Βασικοί υπολογισμοί ανανεωτικής θεωρίας (ανανεωτική συνάρτηση)

① Ασκ. 2.1 /Φυλ.2

$\{N(t)\}$ αναν. διαδ. με σ.π.π. ενδιαμ. χρόνων

$\sim f_X(t) = p\lambda e^{-\lambda t} + (1-p)\lambda^2 t e^{-\lambda t}, \ t \geq 0 \qquad p \in (0,1), \ \lambda > 0$

$\tilde{F}_{S_k}(s) = ;\quad \tilde{P}_k(s) = ;\quad \tilde{m}(s) = ;\quad , m(t).$

Λύση:

$$\text{Ενδιάμεσος χρόνος} = \begin{cases} Y \sim \mathcal{E}xp(\lambda) & \text{με πιθ. } p \\ Z \sim \mathcal{E}rlang(2,\lambda) & \text{με πιθ. } 1-p. \end{cases}$$

$$\tilde{F}_X(s) = p \cdot \frac{\lambda}{\lambda+s} + (1-p)\left(\frac{\lambda}{\lambda+s}\right)^2 = \frac{p\lambda(\lambda+s)+(1-p)\lambda^2}{(s+\lambda)^2} = \frac{p\lambda \cdot s + \lambda^2}{(s+\lambda)^2}$$

$$\tilde{F}_{S_k}(s) = \left(\frac{p\lambda \cdot s + \lambda^2}{(s+\lambda)^2}\right)^k, \quad \tilde{P}_k(s) = \left(1-\tilde{F}_X(s)\right)\left(\tilde{F}_X(s)\right)^k = \left(1 - \frac{p\lambda s + \lambda^2}{(s+\lambda)^2}\right)\left(\frac{p\lambda s + \lambda^2}{(s+\lambda)^2}\right)^k$$

$$\tilde{m}(s) = \frac{\tilde{F}(s)}{1-\tilde{F}(s)} = \frac{\frac{p\lambda s+\lambda^2}{(s+\lambda)^2}}{1-\frac{p\lambda s+\lambda^2}{(s+\lambda)^2}} = \frac{p\lambda s+\lambda^2}{(s+\lambda)^2 - p\lambda s - \lambda^2} = \frac{p\lambda s+\lambda^2}{s^2+2\lambda s+\cancel{\lambda^2}-p\lambda s-\cancel{\lambda^2}}$$

$$= \frac{p\lambda s+\lambda^2}{s(s+\lambda(2-p))} = \frac{A}{s} + \frac{B}{s+\lambda(2-p)} = \frac{A}{s} + \frac{B}{\lambda(2-p)} \cdot \frac{\lambda(2-p)}{s+\lambda(2-p)}$$

$$\xRightarrow[s=0]{\times s} \quad A = \frac{\lambda^2}{\lambda(2-p)} = \frac{\lambda}{2-p}$$

$$\xRightarrow[s=-\lambda(2-p)]{\times(s+\lambda(2-p))} \quad B = \frac{-p\lambda^{\cancel{2}}(2-p)+\lambda^{\cancel{2}}}{-\cancel{\lambda}(2-p)} = \frac{\lambda(2p-p^2-1)}{2-p} = -\frac{\lambda(1-p)^2}{2-p}$$

Άρα

$$m(t) = \frac{\lambda}{2-p}\cdot t - \frac{(1-p)^2}{(2-p)^2}\left(1-e^{-\lambda(2-p)t}\right), \quad t \geq 0.$$

② Ασκ. 2.2 / Φυλ. 2

$\{N(t)\}$ αναν. διαδικ. με σ.κ. ενδιαμ. χρονων

$$F_X(t) = \begin{cases} 1 - e^{-\lambda(t-2)} & , \ t \geq 2 \\ 0 & , \ 0 \leq t < 2 \end{cases} \quad , \ \lambda > 0$$

$\tilde{F}_{S_k}(s) = ;$ $\tilde{P}_k(s) = ;$ $\tilde{m}(s) = ;$ $m(t) = ;$

Λύση:

$X = 2 + \mathcal{E}xp(\lambda)$.

$$\tilde{F}_X(s) = \int_0^\infty e^{-st}\, dF_X(t) = \int_0^\infty e^{-st} F_X'(t)\, dt = \int_2^\infty e^{-st} \lambda e^{-\lambda(\overbrace{t-2}^{u})} dt$$

$$= \int_0^\infty e^{-s(u+2)} \lambda e^{-\lambda u} du = e^{-2s} \int_0^\infty e^{-su} \lambda e^{-\lambda u} du = \underbrace{e^{-2s}}_{\text{μετ. L-S της σταθερής} = 2} \cdot \underbrace{\frac{\lambda}{\lambda + s}}_{\text{μετ. L-S } \mathcal{E}xp(\lambda)}$$

$$\tilde{F}_{S_k}(s) = (\tilde{F}_X(s))^k = e^{-2ks} \cdot \left(\frac{\lambda}{\lambda+s}\right)^k$$

$$\tilde{P}_k(s) = (1 - \tilde{F}_X(s))(\tilde{F}_X(s))^k = \left(1 - e^{-2s}\frac{\lambda}{\lambda+s}\right) e^{-2ks}\left(\frac{\lambda}{\lambda+s}\right)^k$$

$$\tilde{m}(s) = \frac{\tilde{F}_X(s)}{1-\tilde{F}_X(s)} = \frac{e^{-2s}\frac{\lambda}{\lambda+s}}{1-e^{-2s}\frac{\lambda}{\lambda+s}} = \frac{\lambda e^{-2s}}{\lambda+s-\lambda e^{-2s}}.$$

$$m(t) = \sum_{k=1}^{\infty} F_X^{*k}(t)$$

Η σ.κ. $F_{S_k}(t) = F_X^{*k}(t)$ αντιστοιχεί σε τ.μ. $\sim 2k + Erlang(k, \lambda)$

Άρα $F_{S_k}(t) = \begin{cases} \text{σ.κ. ως } Erlang(k,\lambda) \text{ στο } t-2k, & t \geq 2k \\ 0, & 0 \leq t < 2k \end{cases}$

$\parallel$

$Pr[S_k \leq t] = Pr[2k + Erlang(k,\lambda) \leq t] = Pr[Erlang(k,\lambda) \leq t-2k]$

Υπενθύμιση: σ.π.π. $Erlang(n,\mu)$: $\frac{\mu^n}{(n-1)!} t^{n-1} e^{-\mu t}$, $t \geq 0$

σ.κ. —″— : $1 - \sum_{j=0}^{n-1} e^{-\mu t} \frac{(\mu t)^j}{j!}$, $t \geq 0$

Άρα $$F_X^{*k}(t) = F_{S_k}(t) = \begin{cases} 1 - \sum_{j=0}^{k-1} e^{-\lambda(t-2k)} \cdot (\lambda(t-2k))^j / j!\,, & t \geq 2k \\ 0 & 0 \leq t < 2k \end{cases}$$

$$= \begin{cases} \int_0^{t-2k} \frac{\lambda^k}{(k-1)!} u^{k-1} e^{-\lambda u}\, du & , \ t \geq 2k \\ 0, & , \ 0 \leq t < 2k \end{cases}$$

$$m(t) = \sum_{k=1}^{\infty} F_X^{*k}(t) = \sum_{k=1}^{\lfloor t/2 \rfloor} \left(1 - \sum_{j=0}^{k-1} e^{-\lambda(t-2k)} \cdot \frac{(\lambda(t-2k))^j}{j!} \right)$$

$$F_X^{*k}(t) = \begin{cases} * & , \quad t \geq 2k \\ 0 & , \quad 0 \leq t < 2k. \end{cases}$$ ← Για $k > t/2$ τότε ο όρος είναι 0

$$\sum_{k=1}^{\infty} F_X^{*k}(t) = \sum_{k=1}^{\lfloor t/2 \rfloor} * + \cancel{\sum_{k=\lfloor t/2 \rfloor + 1}^{\infty} 0}$$