}(Y)\right) + E_Y\left(m_{X|Y}(Y)^2\right) - \left[E_Y(m_{X|Y}(Y))\right]^2$$

$$= E_Y\left(\sigma^2_{X|Y}(Y)\right) + Var_Y\left(m_{X|Y}(Y)\right)$$

Άσκηση 2

$E_Y = \mu_Y, \quad Var_Y(Y) = \sigma_Y^2$

$$m(X|Y) = E(X|Y) = aY, \quad \sigma^2_{X|Y}(Y) = Var(X|Y) = \sigma^2 Y$$

Από Ασκ 1: $E(X) = E_Y\left(m(X|Y)\right) = E_Y(aY) = a\mu_Y$

$$Var(X) = E_Y\left(\sigma^2_{X|Y}(Y)\right) + Var_Y\left(m_{X|Y}(Y)\right)$$

$$= E_Y(\sigma^2 Y) + Var_Y(aY) = \sigma^2\mu_Y + a^2\sigma_Y^2$$

Άσκηση 3 (α) Η N δεν είναι ανεξάρτητη των $X_1, X_2, \dots$

(π.χ. $P(N=1 \mid X_1=k)=1$, $P(N=1 \mid X_1 \neq k)=0$).

(β) $E(S_N) = \sum_{n=1}^{\infty} E(S_N \mid N=n) \cdot P(N=n)$

$P(N=n) = \left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right) \quad n=1,2,\dots$ (αρ. δοκιμών έως 1η επιτυχία)

$$E(S_N \mid N=n) = E(X_1+\dots+X_n \mid X_1, X_2, \dots, X_{n-1} \neq k,\ X_n = k)$$

$$= \sum_{j=1}^{n-1} E(X_j \mid X_j \neq k) + k = (n-1)\, E(X_1 \mid X_1 \neq k) + k$$

Όμως $P(X_j = \ell \mid X_j \neq k) = \dfrac{P(X_j=\ell)}{P(X_j \neq k)} = \dfrac{1/6}{5/6} = \dfrac{1}{5}$ για $\ell = 1,\dots,6$, $\ell \neq k$

Επομένως $E(X_j \mid X_j \neq k) = \sum_{\substack{\ell=1 \\ \ell \neq k}}^{6} \ell \cdot \frac{1}{5} = \frac{1}{5}\left(\frac{6 \cdot (6+1)}{2} - k\right) = \frac{1}{5}(21-k)$

$$E(S_N \mid N=n) = (n-1) \cdot \frac{1}{5}(21-k) + k = \frac{21}{5} \cdot (n-1) - \frac{n-1}{5} k + k =$$

$$= \frac{21}{5}(n-1) - k\, \frac{n-6}{5}$$

$$E(N) = \sum_{n=1}^{\infty} n \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6} = 6$$

Επομένως $E(S_N) = \frac{21}{5}(6-1) - k \cdot 0 = 21$

Επίσης $E(X_1) = \frac{1}{6} \sum_{\ell=1}^{6} \ell = \frac{21}{6}$, $E(N) = 6 \Rightarrow$

$\Rightarrow E(S_N) = E(N) \cdot E(X_1)$ συνεχίζει να ισχύει.

Άσκηση 4 $E(\Pi_N) = \sum_{k=1}^{\infty} P_N(k) \cdot E(\Pi_N | N=k) =$

$$= \sum_{k=1}^{\infty} P_N(k)\, E(X_1) \cdots E(X_k) = \sum_{k=1}^{\infty} P_N(k)\, \mu_X^k = \tilde{P}_N(\mu_X)$$

Άσκηση 5 $E(M_N) = \sum_{n=1}^{\infty} P_N(k)\, E(X^N | N=k) = \sum_{n=1}^{\infty} P_N(k)\, E_X(X^k)$

$$= E_X\left(\sum_{n=1}^{\infty} P_N(k) X^k \right) = E_X\left(\tilde{P}_N(X)\right)$$

Άσκηση 6 (α) $\tilde{P}_X(1) = 1 \Rightarrow c e^{\lambda} = 1 \Rightarrow c = e^{-\lambda}$

(β) $\tilde{P}_X(z) = e^{-\lambda} e^{\lambda z^k} = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n z^{kn}}{n!}$

$$= \frac{e^{-\lambda}\lambda^0}{0!} z^0 + e^{-\lambda}\frac{\lambda^1}{1!} z^k + e^{-\lambda}\frac{\lambda^2}{2!} z^{2k} + \cdots$$

$$= \sum_{j=0}^{\infty} p_j z^j, \quad \text{όπου } p_j = \begin{cases} e^{-\lambda}\dfrac{\lambda^{(j/k)}}{(j/k)!}, & j = 0, k, 2k, 3k, \dots \\ 0, & \text{διαφορετικά} \end{cases}$$

Επομένως η $X \overset{d}{=} kY$, όπου $Y \sim \text{Poisson}(\lambda)$.

Εναλλακτικά: Έστω Y τμ με $P(Y=j) = P_Y(j)$ και $X = kY$, $k \in \mathbb{N}$.

Τότε $\tilde{P}_X(z) = E(z^X) = E(z^{kY}) = E\left((z^k)^Y\right) = \tilde{P}_Y(z^k)$

Εδώ $\tilde{p}_X(z) = e^{-\lambda} e^{\lambda z^k} = \tilde{P}_Y(z^k)$, όπου $\tilde{P}_Y = e^{-\lambda} e^{\lambda z}$

η πιθανογεννήτρια της $Y \sim \text{Poisson}(\lambda)$.

Άσκηση 7 Γνωρίζουμε $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ για $|z| < 1$

Παραγωγίζοντας n φορές: Στο αριστερό μέρος

$$\frac{d^n}{dz^n}(z^k) = \begin{cases} k(k-1)\cdots(k-n+1)\, z^{k-n}, & k \geq n \\ 0, & k < n \end{cases}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{d^n}{dz^n} z^k = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)\, z^{k-n} =$$

$$= \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} z^{k-n} = \frac{n!}{z^n} \sum_{k=n}^{\infty} \binom{k}{n} z^k = \frac{n!}{z^n} \sum_{k=0}^{\infty} \binom{k}{n} z^k$$

$\left[\binom{k}{n} = 0 \text{ για } k<n \right]$

Για το δεξιό μέρος:

$$\frac{d}{dz}\left(\frac{1}{1-z}\right) = (-1)\left(-\frac{1}{(1-z)^2}\right) = \frac{1}{(1-z)^2}$$

$$\frac{d^2}{dz^2}\left(\frac{1}{1-z}\right) = (-1) \cdot \frac{-2}{(1-z)^3} = \frac{2}{(1-z)^3}$$

Επαγωγικά μπορούμε ν.δ.ο

$$\frac{d^n}{dz^n}\left(\frac{1}{1-z}\right) = \frac{n!}{(1-z)^{n+1}}$$

Επομένως $\displaystyle \frac{n!}{z^n} \sum_{k=0}^{\infty} \binom{k}{n} z^k = \frac{n!}{(1-z)^{n+1}} \Rightarrow \sum_{k=0}^{n} \binom{k}{n} z^k = \frac{z^n}{(1-z)^{n+1}}$

Άσκηση 8

$$\tilde{P}_x(z) = \frac{5-2z}{z^2-6z+8} = \frac{5-2z}{(z-2)(z-4)} = \frac{A}{z-2} + \frac{B}{z-4}$$

$$A = \left.\frac{5-2z}{z-4}\right|_{z=2} = -\frac{1}{2}, \quad B = \left.\frac{5-2z}{z-2}\right|_{z=4} = -\frac{3}{2}$$

$$\Rightarrow \tilde{P}_x(z) = -\frac{1}{2}\frac{1}{z-2} - \frac{3}{2}\frac{1}{z-4} = \frac{1}{2}\frac{1}{2(1-z/2)} + \frac{3}{2} \cdot \frac{1}{4(1-z/4)}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \frac{3}{8}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n = \sum_{n=0}^{\infty}\left[\frac{1}{4}\cdot\left(\frac{1}{2}\right)^n + \frac{3}{8}\cdot\left(\frac{1}{4}\right)^n\right] z^n$$

Επομένως $\displaystyle P(X=n) = \frac{1}{4}\cdot\left(\frac{1}{2}\right)^n + \frac{3}{8}\left(\frac{1}{4}\right)^n \quad n=0,1,\ldots$

$$P(X=n) = \frac{1}{2}\cdot\left(\frac{1}{2}\right)^n\cdot\frac{1}{2} + \frac{1}{2}\cdot\left(\frac{1}{4}\right)^n\cdot\left(\frac{3}{4}\right) =$$

$$= \frac{1}{2}P(X_1=n) + \frac{1}{2}P(X_2=n),$$ όπου $X_1 \sim Geom(1/2)$, $X_2 \sim Geom(3/4)$ (αρ. αποτυχιών)

Άσκηση 9

$$\tilde{P}_X(1) = \frac{\alpha}{1} = 1 \Rightarrow \alpha = 1$$

$$\tilde{P}_X(z) = \frac{1}{3z^2-10z+8} = \frac{1}{3(z-2)(z-\frac{4}{3})} = \frac{1}{2}\cdot\frac{1}{z-2} - \frac{1}{2}\frac{1}{z-\frac{4}{3}}$$

Προχωρώντας όμοια με την Άσκηση 8 προκύπτει

$$P(X=n) = \frac{3}{8}\left(\frac{3}{4}\right)^n - \frac{1}{4}\left(\frac{1}{2}\right)^n, \quad n=0,1,2,\ldots$$

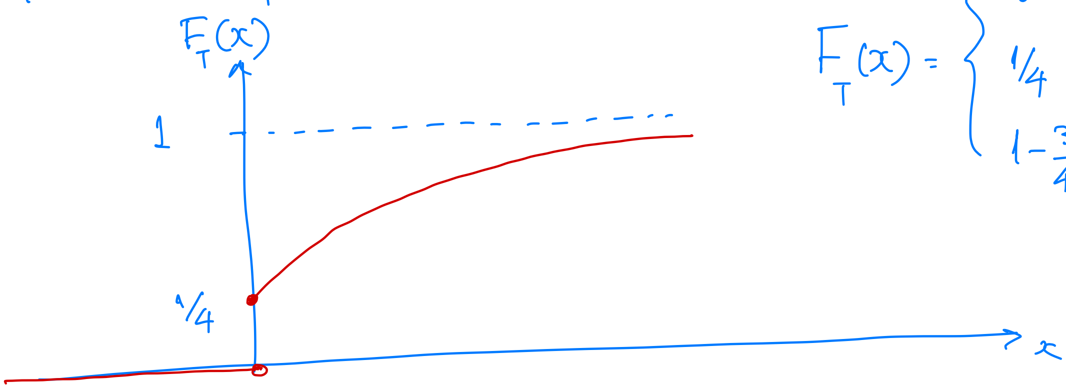
Άσκηση 10

$$P(T=0) = \frac{1}{4}, \quad P(T>0) = \frac{3}{4} \Rightarrow F_T(0) = \frac{1}{4}$$

$$P(T\le x \mid T>0) = 1-e^{-\lambda x},\ x>0, \quad P(T\le x\mid T=0) = 1,\ x>0$$

Επομένως για $x>0$: $F_T(x) = \frac{1}{4}\cdot 1 + \frac{3}{4}(1-e^{-\lambda x}) = 1-\frac{3}{4}e^{-\lambda x}$

Προφανώς $F_T(x)=0$ για $x<0$



$$F_T(x) = \begin{cases} 0, & x<0 \\ 1/4, & x=0 \\ 1-\frac{3}{4}e^{-\lambda x}, & x>0 \end{cases}$$

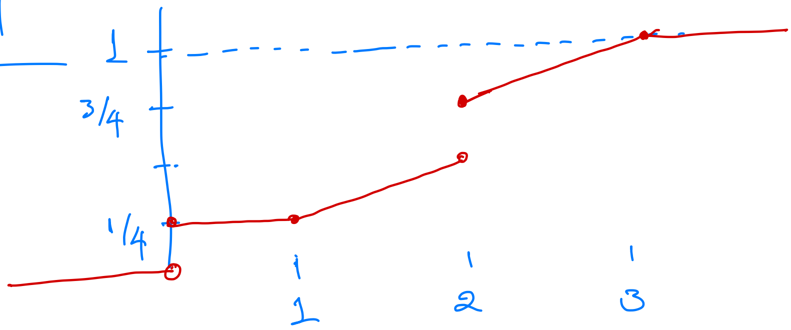
$$\mu_T = \int_0^{\infty} x\, dF_T(x) = 0 \cdot \frac{1}{4} + \int_0^{\infty} x \cdot \frac{3}{4} \lambda e^{-\lambda x} dx = \frac{3}{4\lambda}$$

Η μέση τιμή θα μπορούσε να υπολογιστεί κ' απευθείας:

$$\mu_T = E(T) = E(T/T=0) \cdot P(T=0) + E(T/T>0) \cdot P(T>0)$$

$$= 0 \cdot \frac{1}{4} + \frac{1}{\lambda} \cdot \frac{3}{4}$$

Άσκηση 11



$P(X=0) = \frac{1}{4}$, $P(X=2) = \frac{1}{4}$,

$$f_X(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1/4, & 1 \leq x < 2 \\ 1/4, & 2 < x \leq 3 \\ 0, & x \geq 3 \end{cases}$$

$$E(X) = 0 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + \int_1^2 \frac{1}{4} x\,dx + \int_2^3 \frac{1}{4} x\,dx = \frac{3}{2}$$

Άσκηση 12

$$\int \frac{s^n t^{n-1} e^{-st}}{(n-1)!} = 1$$

① $F'(t) = 1$, $t \in [0, \infty)$ $\Rightarrow$ $\tilde{F}(s) = \int_0^{\infty} e^{-st} dt = \frac{1}{s}$

② $F'(t) = n t^{n-1}$, $t \in [0, \infty)$ $\Rightarrow$ $\tilde{F}(s) = \int_0^{\infty} e^{-st} n t^{n-1} dt = n \cdot \frac{(n-1)!}{s^n} = \frac{n!}{s^n}$

③ $F'(t) = a e^{at}$, $t \geq 0$ $\Rightarrow$ $\tilde{F}(s) = \int_0^{\infty} e^{-st} a e^{at} dt = \frac{a}{s-a}$, $a < Re(s)$

④ $F(c^+) - F(c^-) = 1$, $F'(t) = 0$, $t \geq 0, t \neq c$

$$\tilde{F}(s) = \int e^{-st} dF(t) = e^{-sc}\left[F(c^+) - F(c^-)\right] + \int_0^{\infty} e^{-st} F'(t) dt = e^{-sc}$$

⑤ $F'(t) = f(t)$

$$\tilde{F}(s) = \int_0^{\infty} e^{-st} f(t) dt = -\frac{1}{s} \int_{t=0}^{\infty} f(t)\, de^{-st} = -\frac{1}{s}\left[e^{-st} f(t) \Big|_0^{\infty} - \int_0^{\infty} e^{-st} df(t) \right]$$

$$= \frac{1}{s} f(0) + \frac{1}{s} \tilde{f}(s)$$

(6) $F(t) = f'(t)$, $f(0) = 0$ $\Rightarrow f(t) = \int_0^t F(u)\,du$

$$\Rightarrow \tilde{f}(s) = \frac{F(0)}{s} + \frac{1}{s} \tilde{F}(s) \Rightarrow \tilde{F}(s) = s\tilde{f}(s) - F(0)$$

Ασκηση 13 (a) $P(Z \leq z) = P(X + Y \leq z) = \int_0^z P(Y \leq z - x) f_X(x)\,dx$

$$= \int_0^z \left(1 - e^{-\mu(z-x)}\right) \lambda e^{-\lambda x} dx = \int_0^z \lambda e^{-\lambda x} dx - \lambda e^{-\mu z} \int_0^z e^{(\mu-\lambda)x} dx$$

$$= 1 - e^{-\lambda z} - \frac{\lambda e^{-\mu z}}{\mu - \lambda}\left(e^{(\mu-\lambda)z} - 1\right) = 1 - e^{-\lambda z} - \frac{\lambda}{\mu - \lambda}\left(e^{-\lambda z} - e^{-\mu z}\right)$$

$$= 1 + \frac{\mu}{\mu - \lambda} e^{-\lambda z} - \frac{\lambda}{\mu - \lambda} e^{-\mu z} = \frac{\mu}{\mu - \lambda}\left(1 - e^{-\lambda z}\right) - \frac{\lambda}{\mu - \lambda}\left(1 - e^{-\mu z}\right)$$

(b) $X \sim Exp(\lambda) \Rightarrow \tilde{F}_X(s) = \frac{\lambda}{\lambda + s}$, $Y \sim Exp(\mu) \Rightarrow \tilde{F}_Y(s) = \frac{\mu}{\mu + s}$ $\Rightarrow \tilde{F}_Z(s) = \tilde{F}_X(s)\tilde{F}_Y(s) = \frac{\lambda\mu}{(\lambda+s)(\mu+s)}$

$$= \frac{\lambda\mu}{\mu - \lambda}\left(\frac{1}{\lambda + s} - \frac{1}{\mu + s}\right) = \frac{\mu}{\mu - \lambda}\frac{\lambda}{\lambda + s} - \frac{\lambda}{\mu - \lambda}\frac{\mu}{\mu + s} =$$

$$= \frac{\mu}{\mu - \lambda} \int_0^\infty e^{-st} dF_X(t) - \frac{\lambda}{\mu - \lambda} \int_0^\infty e^{-st} dF_Y(t) = \frac{\mu}{\mu - \lambda} \tilde{F}_X(s) - \frac{\lambda}{\mu - \lambda} \tilde{F}_Y(s)$$

Επομένως $\tilde{F}_Z(s) = \frac{\mu}{\mu - \lambda} \tilde{F}_X(s) - \frac{\lambda}{\mu - \lambda} \tilde{F}_Y(s) \Rightarrow$

$$\Rightarrow F_Z(t) = \frac{\mu}{\mu - \lambda} F_X(t) - \frac{\lambda}{\mu - \lambda} F_Y(t)$$

Ασκηση 14 $P(X < Y) = \int_0^\infty P(X < Y / X = x)\, dF_X(x)$

$$= \int_0^\infty P(Y > x)\, dF_X(x) = \int_0^\infty e^{-\lambda x} dF_X(x) = \tilde{F}_X(\lambda)$$

Άσκηση 15

$$P(X>Y+t \mid X>Y) = \int_y P(X>Y+t \mid X>Y, Y=y)\, dF_Y(y)$$

$$= \int_y P(X>y+t \mid X>y, Y=y)\, dF_Y(y) = \int_y P(X>y+t \mid X>y)\, dF_Y(y)$$

$$= \int_y P(X>t)\, dF_Y(y) = P(X>t)$$

Άσκηση 16 $F_Z(z) = P(Z\le z) = P(X\le z)\,P(Y\le z) = (1-e^{-\lambda z})(1-e^{-\mu z})$

$$= 1 - e^{-\lambda z} - e^{-\mu z} + e^{-(\lambda+\mu)z}$$

$$f_Z(z) = \lambda e^{-\lambda z} + \mu e^{-\mu z} - (\lambda+\mu)\, e^{-(\lambda+\mu)z}$$

$$\tilde{F}_Z(s) = \int_0^\infty e^{-st} f_Z(t)\, dt = \frac{\lambda}{\lambda+s} + \frac{\mu}{\mu+s} - \frac{\lambda+\mu}{\lambda+\mu+s}$$

Άσκηση 17 Έστω $T_{A_i} \sim Exp(\lambda_i)$, $T_{B_i} \sim Exp(\mu_i)$

Τότε ο χρόνος του αρχείου A: $T_A = \min(T_{A1}, \dots, T_{An})$

κ' του αρχείου B: $T_B = \min(T_{B1}, \dots, T_{Bm})$

ενώ ο συνολικός χρόνος ολοκλήρωσης: $T = \max(T_A, T_B)$

Έχουμε $T_A \sim Exp(\lambda)$, $\lambda = \lambda_1 + \dots + \lambda_n$ (ανεξάρτητα)

$T_B \sim Exp(\mu)$, $\mu = \mu_1 + \dots + \mu_m$

$T = \max(T_A, T_B)$

$$\Rightarrow F_T(t) = 1 - e^{-\lambda t} - e^{-\mu t} + (\lambda+\mu)\, e^{-(\lambda+\mu)t}$$

① $P(T_A < T_B) = \dfrac{\lambda}{\lambda+\mu}$

② $$E(T) = \int_0^\infty t f_T(t)\, dt = \int_0^\infty t \lambda e^{-\lambda t} dt + \int_0^\infty \mu e^{-\mu t} dt - \int_0^\infty (\lambda+\mu) e^{-(\lambda+\mu)t} dt$$

$$= \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda+\mu}$$