Διαδικασία Poisson: Γενικεύσεις - Επεκτάσεις

1 Βασικά σημεία

- Μη-ομογ. διαδ. Poisson

  $\lambda(t)$: Ρυθμός γεγον. εξαρτάται από το t

  $N(t)$ ανεξ. αλλά όχι ομογ. προσαυξήσεις

  $$\Lambda(t) = \int_0^t \lambda(u)\,du$$

  $$Pr[n \text{ γεγ. στο } (s, s+t]] = Pr[N(s+t) - N(s) = n] = e^{-(\Lambda(s+t)-\Lambda(s))} \frac{(\Lambda(s+t)-\Lambda(s))^n}{n!}$$

  $$N(s+t) - N(s) \sim \text{Poisson}\,(\Lambda(s+t) - \Lambda(s)) = \text{Poisson}\left(\int_s^{s+t} \lambda(u)\,du\right)$$

  Ειδική περ: $N(t) \sim \text{Poisson}\,(\Lambda(t))$.

- Σύνθετη διαδ. Poisson $\{Y(t)\} \longrightarrow \{N(t)\}$ διαδ. Poisson, $Y_1, Y_2, \ldots$ ανεξ.+ισον.

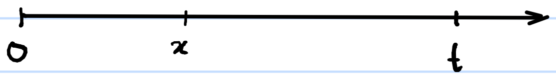
  $Y(t) = \sum_{i=1}^{N(t)} Y_i$.

② Ασκ 3.17 / Φυλ. 1

$\{N(t)\}$ σ.δ. Poisson με συνάρτηση ρυθμού $\lambda(t), t \geq 0$.

$F_{(S_1|N(t)=1)}(x) = Pr[S_1 \leq x \mid N(t)=1], \quad x \geq 0.$

Λύση:



$N(t)=1 \longrightarrow 1$ γεγ. στο $(0,t]$.

Στην συνήθη διαδ. Poisson $(\lambda(t)=\lambda,\ t \geq 0)$ $(S_1 \mid N(t)=1) \sim Unif([0,t])$

οπότε $Pr[S_1 \leq x \mid N(t)=1] = \begin{cases} \frac{x}{t}, & 0 \leq x \leq t \\ 1, & x \geq t \end{cases}$,

Στην περίπτωση μη-ομογ.: Για $0 \leq x \leq t$: $(\Longrightarrow N(x) \leq N(t))$

$$Pr[S_1 \leq x \mid N(t)=1] = Pr[N(x) \geq 1 \mid N(t)=1] = Pr[N(x)=1 \mid N(t)=1]$$

$$= \frac{Pr[N(x)=1,\ N(t)=1]}{Pr[N(t)=1]} = \frac{Pr[N(x)=1,\ N(t)-N(x)=0]}{Pr[N(t)=1]} = \frac{Pr[N(x)=1]\,Pr[N(t)-N(x)=0]}{Pr[N(t)=1]}$$

$$= \frac{\cancel{e^{-\Lambda(x)}}\,\Lambda(x)^1/1! \cdot \cancel{e^{-(\Lambda(t)-\Lambda(x))}}\,(\Lambda(t)-\Lambda(x))^0/0!}{\cancel{e^{-\Lambda(t)}} \cdot \Lambda(t)^1/1!} = \frac{\Lambda(x)}{\Lambda(t)}$$

Τελικά στην μη-ομογ. Poisson

$$F_{(S_1 | N(t)=1)}(x) = \begin{cases} \Lambda(x)/\Lambda(t), & 0 \le x \le t \\ 1, & x \ge t. \end{cases}$$

Σχόλιο: Στην μη-ομογ. διαδ. Poisson ισχύει η γενίκευση του Θ. Campbell:

$$(S_1, S_2, \dots, S_n \mid N(t)=n) \stackrel{d}{=} (Y_{1:n}, Y_{2:n}, \dots, Y_{n:n})$$

όπου $Y_{1:n}, Y_{2:n}, \dots, Y_{n:n}$ οι διατ. τ.μ. από δείγμα n ανεξ. ισον. με

$$F_{Y_i}(x) = \begin{cases} \Lambda(x)/\Lambda(t), & 0 \le x \le t \\ 1, & x \ge t \end{cases}$$

③ Ασκ. 3.18 / φυλ. 1

$\{N(t)\}$ μη-ομογ. διαδ. Poisson με $\lambda(t) = \lambda t$, $t \ge 0$. S_1: χρόνος $1^{ου}$ γεγ.

$E[N(t)] =$; $E[S_1] =$;

Λύση:

$$\Lambda(t) = \int_0^t \lambda(u)\,du = \int_0^t \lambda u\,du = \lambda\, t^2/2.$$

$N(t) \sim \text{Poisson}(\Lambda(t)) = \text{Poisson}(\lambda t^2/2)$.

Άρα $E[N(t)] = \dfrac{\lambda t^2}{2}$.

$$E[S_1] = \int_0^\infty (1 - F_{S_1}(x))\,dx \quad \left(\begin{array}{l} \text{Γενικά} \quad X \geq 0 \text{ τ.μ. με } \begin{array}{l} \text{σ.π.π. } f_X(x) \\ \text{σ.κ. } F_X(x) \end{array} \\ E[X] = \int_0^\infty x f_X(x) = \int_0^\infty (1 - F_X(x))\,dx \end{array} \right)$$

$$= \int_0^\infty \Pr[S_1 > x]\,dx$$

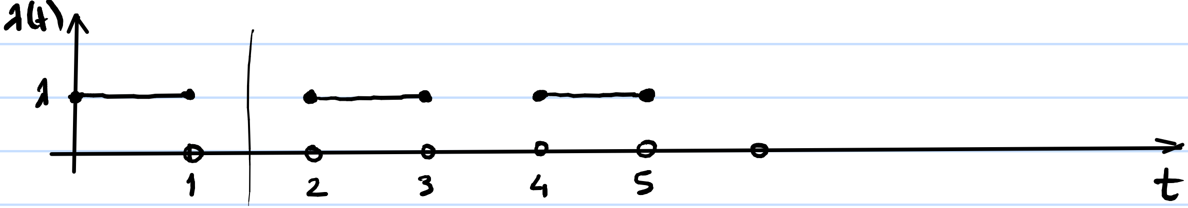
$$= \int_0^\infty \Pr[N(x) = 0]\,dx = \int_0^\infty e^{-\frac{\lambda x^2}{2}}\,dx \overset{\sqrt{\lambda}x = u}{=\!=\!=} \int_0^\infty e^{-u^2/2} \frac{1}{\sqrt{\lambda}}\,du$$

$$= \frac{1}{\sqrt{\lambda}} \sqrt{2\pi} \underbrace{\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du}_{P(Z \geq 0),\ Z \sim \mathcal{N}(0,1)} = \frac{1}{\sqrt{\lambda}} \sqrt{2\pi} \cdot \frac{1}{2}$$

$$= \sqrt{\frac{\pi}{2\lambda}}.$$

④ Ασκ. 3.13 / Φυλ. 1

$\{N(t)\}$ μη-ομογ. διαδ. Poisson με $\lambda(t)=\lambda$ για $t\in[0,1]\cup[2,3]\cup[4,5]\cup\ldots$
και $\lambda(t)=0$ για $t\in(1,2)\cup(3,4)\cup(5,6)\cup\ldots$



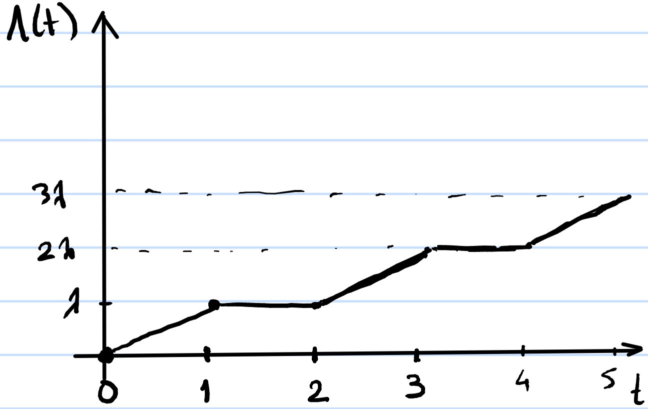
S_1 : χρόνος $1^{ου}$ γεγον.

1) $Pr[S_1\le x]=;$

2) $Pr[N(t)=n]=;$

3) $E[S_1 \mid N(t)=n]$ για $0\le t\le 2$.

Λύ:



$$\Lambda(t)=\int_0^t \lambda(u)\,du=\begin{cases}\lambda t, & t\in[0,1]\\ \lambda, & t\in[1,2]\\ \lambda+\lambda(t-2), & t\in[2,3]\\ 2\lambda, & t\in[3,4]\end{cases}$$

Άρα:

$$\Lambda(t) = \begin{cases} \lambda(t-k), & \text{αν} \quad t \in [2k, 2k+1], \; k=0,1,2,\dots \\ \lambda(k+1), & \text{αν} \quad t \in [2k+1, 2k+2], \; k=0,1,2,\dots \end{cases}$$

1) $Pr[S_1 \leq x] = Pr[N(x) \geq 1] = 1 - Pr[N(x) = 0] = 1 - e^{-\Lambda(x)}$

2) $Pr[N(t) = n] = e^{-\Lambda(t)} \dfrac{(\Lambda(t))^n}{n!}$

3) $0 \leq t \leq 1$:

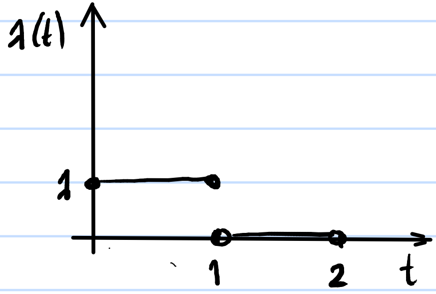
$$E[S_1 | N(t) = n] = \frac{1 \cdot t}{n+1} = \frac{t}{n+1}$$

Θ. Campbell

$1 \leq t \leq 2$:

$$E[S_1 | N(t) = n] = E[S_1 | N(1) = n] \stackrel{\cdot}{=} \frac{1}{n+1}$$

Στο $(1,t)$ ο ρυθμός είναι 0
άρα δεν γίνονται γεγονότα



⑤ Ασκ. 3.20 / φυλ. 1

$\{N(t)\}$ σ.δ. Poisson ρυθμού λ, $Z_1, Z_2, \dots$ ανεξ. ισον. ≥ 0, ακέραιες

$p_j = \Pr[Z_n = j]$, $j = 0, 1, \dots$: Συν. πιθ. των Z_n

$P_Z(z) = \sum_{j=0}^{\infty} p_j z^j$: Πιθανογ. των Z_n $\longleftarrow$

$E[Z_n] = \mu_Z$, $\quad Var[Z] = \sigma_Z^2$

$Z(t) = \sum_{i=1}^{N(t)} Z_i$: Συνθ. διαδ. Poisson

1| $E[Z(t)] = ;$ $\quad Var[Z(t)] = ;$

2| $P_{Z(t)}(z) = ;$

3| $r_k(t) = \Pr[Z(t) = k]$, $k = 0, 1, \dots$ $\longrightarrow$ Αναδρομικό σχήμα

Λύση:

2) $Z(t) = \sum_{i=1}^{N(t)} Z_i \implies P_{Z(t)}(z) = \underset{\sim \text{Poisson}(\lambda t)}{P_{N(t)}}(P_Z(z)) = \underbrace{e^{-\lambda t(1-P_Z(z))}}_{e^{-\lambda t}\cdot e^{\lambda t P_Z(z)}}$

παραγ. ως προς z

1) $E[Z(t)] = P'_{Z(t)}(1) = \underbrace{e^{-\lambda t} e^{\lambda t P_Z(z)} \cdot \lambda t P_Z'(z)}\Big|_{z=1}$

$= \cancel{e^{-\lambda t}} \cancel{e^{\lambda t P_Z(1)}}^{=1} \cdot \lambda t P_Z'(1) = \lambda t \mu_Z.$

$E[Z(t)(Z(t)-1)] = P''_{Z(t)}(1) = e^{-\lambda t}\left(e^{\lambda t P_Z(z)}(\lambda t P_Z'(z))^2 + e^{\lambda t P(z)} \cdot \lambda t P_Z''(z)\right)\Big|_{z=1}$

$= \cancel{e^{-\lambda t}}\left(\cancel{e^{\lambda t}}(\lambda t \mu_Z)^2 + \cancel{e^{\lambda t}} \cdot \lambda t\, E[Z(Z-1)]\right)$

$= (\lambda t \mu_Z)^2 + \lambda t\left(E[Z^2] - E[Z]\right)$

$= \lambda^2 t^2 \mu_Z^2 + \lambda t\left(\sigma_Z^2 + \mu_Z^2 - \mu_Z\right).$

Άρα:

$$\mathrm{Var}[Z(t)] = E[Z(t)^2] - E[Z(t)]^2$$

$$= E[Z(t)(Z(t)-1)] + E[Z(t)] - E[Z(t)]^2$$

$$= \cancel{\lambda^2 t^2 \mu_Z^2} + \lambda t(\sigma_Z^2 + \mu_Z^2 - \cancel{\mu_Z}) + \cancel{\lambda t \mu_Z} - \cancel{\lambda^2 t^2 \mu_Z^2}$$

$$= \lambda t(\sigma_Z^2 + \mu_Z^2)$$

3]

$$P_{Z(t)}(z) = e^{-\lambda t(1-P_Z(z))} = e^{-\lambda t}\, e^{\lambda t P_Z(z)} \xRightarrow{\log}$$

$$\log P_{Z(t)}(z) = -\lambda t + \lambda t P_Z(z) \xRightarrow{d/dz}$$

$$\frac{P'_{Z(t)}(z)}{P_{Z(t)}(z)} = \lambda t P_Z'(z) \Rightarrow$$

$$P'_{Z(t)}(z) = \lambda t P_Z'(z)\, P_{Z(t)}(z) \Rightarrow$$

$$\sum_{k=1}^{\infty} k \underbrace{\Pr[Z(t)=k]}_{\pi_k(t)} z^{k-1} = \lambda t \sum_{j=1}^{\infty} j P_j z^{j-1} \cdot \sum_{i=0}^{\infty} \Pr[Z(t)=i] z^i \xRightarrow{\cdot z}$$

$$\sum_{k=1}^{\infty} k\, \pi_k(t) z^k = \lambda t \sum_{j=1}^{\infty} j P_j z^j \cdot \sum_{i=0}^{\infty} \pi_i(t) z^i \xRightarrow[k=1,2,\dots]{(*)}$$

(✱: Βλ. επομ. σελ.)

$$k\,\pi_k(t) = \lambda t \sum_{i=0}^{k-1} (k-i) P_{k-i}\, \pi_i(t) \Rightarrow \pi_k(t) = \frac{\lambda t}{k} \sum_{i=0}^{k-1} (k-i) P_{k-i}\, \pi_i(t)$$

Για την αρχική τιμή:

$$r_0(t) = P_{Z(t)}(0) = e^{-\lambda t(1-P_Z(0))} = e^{-\lambda t(1-p_0)}$$

Άρα το αναδρομικό σχήμα είναι:

$$r_0(t) = e^{-\lambda t(1-p_0)}$$

$$r_k(t) = \frac{\lambda t}{k} \sum_{i=0}^{k-1} (k-i) p_{k-i}\, r_i(t), \quad k = 1, 2, \ldots$$

(*): Χρησιμοποιήσαμε το:

Αν $A(z) = \sum_{k=0}^{\infty} a_k z^k$, $B(z) = \sum_{k=0}^{\infty} b_k z^k$, $C(z) = \sum_{k=0}^{\infty} c_k z^k$.

Τότε $A(z) = B(z) \cdot C(z) \Rightarrow a_k = \sum_{i=0}^{k} b_{k-i} c_i, \quad k = 0, 1, \ldots$

Διότι:

$$a_0 + a_1 z + a_2 z^2 + a_3 z^2 + \dots \quad \longleftarrow A(z)$$

$$= (b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \dots) \quad \longleftarrow B(z)$$

$$\bullet\, (c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \dots) \quad \longleftarrow C(z)$$

$$\Longleftrightarrow \quad \begin{aligned} a_0 &= b_0 c_0 \\ a_1 &= b_0 c_1 + b_1 c_0 \\ a_2 &= b_0 c_2 + b_1 c_1 + b_2 c_0 \\ &\vdots \end{aligned}$$